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Closed Geodesics on Non-Compact Riemannian Manifolds

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§ 1. Introduction

In this paper we investigate the existence of closed geodesics on non-compact Riemannian manifolds. In § 2 we show that there do not exist any restrictions on homotopy or homology groups, implying the existence of closed geodesics with respect to all Riemannian metrics, which are independent of the dimension of the manifold. In § 3 we give an answer to the question in the case of non-compact surfaces. We show that the only surfaces which may have no closed geodesics are the plane and the cylinder, and that there are infinitely many closed geodesics on all non-compact surfaces with the exception of these two and the Möbius strip. We also investigate the existence of closed geodesics without self-intersections. In § 4 we place geometric restrictions on the complete Riemannian manifold. We show that the existence of a compact convex set which is not homotopically trivial implies the existence of a closed geodesic. Then we use this theorem to prove that a complete non-contractible Riemannian manifold with non-negative sectional curvature outside a compact set has a closed geodesic.

The present paper contains the main results of the author's thesis at the University of Bonn.

§ 2. Examples

In this section we will discuss examples of complete Riemannian manifolds without closed geodesics. A trivial example is a Euclidean space. A theorem of Gromoll and Meyer [6] says that there are no closed geodesics on complete Riemannian manifolds of positive curvature (but geodesic loops may exist). Gromoll and Meyer also prove in their paper that manifolds satisfying this curvature condition are necessarily homeomorphic to $\mathbb{R}^n$. Examples with non-trivial fundamental group are surfaces of revolution of funnel type (e.g. $z(x^2+y^2)=1$).

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It is possible to construct examples with arbitrary complicated homotopy and homology groups. Let M be an arbitrary manifold. Then there exists a complete Riemannian metric on $\mathbb{R} \times M$ without closed geodesics as we now want to show.

Let g^* be a complete Riemannian metric on M. On $\mathbb{R} \times M$ we define a metric as follows:

$$\langle X, Y\rangle := xy + e^r g^*(X^*, Y^*); \qquad X=(x, X^*), \ Y=(y, Y^*) \in T_{(r,p)}(\mathbb{R} \times M).$$

The Riemannian metric $\langle\ ,\ \rangle$ is complete.

Let $c(t)=(r(t), u(t))$ be a geodesic on $\mathbb{R} \times M$. We will prove that the function $r(t)$ has no maximum which implies that there are no closed geodesics on $\mathbb{R} \times M$.

Suppose that t' is a maximum of $r(t)$. Let $(U,(u^1, \ldots, u^n))$ be local coordinates around $u(t')$ and let (g^*_{ik}) be the local representation of g^* in U. In the local coordinates $(\mathbb{R} \times U, (\mathrm{id}, u^1, \ldots, u^n))$ the Riemannian metric $\langle\ ,\ \rangle$ has the following form:

$$\begin{aligned} g_{00} &= 1, && \\ g_{i0} &= 0 && \text{for } i \geqq 1, \\ g_{ik} &= e^r g^*_{ik} && \text{for } i,k \geqq 1. \end{aligned}$$

The Christoffel symbols in the differential equation for $r(t)$ are

$$\begin{aligned} \Gamma^0_{0k} &= 0 && \text{for } k \geqq 0, \\ \Gamma^0_{ik} &= -\frac{e^r}{2} g^*_{ik} && \text{for } i,k \geqq 1. \end{aligned}$$

The differential equation for $r(t)$ is

$$\ddot{r}(t) + \sum_{i,j \geqq 1} \Gamma^0_{ij}\, \dot{u}^i(t)\, \dot{u}^j(t) = 0$$

or

$$\ddot{r}(t) - \frac{e^r}{2} \sum_{i,j \geqq 1} g^*_{ij}\, \dot{u}^i(t)\, \dot{u}^j(t) = 0.$$

This can be written as

$$\ddot{r}(t) = \frac{e^r}{2} g^*(\dot{u}(t), \dot{u}(t)).$$

$\dot{u}(t') \neq 0$ because $\dot{r}(t')=0$. Thus $\ddot{r}(t')>0$ which implies that t' is not a maximum – a contradicition.

Remark. In the two-dimensional case the only counter-examples which we obtain as above are $\mathbb{R} \times S^1$ and $\mathbb{R}^2$. We shall see in the next section that there are no more surfaces without closed geodesics.

In higher dimensions it is necessary to place *geometric* and topological restrictions on the Riemannian manifold. This will be done in § 4.

A question which we do not discuss in this paper is whether some topological restrictions depending on the dimension of the manifold imply the existence of a closed geodesic for all Riemannian metrics.

§ 3. Closed Geodesics on Complete Surfaces

The main result of this section is Theorem (3.2). We first prove a lemma.

3.1. Lemma. *Let M be a non-compact surface without boundary which is neither homeomorphic to the plane nor to the cylinder. Then there is a compact set K in M with the following properties:*

(i) *There is a closed curve in K which is not freely homotopic to a curve outside K.*

(ii) *If M is not homeomorphic to the twice punctured plane, then it is possible to choose the curve in* (i) *without self-intersections.*

(iii) *If M is not homeomorphic to the Möbius strip, then there are infinitely many curves as in* (i) *which are not freely homotopic to each other or to coverings of each other.*

(iv) *If M is not homeomorphic to an open set in the plane or in the projective plane, then it is possible to choose the curves in* (iii) *without self-intersections.*

Proof. (1) In this step we prove (iv). We will use some facts about normal forms of compact surfaces with boundary which can be found in [1] or [8].

We consider a sequence of compact subsets of M

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

which are surfaces with boundary, satisfying $M_n \subset \mathring{M}_{n+1}$ and $M = \bigcup_{n \geqq 1} M_n$. There is a surface M_l in the sequence which is neither homeomorphic to the disk with q holes nor to the projective plane with q holes. If M_l is orientable, then it is homeomorphic to the normal form

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} c_1 h_1 c_1^{-1} \ldots c_q h_q c_q^{-1}.$$

In the non-orientable case M_l is homeomorphic to the normal form

$$a_1 a_1 \ldots a_k a_k c_1 h_1 c_1^{-1} \ldots c_1 h_1 c_1^{-1}.$$

If $k > 2$, it is possible to make a handle out of two cross caps [8, p. 139, Fig. 72]. In other words the normal form is homeomorphic to

$$a_1 b_1 a_1^{-1} b_1^{-1} a_3 a_3 \ldots a_k a_k c_1 h_1 c_1^{-1} \ldots c_q h_q c_q^{-1}.$$

If $k = 2$, it is homeomorphic to

$$a_1 b_1 a_1^{-1} b_1 c_1 h_1 c_1^{-1} \ldots c_q h_q c_q^{-1}.$$

In M_l we define curves K_n for all $n \geqq 1$ as follows: If M_l is orientable or non-orientable with $k > 2$, then it has an handle $a_1 b_1 a_1^{-1} b_1^{-1} \ldots$. On this handle we choose K_n as a curve without self-intersections which winds n-times around the handle and is homotopic to $b_1^n a_1^{-1}$. If M_l is non-orientable with $k=2$, then in a similar way we let K_n wind n-times without self-intersections around the Klein bottle $a_1 b_1 a_1^{-1} b_1 \ldots$ and be homotopic to $b_1^n a_1^{-1}$. (Ballmann uses such curves in [2].) We choose the compact set K as M_l.

We now want to prove that the curves K_n satisfy the claims in the lemma. I.e. $b_1^n a_1^{-1}$ and $(b_1^m a_1^{-1})^l$ are not freely homotopic as curves in M if $n \neq m$, $l \geqq 1$; and that the curve $b_1^n a_1^{-1}$ is not freely homotopic to a curve outside M_l for all n.

Assume that $K_n^* := b_1^n a_1^{-1}$ and $K_{ml}^* := (b_1^m a_1^{-1})^l$ are freely homotopic. Then there is a surface M_m in the sequence $M_1 \subset \cdots \subset M_n \subset \cdots$, containing the whole homotopy between K_n^* and K_{ml}^*. It is not difficult to see that M_m can be obtained by identification of edges in a polygon in such a way that a_1 and b_1 correspond to edges [10]. Instead of the edges corresponding to one of the boundary components of M_m we can remove a small disk from the interior of the polygon. By expanding this disk to the boundary of the polygon one sees that M_m is homotopy equivalent to a graph. The fundamental group of a graph is free [8, § 47], so it follows that $\pi_1(M_m)$ is free and, further, that $\{a_1, b_1\}$ can be extended to a set of free generators. If K_n^* and K_{ml}^* are freely homotopic, then the elements $b_1^n a_1^{-1}$ and $(b_1^m a_1^{-1})^l$ are conjugate in $\pi_1(M_m)$ [8, § 49]. That implies the existence of a relation in the group – a contradiction.

Assume now that $b_1^n a_1^{-1}$ is freely homotopic to a curve d outside M_l for some n. As above the whole homotopy is contained in some M_m for $m \geqq l$. Again we represent M_m as a polygon. In that polygon d is freely homotopic to an edge path neither containing a_1 nor b_1. The contradiction now follows as above.

(2) In this step we prove (iii). Because we have already proved (iv), we can restrict ourselves to open sets in the plane or in the projective plane. As above we choose a sequence $M_1 \subset \cdots \subset M_n \subset \cdots$ of compact surfaces with the properties that $M_n \subset \mathring{M}_{n+1}$; that all connected components of $M - M_n$ are non-compact for all $n \geqq 1$ and that $M = \bigcup_{n \geqq 1} M_n$. There is a surface M_k in the sequence such that it and all the following surfaces are homeomorphic to the disk with at least two holes or to the projective plane with at least two holes. If q is the number of boundary components of M_k and M_k is orientable (resp. non-orientable), then the fundamental group of M_k is generated by $e_1, \ldots, e_q$ with the relation $e_1 \cdot \ldots \cdot e_q = 1$ (resp. $a, e_1, \ldots, e_q$ with the relation $a^2 e_1 \cdot \ldots \cdot e_q = 1$). This shows that the fundamental group is freely generated by $e_1, \ldots, e_{q-1}$ $(q > 2)$ (resp. $a, e_1, \ldots, e_{q-1}$ $(q \geqq 2)$). We consider the curves $e_1^n e_2$ (resp. $a^n e_1$) for all $n \geqq 1$. These curves satisfy the claim of the lemma with K as M_k as we now want to prove. Suppose that $e_1^n e_2$ and $(e_1^m e_2)^l$ are freely homotopic. The homotopy is contained in a surface M_m for $m \geqq n$. The curves $e_1^n e_2$ become $(d_1 \ldots d_r)^n (d_{r+1} \ldots d_s)$, $1 \leqq r < s$, if the fundamental group of M_m is freely generated by $d_1, \ldots, d_u$ (the indices chosen in a proper way). Here we have used the fact that the connected components of $M - M_n$ are non-compact to see that e_1 and e_2 do not become trivial in the fundamental group of M_m. The free homotopy between $e_1^n e_2$ and $(e_1^m e_2)^l$ implies the existence of a non-trivial relation in the fundamental group of M_m which is free – a contradiction. (The proof in the non-orientable case is exactly the same.)

(3) It is left to prove the existence of one curve without self-intersections, satisfying the claim in part (ii) of the lemma, if the surface is the Möbius strip or one of the surfaces we dealt with in (2) with the exception of the twice punctured plane. As above we choose a sequence of compact surfaces exhausting M. In the normal form of M_k, for some large k, we choose our curve in the non-orientable case as a (see (2)). In the orientable case M_k is a disk with at least three holes. In that case we let our curve go once around exactly two of the holes. □

We now come to the main result of this section.

3.2. Theorem. *Let M be a non-compact surface with a complete Riemannian metric which is neither homeomorphic to the plane nor to the cylinder.*

(i) *Then there is a closed geodesic on M.*

(ii) *If M is not the twice punctured plane, then there is a closed geodesic without self-intersections on M.*

(iii) *If M is not the Möbius strip, then there are infinitely many closed geodesics on M.*

(iv) *If M is not a subset of the plane or of the projective plane, then there are infinitely many closed geodesics without self-intersections on M.*

Remark. In [10] stability and instability of closed geodesics are defined and discussed. There it is proved that most of the closed geodesics in Theorem (3.2) are unstable.

Proof. (1) We first prove the existence of closed geodesics in (i) and (iii) which are the cases where the closed geodesics may have self-intersections. According to Lemma (3.1) there exist a compact set K and a closed curve c in K which is not freely homotopic to a curve outside K. We define the set

$$C := \{p \in M \mid d(p, K) \leqq L(c)\}.$$

C is compact and every curve which is freely homotopic to c and shorter than c is completely contained in C.

We set $\sigma := E(c)$ and $\gamma :=$ the infimum of the injectivity radius on C. $\gamma > 0$ because C is compact. Further let k be an even number such that $4\sigma/k < \gamma^2$. For a curve $e: S^1 \to C$ with $E(e) \leqq \sigma$ which is freely homotopic to c we have

$$d\left(e(t), e\left(t+\frac{2}{k}\right)\right) \leqq L\left(e\left|\left[t, t+\frac{2}{k}\right]\right.\right) \leqq \sqrt{2\frac{2}{k}E(e)} < \gamma.$$

This implies that there is a unique minimal geodesic segment connecting $e(t)$ and $e\left(t+\frac{2}{k}\right)$. We define $D_1 e$ (resp. $D_2 e$) as the curve which one obtains from e, if the part of e between $e\left(\frac{l}{k}\right)$ and $e((l+2)/k)$ is replaced by the minimal geodesic connecting the two points for all even l (resp. for all odd l). $D_1 e$ and $D_2 e$ are freely homotopic to c and shorter than c, so that they are also contained in C. Therefore we can define $De := D_2(D_1 e)$.

D is defined on the set F of piecewise differentiable curves $e: S^1 \to C$ which are freely homotopic to c and have less energy than c. D maps F into F because $E(De) \leqq E(e) \leqq \gamma$ and e and De are freely homotopic for every $e \in F$. $D: E \to E$ is continuous in the compact-open topology. Further $E(De) = E(e)$ if and only if e is a closed geodesic.

We consider the sequence $\{D^n c\}$ of closed broken geodesics in C. $(D^n c)|[l/k, (l+2)/k]$ is a geodesic segment for all n and all odd l. From the compactness of C and the continuity of the exponential mapping it follows that there is a converging subsequence of $\{(D^n c)|[l/k, (l+2)/k]\}$ for all odd l. This implies the existence of a subsequence $\{D^{n_k} c\}$ which converges to a broken geodesic $d: S^1 \to C$. The sequence $\{E(D^n c)\}$ is decreasing and >0 and has consequently a limit point. This and the continuity of E on the set of broken geodesics implies:

$$\begin{aligned} E(Dd) &= E(D \lim D^{n_k} c) = E(\lim D^{n_k+1} c) = \lim E(D^{n_k+1} c) \\ &= \lim E(D^{n_k} c) = E(d). \end{aligned}$$

$E(Dd) = E(d)$ implies that d is a closed geodesic. d is freely homotopic to c, so that the existence of the closed geodesics in (i) and (iii) follows from Lemma (3.1).

(2) We now want to prove (ii) and (iv); i.e. we want to find the closed geodesics without self-intersections. For that purpose we modify the deformation D in (1) (see the Appendix in [7]). As in (1) we can find with help of Lemma (3.1) a compact set K and a closed curve c without self-intersections which is not freely homotopic to a curve outside K. It is not difficult to prove that c can be chosen as a broken geodesic [10, p. 24]. We define C as in (1) and $\sigma := E(c)$. The modified deformation will only be defined on the set F' of closed broken geodesics in C with energy $\leqq \sigma$ which can be approximated by closed curves without self-intersections and which are freely homotopic to c.

Let γ and k be as in (1). Let $e \in F'$. Let D_m, $0 < m \leqq (k/2)$, be the deformation which replaces the segment $e|[(2m-2)/k, 2m/k]$ of the curve e by the unique minimal geodesic α joining $e((2m-2)/k)$ and $e(2m/k)$ and leaves the rest of the curve $e^* := e|[2m/k, 1+(2m-2)/k]$ unchanged, if α and e^* have no points in common. If α and e^* have common points, we let D_m replace parts of e^* by corresponding parts of α in such a way that $D_m e$ will be an element of F'. $D_m e$ for $(k/2) < m \leqq k$ is defined just as in the case $m \leqq (k/2)$, except that we first replace $e|[(2m-1)/k, (2m+1)/k]$ by a geodesic segment instead of $e|[(2m-2)/k, 2m/k]$.

We now define $\tilde{D} e := D_k \circ \cdots \circ D_1 e$.

It is easy to see that e and $\tilde{D} e$ are freely homotopic and that $E(\tilde{D} e) \leqq E(e)$ with equality sign if and only if e is a closed geodesic. $\tilde{D} c$ is an element in F', so we can define $\tilde{D}^2 c$ and by induction we define $c_n := \tilde{D}^n c$. The sequence $\{c_n\}$ satisfies $\lim E(\tilde{D} c_n) = \lim E(c_n)$ which implies that $\lim E(D c_n) = \lim E(c_n)$ (D is the deformation in (1)). From [2] or [7, (A.1.3)] it now follows that $\{c_n\}$ has a converging subsequence with limit d which must be a closed geodesic because of the continuity of D and the energy function on the set of broken geodesics (see the end of (1)). It is left to prove that d has no self-intersections. This follows from the fact that M is two-dimensional. If d has some *transversal* self-intersection, then every curve in a small neighbourhood of d in the compact-open topology must also have transversal self-intersections which is not the case for the whole sequence $\{c_n\}$ converging to d. □

Remark. Examples of higher dimensional complete Riemannian manifolds where the methods of this section still work can be found in [10].

§ 4. Closed Geodesics and Convex Sets

We have seen in § 2 that there are no purely homological or homotopical conditions implying the existence of closed geodesics which do not depend on the dimension of the manifold. In this section we therefore investigate the existence of closed geodesics under (intrinsic) geometric restrictions on complete non-compact Riemannian manifolds. In [4, p. 144] and [11] conditions on the second fundamental form of an embedding are given which ensure the existence of a closed geodesic. If the soul of a Riemannian manifold of non-negative sectional curvature is not a point, then it follows from the Theorem of Fet and Lusternik that it has a closed geodesic. We shall generalize this in Theorem (4.3).

We begin with a definition.

4.1. Definition. Let M be a complete Riemannian manifold.

(i) A non-void set S in M is called *strongly convex*, if for any $p, q \in S$ there is a unique minimal geodesic joining p and q with image in S. A non-void set K in M is called *convex*, if the sets $B_{r(p)}(p) \cap K$ are strongly convex for all $p \in K$. ($r(p)$ is the radius of convexity which depends continuously on p [5, p. 162]. $B_r(p) := \{q \in M \mid d(p,q) \leqq r\}$.)

(ii) A non-void set K in M is called *totally convex*, if for any $p, q \in K$ and any geodesic segment c joining p and q, we have that the image of c lies in K.

Remark. Our definition of "convex" and "strongly convex" is different from [5], but nearly the same as in [3].

4.2. Theorem. *Let M be a complete Riemannian manifold, let $K \subset M$ be compact and convex. If there is an $i > 0$ such that $\pi_i(K) \neq 0$, then there is a closed geodesic in M.*

We will use this theorem and methods from the paper [3] by Cheeger and Gromoll to prove the following theorem.

4.3. Theorem. *Let M be a complete non-contractible Riemannian manifold with non-negative sectional curvature outside some compact set. Then there is a closed geodesic in M.*

Remark. The above theorems generalize the Theorem of Fet and Lusternik (which says that there exists a closed geodesic on every compact Riemannian manifold). Note that Theorem (4.2) is more general than (4.3) (an example which shows this is the hyperboloid of revolution). Our proof of Theorem (4.2) is simpler than the proofs of the Fet-Lusternik Theorem known to us.

Proof of Theorem (4.2). Let $i > 0$ be the smallest integer with $\pi_i(K) \neq 0$. Let $F: (I^i, \partial I^i) \to (K, p)$ represent a non-trivial element of $\pi_i(K)$. We can assume that F is differentiable. F induces in a natural way a map f: $(I^{i-1}, \partial I^{i-1}) \to (C(S^1, M), p)$ which is continuous in the compact-open topology of $C(S^1, M)$. Let $\sigma := \sup\{E(f(x)) | x \in I^{i-1}\}$, 2γ: $=$minimum of the radius of convexity in K and let k be an even number satisfying $4\sigma/k < \gamma^2$. Then we have $d(c(t), c(t+(2/k))) < \gamma$ for curves in K with $E(c) \leqq \sigma$. It is therefore possible to define the deformation D on the set of curves $c: S^1 \to K$ with $E(c) \leqq \sigma$ exactly as in part (1) of Theorem (3.2). Dc lies in K because of the convexity of K. So it is possible to define $D^n c$ by induction. We define

$$\alpha := \lim_{n\to\infty} (\max\{E(D^n f(x)) | x \in I^{i-1}\}).$$

$\alpha > 0$ because otherwise there would be an $n_0 > 0$ with the property that $L(D^{n_0} f(x))$ is smaller than the minimum of the radius of injectivity on K for all $x \in I^{i-1}$. Then it would be possible to deform f into a map g: $(I^{i-1}, \partial I^{i-1}) \to (K, p) \subset (C(S^1, M), p)$ which is homotopically trivial because $\pi_{i-1}(K) = 0$. If f is freely homotopic to a constant map, then the same is true for F. Thus $\alpha > 0$.

Put $c_n := D^n f(x_n)$ where $x_n \in I^{i-1}$ satisfies

$$E(D^{n+1} f(x_n)) = \max\{E(D^{n+1} f(x)) | x \in I^{i-1}\}.$$

We have $\lim E(Dc_n) = \lim E(c_n) = \beta$. It follows from the fact that the c_n are broken geodesics with corners at the same place in the domain of definition, the compactness of K and the continuity of the exponential map that a subsequence $\{c_{n_i}\}$ converges to a broken geodesic d with $E(d) = \beta$.

$$\begin{aligned} E(Dd) &= E(D \lim c_{n_i}) = E(\lim D c_{n_i}) = \lim E(D c_{n_i}) \\ &= \beta = E(d). \end{aligned}$$

$E(Dd) = E(d)$ implies that d is a closed geodesic. □

Proof of Theorem (4.3). Let c: $[0, +\infty) \to M$ be a geodesic ray with $\|\dot{c}(t)\| = 1$. (A geodesic c: $[0, +\infty) \to M$ is called a geodesic ray, if $c|[t_1, t_2]$ is a minimal geodesic joining $c(t_1)$ and $c(t_2)$ for any $0 \leqq t_1 \leqq t_2 < +\infty$. On complete non-compact Riemannian manifolds every point has at least one geodesic ray emanating from it.) We define $B_t(c) := \bigcup_{s>t} B_{s-t}(c(s))$. Let $H_t(c)$ be the compliment of $B_t(c)$.

(1) Let C be a compact set which contains every point of M with negative sectional curvature. In this step we prove that there is a $t_0 > 0$ such that $C \cap B_{t_0}(c) = \varnothing$.

We first prove that $\bigcap_{t>0} \overline{B_t(c)} = \varnothing$. Assume that $q \in \bigcap \overline{B_t(c)}$. Then we have for every $t' > 0$ and every $\varepsilon > 0$ a $t'' > t'$ with $B_\varepsilon(q) \cap B_{t''-t'}(c(t'')) \neq \varnothing$. This implies that

$$\begin{aligned} d(q, c(t)) &\leqq d(q, c(t'')) + d(c(t''), c(t)) \\ &< \varepsilon + (t'' - t') + (t - t'') = \varepsilon + t - t' \end{aligned}$$

for all $t>t''$. Thus we have $d(q, c(t))-t<-t'+\varepsilon$ for all $t>t''$. On the other hand $t=d(c(0), c(t))\leqq d(c(0), q)+d(q, c(t))$ for all $t\geqq 0$. I.e. $d(q, c(t))-t\geqq -d(c(0), q)$ for all $t\geqq 0$ – a contradiction.

Suppose that $C\cap \overline{B_t(c)}\neq\varnothing$ for all $t>0$. Then we have a decreasing sequence $\{C\cap \overline{B_t(c)}\}$ of non-void closed subsets of C with $\bigcap(C\cap \overline{B_t(c)})=\varnothing$ – a contradiction with the compactness of C.

(2) We now prove that the compliment $H_t(c)$ of $B_t(c)$ is totally convex for $t>t_0$.

Suppose that $H_{t_1}(c)$ is not totally convex for some $t_1>t_0$. Then there is a geodesic c_0: $[0,1]\to M$ which leaves $H_{t_1}(c)$ and comes back. There exists a $t_2>t_1$ with the property that $c_0([0,1])\cap B_{t-t_1}(c(t))\neq\varnothing$ for all $t>t_2$. Further there is for every $t>t_2$ an $s_t\in(0,1)$ such that $d(c_0(s_t), c(t))=d(c(t), c_0([0,1]))$. Let c_2^t be the minimal geodesic joining $c(t)$ and $c_0(s_t)$. Let $r_t\in[0,1)$ be smaller than s_t and have the property that $c_0(r_t)\in\partial B_{t-t_1}(c(t))$ and $c_0((r_t, s_t))\subset B_{t-t_1}(c(t))$. Let c_1^t be a minimal geodesic joining $c(t)$ and $c_0(r_t)$. Define $c_0^t:=c|[r_t, s_t]$.

The argument we are now going to use is due to Cheeger and Gromoll [3].

As $\lim L(c_1^t)=+\infty$ and $L(c_0^t)\leqq L(c_0)$, it is possible to choose t so big that $L(c_1^t)+L(c_2^t)>L(c_0^t)$. ($c_0$ is not necessarily a minimal geodesic.) We can therefore use the Theorem of Toponogov to compare the triangle (c_0^t, c_1^t, c_2^t) which is contained in the ball $B_{t-t_1}(c(t))$ of non-negative sectional curvature with a triangle in the Euclidean plane. Let α (resp. α^*) be the angle between c_0^t and c_2^t in M (resp. in the Euclidean plane). Then we have from the Theorem of Toponogov that $\alpha^*\leqq\alpha=\pi/2$. From the Law of Cosines we have

$$\cos\alpha^*=\frac{L(c_0^t)^2+L(c_2^t)^2-L(c_1^t)^2}{2L(c_0^t)\,L(c_2^t)}$$
$$=\frac{1}{2L(c_2^t)}\left[\frac{(L(c_2^t)+L(c_1^t))(L(c_2^t)-L(c_1^t))}{L(c_0^t)}+L(c_0^t)\right].$$

From $(L(c_2^t)-L(c_1^t))<0$, $\lim(L(c_2^t)+L(c_1^t))=+\infty$ and $L(c_0^t)<L(c_0)$ it follows that $\cos\alpha^*<0$ for big t and thus $\alpha^*>\pi/2$ – a contradiction.

(3) In this last step we prove the existence of a compact totally convex set K with $\pi_i(K)\neq 0$ for some $i>0$. Then the claim of the theorem follows from (4.2).

For that purpose we show that there is a family $\{K_t|t>0\}$ of compact totally convex sets with $M=\bigcup K_t$ and $K_t\subset K_s$ for $t\leqq s$. M is not contractible so that there exists a homotopically non-trivial map f: $(I^i, \partial I^i)\to(M, p)$ for some $i>0$ [9, p. 405, Cor. 24]. The image of f is contained in K_t for t big enough. $\pi_i(K_t)\neq 0$ because f is of course homotopically trivial in M if it is it in K_t.

So we have only to find the family $\{K_t\}$. Let p be an arbitrary point in M, G the set of all geodesic rays with arc length parameter which emanate from p. We have for every c a $t_c>0$ such that $H_t(c)$ is totally convex for every $t>t_c$.

Set $K_t:=\bigcap_{c\in G} H_{t_c+t}(c)$ for $t>0$. Every K_t is totally convex because the intersection of totally convex sets is totally convex. Further every K_t is compact because the existence of an unbounded sequence $\{p_n\}$ in some K_s would imply that the sequence of minimal geodesics $\{c_{pp_n}\}$ clusters around a geodesic ray which must be completely contained in K_s. This obviously contradicts the construction of K_s. $M=\bigcup K_t$ because the balls $B_t(p)$ are contained in $H_t(c)$ for all $c\in G$. □

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