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On the Integral Representation in Convex Noncompact Sets of Tight Measures

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1. Introduction

We prove a representation theorem of the Choquet type. Let (X, t) be a topological space and H a weakly closed – not necessarily compact – bounded convex set of nonnegative tight Borel measures on X. For $\mu \in H$ there is a probability measure p on the set $\operatorname{ex} H$ of extreme points of H with barycenter μ. The problem is, what "barycenter" means. In [8] Weizsäcker essentially proves the existence of a probability measure p such that

$$\mu(A) = \int_{\operatorname{ex} H} v(A)\, dp$$

for every Baire subset A of X. If X is completely regular and Souslin, this result can easily be obtained from [4], ex. 5.10, if it is Polish, from a remark in [5], § 3.

It is natural to ask for a representing measure p such that the formula holds for every Borel set or even for every μ-measurable set A (compare [8], 5.2). Even in the case where X is compact there does not seem to be a simple way from the results mentioned above to the answer of this question. In this paper we give a positive answer by proving the desired result directly.

2. Preliminaries

If A is a set and $\mathscr{F}$ a set of real functions on A, then $\sigma(\mathscr{F})$ denotes the σ-algebra generated on A by $\mathscr{F}$. If $B \subset A$, then 1_B is the characteristic function of B on A, and if f is a function on A, then $f|B$ is the restriction of f on B. If $\mathscr{C}$ is a system of subsets of A, then we define $B \cap \mathscr{C} := \{B \cap C\colon C \in \mathscr{C}\}$. If C is a convex subset of a linear space, then $x \in C$ will be called extreme point, iff there do not exist distinct points $y, z \in C$ with $x = 1/2(y+z)$. Ex C is the set of extreme points of C.

Let (X, t) be a topological space. $\mathscr{C}(X, t)$ denotes the set of bounded continuous functions on X, $\mathfrak{B}_0(X, t) := \sigma(\mathscr{C}(X, t))$ ($\mathfrak{B}(X, t)$) the σ-algebra of Baire (Borel) sets and $\mathscr{B}_0(X, t)$ ($\mathscr{B}(X, t)$) the set of real bounded $\mathfrak{B}_0(X, t)$ ($\mathfrak{B}(X, t)$)-measurable

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functions on X. A nonnegative measure μ on $\mathfrak{B}(X, t)$ is said to be tight iff, for every $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\mu(X - K) < \varepsilon$. The set of bounded tight measures on X will be denoted by $\mathcal{M}_+(X, t)$. If $f \in \mathscr{B}(X, t)$ and $\mu \in \mathcal{M}_+(X, t)$, then the integral $\int_X f \, d\mu$ is denoted by $\mu(f)$; f' is the mapping $v \mapsto v(f)$ on $\mathcal{M}_+(X, t)$. $\mathscr{I}(X, t)$ is the set of bounded lower semicontinuous functions on X.

Throughout the paper subsets M of $\mathcal{M}_+(X, t)$ will be endowed with the weak topology $\upsilon(M, \mathscr{I}(X, t))$, the smallest topology for which every function $f'|M$ with $f \in \mathscr{I}(X, t)$ is lower semicontinuous.

3.

We now formulate the main result:

Theorem. *Suppose that (X, t) is a topological space and H a convex and weakly closed set of tight measures with $\sup\{v(X)\colon v \in H\} < \infty$. Then for every $\mu \in H$ there is a probability measure p on the σ-algebra $\sigma(\{f'|\mathrm{ex}\, H : f \in \mathscr{B}(X, t)\})$ on the set of extreme points of H such that*

$$\mu(f) = \int_{\mathrm{ex}\, H} v(f) \, dp \qquad \text{for every } f \in \mathscr{B}(X, t). \tag{1}$$

This result can be extended to a wider class of functions. A function on X is called μ-measurable, iff it is measurable with respect to the μ-completion of $\mathfrak{B}(X, t)$.

Corollary. *In the theorem bounded μ-measurable functions f on X can be substituted for the functions $f \in \mathscr{B}(X, t)$.*

Before we give the proofs, we recall some well known facts. Let E be a real locally convex linear space and $K \subset E$ a convex compact subset with the relative topology s. A measure $p \in \mathcal{M}_+(K, s)$ represents $x \in E$, which is denoted by $x = r(p)$, iff $x'x = \int_K x' \, dp$ for every $x' \in E'$, the topological dual of E. If p is a probability measure, this is equivalent to $a(x) = \int_K a \, dp$ for every lower semicontinuous function on K which is affine (i.e. if $c \in (0, 1)$ and $y, z \in K$, then $f(cy + (1-c)z) = cf(y) + (1-c)f(z)$) ([6], 9.7). On $\mathcal{M}_+(K, s)$ the Choquet ordering is defined by $p \prec q$ iff $p(f) \leqq q(f)$ for every $f \in \mathscr{C}(K, s)$ which is convex (substitute "$\leqq$" for "$=$" in the definition of affine). This ordering is inductive. A convenient reference for Choquet theory is [1].

Suppose now that X is compact. $\mathscr{C}(X, t)$ with the norm of uniform convergence is a Banach space. We define the linear space of signed measures $\mathcal{M}(X, t) := \mathcal{M}_+(X, t) - \mathcal{M}_+(X, t)$. By the Riesz representation theorem $\mathcal{M}(X, t)$ can be identified with $\mathscr{C}(X, t)'$ and is locally convex in the weak*-topology. On $\mathcal{M}_+(X, t)$ the weak*-topology coincides with $\upsilon(\mathcal{M}_+(X, t), \mathscr{I}(X, t))$ ([7], Th. 8.1). Thus H is a convex weak*-compact subset of $\mathcal{M}(X, t)$ and Choquet theory can be applied.

For this compact case we reformulate the notions of barycenter and Choquet ordering in a way which is more convenient in our special situation. First, we define

$$\mathscr{B}_0^{\vee}(H) := \{(\bigvee_{i \leqq n} (f_i' + c_i))|H \colon f_i \in \mathscr{B}_0(X, t);\ c_i \in \mathbb{R};\ i = 1, \ldots, n;\ n \in \mathbb{N}\}.$$

Lemma. *Suppose (X,t) to be compact and let $\mu\in\mathcal{M}_+(X,t)$ and $p,q\in\mathcal{M}_+(H, v(H, \mathcal{I}(X,t)))$. Then:*

a) *If $f\in\mathcal{B}(X,t)$, then $f'|H$ is measurable with respect to $\mathfrak{B}(H, v(H, \mathcal{I}(X,t)))$.*

b) *$\mu=r(p)$ if and only if*

$$\mu(f)=\int_H v(f)\,dp \qquad \text{for every } f\in\mathcal{B}(X,t). \tag{2}$$

c) *$p\prec q$ if and only if $p(b)\leq q(b)$ for every $b\in\mathcal{B}_0^{\vee}(H)$.*

Proof. Define the sets

$$\mathcal{H}:=\{f\in\mathcal{B}(X,t):\ f'|H \text{ is Borel measurable}\},$$

$$\mathcal{H}':=\{f\in\mathcal{B}(X,t):\ \mu(f)=\int_H v(f)\,dp\}.$$

According to the definition of the weak topology $(1_A)'$ is lower semicontinuous on H for every $A\in t$. Hence $1_A\in\mathcal{H}$ and if $\mu=r(p)$, then $1_A\in\mathcal{H}'$ too. Since $\mathcal{H}$ and $\mathcal{H}'$ are closed under bounded sequential convergence, this implies $\mathcal{B}(X,t)\subset\mathcal{H}$ and $\mathcal{B}(X,t)\subset\mathcal{H}'$ ([3], § 0). Thus we have proved a) and the nontrivial implication of b).

$\mathcal{B}_0(X,t)$ is the smallest class of bounded functions on X which contains $\mathcal{C}(X,t)$ and is closed under bounded sequential convergence ([2], ex. 56.9). By applying this argument for each $i\leq n$ separately, we see that $p(b)\leq q(b)$ holds for every b of the form

$$b=\bigvee_{i\leq n}(f_i'+c_i)|H$$

with $c_i\in\mathbb{R}$ and $f_i\in\mathcal{B}_0(X,t)$ if and only if it holds for every b with $f_i\in\mathcal{C}(X,t)$. The latter condition is equivalent to $p\prec q$ by [1], I.1.3.

Proof of the Theorem. I. First we assume (X,t) to be compact.

1. H is a convex weak*-compact subset of $\mathcal{M}(X,t)$. Choquet's theorem ([1], I.4.8) provides the existence of a probability measure $q\in\mathcal{M}_+(H)$ (we omit the topology) which represents μ and is maximal in the Choquet ordering. By the lemma, (2) holds for q. If $M\subset\mathcal{M}_+(X,t)$ define

$$\mathfrak{W}(M):=\sigma(\{f'|M:\ f\in\mathcal{B}(X,t)\}).$$

We will show that $q(W)=0$ for every $W\in\mathfrak{W}(H)$ with $W\cap\operatorname{ex}H=\varnothing$. Hence the outer measure of $q|\mathfrak{W}(H)$ induces a probability measure p on $\mathfrak{W}(\operatorname{ex}H)$ for which (1) holds.

2. Let $W\in\mathfrak{W}(H)$ be disjoint from $\operatorname{ex}H$. It is easily verified that

$$\mathfrak{W}(H)=\bigcup\{\sigma(\{f'|H:\ f\in\mathcal{F}\}):\ \mathcal{F}\subset\mathcal{B}(X,t) \text{ countable}\}.$$

Hence there is a countable family $\mathcal{F}\subset\mathcal{B}(X,t)$ such that

$$W\in\sigma(\{f'|H:\ f\in\mathcal{F}\}).$$

By use of Lusin's theorem ([2], ex. 68.5) a sequence of mutually disjoint compact sets $(K_n:\ n\in\mathbb{N})$ in X can be constructed which has the following properties:

(a) $\mu(X - \bigcup_{n\in\mathbb{N}} K_n) = 0$;

(b) $f|K_n \in \mathscr{C}(K_n, K_n \cap t)$ for every $f \in \mathscr{F}$ and $n \in \mathbb{N}$.

This is already done in [8], proof of Satz 1.

3. We define

$$X_1 := \bigcup_{n\in\mathbb{N}} K_n, \quad t_0 := X_1 \cap t.$$

t_1 denotes the topology on X_1 characterised by: $G \in t_1$ iff $G \cap K_n \in K_n \cap t$ for every $n \in \mathbb{N}$.

(X_1, t_1) is a locally compact σ-compact space. Furthermore

$$\mathfrak{B}(X_1, t_0) = \mathfrak{B}(X_1, t_1)$$

and the sets $\mathscr{M}_+(X_1, t_0)$ and $\mathscr{M}_+(X_1, t_1)$ are equal.

Define (X_2, t_2) to be the Alexandrov compactification of (X_1, t_1),

$$\mathscr{M}_{X_1}(X, t) := \{\nu \in \mathscr{M}_+(X, t) \colon \nu(X - X_1) = 0\},$$

$$\mathscr{M}_{X_1}(X_2, t_2) := \{\nu \in \mathscr{M}_+(X_2, t_2) \colon \nu(X_2 - X_1) = 0\}$$

and $i \colon X_1 \to X_2$ the natural embedding.

By a well known lemma ([4], 5.3) the mappings

$$\iota_1 \colon \mathscr{M}_{X_1}(X, t) \to \mathscr{M}_+(X_1, t_0), \qquad \nu \mapsto \nu|\mathfrak{B}(X_1, t_0),$$

$$\iota_2 \colon \mathscr{M}_+(X_1, t_1) \to \mathscr{M}_{X_1}(X_2, t_2), \qquad \nu \mapsto \nu \circ i^{-1}$$

are homeomorphisms.

Further we define

$$\iota := \iota_2 \circ \iota_1, \quad H_0 := H \cap \mathscr{M}_{X_1}(X, t), \quad H_2 := \iota H_0,$$

$\bar{H}_2$ the closure of H_2 in $\mathscr{M}_+(X_2, t_2)$,

$$\mathfrak{A} := \sigma(\{f'|H \colon f \in \mathscr{B}(X, t), f|K_n \in \mathscr{C}(K_n, K_n \cap t) \text{ for every } n \in \mathbb{N}\})$$

and state some properties:

(c) H_0, H_2 and $\bar{H}_2$ are convex; $\bar{H}_2$ is compact.

(d) $\operatorname{ex} H_0 = \operatorname{ex} H \cap \mathscr{M}_{X_1}(X, t)$, $H_2 = \bar{H}_2 \cap \mathscr{M}_{X_1}(X_2, t_2)$,
$\operatorname{ex} H_2 = \operatorname{ex} \bar{H}_2 \cap \mathscr{M}_{X_1}(X_2, t_2)$.

(e) $\iota \operatorname{ex} H_0 = \operatorname{ex} H_2$.

(f) $\iota^{-1}(\mathfrak{B}_0(\bar{H}_2)) = \mathfrak{A} \cap H_0 \supset \mathfrak{B}_0(H) \cap H_0$; $\iota^{-1}|H_2 \colon H_2 \to H_0$ is continuous.

We prove (c)–(f).

(c) The convexity of the sets is clear. $\bar{H}_2$ is bounded and closed in the weak*-topology, hence compact.

(d) If $\nu \in \operatorname{ex} H \cap H_0$, then $\nu \in \operatorname{ex} H_0$. Conversely, let $\nu \in \operatorname{ex} H_0$ and $\nu_1, \nu_2 \in H$ with $\nu = 1/2(\nu_1 + \nu_2)$. $\nu(X - X_1) = 0$ implies $\nu_i(X - X_1) = 0$ for $i = 1, 2$, hence $\nu_i \in H_0$, $\nu = \nu_1 = \nu_2$ and $\nu \in \operatorname{ex} H$.

As H_0 is relatively closed in $\mathcal{M}_{X_1}(X,t)$ and ι_1 is a homeomorphism, H_1 is closed with respect to $\upsilon(\mathcal{M}_+(X_1,t_0),\mathcal{I}(X_1,t_0))$, hence with respect to the finer topology $\upsilon(\mathcal{M}_+(X_1,t_1),\mathcal{I}(X_1,t_1))$. Since ι_2 is a homeomorphism too, H_2 is relatively closed in $\mathcal{M}_{X_1}(X_2,t_2)$ which means

$$H_2=\bar{H}_2\cap\mathcal{M}_{X_1}(X_2,t_2).$$

From this the last equation follows exactly like the first one.

(e) holds, since ι is affine and injective.

(f) We recall that H and $\bar{H}_2$ are compact. By use of the Stone-Weierstraß theorem we find

$$\mathfrak{B}_0(H)=\sigma(\{f'|H\colon f\in\mathscr{C}(X,t)\}),\qquad \mathfrak{B}_0(\bar{H}_2)=\sigma(\{f'|\bar{H}_2\colon f\in\mathscr{C}(X_2,t_2)\}).$$

Hence $\mathfrak{B}_0(H)\cap H_0\subset\mathfrak{A}\cap H_0$ and

$$\begin{aligned}\mathfrak{A}\cap H_0&=\iota^{-1}(\sigma(\{f'|\bar{H}_2\colon f\in\mathscr{B}(X_2,t_2), f|K_n\in\mathscr{C}(K_n,K_n\cap t_2) \text{ for every } n\in\mathbb{N}\}))\\&=\iota^{-1}(\mathfrak{B}_0(\bar{H}_2)).\end{aligned}$$

$\iota^{-1}|H_2$ is continuous because of

$$\upsilon(H_1,\mathcal{I}(X_1,t_0))\subset\upsilon(H_1,\mathcal{I}(X_1,t_1)).$$

4. The relation

$$0=\mu(X-X_1)=\int_H \nu(X-X_1)\,dq$$

implies $q(H-H_0)=0$. Because of this and (f) a probability measure is defined by

$$\bar{q}(B):=q(\iota^{-1}(B))\qquad\text{for every } B\in\mathfrak{B}_0(H_2).$$

We denote the unique tight extension of $\bar{q}$ to $\mathfrak{B}(\bar{H}_2)$ by $\bar{q}$ as well and show that it is maximal in the Choquet ordering. Suppose $\bar{m}\in\mathcal{M}_+(\bar{H}_2)$ with $\bar{q}\prec\bar{m}$. The compact sets K_n being closed and open in X_2, we have $X_1\in\mathfrak{B}_0(X_2,t_2)$ which implies

$$(1_{X_2-X_1})'\in\mathfrak{B}_0^{\vee}(\bar{H}_2)\cap(-\mathfrak{B}_0^{\vee}(\bar{H}_2)),$$

hence

$$\int_{\bar{H}_2}(1_{X_2-X_1})'\,d\bar{m}=\int_{\bar{H}_2}(1_{X_2-X_1})'\,d\bar{q}$$

which means $\bar{m}(\bar{H}_2-H_2)=0$.

In (f) we showed that ι^{-1} is continuous on H_2; thus a tight measure m is defined on H by

$$m(B):=\bar{m}(\iota(B\cap H_0))\qquad\text{for every } B\in\mathfrak{B}(H).$$

If $f\in\mathscr{B}_0(X,t)$ let $\bar{f}\in\mathscr{B}_0(X_2,t_2)$ be an extension of $f|X_1$. Suppose $f_1,\dots,f_n\in\mathscr{B}_0(X,t)$ and $c_1,\dots,c_n\in\mathbb{R}$. By part c) of the lemma

$$\int_{\bar{H}_2} \bigvee_{i \leqq n} (\bar{f}_i' + c_i)\, d\bar{q} \leqq \int_{\bar{H}_2} \bigvee_{i \leqq n} (\bar{f}_i' + c_i)\, d\bar{m},$$

hence

$$\int_{H} \bigvee_{i \leqq n} (f_i' + c_i)\, dq \leqq \int_{H} \bigvee_{i \leqq n} (f_i' + c_i)\, dm.$$

By the lemma this is equivalent to $q \prec m$. By hypothesis q is maximal which implies $q = m$, hence $\bar{q} = \bar{m}$. Thus $\bar{q}$ is maximal.

5. If $W \in \mathfrak{W}(H)$ is disjoint from $\operatorname{ex} H$ we have $W \cap \operatorname{ex} H_0 = \varnothing$ by (d) and $\imath(W \cap H_0) \cap \operatorname{ex} \bar{H}_2 = \varnothing$ by (d) and (e). From the construction of X_1 and t_1 follows $W \in \mathfrak{A}$, hence $\imath(W \cap H_0) \in \mathfrak{B}_0(\bar{H}_2)$ by (f). The maximal measure $\bar{q}$ vanishes on every Baire set disjoint from $\operatorname{ex} \bar{H}_2$ ([1], p. 38), especially $\bar{q}(\imath(W \cap H_0)) = 0$ which implies $q(W) = 0$.

This completes the proof in the compact case.

II. Now we consider an arbitrary topological space.

1. As μ is tight there is a sequence of mutually disjoint compact sets $(K_n\colon n \in \mathbb{N})$ in X such that

$$X_1 := \bigcup_{n \in \mathbb{N}} K_n$$

has measure $\mu(X)$. Exactly like in part I we can construct the compact space (X_2, t_2), the mapping $\imath$ and the set H_2 such that (c), (d) and (e) keep valid. Again we have $X_1 \cap \mathfrak{B}(X, t) = X_1 \cap \mathfrak{B}(X_2, t_2)$, hence

(g) $\imath \mathfrak{W}(\operatorname{ex} H_0) = \mathfrak{W}(\operatorname{ex} H_2)$.

According to part I, for $\imath\mu \in \bar{H}_2$ there is a probability measure $\bar{p}$ on $\mathfrak{W}(\operatorname{ex} \bar{H}_2)$ for which

$$\imath\mu(A) = \int_{\operatorname{ex} H_2} \nu(A)\, d\bar{p} \qquad \text{for every } A \in \mathfrak{B}(X_2, t_2).$$

$\imath\mu(X_2 - X_1) = 0$ implies

$$1 = \bar{p}(\{\nu \in \operatorname{ex} \bar{H}_2\colon \nu(X_2 - X_1) = 0\}) = \bar{p}(\operatorname{ex} H_2).$$

Using (g) we define the image measure $\bar{p} \circ \imath$ on $\mathfrak{W}(\operatorname{ex} H_0)$ and denote its canonical extension to $\mathfrak{W}(\operatorname{ex} H)$ by p. If $A \in \mathfrak{B}(X, t)$, then

$$\mu(A) = \imath\mu(A \cap X_1) = \int_{\operatorname{ex} H_2} \nu(A \cap X_1)\, d\bar{p} = \int_{\operatorname{ex} H} \nu(A)\, dp.$$

Hence (1) holds and the theorem is proved.

Proof of the Corollary. Let f be a bounded μ-measurable function on X. There are $g \in \mathscr{B}(X, t)$ and $N \in \mathfrak{B}(X, t)$ with $f | X - N = g | X - N$ and $\mu(N) = 0$. From

$$\int_{\operatorname{ex} H} \nu(N)\, dp = \mu(N) = 0$$

follows

$$p(\{\nu \in \operatorname{ex} H\colon \nu(N) > 0\}) = 0.$$

We denote this null-set by M. f' is defined on $\mathrm{ex}\,H-M$ (and can be trivially extended to $\mathrm{ex}\,H$) and equal to g' there. Thus

$$\mu(f)=\mu(g)=\int_{\mathrm{ex}\,H-M} v(g)\,dp=\int_{\mathrm{ex}\,H-M} v(f)\,dp=\int_{\mathrm{ex}\,H} v(f)\,dp.$$

This completes the proof.

4. Remarks

1. The proof yields a stronger result as stated in the theorem: There is a tight probability measure q on the Borel sets of H such that (2) holds for q and $q|\sigma(\{f'|H: f\in\mathfrak{B}(X,t)\})$ induces p by outer measure construction.
2. Our final remark concerns uniqueness of representing measures. We introduce

$$\mathcal{M}:=\{q\in\mathcal{M}_+(H)\text{: there is } \lambda\in\mathcal{M}_+(X,t) \text{ such that } \lambda(f)=\int_H v(f)\,dq \text{ for every } f\in\mathcal{B}(X,t)\}$$

and

$$\mathcal{B}^{\vee}(H):=(\bigvee_{i\leq n}(f_i'+c_i)|H: f_i\in\mathcal{B}(X,t);\ c_i\in\mathbb{R};\ i=1,\dots,n;\ n\in\mathbb{N}\}.$$

On $\mathcal{M}$ an inductive ordering is defined by

$$p\prec\!\cdot\, q \quad \text{iff} \quad p(b)\leqq q(b) \qquad \text{for every } b\in\mathcal{B}^{\vee}(H).$$

If X is compact, then it is equivalent to the Choquet ordering. In $(\mathcal{M}, \prec\!\cdot)$ we can prove analoga of the classical uniqueness theorems, e.g. [1], II.3.6 ([9], p. 62).

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