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Partial Geometries in Finite Affine Spaces

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1. Introduction

A finite partial geometry [2] is an incidence structure $S=(P,B,\mathrm{I})$ with a symmetric incidence relation satisfying the following axioms:

(i) each point is incident with $t+1$ lines ($t \geqq 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $s+1$ points ($s \geqq 1$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there are exactly α ($\alpha \geqq 1$) points $x_1, x_2, \ldots, x_\alpha$ and α lines $L_1, L_2, \ldots, L_\alpha$ such that $x \mathrm{I} L_i \mathrm{I} x_i \mathrm{I} L$, $i=1,2,\ldots,\alpha$.

If $|P|=v$ and $|B|=b$, then $v=(s+1)(st+\alpha)/\alpha$ and $b=(t+1)(st+\alpha)/\alpha$ [2]. Also $\alpha(s+t+1-\alpha) | st(s+1)(t+1)$ [7] and $(t+1-2\alpha)s \leqq (t+1-\alpha)^2(t-1)$ [4].

The partial geometries with $\alpha=1$ are the generalized quadrangles [7]. If $\alpha=s+1$, then the partial geometry is the same as a $2-(v, s+1, 1)$ design. If $\alpha=t$, the geometry is nothing else than a net of order $s+1$ and deficiency $s-t+1$ [6].

If the points x, y (resp. lines L, M) of S are incident with a common line (resp. point) of S, then we write $x \sim y$ (resp. $L \sim M$); otherwise we write $x \not\sim y$ (resp. $L \not\sim M$).

Recently all partial geometries with B a lineset of $PG(n,q)$, $n \geqq 2$, P the set of all points of $PG(n,q)$ on these lines, and I the natural incidence relation, were determined (the case $\alpha=1$ was handled by Buekenhout and Lefèvre ([3], [18]), the case $\alpha>1$ by De Clerck and Thas [5]).

In this paper we determine all partial geometries with B a lineset of $AG(n,q)$, $n \geqq 2$, P the set of all points of $AG(n,q)$ on these lines, and I the natural incidence relation.

2. Lemma

Let $S=(P,B,\mathrm{I})$ be a partial geometry with parameters s, t, α, where $\alpha>1$, and suppose that $S'=(P',B',\mathrm{I}')(P' \subset P, B' \subset B, \mathrm{I}'=\mathrm{I} \cap ((P' \times B') \cup (B' \times P')))$ is a substructure of S for which the following conditions are satisfied:

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(i) $|B'|>1$ *and any element of* B' *is incident with* $s+1$ *elements of* P';

(ii) *if the line* $L\in B$ *is incident with the points* $x, y\in P'$, $x\neq y$, *then* $L\in B'$.

Then S' *is a partial geometry with parameters* s, t', α.

Proof. It is sufficient to prove that any point of P' is incident with $t'+1$ ($t'\geqq 1$) lines of B'. Consider distinct points x, y of P', and let t'_x+1, resp. t'_y+1, be the number of lines of B' which are incident with x, resp. y. Now we count in two different ways the number of ordered pairs (L_x, L_y), where $L_x\neq L_y$, $x\,\mathrm{I}'\,L_x$, $y\,\mathrm{I}'\,L_y$, $L_x\sim L_y$. If $x\sim y$, then we obtain $t'_x(\alpha-1)=t'_y(\alpha-1)$; if $x\nsim y$, then we obtain $(t'_x+1)\,\alpha=(t'_y+1)\,\alpha$. Consequently $t'_x=t'_y$. Hence any point of P' is incident with $t'+1$ lines of B'. From $|B'|>1$ and $\alpha>1$, it follows that $t'\geqq 1$. We conclude that S' is a partial geometry with parameters s, t', α.

3. Embedding in $AG(2, s+1)$

Theorem. *If the partial geometry* $S=(P, B, \mathrm{I})$ *with parameters* s, t, α *is embedded in* $AG(2, s+1)$, *then* S *is a net of order* $s+1$ *and deficiency* $s-t+1$, *or* $B\cup\{$*line at infinity of* $AG(2, s+1)\}$ *is a complete oval of the dual projective plane of* $PG(2, s+1)$, *where* $PG(2, s+1)$ *is the projective completion of* $AG(2, s+1)$ (*here necessarily* $s=2^h-1$).

Proof. If the partial geometry $S=(P, B, \mathrm{I})$ with parameters s, t, α is embedded in $AG(2, s+1)$, then evidently $\alpha\in\{t, t+1\}$. Consequently S is a net of order $s+1$ and deficiency $s-t+1$, or a dual design. Let S be a dual design. Since no two distinct lines of S are parallel, we have $b\leqq s+2$ and so $t+1=2$. Consequently B is a set of $s+2$ lines, no three of which are concurrent and no two of which are parallel. This proves the theorem.

4. Embedding in $AG(3, s+1)$

4.1. **Theorem.** *Suppose that the partial geometry* $S=(P, B, \mathrm{I})$ *with parameters* s, t, α, *where* $\alpha>1$, *is embedded in* $AG(3, s+1)$, *and that* P *is not contained in a plane of* $AG(3, s+1)$. *Then the following cases can occur.*

(a) $s=1$, $\alpha=2$, $t\in\{2, 3, 4, 5\}$ (S *is a* $2-(t+2, 2, 1)$ *design in* $AG(3, 2)$).

(b) P *is the pointset of* $AG(3, s+1)$, *and* B *is the set of all lines of* $AG(3, s+1)$ *whose points at infinity are the points of a (maximal)* $\{(s+1)n-(s+1)+n; n\}$*-arc of the plane at infinity of* $AG(3, s+1)$ (*here* $n-1=\alpha$, $t=(s+2)(n-1)$, $2<n\leqq s+2$).

Proof. Suppose that $x\in P$, $L\in B$ and $x\,\not\mathrm{I}\,L$. Then a substructure $S_\omega=(P_\omega, B_\omega, \mathrm{I}_\omega)$ of S is induced in the plane $xL=\omega$. From the lemma follows that S_ω is a partial geometry with parameters s, t', α. Since S_ω is embedded in a plane, S_ω is a net of order $s+1$ and deficiency $s-t'+1$ ($\alpha=t'$), or $B_\omega\cup\{$line at infinity of $\omega\}$ is a complete oval of the dual of the projective completion of ω.

Let us suppose that S_ω is of the second type. Then $s=2^h-1$ and $\alpha=2$. Now we assume that there exists a point $x'\in P$, a line $L'\in B$, $x'\,\not\mathrm{I}\,L'$, such that the corresponding geometry $S_{\omega'}$ is a net (necessarily of deficiency $s-1$). If $z \notin P_\omega$ (resp. $z \notin P_{\omega'}$), then the number of lines M of B which are incident with z and a point of P_ω (resp. $P_{\omega'}$) equals $s+2$ (resp. $2s+2$). Now we consider the lines of S incident with z and parallel to the plane ω (resp. ω'). If there are at least two such lines, then their plane defines a subgeometry $S_{\omega''}$. For such a subgeometry $t'' \in \{2, 1\}$, and so there are at most three lines of S which are incident with z and parallel to ω (resp. ω'). Hence $s+2 \leqq t+1 \leqq s+5$ resp. $2s+2 \leqq t+1 \leqq 2s+5$, and so $s \in \{1, 2, 3\}$. Since s is odd, we have $s \in \{1, 3\}$. If $s=3$, then $t=7$, in contradiction with $\alpha(s+t+1-\alpha) \mid st(s+1)(t+1)$. So $s=1$ and $t \in \{3, 4, 5\}$. Next we assume that for any point $x' \in P$ and any line $L' \in B$, $x' \not\mathrm{I} L'$, the corresponding geometry $S_{\omega'}$ is not a net. From the preceding case there follows that $t+1 \geqq s+2$. Now we remark that no two lines of S are parallel (if two lines of S are parallel and if ω' is the plane of these lines, then $S_{\omega'}$ is a net). Consequently $b \leqq (s+1)^2+(s+1)+1$. Hence $(t+1)(st+2)/2 \leqq (s+1)^2+(s+1)+1$, and so $(s+2)(s(s+1)+2) \leqq (s+1)^2 +(s+1)+1$. So $s=1$ and $t=2$.

Hence, if S_ω is of the second type, then $s=1$, $\alpha=2$ and $t \in \{2, 3, 4, 5\}$.

Finally we suppose that for any plane ω, $\omega = xL$ with $x \in P$, $L \in B$, $x \not\mathrm{I} L$, the geometry S_ω is a net. Let $L' \in B$ and let ω'' be a plane containing L'. Suppose that L' is the only line of B in ω''. If $x' \in P$, $x' \not\mathrm{I} L'$, then any line of S incident with x', is incident with a point of L' or is parallel to ω''. Since $S \neq S_{\omega'}$, with $\omega' = x'L'$, there is a line M in B which is incident with x', which is not contained in ω', and which is parallel to ω''. If $y \mathrm{I} L'$, then S_{yM} is a net and so the intersection of yM and ω'' is a line of B, a contradiction. Hence the plane ω'' contains a second line of S, and so all points of ω'' are points of P ($S_{\omega''}$ is a net) and all lines of ω'' parallel to L' are lines of B. Consequently P is the pointset of $AG(3, s+1)$, and if $L' \in B$ then B contains all lines parallel to L'. Let $PG(2, s+1)$ be the plane at infinity of $AG(3, s+1)$, and let us consider the points at infinity of the lines of B. The set of these points intersects any line of $PG(2, s+1)$ in $\alpha+1$ points or in none at all. Consequently this set is a (maximal) $\{(s+1)(\alpha+1)-(s+1)+(\alpha+1); \alpha+1\}$-arc [1] of $PG(2, s+1)$. This proves completely the theorem.

4.2. **Theorem.** *Suppose that the generalized quadrangle $S=(P, B, \mathrm{I})$ with parameters s, t is embedded in $AG(3, s+1)$, and that P is not contained in a plane of $AG(3, s+1)$. Then the following cases can occur.*

(a) *$s=1$, $t=2$ (trivial case).*

(b) *$t=1$ and the elements of S are the affine points and affine lines of an hyperbolic quadric of $PG(3, s+1)$, the projective completion of $AG(3, s+1)$, which is tangent to the plane at infinity of $AG(3, s+1)$.*

(c) *$s=2$, $t=2$ (an embedding of the generalized quadrangle with 15 points and 15 lines in $AG(3, 3)$).*

(d) *P is the pointset of $AG(3, s+1)$, and B is the set of all lines of $AG(3, s+1)$ whose points at infinity are the points of a complete oval of the plane at infinity of $AG(3, s+1)$ (here $s+1=2^h$ and $t=s+2$).*

(e) *P is the pointset of $AG(3, s+1)$ and $B=B_1 \cup B_2$, where B_1 is the set of all affine totally isotropic lines with respect to a symplectic polarity π of the projective completion $PG(3, s+1)$ of $AG(3, s+1)$ and where B_2 is the class of parallel lines defined by the pole (the image with respect to π) of the plane at infinity of $AG(3, s+1)$ (here $t=s+2$).*

Proof. Suppose that $x \in P$, $L \in B$ and $x \not\mathrm{I} L$. Then a substructure $S_\omega=(P_\omega, B_\omega, \mathrm{I}_\omega)$ is induced in the plane $xL=\omega$. From [16] follows that S_ω is a net with parameters s, $t'=1$ (consisting of two classes of parallel lines in ω) or that B_ω is a set of lines with common point y. Suppose that B_ω is a set of lines with common point y, and that there exists a line M in B which is incident with y and which is not contained in ω (then we have $t>1$). Let $z \mathrm{I} M$, $z \in P$, $z \neq y$. The lines of B through z are necessarily the line M and t lines in a plane ω' parallel to ω. Suppose a moment that $S_{\omega'}$ is a net. Then $t=2$ and the number of lines of B which are incident with y and have a point in common with the net $S_{\omega'}$, equals $s+1$. So there are at least $(s+1)+2>3$ lines of B which are incident with y, a contradiction. Consequently $B_{\omega'}$ is a set of lines with common point z. Analogously (interchange y and z) B_ω is a set of t lines with common point y. If S_ω is a net (where ω is a plane containing at least two lines of B), then we say that ω is of type I; if B_ω is a set of t lines having a common point y, then we say that ω is of type II (if M is the line defined by $y \mathrm{I} M$, $M \in B-B_\omega$, and if $z \mathrm{I} M$, then the $t+1$ lines of B incident with z are M and t lines in a plane ω' parallel to ω; moreover ω' is also of type II); if B_ω is a set of $t+1$ lines having a common point y, then we say that ω if of type III. Now we consider three cases.

(a) $t>2$. Suppose that the plane ω contains a line L of B. Assume a moment that L is the only line of B in ω. Let $L \mathrm{I} y \mathrm{I} M \mathrm{I} x$, $M \in B-\{L\}$, $x \neq y$. The lines of B which are incident with x are M and t lines in a plane ω' parallel to ω. Since $t>2$ the plane ω' is of type II. Consequently the lines of B which are incident with y are M and t lines in the plane ω, a contradiction. There follows that ω contains at least two lines of B.

Next we suppose that ω is a plane of type III, and let L be a line of B_ω. The common point of the $t+1$ lines of B_ω is denoted by y. Now we assume that any plane through L is of type II or III. Since there are $s+2$ planes through L and only $s+1$ points on L, there is some point z on L which is incident with at least $2t-1$ lines, a contradiction. So there is a plane ω' through L for which $S_{\omega'}$ is a net. There are two lines L, N of $S_{\omega'}$ which are incident with y, and hence y is incident with at least $t+2$ lines of B, a contradiction. There follows that there are no planes of type III.

Now we assume that there is at least one plane ω of type II. The common point of the lines of B_ω is denoted by y_0, and let M be the line for which $y_0 \mathrm{I} M$ and $M \in B-B_\omega$. Suppose that $y_0, y_1, \ldots, y_s$ are the points of M, and that $L_{i1}, \ldots, L_{it}$, M are the $t+1$ lines of B incident with y_i $(i=0,1,\ldots,s)$. If ω' is a plane which contains L_{ij}, which does not contain M, and which is not parallel to ω, then ω' is of type II (otherwise the common point of L_{ij} and M is incident with at least $t+2$ lines of B, a contradiction). If ω'' is a plane which contains M, then ω'' is of type I (if ω'' is of type II and if y_i is the common point of the lines of $B_{\omega''}$, then y_i is incident with the t lines of $B_{\omega''}$ and also with the t lines $L_{i1}, \ldots, L_{it}$, a contradiction). Consequently for any $i \in \{0,1,\ldots,s\}$ there is a line L_{ij} which is contained in $B_{\omega''}$. Hence the number of planes ω'' through M is equal to $|\{L_{i1}, \ldots, L_{it}\}|=t$. There follows that $t=s+2$, and so $v=(s+1)^3$, or P is the pointset of $AG(3,s+1)$. From the preceding there also follows that any line of $AG(3,s+1)$ which is parallel to M, is an element of B. We also remark that any plane parallel to M is of type I, and that any plane not parallel to M is of type II (since such a plane does not contain M, but contains at least one of the lines L_{ij}). Now it is easy to see that all lines parallel to M play exactly the same role. The plane at infinity of $AG(3,s+1)$ is denoted by π_∞, and the point at infinity of M is denoted by y_∞. Let y_i' be a point of M', where M' is parallel to M, and let $L'_{i1},\dots,L'_{it}$, M' be the lines of B which are incident with y_i'. The lines $L'_{i1},\dots,L'_{it}$ are contained in a plane ω_i', and the line at infinity M'_∞ of ω_i' is independent of the choice of the point y_i' on M' (remark that y_∞ is not on M'_∞). If the lines M' and M'', $M' \neq M''$, are both parallel to M, then $M'_\infty \neq M''_\infty$ (if $M'_\infty = M''_\infty$, then any plane with line at infinity M'_∞ contains at least $2t-1$ lines of B, a contradiction). So with the $(s+1)^2$ lines parallel to M' correspond the $(s+1)^2$ lines of π_∞ which does not contain y_∞. Now we consider a line N_∞ of π_∞ through y_∞. A plane ω'' with line at infinity N_∞ is of type I, and the lines of B in ω'' define two points at infinity y_∞ and z_∞ on N_∞. Consequently with the $s+1$ lines of ω'' which are parallel to M, there correspond the $s+1$ lines of π_∞ which contain y_∞. Now we define as follows an incidence structure $S'=(P',B',\mathrm{I}')$: $P' = P \cup P_\infty$ with P_∞ the pointset of π_∞; $B'=(B-B_M)\cup B_\infty$, where B_M is the set of lines parallel to M and where B_∞ is the set of the lines of π_∞ which contain y_∞; I' is the natural incidence relation. From the preceding considerations it follows readily that S' is a generalized quadrangle with parameters $s'=t'=s+1$, which is embedded in the projective completion $PG(3,s+1)$ of $AG(3,s+1)$. Now the theorem of Buekenhout and Lefèvre [3] tells us that B' is the set of totally isotropic lines with respect to a symplectic polarity of $PG(3,s+1)$. Consequently S is the generalized quadrangle with parameters s, $s+2=t$ defined by the regular point y_∞ of S' ([14], p. 20). We conclude that we have here case (e) in the statement of the theorem.

Finally we assume that there are no planes of type II. Let L be a line of B, and let ω be an arbitrary plane containing L. Then ω contains at least two lines of B, and S_ω is a net with parameters s, $t'=1$. Consequently any point of ω is in P, and any line of ω parallel to L belongs to B. Hence P is the pointset of $AG(3,s+1)$ and if $L\in B$, then B contains all lines parallel to L. Let $PG(2,s+1)$ be the plane at infinity of $AG(3,s+1)$, and let us consider the points at infinity of the lines of B. The set of these points intersects any line of $PG(2,s+1)$ in 2 points or in none at all.

Consequently this set is a complete oval of $PG(2,s+1)$. We conclude that we have here case (d) in the statement of the theorem.

(b) $t=1$. Suppose that $B=\{L_0,\dots,L_s,M_0,\dots,M_s\}$, $L_i \sim M_j$, and let us consider the projective completion $PG(3,s+1)$ of $AG(3,s+1)$. If $s \geqq 2$, then the $s+2$ lines of $PG(3,s+1)$ which are concurrent with the projective lines M_0, M_1, M_2 constitute a regulus R of an hyperbolic quadric Q (the projective lines M_i, M_j $(i \neq j)$ are non-concurrent, since P is not contained in an $AG(2,s+1)$). Consequently $L_0,\dots,L_s$ are elements of R, and $M_0,\dots,M_s$ are elements of the complementary regulus R' of Q. There also follows that Q contains two lines at infinity. Se we have case (b) in the statement of the theorem. If $s=1$, then it is easy to check that we have also case (b).

(c) $t=2$. We shall prove that $s\in\{1,2\}$. First of all we assume that there is a plane ω of type I. If x is a point of $P-P_\omega$, then the number of lines of B which are incident with x and a point of S_ω equals $s+1$. Hence $s+1\leqq t+1=3$, or $s\in\{1,2\}$.

Now we suppose that there is no plane of type I. Let $L\in B$ and assume a moment that the plane ω contains only the line L of B. If x is a point of P which is not in ω, then the lines of B which are incident with x are the line M defined by $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$, and two lines in a plane ω' parallel to ω. Evidently ω' is of type II. Consequently the lines of B which are incident with y are M and two lines in ω, a contradiction. There follows that the plane ω is always of type II or III. Now we suppose that any plane ω through L is of type III. Since there are $s+2$ planes through L and only $s+1$ points on L, there is a point on L which is incident with at least 5 lines of B, a contradiction. Consequently there exists a plane ω of type II. Let ω be of type II and suppose that L_1, $L_2\in B_\omega$, $L_1\mathrm{I}x\mathrm{I}L_2$, $x\mathrm{I}M$, $M\in B-B_\omega$. If $y\mathrm{I}M$, then the lines of B which are incident with y are M and two lines in a plane ω' parallel to ω. A plane ω'' through M is necessarily of type II (if ω'' is of type III, then there is a point on M which is incident with at least 4 lines of B). Hence the number of lines of B having one point in common with M equals $s+2$. This number also equals $(s+1)t=2(s+1)$, a contradiction.

So we conclude that there is at least one plane of type I, and that $s\in\{1,2\}$. The case $s=1$, $t=2$ is a trivial case. So there remains only the case $s=t=2$.

Let ω be a plane of type I, with

$$B_\omega=\{L_0,L_1,L_2,M_x,M_y,M_z\},$$

$$P_\omega=\{x_0,y_0,z_0,x_1,y_1,z_1,x_2,y_2,z_2\},$$

$x_i\mathrm{I}M_x$, $y_i\mathrm{I}M_y$, $z_i\mathrm{I}M_z$, $x_i\mathrm{I}L_i$, $y_i\mathrm{I}L_i$, $z_i\mathrm{I}L_i$. Suppose that $x_0\mathrm{I}N_x$, $y_0\mathrm{I}N_y$, $z_0\mathrm{I}N_z$ ($N_x,N_y,N_z\in B$), where $N_x\notin\{M_x,L_0\}$, $N_y\notin\{M_y,L_0\}$, $N_z\notin\{M_z,L_0\}$, that x_0, x_3, x_4 are the points of N_x, that y_0, y_3, y_4 are the points of N_y, and that z_0, z_3, z_4 are the points of N_z. Then $P=\{x_i,y_i,z_i\,\|\,i=0,1,2,3,4\}$. We remark that the planes N_xM_x, N_yM_y, N_zM_z are parallel (since these planes are of type II), and that the planes ω, L_0N_x, L_0N_y, L_0N_z are exactly the 4 planes which contain L_0 (if $L_0N_x=L_0N_y$, then $L_0N_x=L_0N_y=L_0N_z=\omega'$ is of type I, and so, if V is one of the lines of $B_{\omega'}$ parallel to L_0, then L_1V is of type I, a contradiction). Now we shall speak about the six lines of the set $B-\{L_0, L_1, L_2, M_x, M_y, M_z, N_x, N_y, N_z\}$. Since the planes N_xM_x, N_yM_y, N_zM_z are parallel we may assume that x_3, y_3, z_3 resp. x_4, y_4, z_4 are collinear in $AG(3,3)$ (the lines x_3y_3 and x_4y_4 are parallel to ω). As any line of B is incident with a point of P_ω, the lines x_3y_4, x_4y_3, x_3z_4, x_4z_3, y_3z_4, y_4z_3 are the remaining six lines of B. Let us suppose that z_1 is on y_3x_4, then z_2 is on x_3y_4, x_2 is on y_3z_4, x_1 is on y_4z_3, y_1 is on x_3z_4, y_2 is on x_4z_3. Now S is completely described. From this detailed description it easily follows that the generalized quadrangle with 15 points and 15 lines is effectively embeddable in an $AG(3,3)$.

Remark. If S is of type 4.2.e with $s=2$, then S is the unique generalized quadrangle with 27 points and 45 lines ([10], [11], [17]). In such a case the subquadrangles of S with parameters $s=2$, $t'=2$ are of type 4.2.c.

5. Embedding in $AG(d, s+1)$, $d \geqq 4$

5.1. Embedding of Partial Geometries with $\alpha > 1$

Theorem. *Suppose that the partial geometry $S=(P, B, \mathrm{I})$ with parameters s, t, α ($\alpha > 1$) is embedded in $AG(d, s+1)$, where $d \geqq 4$, and that P is not contained in an $AG(d', s+1)$, with $d' < d$. Then the following cases can occur.*

(a) *$s=1$, $\alpha=2$, $t \in \{d-1, d, \ldots, 2^d\}$ and then S is a $2-(t+2, 2, 1)$ design (P is an arbitrary pointset of $AG(d, 2)$ which is not contained in an $AG(d', 2)$, $d' < d$).*

(b) *S is the design of points and lines of $AG(d, s+1)$.*

(c) *P is the pointset of $AG(d, s+1)$, and B is the set of all lines of $AG(d, s+1)$ whose points at infinity constitute the complement of a hyperplane $PG(d-2, s+1)$ of the space at infinity of $AG(d, s+1)$.*

Proof. Suppose that $s=1$. Then $\alpha=2$ and S is a $2-(t+2, 2, 1)$ design. Evidently P is an arbitrary pointset of $AG(d, 2)$ which is not contained in an $AG(d', 2)$, $d' < d$. Here $t \in \{d-1, d, \ldots, 2^d\}$.

Now we suppose that $s \geqq 2$. Let L, M be two non-concurrent lines of S which are not parallel in $AG(d, s+1)$, and suppose that $AG(3, s+1)$ is the affine subspace containing these lines. From 2. follows that S induces a partial subgeometry $S'=(P', B', \mathrm{I}')$ in $AG(3, s+1)$, with parameters s, t', α. From 4.1 we know that P' is the pointset of $AG(3, s+1)$, and that B' is the set of all lines of $AG(3, s+1)$ whose points at infinity are the points of a (maximal) $\{(s+1)n-(s+1)+n; n\}$-arc of the plane at infinity of $AG(3, s+1)$($n-1=\alpha$, $t'=(s+2)(n-1)$, $2 < n \leqq s+2$). Let p be a point of $P-P'$, and suppose that $AG(4, s+1)$ contains p and $AG(3, s+1)$. In $AG(4, s+1)$ a partial subgeometry $S''=(P'', B'', \mathrm{I}'')$, with parameters s, t'', α, is induced. If L_1 is a line of $AG(3, s+1)$ parallel to L ($L_1 \neq L$), then in the affine threespace pLL_1 a partial subgeometry with parameters s, t_1, α is induced. Consequently all the points of pLL_1 belong to P'' (see 4.1). There follows immediately that P'' is the pointset of $AG(4, s+1)$. Let N be a line of B'' and let q be a point of P'', which is not on N. Since any point of the plane qN belongs to P'', the partial subgeometry induced in qN is a net of order $s+1$ and deficiency $s-\alpha+1$ (see 3). Consequently any line of $AG(4, s+1)$ parallel to N, belongs to B''. If Ω is the set of the points at infinity of the lines of B'', then any line of the space at infinity $PG(3, s+1)$ of $AG(4, s+1)$ contains exactly 0 or $\alpha+1$ points of Ω. Hence Ω is the pointset of $PG(3, s+1)$ or is the complement of a plane of $PG(3, s+1)$ [12]. So $\alpha=s+1$ or $\alpha=s$. Next, let r be a point of $P-P''$, and suppose that $AG(5, s+1)$ contains r and $AG(4, s+1)$. In $AG(5, s+1)$ a partial subgeometry $S'''=(P''', B''', \mathrm{I}''')$ is induced. If L_1 is a line of $AG(3, s+1)$ parallel to L ($L_1 \neq L$), then in the fourdimensional affine space $rpLL_1$ a partial subgeometry is induced. Consequently all the points of $rpLL_1$ belong to P'''. There follows immediately that P''' is the pointset of $AG(5, s+1)$. If N is a line of B''', then any line of $AG(5, s+1)$ parallel to N belongs to B''', and if Ω' is the set of the points at infinity of the lines of B''', then any line of the space at infinity $PG(4, s+1)$ of $AG(5, s+1)$ contains exactly 0 or $\alpha+1$($\alpha \in \{s, s+1\}$) points of Ω'. Hence Ω' is the pointset of $PG(4, s+1)$ or is the complement of a threespace of $PG(4, s+1)$. By repeating this reasoning, the desired result follows.

5.2. *Embedding of Generalized Quadrangles*

Theorem. *Suppose that the generalized quadrangle $S=(P, B, \mathrm{I})$ with parameters s, t is embedded in $AG(4, s+1)$, and that P is not contained in an $AG(3, s+1)$. Then the following cases can occur.*

(a) $s=1$, $t\in\{2, 3, 4, 5, 6, 7\}$ *(trivial cases).*

(b) $s=t=2$ *(an embedding of the generalized quadrangle with* 15 *points and* 15 *lines in $AG(4, 3)$).*

(c) $s=t=3$ *and S is isomorphic to the generalized quadrangle $Q(4, 3)$ arising from a non-singular hyperquadric of $PG(4,3)$.*

(d) $s=2$, $t=4$ *(an embedding of the generalized quadrangle with* 27 *points and* 45 *lines in $AG(4, 3)$).*

Proof. Suppose that $s=1$. Let $x_0, x_1, \ldots, x_t, y_0, y_1, \ldots, y_t$, $t\in\{2, \ldots, 7\}$, be distinct points of $AG(4,2)$ which are not contained in an hyperplane. Then the sets $P=\{x_i, y_j \| i,j\in\{0, \ldots, t\}\}$ and $B=\{\{x_i, y_j\} \| i,j\in\{0, \ldots, t\}\}$ define a generalized quadrangle with parameters $s=1$ and t. From now on we suppose $s\geqq 2$.

Let L, M be two non-concurrent lines of S which are not parallel in $AG(4, s+1)$, and suppose that $AG(3, s+1)$ is the affine subspace containing these lines. From [16] follows that S induces a generalized subquadrangle $S'=(P', B', \mathrm{I}')$, with parameters s, t', in $AG(3, s+1)$.

Suppose that S' is of type 4.2.d or 4.2.e. Then $t'=s+2$. In [16] it is proved that $st'\leqq t$. Since $s\neq 1$, we have also $t\leqq s^2$ [7]. Consequently $s(s+2)\leqq s^2$, a contradiction.

Next, we suppose that S' is of type 4.2.c. Then $s=t'=2$. Since $st'\leqq t\leqq s^2$, we have $t=4$. So S is necessarily the unique generalized quadrangle with 27 points and 45 lines. For the points and lines of S' we use the notations of the final part of the proof of 4.2. Let N_x, N'_x, N''_x, M_x, L_0 be the lines of B which contain x_0. The hyperplane $AG(3, 3)$ defined by ω and N_x is denoted by H, the hyperplane $\omega N'_x$ is denoted by H', and the hyperplane $\omega N''_x$ is denoted by H''. The subquadrangle $S''=(P'', B'', \mathrm{I}'')$ resp. $S'''=(P''', B''', \mathrm{I}''')$ induced in H' resp. H'' has parameters s, t'' resp. s, t''', with $s=t''=t'''=2$. Suppose that $N'_y, N'_z\in B''$, $y_0 \mathrm{I} N'_y$, $z_0 \mathrm{I} N'_z$, $N'_y\notin\{M_y, L_0\}$, $N'_z\notin\{M_z, L_0\}$, and that $N''_y, N''_z\in B'''$, $y_0 \mathrm{I} N''_y$, $z_0 \mathrm{I} N''_z$, $N''_y\notin\{M_y, L_0\}$, $N''_z\notin\{M_z, L_0\}$. Any point of S is on one of the lines L_0, M_x, M_y, M_z, N_x, N_y, N_z, N'_x, N'_y, N'_z, N''_x, N''_y, N''_z. The point at infinity of L_0 is denoted by (L_0), etc. Then $(M_x)=(M_y)=(M_z)$. Moreover the points (N_x), (N_y), (N_z), (M_x) are on a line N_∞, the points (N'_x), (N'_y), (N'_z), (M_x) are on a line N'_∞, and the points (N''_x), (N''_y), (N''_z), (M_x) are on a line N''_∞. So there arise three lines N_∞, N'_∞, N''_∞ which contain the point (M_x). Let us consider the lines N_a and N'_b, a, $b\in\{x, y, z\}$ and $a\neq b$. There are three lines L_0, L_{abc}, L'_{abc} of B concurrent with N_a and N'_b, and there are also three lines N_a, N'_b, T concurrent with each of L_0, L_{abc}, L'_{abc} [14]. We have necessarily $T=N''_c$, with $\{a, b, c\}=\{x, y, z\}$. Consequently the lines N_a, N'_b, N''_c, L_0, L_{abc}, L'_{abc} constitute a generalized quadrangle which is embedded in the affine threespace defined by N_a, N'_b, and this quadrangle necessarily is of type 4.2.b. There follows that (N_a), (N'_b), (N''_c) are on a line V_∞, that (L_0), (L_{abc}), (L'_{abc}) are on a line W_∞, and that V_∞ and W_∞ have a point u_∞ in common ((N_a), (N'_b), (N''_c), u_∞ are the four points of V_∞, and (L_0), (L_{abc}), (L'_{abc}), u_∞ are the four points of W_∞). If L_0, D_{ab}, E_{ab} (resp. L_0, D'_{ab}, E'_{ab}, resp. L_0, D''_{ab}, E''_{ab}), $a \neq b$ and $a, b \in \{x, y, z\}$, are the lines of S which are concurrent with N_a, N_b (resp. N'_a, N'_b, resp. N''_a, N''_b), then the lines L_0, L_1, L_2, M_x, M_y, M_z, N_x, N_y, N_z, N'_x, N'_y, N'_z, N''_x, N''_y, N''_z, D_{ab}, E_{ab}, D'_{ab}, E'_{ab}, D''_{ab}, E''_{ab}, L_{abc}, L'_{abc} are the 45 lines of S.

So we have the following construction of the generalized quadrangle with 27 points and 45 lines in $AG(4, 3)$. First of all we choose the points (L_0), (M_x), (N_x), (N_y), (N_z), (N'_x), (N'_y), (N'_z), (N''_x), (N''_y), (N''_z) in such a way that the conditions above are satisfied. Next we choose the line L_0 through (L_0), and we label the three points on L_0. Such a choice determines a configuration of 27 points and 45 lines, and it is not a difficult but a tedious work to check that this incidence structure is indeed a generalized quadrangle. We also remark that in the hyperplane at infinity coordinates can be chosen in such a way that we have $(L_0) = (0, 0, 0, 1)$, $(M_x) = (1, 0, 0, 0)$, $(N_x) = (0, 1, 0, 0)$, $(N_y) = (1, 1, 0, 0)$, $(N_z) = (1, -1, 0, 0)$, $(N'_x) = (0, 0, 1, 0)$, $(N'_y) = (1, 0, 1, 0)$, $(N'_z) = (1, 0, -1, 0)$, $(N''_x) = (0, 1, 1, 0)$, $(N''_y) = (1, -1, -1, 0)$, $(N''_z) = (1, 1, 1, 0)$.

Finally we suppose that any two non-coplanar lines of S define a subquadrangle of type 4.2.b. If the lines L, M of S contain the point p of S, then the plane LM contains just the lines L, M of S (if N is a line of S which is concurrent with L, but not coplanar with M, then in the threespace MN a quadrangle of type 4.2.b is induced, and so L, M are the only lines of S in the plane LM). Let L be a line of S, let $p_0, \ldots, p_s$ be the points of L, and let L, $M_{i1}, \ldots, M_{it}$ be the $t+1$ lines of S through p_i. Then the t^2+s+1 hyperplanes $M_{1k}M_{2l}$, $LM_{i1}M_{i2}$ are distinct. The number of hyperplanes containing L equals $(s+1)^2+(s+1)+1$, and so $t^2 \leqq (s+1)^2+1$ or $t \leqq s+1$. Since any pair of distinct lines of S is regular, we have $t=1$ or $t \geqq s$ [14]. Since $t \neq 1$, we have $t \in \{s, s+1\}$. Since $s \neq 1$ and since $s+t$ divides $st(s+1)(t+1)$ [7], there holds $s=t$. As any line of S is regular, S is isomorphic to the classical quadrangle $Q(4,s)$ [14].

Let H be the threespace defined by three concurrent lines L_0, L_1, L_2 of S. The common point of these lines is denoted by p. Then all the lines of S in H contain p and any point of S in H is on one of these lines (if there is a line of S in H which does not contain p, then a subquadrangle S' of type 4.2.b is induced in H, a contradiction since in S' there are at least three lines which contain p). The lines of S in H are denoted by $L_0, \ldots, L_{t'}$. Suppose that $t' < t$, and let L_t be a line of S through p which is not in H (necessarily $t > 2$). Let $q \mathrm{I} L_t$, $q \neq p$. The $t+1$ lines of S through q are L_t and t lines in the threespace $\bar{H}$ through q and parallel to H. Analogously the $t+1$ lines of S through p are L_t and t lines in H. So we have $t' = t-1$. Now we consider the threespace $\bar{\bar{H}}$ defined by L_0, L_1, L_t ($\bar{\bar{H}}$ does not contain $L_{t'} = L_{t-1}$). Since H and $\bar{\bar{H}}$ contain t lines of S through p, their intersection contains $t-1$ lines of S through p. If $s = t > 3$, then $t-1 \geqq 3$, and so $H = \bar{\bar{H}}$, a contradiction. So we have necessarily $s = t = 3$. The points of L_t are denoted by $p_0, \ldots, p_3$, and let L_t, $M_{i1}, \ldots, M_{i3}$ be the lines of S through p_i (the lines L_t, M_{i1}, $\ldots$, M_{i3} are not contained in a hyperplane). The point at infinity of M_{i1} is denoted by (M_{i1}), etc. The points (M_{i1}), (M_{i2}), (M_{i3}) are not collinear, and the set $\{(M_{i1}), (M_{i2}), (M_{i3})\}$ is denoted by V_i. Remark that the 12 points (M_{ij}) are coplanar ((M_{ij}) is always a point of the plane at infinity ω_∞ of the threespace H). If T is a line of ω_∞ which intersects V_i and V_j, $i \neq j$, then T also intersects V_k and

V_l, $\{i, j, k, l\} = \{0, 1, 2, 3\}$ (since any two lines M_{ia} and M_{jb} define a quadrangle of type 4.2.b). Moreover a line which contains two points of V_i, has no point in common with V_j, $i \neq j$. The number of lines of these two types equals $9+12=21$, and so any line of ω_∞ has 2 or 4 points in common with $V_0 \cup V_1 \cup V_2 \cup V_3 = V$. Let W be the set of the points of ω_∞ which are not in V. Any line of ω_∞ has 1 or 3 points in common with W, and since $|W|=9$ the set W is a unital of ω_∞ [13]. Since ω_∞ has order 4 the unital evidently is a hermitian curve. In ω_∞ there are exactly 4 triangles whose vertices are exterior points of W and whose sides are secants of W. So we have a complete description of the sets V_0, V_1, V_2, V_3. We remark that the 40 points of S are on the lines L_t, M_{ij} and that the 40 lines of S are the lines of the 9 subquadrangles defined by the pairs $\{M_{ia}, M_{jb}\}$, $i \neq j$ (the lines at infinity defined by these 9 subquadrangles are the 9 tangents of the hermitian curve W and the 9 lines which join $(L_t)=(L_3)$ to the points of W). Now we have a detailed description of the quadrangle S. From this description follows the construction of a model of the classical quadrangle $Q(4,3)$ in $AG(4,4)$.

Now we assume that for any point p of S, the lines of S through p are contained in a hyperplane. Suppose a moment that $s>2$. Let L and M be lines of S which contain the point p of S. Now we consider the $s+2$ threespaces which contain the plane LM. If H is such a threespace, then H contains only the lines L, M of S, or H contains any line of S through p, or a subquadrangle of type 4.2.b is induced in H. There are s hyperplanes of the third type, one hyperplane $\bar{H}$ of the second type, and consequently there is one hyperplane $\bar{\bar{H}}$ which contains only the lines L, M of S. Let N be a line of S through p, which is not contained in $\bar{\bar{H}}$, and let $q \mathrm{I} N$, $q \neq p$. The $s+1$ lines of S through q are N and s lines in the threespace through q and parallel to $\bar{\bar{H}}$. Analogously the $s+1$ lines of S through p are N and s lines in $\bar{\bar{H}}$, a contradiction since $s>2$. There follows that $s=t=2$.

So we suppose that $s=t=2$. Let L be a line of S, let p_0, p_1, p_2 be the points of L, and let L, M_{i1}, M_{i2} be the lines of S through p_i. Through the plane $M_{01}M_{02}$ there is exactly one hyperplane H_0 which contains only the lines M_{01}, M_{02} of S. Evidently the lines M_{11}, M_{12}, M_{21}, M_{22} are parallel to H_0. The point at infinity of M_{01} is denoted by (M_{01}), etc. Then $(M_{01}), \ldots, (M_{22})$ are points of the plane at infinity ω_∞ of H_0. In the threespace $M_{ia}M_{jb}$, $i \neq j$, a subquadrangle of type 4.2.b is induced, and so we may assume that (M_{01}), (M_{21}), (M_{11}) are on a line N_1, that (M_{11}), (M_{02}), (M_{22}) are on a line N_2, that (M_{21}), (M_{02}), (M_{12}) are on a line N_3, and that (M_{01}), (M_{22}), (M_{12}) are on a line N_4. The fourth point on the line N_i is denoted by n_i. The points on the line $(L)n_i$ are denoted by (L), n_i, u_{i1}, u_{i2}. Then u_{i1}, u_{i2}, (L), (M_{j1}), (M_{j2}), $i=1,\ldots,4$ and $j=0,1,2$, are the points at infinity of the 15 lines of S. Now we have a detailed description of the quadrangle S. From this description follows the construction of a model of the classical quadrangle with 15 points and 15 lines in $AG(4,3)$ (by the choice of the points (M_{ij}) in ω_∞, and the point (L) not in ω_∞, the points at infinity of the 15 lines are determined; next, we choose the line L through (L); by the labeling of the 3 points on L, the generalized quadrangle is completely determined).

Theorem. *Suppose that the generalized quadrangle $S=(P, B, \mathrm{I})$ with parameters s, t is embedded in $AG(d, s+1)$, $d \geqq 5$, and that P is not contained in an $AG(d-1, s+1)$. Then the following cases can occur.*

(a) $s=1$ *and* $t\in\{[d/2],\ldots,2^{d-1}-1\}$ *(trivial case)*

(b) $s=1$, $t=4$, $d=5$ (*an embedding of the generalized quadrangle with* 27 *points and* 45 *lines in* $AG(5,3)$).

Proof. Suppose that $s=1$. Let $x_0, x_1, \ldots, x_t, y_0, y_1, \ldots, y_t$, $t\in\{[d/2],\ldots,2^{d-1}-1\}$, be distinct points of $AG(d,2)$ which are not contained in an hyperplane. Then the sets $P=\{x_i, y_j \| i,j\in\{0,\ldots,t\}\}$ and $B=\{\{x_i,y_j\}\| i,j\in\{0,\ldots,t\}\}$ define a generalized quadrangle with parameters $s=1$ and t.

Now we suppose that $s\geqq 2$. Let L, M be two non-coplanar lines of S, and call $AG(3, s+1)$ the affine threespace which contains these lines. Suppose that p is a point of S which does not belong to $AG(3,s+1)$, and call $AG(4,s+1)$ the fourdimensional affine space defined by $AG(3, s+1)$ and p. Suppose that q is a point of S which does not belong to $AG(4, s+1)$, and call $AG(5, s+1)$ the affine space defined by $AG(4,s+1)$ and q. In $AG(3,s+1)$ (resp. $AG(4,s+1)$, resp. $AG(5,s+1)$) a subquadrangle S' (resp. S'', resp. S''') with parameters s,t' (resp. s,t'', resp. s,t''') is induced. We have $t'<t''<t'''\leqq t\leqq s^2$. From [16] follows that necessarily $t'=1$, $t''=s$, $t'''=s^2$, and so $t=s^2$ and $d=5$. From the preceding theorem follows that $t'=1$, $t''=s=2$, $t=4$, $d=5$ or $t'=1$, $t''=s=3$, $t=9$, $d=5$.

Let us suppose that $s=2$, $t=4$, $d=5$. Let S' be a subquadrangle with parameters 2, 2 of S, which is embedded in the hyperplane H of $AG(5,3)$. Let L_0 be a line of S', suppose that x_0, y_0, z_0 are the points of L_0, that N_x, M_x, L_0 are the lines of S' containing x_0, that N_y, M_y, L_0 are the lines of S' containing y_0, that N_z, M_z, L_0 are the lines of S' containing z_0, and that M_x, M_y, M_z belong to a threedimensional affine space T. Let N_x, N_x', N_x'', M_x, L_0 be the lines of B which contain x_0. The hyperplane defined by T and N_x' is denoted by H', and the hyperplane defined by T and N_x'' is denoted by H''. The subquadrangle $S''=(P'',B'',\mathrm{I}'')$ resp. $S'''=(P''',B''',\mathrm{I}''')$ induced in H' resp. H'' has parameters s,t'' resp. s,t''', with $s=t''=t'''=2$. Suppose that $N_y', N_z'\in B''$, $y_0\mathrm{I}N_y'$, $z_0\mathrm{I}N_z'$, $N_y'\notin\{M_y,L_0\}$, $N_z'\notin\{M_z,L_0\}$, and that N_y'', $N_z''\in B'''$, $y_0\mathrm{I}N_y''$, $z_0\mathrm{I}N_z''$, $N_y''\notin\{M_y,L_0\}$, $N_z''\notin\{M_z,L_0\}$. Any point of S is on one of the lines L_0, M_x, M_y, M_z, N_x, N_y, N_z, N_x', N_y', N_z', N_x'', N_y'', N_z''. The point at infinity of L_0 is denoted by (L_0), etc. Then (M_x), (M_y), (M_z) are on a line M_∞. Moreover the points (M_x), (M_y), (M_z), (N_x), (N_y), (N_z) are in a plane ω_∞, the points (M_x), (M_y), (M_z), (N_x'), (N_y'), (N_z') are in a plane ω_∞', and the points (M_x), (M_y), (M_z), (N_x''), (N_y''), (N_z'') are in a plane ω_∞''. We remark that ω_∞, ω_∞', ω_∞'' are three distinct planes, and that (L_0) is in no one of these planes. Moreover, if $\{a,b,c\}=\{x,y,z\}$, the points (M_a), (N_b), (N_c) are collinear, the points (M_a), (N_b'), (N_c') are collinear, and the points (M_a), (N_b''), (N_c'') are collinear. There are three lines L_0, L_{abc}, L_{abc}' of B concurrent with N_a and N_b', and there are also three lines N_a, N_b', U concurrent with each of L_0, L_{abc}, L_{abc}'. We have necessarily $U=N_c''$, with $\{a,b,c\}=\{x,y,z\}$. There follows that (N_a), (N_b'), (N_c'') are on a line V_∞, that (L_0), (L_{abc}), (L_{abc}') are on a line W_∞, and that V_∞ and W_∞ have a point u_∞ in common ((N_a), (N_b'), (N_c''), u_∞ are the four points of V_∞, and (L_0), (L_{abc}), (L_{abc}'), u_∞ are the four points of W_∞). If L_0, D_{ab}, E_{ab} (resp. L_0, D_{ab}', E_{ab}', resp. L_0, D_{ab}'', E_{ab}''), $a\neq b$ and $a, b\in\{x,y,z\}$, are the lines of S which are concurrent with N_a, N_b (resp. N_a', N_b', resp. N_a'', N_b''), and if L_0, L_0', L_0'' are the lines of S concurrent with M_x, M_y, M_z, then the lines L_0, L_0', L_0'', M_x, M_y, M_z, N_x, N_y, N_z, N_x', N_y', N_z', N_x'', N_y'', N_z'', D_{ab}, E_{ab}, D_{ab}', E_{ab}', D_{ab}'', E_{ab}'', L_{abc}, L_{abc}' are the 45 lines of S.

So we have the following construction of the generalized quadrangle with 27 points and 45 lines in $AG(5,3)$. First of all we choose the points (L_0), (M_x), (M_y), (M_z), (N_x), (N_y), (N_z), (N_x'), (N_y'), (N_z'), (N_x'), (N_y''), (N_z'') in such a way that the conditions above are satisfied. Next we choose the line L_0 through (L_0), and we label the three points on L_0. Such a choice determines a configuration of 27 points and 45 lines, and it is not a difficult but tedious work to check that this incidence structure is indeed a generalized quadrangle. We also remark that in the hyperplane at infinity coordinates can be chosen in such a way that we have

$(L_0)=(0,0,0,0,1)$,

$(M_x)=(1,0,0,0,0)$, $(M_y)=(0,1,0,0,0)$, $(M_z)=(1,1,0,0,0)$,

$(N_x)=(0,0,1,0,0)$, $(N_y)=(1,1,1,0,0)$, $(N_z)=(0,1,1,0,0)$,

$(N_x')=(0,0,0,1,0)$, $(N_y')=(1,1,0,1,0)$, $(N_z')=(0,1,0,1,0)$,

$(N_x'')=(1,-1,1,1,0)$, $(N_y'')=(0,1,1,1,0)$, $(N_z'')=(1,1,1,1,0)$.

Finally we assume that $s=3$, $t=9$, $d=5$. Then S contains subquadrangles with parameters 3, 3, and described in the preceding theorem. So in S we can choose three concurrent lines L, M, N, with common point p, in such a way that L, M, N are the only lines of S in the threespace T defined by L, M, N. There are 5 hyperplanes containing T and in 3 of these hyperplanes a subquadrangle with parameters 3, 3 is induced. Let H_1, H_2 be the other hyperplanes through T. The lines of S in H_i all contain p. The number of lines of S in H_i equals $3+a_i$, with $a_1+a_2=4$. Let L_1 be a line of S through p and not in H_1, and let $q\,\mathrm{I}\,L_1$, $q \neq p$. The 10 lines of S through q are L_1 and 9 lines in the hyperplane H_3 through q and parallel to H_1. Analogously the 10 lines of S through p are L_1 and 9 lines in the hyperplane H_1. Consequently $3+a_1=9$, a contradiction.

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