Math. Z. 176, 311–318 (1981)

Mathematische Zeitschrift


# The Peripheral Point Spectrum of Schwarz Operators on $C^*$-Algebras

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The spectral theory of positive operators has its origin in the classical work of Perron and Frobenius, who considered matrices with positive entries on finite-dimensional vector spaces and obtained results on the cyclic structure of the set of eigenvalues of such operators.

Frobenius' main result rests on the concept of irreducibility, which is usually defined in matrix terms. But it is easy to see that this concept involves the ideals of the commutative $C^*$-algebra $\mathbb{C}^n$.

In this paper we extend these results to the case of positive operators on arbitrary $C^*$-algebras. The extensions to commutative $C^*$-algebras has been studied, among many others, by Schaefer [10].

We shall now briefly discuss the content of this paper. In Sect. 1 and 2 we introduce the basic definitions (e.g., irreducibility) and present the classical results of Krein-Rutman on the peripheral point spectrum of the adjoint of a positive operator in the context of $C^*$-algebras.

In Sect. 3 we consider a $C^*$-algebra $\mathfrak{A}$ and a linear operator $T$ on $\mathfrak{A}$ satisfying $T(x^*x) \geqq T(x)^* T(x)$ for all $x \in \mathfrak{A}$ and $T(\mathbb{1}) = \mathbb{1}$. We show that the results of Frobenius can be extended to irreducible operators of that type. We also study the action of $T$ on the *-algebra generated by the eigenvectors pertaining to the peripheral eigenvalues of $T$.

## 1. Notation

By $\mathfrak{A}$ we shall denote a $C^*$-algebra with unit $\mathbb{1}$. $\mathfrak{A}^{sa} := \{x \in \mathfrak{A} \mid x^* = x\}$ is the *self-adjoint* part of $\mathfrak{A}$ and $\mathfrak{A}_+ := \{x^* x \mid x \in \mathfrak{A}\}$ the *positive cone* in $\mathfrak{A}$. If $\mathfrak{A}^*$ is the *dual* of $\mathfrak{A}$, then $\mathfrak{A}_+^* := \{\varphi \in \mathfrak{A} \mid \varphi(x) \geqq 0$ for all $x \in \mathfrak{A}_+\}$ is a weakly closed generating cone in $\mathfrak{A}^*$. $\mathfrak{S}(\mathfrak{A}) := \{\varphi \in \mathfrak{A}_+^* \mid \varphi(\mathbb{1}) = \mathbb{1}\}$ is called the *state space* of $\mathfrak{A}$.

A *face* $\mathfrak{F}$ in $\mathfrak{A}_+$ is a subcone of $\mathfrak{A}_+$ such that for all $x \in \mathfrak{A}_+$, the conditions $x \leqq y$ and $y \in \mathfrak{F}$ imply that $x \in \mathfrak{F}$. We call a subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ *solid* if $\mathfrak{M} \cap \mathfrak{A}_+$ is a face.

0025-5874/81/0176/0311/$01.60

For a face $\mathfrak{F}$ in $\mathfrak{A}_+$ we have:

1. $\mathfrak{M} := \mathrm{lin}_{\mathbb{C}}(\mathfrak{F} - \mathfrak{F})$ is a solid *-subalgebra satisfying $\mathfrak{M} \cap \mathfrak{A}_+ = \mathfrak{F}$.
2. $\mathfrak{I} := \{x \in \mathfrak{A} \mid x^* x \in \mathfrak{F}\}$ is a left ideal of $\mathfrak{A}$ which is closed iff $\mathfrak{F}$ is closed.
3. $\mathfrak{I}^* \cdot \mathfrak{I} = \mathfrak{M}$.
4. $(\overline{\mathfrak{I}^*}) \cap \overline{\mathfrak{I}} = \overline{\mathfrak{M}}$, where the bar denotes the norm closure.

For these facts see, e.g., [4, 5, 9].

If $x \in \mathfrak{A}_+$ and $[0, x] := \{y \in \mathfrak{A}_+ \mid 0 \leqq y \leqq x\}$ then $\mathfrak{F}_x := \bigcup_{n \in \mathbb{N}} n[0, x]$ is a face in $\mathfrak{A}_+$ and therefore the norm closure $\overline{\mathfrak{F}}_x$ ([3] 2.3).

$T \in \mathscr{L}(\mathfrak{A})$ is called *positive*, if $T(\mathfrak{A}_+) \subset \mathfrak{A}_+$. For such an operator one has $T(x^*) = T(x)^*$ for all $x \in \mathfrak{A}$. $\sigma(T)$ is the *spectrum* of $T$ and $r(T)$ the *spectral radius*. The subset of $\sigma(T)$ located on the spectral-circle $\{\lambda \in \mathbb{C} \mid |\lambda| = r(T)\}$ will be called the *peripheral spectrum* and its intersection with the point spectrum the *peripheral point spectrum*. For $\lambda$ in the resolvent set $\rho(T)$ of $T$ we set $(\lambda - T)^{-1} = R(\lambda, T)$. For $T$ positive and $\lambda > r(T)$ one has $R(\lambda, T)$ positive as one can see using the C. Neumann's series. For a brief introduction to spectral theory of positive operators see [11], Appendix.

## 2. Positive Operators

In this section we collect some more or less well known spectral properties of positive operators on $C^*$-algebras. First let us note that $r(T) \in \sigma(T)$ for such operators ([11], Appendix 2.2). In particular, the following property follows from the famous Krein-Rutman theorem ([10], Satz A).

2.1 **Eigenvalue Theorem.** *Let $T$ be a positive operator, with spectral radius $r(T)$, on a $C^*$-algebra $\mathfrak{A}$. Then $r(T)$ is an eigenvalue of the adjoint $T^*$ with an eigenvector $\varphi \in \mathfrak{S}(\mathfrak{A})$.*

*Proof.* Since $r(T) \in \sigma(T)$ and the dual cone is generating, there exists a linear functional $\psi > 0$, such that $R(\lambda, T^*)(\psi)$ is not bounded as $\lambda \searrow r(T)$. Define

$$\psi_\lambda := \|R(\lambda, T^*)\psi\|^{-1} R(\lambda, T^*)(\psi)$$

for $\lambda > r(T)$. Then $\psi_\lambda$ is a state on $\mathfrak{A}$ for all $\lambda > r(T)$ and the directed family $(\psi_\lambda)_{\lambda > r(T)}$ has a $\sigma(\mathfrak{A}^*, \mathfrak{A})$ accumulation point $\varphi \in \mathfrak{S}(\mathfrak{A})$. On the other hand

$$\lim_{\lambda \to r(T)} \|(\lambda - T^*)(\psi_\lambda)\| = 0$$

hence $T^*(\varphi) = r(T)\varphi$, since $T^*$ is $\sigma(\mathfrak{A}^*, \mathfrak{A})$-continuous. □

2.2 **Definition.** Let $T$ be a positive operator on $\mathfrak{A}$. We call $T$ *irreducible* if no closed, solid *-subalgebra of $\mathfrak{A}$, distinct from $\{0\}$ and $\mathfrak{A}$, is invariant under $T$.

*Remarks.* 1. $T$ is irreducible iff no closed face of $\mathfrak{A}_+$, different from $\{0\}$ and $\mathfrak{A}_+$, is invariant under $T$ (see Sect. 1).

2. The Definition 2.2 is in accord with the concept of irreducibility given by Frobenius [6] (see also [13]) for positive matrices and by Schaefer [10] for positive operators on commutative $C^*$-algebras.

3. Let $\varphi \in \mathfrak{S}(\mathfrak{A})$ with $T^*(\varphi)=r(T)\varphi$. Then

$$\mathfrak{F} := \{x \in \mathfrak{A}_+ \mid \varphi(x)=0\}$$

is a closed face of $\mathfrak{A}_+$ which is $T$-invariant. Therefore, if $T$ is irreducible, such a linear form is faithful.

4. The results of Frobenius cannot be extended to non irreducible positive operators, even in the finite-dimensional case. To see this one has only to consider a matrix algebra and a *-automorphism $T$ given by a unitary $u$. Then

$$\sigma(T)=\{\lambda_1^* \cdot \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(u)\},$$

hence in general $\sigma(T)$ is not cyclic (see e.g. [12], I.2.6).

2.3 **Proposition.** *If $T$ is an irreducible positive operator on a $C^*$-algebra $\mathfrak{A}$ one has $r(T)>0$, and $\|T\|=r(T)$ is equivalent with $T(\mathbb{1})=r(T)\mathbb{1}$.*

*Proof.* If $0<\varphi \in \mathfrak{A}^*$ with $T^*(\varphi)=r(T)\varphi$ one has $\varphi$ faithful, since $T$ is irreducible. $r(T)\varphi(\mathbb{1})=\varphi(T(\mathbb{1}))$ implies that $r(T)>0$, for otherwise $\|T\|=\|T(\mathbb{1})\|=0$ since $\varphi$ is faithful. The rest of the proposition is obvious because of the faithfulness of $\varphi$. □

## 3. The Peripheral Point Spectrum of Schwarz Operators

In order to obtain a generalization of the results of Frobenius, we study operators $T \in \mathscr{L}(\mathfrak{A})$, $\mathfrak{A}$ a $C^*$-algebra, satisfying

$$\|T\|\, T(x^* x) \geqq T(x)^* T(x) \qquad \text{for all } x \in \mathfrak{A}.$$

Such an operator, which is automatically positive, we call a *Schwarz operator.* Every positive map on a commutative $C^*$-algebra satisfies the above inequality ([3], I.4.2.).

3.1 **Theorem.** *Let $\mathfrak{A}$ be a $C^*$-algebra and suppose that $T$ is an irreducible Schwarz operator on $\mathfrak{A}$ satisfying $T(\mathbb{1})=\mathbb{1}$. Then the following assertions hold:*

1. *The fixed space of $T$ is one-dimensional.*
2. *The peripheral point spectrum of $T$ is a subgroup of the circle group.*
3. *Each peripheral eigenvalue $\alpha$ of $T$ is simple and $\sigma(T)=\alpha \cdot \sigma(T)$.*
4. *1 is the unique eigenvalue of $T$ with a positive eigenvector.*

*Proof.* For $x, y \in \mathfrak{A}$ define

$$B(x, y) := T(x^* y) - T(x)^* T(y).$$

Then $B(\cdot, \cdot)$ is a sesquilinear, positive map of $\mathfrak{A} \times \mathfrak{A}$ in $\mathfrak{A}$ satisfying

$$B(x, x)=0 \quad \text{iff } B(x, y)=0 \text{ for all } y \in \mathfrak{A}.$$

To prove this one has only to note that for $\psi \in \mathfrak{A}_+^*$ $\psi \circ B$ is a positive, hermitian sesquilinear form on $\mathfrak{A}$. Hence $\psi(B(x, x))=0$ iff $\psi(B(x, y))=0$ for all $y \in \mathfrak{A}$. Since $\mathfrak{A}_+^*$ is generating the assertion follows.

By 2.1 there exists a state $\varphi \in \mathfrak{A}^*$ with $T^*(\varphi)=\varphi$ and we have by remark 3 of Sect. 2 that $\varphi$ is faithful.

1. Choose $x \in \mathrm{Fix}(T)$, the fixed space of $T$. We have, since $T$ is real, $T(x^*) = T(x)^* = x^*$, hence $x^* \in \mathrm{Fix}(T)$ and $T(x^* x) \geqq T(x)^* T(x) = x^* x$. By the faithfulness of $\varphi$ we obtain from

$$0 \leqq \varphi(T(x^* x) - x^* x) = \varphi(x^* x) - \varphi(x^* x) = 0$$

that $T(x^* x) = x^* x$. Consequently $B(x, x)=0$ and therefore $B(x, y)=0$ for all $y \in \mathfrak{A}$, in particular for all $y \in \mathrm{Fix}(T)$. This implies $T(x^* y) = x^* y$ and $\mathrm{Fix}(T)$ is a $*$-subalgebra of $\mathfrak{A}$ containing $\mathbb{1}$.

Next choose $0 \leqq x \in \mathrm{Fix}(T)$ with $\|x\|=1$. Then $0 \notin \sigma(x)$; for if $0 \in \sigma(x)$ there exists $0 < \psi \in \mathfrak{A}^*$ such that $\psi(x)=0$. For the closed face $\mathfrak{F}_x$ generated by $x$ we have $T(\mathfrak{F}_x) \subseteq \mathfrak{F}_x$ and $\psi(\mathfrak{F}_x)=\{0\}$. Since $T$ is irreducible we have $\mathfrak{F}_x = \mathfrak{A}_+$ whence $\psi=0$, a contradiction.

If $\mathbb{1} \neq x$ then $0 < y := \mathbb{1} - x \in \mathrm{Fix}(T)$ and the above consideration implies $0 \notin \sigma(y)$. Hence there exists $n \in \mathbb{N}$ with $\mathbb{1} \leqq n \cdot y = n\mathbb{1} - nx$ or $x \leqq \left(\frac{n-1}{n}\right) \mathbb{1}$ which implies $\|x\| < 1$, a contradiction. Therefore $x = \mathbb{1}$ and, since $\mathrm{Fix}(T)_+$ generates $\mathrm{Fix}(T)$, the fixed space is one-dimensional.

2. Let $\alpha$ be a peripheral eigenvalue with normalized eigenvector $x_\alpha$. Then $T(x_\alpha^* x_\alpha) \geqq x_\alpha^* x_\alpha$. Since $\varphi$ is faithful we have $x_\alpha^* x_\alpha \in \mathrm{Fix}(T)$ with $\|x_\alpha^* x_\alpha\|=1$. Therefore $x_\alpha^* x_\alpha = \mathbb{1}$ and $x_\alpha$ is unitary. Since $B(x_\alpha, x_\alpha)=0$ we have $B(x_\alpha, y)=0$ for all $y \in \mathfrak{A}$. Hence if $\beta$ is a peripheral eigenvalue with normalized eigenvector $x_\beta$, we have $T(x_\alpha^* x_\beta) = \alpha^* \beta x_\alpha^* x_\beta$. Since $0 \neq x_\alpha^* x_\beta$ as a product of unitaries, we obtain that $\alpha^* \beta$ is a peripheral eigenvalue.

3. If $\alpha$ is a peripheral eigenvalue with unitary eigenvector $x_\alpha$ then we have $T(x_\alpha y) = \alpha x_\alpha T(y)$ for all $y \in \mathfrak{A}$, or $\alpha T(y) = x_\alpha^* T(x_\alpha y)$. Hence in terms of the isometry $M$: $y \to x_\alpha y$ we obtain $\alpha T = M^{-1} \circ T \circ M$. The assertion is now clear, since $\sigma(T) = \sigma(M^{-1} \circ T \circ M)$ and $\sigma(\alpha T) = \alpha \sigma(T)$.

4. Suppose that $\alpha x = T(x)$ for some $x > 0$. Since $\varphi$ is faithful, we obtain

$$\alpha \varphi(x) = \varphi(T(x)) = \varphi(x) > 0$$

whence $\alpha = 1$. □

*Example.* Consider the left regular representation $\lambda$ of a discrete group $\mathsf{G}$ on $l^2(\mathsf{G})$ ([4] 13.1.6.). If $\delta_t$, $t \in \mathsf{G}$, denotes the canonical unit vectors in $l^2(\mathsf{G})$, we have

$$\lambda(s)(\delta_t) = \delta_{st} \quad \text{for every } s, t \in \mathsf{G}.$$

$\mathfrak{A}$ denotes the $C^*$-algebra given by the norm closure of $\mathrm{lin}\{\lambda(s) | s \in \mathsf{G}\}$ in $\mathscr{L}(l^2(\mathsf{G}))$.

Let $\varphi$ be the state on $\mathfrak{A}$ given by $\varphi(x):=(x(\delta_e)|\delta_e)$ where $(\cdot|\cdot)$ is the scalar product on $l^2(\mathsf{G})$ and $e\in\mathsf{G}$ the unit. It is not difficult to see that $\varphi$ is a faithful trace on $\mathfrak{A}$ with $\varphi(\lambda(s))=0$ for $s\neq e$ and $\varphi(\lambda(e))=1$.

Now let $h$ be a positive definite function on $\mathsf{G}$ with $h(e)=1$ and $0\neq h(s)\neq 1$ for every $e\neq s\in\mathsf{G}$. Then the extension of $T_h$: $\lambda(s)\to h(s)\,\lambda(s)$ to $\mathfrak{A}$ defines a Schwarz operator on $\mathfrak{A}$ with $T_h(\mathbb{1})=\mathbb{1}$ ([7], 1.1).

We assert, that $T_h$ is irreducible. First we show, that the fixed space of $T_h^*$ in $\mathfrak{A}^*$ is one-dimensional: Let $\psi\in\mathfrak{S}(\mathfrak{A})$ be $T_h^*$-invariant, i.e.

$$\psi(\lambda(s))=\psi(T_h(\lambda(s))=h(s)\,\psi(\lambda(s))$$

for every $s\in\mathsf{G}$. Since $0\neq h(s)\neq 1$ for $s\neq e$ we must have $\psi(\lambda(s))=0$ for $s\neq e$ and $\psi(\lambda(e))=1$ whence $\psi=\varphi$.

Next we conclude that the cyclic semi-group $\mathsf{S}$ generated by $T_h$ is mean-ergodic in $\mathscr{L}_s(\mathfrak{A})$ with mean-ergodic projection $P$ given by $x\to\varphi(x)\mathbb{1}$ for $x\in\mathfrak{A}$ ([8] 1.7). Hence, if $\mathfrak{F}$ is a closed $T_h$-invariant face in $\mathfrak{A}_+$ and if $0<x\in\mathfrak{F}$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}T_h^i(x)=\varphi(x)\mathbb{1}\in\mathfrak{F}.$$

Since $\varphi$ is faithful, we conclude $\mathfrak{F}=\mathfrak{A}_+$. Therefore $T_h$ is irreducible.

For a concrete example we consider $\mathsf{G}_1:=\mathbb{Z}$ the group of all integers and a character $\hat{x}\in\hat{\mathsf{G}}_1$ with $\hat{x}(n)\neq 1$ for all $0\neq n\in\mathbb{Z}$. Let $\mathsf{G}_2$ be a discrete group and let $\chi$ be the characteristic function of the set $\{e\}$, $e$ the unit of $\mathsf{G}_2$. For $a\in\mathbb{R}$ with $0<a<1$ and for $s\in\mathsf{G}_2$ we set $f(s):=a(1-\chi(s))+\chi(s)$, then $f$ is a positive definite function on $\mathsf{G}_2$. The group $\mathsf{G}:=\mathsf{G}_1\times\mathsf{G}_2$ is discrete and the function

$$h:=((s,n)\to f(s)\,\hat{x}(n)),\qquad (s,n)\in\mathsf{G}$$

has the desired properties. Especially, the peripheral point spectrum of the corresponding operator $T_h$ is given by the set $\{\hat{x}(n)|n\in\mathbb{Z}\}$ with eigenvectors $\{\lambda(e,n)|n\in\mathbb{Z}\}$. □

For irreducible Schwarz operators, we obtain not only information about the structure of the peripheral point spectrum but also on the action of $T$ on the subalgebra spanned by the eigenvectors pertaining to the peripheral eigenvalues.

3.2 **Proposition.** *Let $T\in\mathscr{L}(\mathfrak{A})$ satisfy the assumptions of Theorem* 3.1 *and denote by $\varphi$ the (unique) state on $\mathfrak{A}$ with $T^*(\varphi)=\varphi$. If $\mathfrak{M}$ is the closed subspace of $\mathfrak{A}$ generated by the eigenvectors pertaining to the peripheral eigenvalues of $T$, then*

1. *$\mathfrak{M}$ is a *-subalgebra of $\mathfrak{A}$.*
2. *$T_0:=T_{|\mathfrak{M}}$ is an irreducible *-automorphism on $\mathfrak{M}$.*
3. *$\tau:=\varphi_{|\mathfrak{M}}$ is a faithful trace on $\mathfrak{M}$.*

*Proof*. Let $\mathsf{E}$ be the set of all eigenvectors of $T$ pertaining to the peripheral eigenvalues. Then it follows from the proof of 3.1, that $\mathfrak{M}$ is a *-subalgebra of $\mathfrak{A}$ containing $\mathbb{1}$. Also we have shown, that $T_0$ is a *-homomorphism on $\mathfrak{M}$. As in the proof of [12] III.10.5 we conclude, that the closure $\mathsf{G}$ of the semigroup $\mathsf{S}:=\{T_0^n|n\in\mathbb{N}\}$ in $\mathscr{L}_s(\mathfrak{M})$ is a compact group with identity $\mathbb{1}_{\mathfrak{M}}$. Hence $T_0$ is a *-automorphism.

Since $\mathsf{G}$ as a compact group in $\mathscr{L}_s(\mathfrak{M})$ is mean-ergodic ([12] III.7.9 Cor. 1), there exists a projection $P \in \overline{\text{conv}}\, \mathsf{G} \subseteq \mathscr{L}_s(\mathfrak{M})$ onto $\text{Fix}(T_0)$ and $P$ is given by $x \to \varphi(x)\mathbb{1}$ for all $x \in \mathfrak{M}$. As in the above example we conclude that $T_0$ is irreducible.

Finally for $x_\alpha, x_\beta \in \mathsf{E}$ we have

$$\tau(x_\alpha^* x_\beta) = \tau(T(x_\alpha^* x_\beta)) = \alpha^* \beta \cdot \tau(x_\alpha^* x_\beta)$$

whence $\tau(x_\alpha^* x_\beta) = 0$ for $\alpha^* \beta \neq 1$. Since for $\alpha^* \beta = 1$ we have $\tau(x_\alpha^* x_\alpha) = \tau(x_\alpha x_\alpha^*)$, we obtain

$$\tau(x^* x) = \tau(x x^*) \qquad \text{for all } x \in \mathfrak{M}$$

and therefore $\tau$ is a faithful trace on $\mathfrak{M}$. $\square$

The following example shows that in general the algebra $\mathfrak{M}$ is not commutative. In particular, the group of all normalized eigenvectors of $T$ pertaining to the peripheral eigenvalues in general is not abelian.

*Example.* Let $\mathscr{H}$ be the Hilbert space $l^2(\mathbb{Z})$, $\mathbb{Z}$ the integers, and let $\xi \in \mathscr{H}$, $k \in \mathbb{Z}$. For $n, m \in \mathbb{Z}$ we define the unitary operators

$$(U(n)\xi)(k) := \xi(k-n)$$
$$(V(m)\xi)(k) := \exp(-ikm)\,\xi(k).$$

Then we have for all $n, m \in \mathbb{Z}$:

$$U(n)\,V(m) = \exp(imn)\,V(m)\,U(n).$$

Therefore the $C^*$-algebra $\mathfrak{A}$ generated by $\{U(n)\,V(m) \mid n, m \in \mathbb{Z}\}$ in $\mathscr{L}(\mathscr{H})$ is not commutative.

We choose a $t \in \mathbb{R}$ such that 1, $t$ and $2\pi$ are independent over the principal ring $\mathbb{Z}$ and let $V(t)$ be the unitary operator

$$(V(t)\xi)(k) := \exp(itk)\,\xi(k).$$

Then $U := U(1)\,V(t)$ is unitary in $\mathscr{L}(\mathscr{H})$, and for $T$ the inner automorphism

$$x \to U^* \circ x \circ U, \qquad x \in \mathscr{L}(\mathscr{H}),$$

we have

$$T(U(n)) = \exp(int)\,U(n)$$
$$T(V(m)) = \exp(-im)\,V(m),$$

whence for all $n, m \in \mathbb{Z}$:

$$T(U(n)\,V(m)) = \exp(i(nt-m))\,U(n)\,V(m).$$

Hence $T$ leaves the $C^*$-algebra $\mathfrak{A}$ globally invariant and is a $*$-automorphism on $\mathfrak{A}$ with $\{\exp(i(nt-m) \mid n, m \in \mathbb{Z}\}$ in the peripheral point spectrum.

Next we define a state $\varphi$ on $\mathfrak{A}$ as follows:

$$\varphi(U(n)V(m)) := \begin{cases} 1 & \text{for } n=m=0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the set $\mathsf{E} := \{U(n)V(m) \mid n, m \in \mathbb{Z}\}$ is linearly independent in $\mathfrak{A}$, $\varphi$ has a unique extension to $\operatorname{lin}\mathsf{E}$ which we denote again by $\varphi$. Since for

$$x = \sum_{i=1}^{n} \alpha_i U(n_i) V(m_i) \in \mathfrak{A}$$

we have

$$\varphi(x^* x) = \sum_{i=1}^{n} |\alpha_i|^2,$$

the extension of $\varphi$ to $\mathfrak{A}$ is a positive trace and $T^*$-invariant.

Next we want to prove that $\varphi$ is faithful. For this let $\mathsf{G}$ be the compact group given by the closure of $\{T^n \mid n \in \mathbb{N}\}$ in $\mathscr{L}_s(\mathfrak{A})$ (see 3.2) and let $dg$ be the Haar measure on $\mathsf{G}$. For $x \in \mathfrak{A}$ we set

$$\tilde{x} := \int_{\mathsf{G}} g(x)\, dg,$$

where the integral exists in the weak sense and defines an element in $\mathfrak{A}$, since $\mathsf{G}$ is compact. Since $\mathsf{G}$ is mean-ergodic in $\mathscr{L}_s(\mathfrak{A})$ and the fixed space of the dual group $\mathsf{G}^*$ in $\mathfrak{A}^*$ is one-dimensional, we have the fixed space of $\mathsf{G}$ one-dimensional. Obviously $\tilde{x} \in \operatorname{Fix}(T)$ whence $\tilde{x} = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$. If we identify $\mathbb{C}$ with $\mathbb{C}\mathbb{1}$ and define for $x \in \mathfrak{A}$

$$\psi(x) := \int_{\mathsf{G}} g(x)\, dg$$

then $\psi$ is a faithful, $T^*$-invariant state on $\mathfrak{A}$. Since the fixed space of $T^*$ is generated by $\varphi$, we must have $\varphi = \psi$ and therefore $\varphi$ faithful. As in the proof of 3.2 we conclude that $T$ is irreducible. □

3.3 **Proposition.** *Let $T \in \mathscr{L}(\mathfrak{A})$ satisfy the assumptions of Theorem 3.1. If there is a primitive n-th root of unity in the point spectrum of T, then there exist mutually orthogonal projections $p_0, p_1, \ldots, p_{n-1}$ in $\mathfrak{A}$ such that*

$$\sum_{i=0}^{n-1} p_i = \mathbb{1} \quad \textit{and} \quad T(p_i) = p_{i-1} \qquad \textit{for } i \in \mathbb{Z}/(n).$$

*Proof.* For $n=1$ there is nothing to prove, so let $n \geqq 2$. If $\varepsilon \in \sigma(T)$ is a primitive $n$-th root of unity with unitary eigenvector $u \in \mathfrak{A}$ then $T(u^n) = u^n$ by 3.2. By 3.1 we can assume $u^n = \mathbb{1}$ and therefore

$$\sigma(u) \subseteq \{\varepsilon^i \mid 0 \leqq i \leqq n-1\}.$$

Since $\varepsilon$ is a primitive $n$-th root of unity we have $u^k \neq \mathbb{1}$ for $1 \leqq k \leqq n-1$ and $\{u^i \mid 0 \leqq i \leqq n-1\}$ is a linearly independent set. Therefore the commutative $C^*$-

algebra $C^*(\mathbb{1}, u)$ generated by $\mathbb{1}$ and $u$ has linear dimension $n$, whence

$$\sigma(u)=\{\varepsilon^i \mid 0 \leqq i \leqq n-1\}.$$

In $C^*(\mathbb{1}, u)$ we have the spectral decomposition $u=\sum_{i=0}^{n-1} \varepsilon^i p_i$ where the $p_i$ are mutually orthogonal projections with $\sum_{i=0}^{n-1} p_i=\mathbb{1}$. Since the restriction of $T$ to $C^*(\mathbb{1}, u)$ is a $*$-automorphism by 3.2 and since

$$\varepsilon u=\sum_{i=0}^{n-1} \varepsilon^{i+1} p_i=\sum_{i=0}^{n-1} \varepsilon^i T(p_i)=T(u)$$

we have $T(p_i)=p_{i-1}$ for $i \in \mathbb{Z}/(n)$ from the uniqueness of such a spectral decomposition. □

*Remark.* If $T$ is an irreducible positive operator on $A$ which is not a Schwarz operator, then Theorem 3.1 is not valid, even in the finite dimensional case. We will investigate these problems in greater detail in a subsequent paper.

## References

1. Albeverio, S., Høegh-Krohn, R.: Frobenius theory for positive maps on von Neumann algebras. Commun. Math. Phys. **64**, 83–94 (1978)
2. Choi, M.D.: Positive linear maps on $C^*$-algebras. Canad. J. Math. **24**, 520–529 (1972)
3. Dixmier, J.: Les Algèbres d'Opérateurs dans l'Espace Hilbertien, 2ème edition. Paris: Gauthier-Villars 1969
4. Dixmier, J.: $C^*$-Algebras. Amsterdam-New York-Oxford: North Holland Publishing Company 1977
5. Effros, E.G.: Order ideals in a $C^*$-algebra and its dual. Duke Math. J. **30**, 391–412 (1963)
6. Frobenius, G.: Über Matrizen aus nichtnegativen Elementen. Sitz-Berichte kgl. Preuss. Akad. Wiss. Berlin, 456–477 (1912)
7. Haagerup, U.: An example of a non nuclear $C^*$-algebra which has the metric approximation property. Inventiones math. **50**, 279–293 (1979)
8. Nagel, R.J.: Mittelergodische Halbgruppen linearer Operatoren. Ann. Inst. Fourier (Grenoble) **23**, 75–87 (1973)
9. Pedersen, G.K.: Measure theory for $C^*$-algebras. Math. Scand. **19**, 131–145 (1966)
10. Schaefer, H.H.: Spektraleigenschaften positiver linearer Operatoren. Math. Z. **82**, 303–313 (1963)
11. Schaefer, H.H.: Topological Vector Spaces, 3rd print. Berlin-Heidelberg-New York: Springer 1971
12. Schaefer, H.H.: Banach Lattices and Positive Operators. Berlin-Heidelberg-New York: Springer 1974
13. Wielandt, H.: Unzerlegbare, nicht negative Matrizen. Math. Z. **52**, 642–648 (1950)

Received October 4, 1979