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The Convergence of Zeta Functions for Certain Geodesic Flows Depends on Their Pressure*

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The Selberg Trace Formula connects the spectrum of the Laplace operator Δ on a compact Riemann surface M to the length spectrum of closed geodesics on M [11], [17]. Selberg defined a zeta function which is associated to this trace formula and gives information on the length spectrum of M. A general trace formula for compact manifolds of variable negative curvature has been obtained by Colin de Verdière [5] and Duistermaat-Guillemin [7]. Millson defined a zeta function in [14] to discuss the η-invariant of a $(4n-1)$-dimensional compact manifold of constant negative curvature. He extended $Z(s)$ from its half-plane of convergence to a meromorphic function on the whole plane. The trace formula in [7] and the definition of $Z(s)$ in [14] both weight the closed geodesics according to the strength of the hyperbolicity of their Poincaré return maps.

A function measuring strength of hyperbolicity has been used by Sinai [19] and Bowen-Ruelle [3] in studying equilibrium states for Anosov and Axiom A flows. In this paper we use their approach to calculate the precise domain of convergence of a zeta function of Millson's type for manifolds of variable negative curvature. It turns out that this domain is given by the pressure of half Sinai's function. This answer reduces in Millson's case to half the entropy of the geodesic flow. We do not know whether Millson's technique of obtaining a functional equation for $Z(s)$ and extending Z to a meromorphic function can be applied in the case of variable negative curvature.

Let M be a compact Riemannian manifold of dimension n with all sectional curvatures negative and let $\mathscr{P}$ be the set of its primitive closed geodesics. Define a zeta function by

$$\log Z(s) = \sum_{\gamma\in\mathscr{P}} \sum_{k=1}^{\infty} \frac{\exp - ks\,L(\gamma)}{k\,|\det(I-p(\gamma)^k)|^{\frac{1}{2}}}.$$

Here $p(\gamma)$ is the Poincaré map of the geodesic flow ϕ around the closed orbit γ and $L(\gamma)$ is the period. It is well-known [1] that ϕ is an Anosov flow on the unit

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tangent bundle $T_1 M$ and $T(T_1 M)$ has invariant $(n-1)$-dimensional subbundles E^s and E^u that are respectively contracted and expanded by the flow. Following [19] and [3] we define $\alpha: T_1 M \to \mathbb{R}$ by

$$\alpha(v) = -\frac{d}{dt} \log \det (D\phi_t \mid E^u)|_{t=0}.$$

This is minus the instantaneous rate of expansion at v.

For a flow $\psi = (\psi_t)_{t\in\mathbb{R}}$ on a metric space (N, d) we define the *pressure* $P(\beta)$ of a function $\beta: N \to \mathbb{R}$ as follows [16], [21]. For large T and small $\delta > 0$ a finite set $Y \subset N$ is said to be (T, δ)-separated if, given $y, y' \in Y$, $y \neq y'$, there is $t \in [0, T]$ with $d(\psi_t y, \psi_t y') \geqq \delta$. Then

$$P(\beta) = \lim_{\delta\to 0} \limsup_{T\to\infty} T^{-1} \log \sup \left\{ \sum_{y\in Y} \exp \int_0^T \beta(\psi_s y)\, ds;\ Y \text{ is } (T, \delta)\text{-separated} \right\}.$$

Thus the pressure of a function, a concept that came into Dynamical Systems from Statistical Mechanics, measures the growth rate of the number of separated orbits weighted according to the value of the function along them.

Theorem. *$Z(s)$ is absolutely convergent if and only if*

$$\mathrm{Re}(s) > P(\tfrac{1}{2}\alpha).$$

Remarks 1. In Millson's zeta function the γ-term in the series is multiplied by a coefficient depending on a representation of the compact group $SO(\dim M - 1)$ and the parallel transport around γ. Any continuous representation of such a compact group will not affect the domain of convergence.

2. The exponent of $\frac{1}{2}$ in the denominator of Z corresponds to the $\frac{1}{2}$ in $P(\frac{1}{2}\alpha)$ and it will be apparent from the proof that if Z were defined with some other exponent t then the domain of convergence would be $\mathrm{Re}(s) > P(t\alpha)$. The zeta function introduced by Smale [20, p. 801] for Axiom A flows corresponds to the case of exponent 0. This was shown by Chen in the case of the geodesic flow for negative curvature [4], [12] to have exponent of convergence equal to the topological entropy, which is $P(0)$.

3. Smale's zeta function was considered by Ruelle [15] and was shown by Gallavotti [9] not to be analytic in general. It is possible that a better definition for an Axiom A flow would weight the terms by some factor $|\det(I - p(\gamma)^k)|^{-t}$ as in Z above. The convergence would again be given by $P(t\alpha)$. However, this idea would only be useful if Z could be shown to have a meromorphic continuation. Here we do not have the half-densities of [7] to suggest $t = \frac{1}{2}$. We might guess that the $E^u(E^s)$ factor should appear with an exponent the Hausdorff dimension of the intersection of the non-wandering set with local strong unstable (stable) manifolds.

Proof of Theorem. First note that the double sum over positive integers k and primitive closed geodesics γ can be considered as a sum over the set Q of *all* closed geodesics

$$\log Z(s) = \sum_{\gamma\in Q} |\det(I - p(\gamma))|^{-\frac{1}{2}} k(\gamma)^{-1} \exp - sL(\gamma)$$

where $k(\gamma)$ is the positive integer for which γ is $k(\gamma)$ times a primitive closed geodesic. For absolute convergence it suffices to consider s real.

Now consider $\det(I-p(\gamma))$. If $p(\gamma)$ has expanding eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ and contracting eigenvalues $\lambda_n, \ldots, \lambda_{2n-2}$ then

$$|\det(I-p(\gamma))| = \prod_{1}^{2n-2} |1-\lambda_i| = \prod_{i=1}^{n-1} |\lambda_i| \cdot \prod_{j=1}^{n-1} |1-\lambda_j^{-1}| \cdot \prod_{k=n}^{2n-2} |1-\lambda_k|.$$

By the hyperbolicity properties of the Anosov flow $\lambda_1^{-1}, \ldots, \lambda_{n-1}^{-1}, \lambda_n, \ldots, \lambda_{2n-2}$ are smaller than $\exp - bL(\gamma)$ for some $b>0$. Thus the ratio of $|\det(I-p(\gamma))|$ to $\prod_{i=1}^{n-1} |\lambda_i|$ tends to 1 as $L(\gamma)\to\infty$ and for purposes of convergence we may replace the first expression by the second in the definition of $Z(s)$.

This product of expanding eigenvalues is the coefficient of expansion of volume in E^u, namely $\exp - \int_\gamma \alpha$, since the function $-\alpha$ is the rate of expansion of volume in E^u (called α_p by Sinai [19, §8] and $-\phi^{(u)}$ by Bowen-Ruelle [3]). Now the summand in Z corresponding to γ is

$$k(\gamma)^{-1} \exp - sL(\gamma) \exp \tfrac{1}{2} \int_\gamma \alpha = k(\gamma)^{-1} \exp \int_\gamma (\tfrac{1}{2}\alpha - s).$$

We break up our summation using a small number ε as follows:

$$\sum_{\gamma\in Q} k(\gamma)^{-1} \exp \int_\gamma (\tfrac{1}{2}\alpha - s) = \sum_{r=0}^{\infty} a_r$$

where

$$a_r = \sum_{r\varepsilon - \frac{1}{2}\varepsilon \leqq L(\gamma) < r\varepsilon + \frac{1}{2}\varepsilon} k(\gamma)^{-1} \exp \int_\gamma (\tfrac{1}{2}\alpha - s).$$

By Lemma 2.8 of [8], $a_r r\varepsilon \exp - r\varepsilon P(\frac{1}{2}\alpha - s)$ lies between two positive constants. (See also Lemma 4 of [2] for this idea that the separated set of periodic orbits is adequate for the definition of pressure.) Now the series converges if and only if $P(\frac{1}{2}\alpha - s) < 0$. By Theorem 2.1 (vii) of [21], $P(\frac{1}{2}\alpha - s) = P(\frac{1}{2}\alpha) - s$ so the series converges absolutely if and only if

$$\mathrm{Re}(s) > P(\tfrac{1}{2}\alpha)$$

as required.

Remarks 4. Pressure is a convex function (Theorem 2.1 of [21]) so $P(\frac{1}{2}\alpha) \leqq \frac{1}{2}(P(0) + P(\alpha))$ and $P(0)$ is the topological entropy $h(\phi)$ of the flow while $P(\alpha)=0$ (with the smooth measure as equilibrium state) by [3]. Thus $P(\frac{1}{2}\alpha) \leqq \frac{1}{2} h(\phi)$. In the special case of constant curvature α is constant so the functions 0 and α have the same equilibrium state (the smooth measure) and $P(\frac{1}{2}\alpha) = \frac{1}{2} h(\phi)$. This agrees with Sinai's formula [18], [13] giving the topological and measure entropy as $(\dim M - 1) K$ when the curvature is constant at $-K^2$ applied to Millson's remark [14, p. 2] that his zeta function converges for $\mathrm{Re}(s) > \frac{1}{2}(\dim M - 1)$ where the implicit assumption is that the curvature is -1.

5. It would be nice to know that the exponent of convergence $P(\frac{1}{2}\alpha)$ is extremal for a metric of constant curvature among Riemannian metrics of negative

curvature on M with the same volume. For surfaces Katok has shown [10] that the topological entropy increases and the measure entropy decreases as we perturb our metric away from constant curvature. The behaviour of $P(\frac{1}{2}\alpha)$ is not clear to us nor is the remark [14, p. 34] that suggests $P(\frac{1}{2}\alpha) \leqq \frac{1}{2}(\dim M - 1)$.

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