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Relative Pseudo-Complements, Join-Extensions, and Meet-Retractions

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Dedicated to Günter Pickert on his sixtieth birthday

In this paper, some classical results about Boolean lattices and Brouwerian (semi)lattices are obtained in a new fashion and strengthened considerably. We are talking about: (1) the Glivenko-Frink result identifying, within a given pseudo-complemented semilattice, a certain Boolean closure retract; (2) the Glivenko-Stone result making the Dedekind-MacNeille completion of a Boolean lattice Boolean; (3) Rasiowa's analogue of that for Brouwerian (semi-) lattices. In our approach, these results become more independent of each other than they used to be in the literature. Our proofs are fairly straightforward (if not trivial).

However, this is only one aspect of our paper. First and above all, we want to stress the importance of certain principles that seem to have been somewhat neglected in the past. We are talking about: (1) the notion of join-extensions of partially ordered sets; (2) the notion of meet-retracts of meet-semilattices. Technically, our observations concerning these principals are laid down in the Lemmas of the paper and their Corollaries.

Join-completions (complete join-extensions) have been introduced by Banaschewski [3]. They are intimately related to (closure) representations of complete lattices as systematically studied by Büchi [7]. Join-extensions in full generality have been introduced by Bruns [6].[1] In spite of the pioneer work of these authors, join-extensions have not been paid much attention in the literature. Recently, one of the present authors has begun to exploit the idea of join-extensions (cf. [22–24]). The present paper may be considered as a joint effort to continue in this direction.

In § 0, we recall the absolute minimum of basic facts about join-extensions necessary for this paper. More facts are found in the literature mentioned above. Note that we are looking at join-extensions, E, of the partially ordered set P as abstract extensions, in particular, as subsets of the largest join-extensions $L(P)$. As a matter of fact, it shortens proofs considerably to look at the elements of

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[1] Pesotan [16] has developed a "symmetric" extension theory coverring both meet- and join-extensions

$L(P)$ (or any E such that $P \subset E \subset L(P)$) as just elements – like the elements of P itself – and not as certain subsets of P. After all, it does not pay to look at the reals as Dedekind cuts of the rationals all the time. As a basic feature of join-extensions, meets in P are preserved, i.e., remain the same in E. As noticed by Ch. Dial, the same preservation principal applies to relative pseudo-complements, as far as they exist in P (§1). A considerably stronger version of Rasiowa's aforementioned result (Theorem 1.4) is obtained as an immediate consequence of this observation. In §2, we apply this preservation principle to pseudo-complements. For the latter, we obtain a result (Proposition 2.2) much stronger even than the strengthened Rasiowa result above. We also get a short direct proof of the Glivenko-Stone result (Theorem 2.3). So far, (relative) pseudo-complements were preserved by going up to join-extensions. In §3, we observe that they are also preserved by going down to meet-retracts. Combining the going up and going down principles, we obtain neat characterizations of Brouwerian join-completions (Corollary 3.8). In §4, we give a new proof of an extended version of the Glivenko-Frink result (Theorem 4.1). Our proof is almost free of calculation and does not use any of the sophisticated axiomatizations of Boolean lattices. In particular, Glivenko's important equation – (4.4) – is obtained without any extra charge. We also arrive at some converse (Proposition 4.2), thus throwing some new light on Glivenko-Frink's result. In this context, the notion of a Brouwerian subact (introduced in §§1, 3) begins to play quite a role. We finish with a considerable generalization of Glivenko-Frink's result (Theorem 4.3).

0. Reminders on Join-Extensions

Let P be a partially ordered set and E an extension of P. I.e., P is a subset of E and the partial order of P is the restriction to P of the partial order of E. In case every element $x \in E$ is the join (l.u.b.) of some subset $M \subset P$ (one may take $M = \{p \in P \mid p \leqq x\}$, for that matter), P is *join-dense* in E and E *a join-extension of* P. The following observation is well-known:

Lemma 0.1. *Let E be a join-extension of P and $M \subset P$. Then*

(0.1) *if* $\inf_P M$ *exists, then* $\inf_E M$ *exists, and* $\inf_E M = \inf_P M$.

Consequently, $\inf_P M$ *exists iff* $\inf_E M$ *exists and belongs to P.*

We use the notation $\inf M$ (more precisely: $\inf_P M$) for the meet (g.l.b.) of the subset M (in P). If $M = \{a_i \mid i \in I\}$, we will also write $\bigwedge_{i\in I} a_i$ (more precisely: ${}^P\bigwedge_{i\in I} a_i$). Analogous notations will be used for joins. (0.1) states that P is *completely meet-faithful* in E, so that actually the reference to the join-extension E we are in (P itself is one of them) can be omitted as far as meets are concerned.

Note that $\inf_P \varnothing$ and the greatest element of P are the same things. Hence applying Lemma 0.1 to the degenerate case $M = \varnothing$, we get:

Corollary 0.2. *If P has a greatest element, e, then e is the greatest element of any join-extension E. And P has a greatest element iff E has and the latter belongs to P.*

Let us now assume that E is a meet-semilattice or a complete lattice respectively. I.e., $x \wedge y = \inf_E \{x, y\}$ exists for each $x, y \in E$, or $\inf_E N$ exists for each subset $N \subset E$ (non-empty or empty). Note that in the latter case $\sup_E N$ exists for each $N \subset E$.

Corollary 0.3. *Let P be a partially ordered set, E a join-extension which is a meet-semilattice or a complete lattice respectively. Then P is a meet-semilattice (or a complete lattice) iff P is a (meet-)subsemilattice of E (or a closure retract of E).*

In an arbitrary partially ordered set E, a *closure operator* is mapping $\gamma: E \to E$ with the usual properties of preserving the order, enlarging ($a \leqq \gamma(a)$), and being idempotent. It is completely determined by its image

(0.2) $\quad C = \operatorname{im} \gamma$

by virtue of the formula

(0.3) $\quad \gamma(a) = \min \{c \in C \mid a \leqq c\}.$

A *closure retract* is any subset $C \subset E$ such that the minima (0.3) exist for all $a \in E$. (0.2), (0.3) establish, in fact, the one-to-one correspondence between all closure operators γ and all closure retracts C of E. Suppose now that P above is a complete lattice. Since P is completely meet-faithful in E, P is, indeed, a closure retract C of E (and completeness of E is not needed for that). Conversely, if P is a closure retract C of E, the completeness of E makes P complete too. Note that the corresponding closure operator γ, as a mapping from E onto P, is completely join-preserving:

(0.4) $\quad$ if $x = {}^E\bigvee_{i \in I} x_i$, $\quad$ then $\quad \gamma(x) = {}^P\bigvee_{i \in I} \gamma(x_i)$.

In particular, for $M = \{x_i \mid i \in I\} \subset P$:

(0.5) $\quad \sup_P M = \gamma(\sup_E M)$,

which comes as a reminder that joins are not preserved the same way as meets are (Lemma 0.1).

A join-extension, E, of the partially ordered set P is a *join-completion* once E is a complete lattice. Corollary 0.3 will help us to establish a survey of all join-completions of P, more generally, of all join-extensions of P *that are meet-semilattices*. This will come as part of the most general survey of *all* join-extensions of P, as obtained from *the largest joint-extension of* P, $L(P)$. Here "largest" means precisely that

(0.6) $\quad$ each join-extension of P is order-embeddable over P into $L(P)$.

"Over P" means, of course, that each element of P is kept fixed by the order-embedding. Note that the latter is uniquely determined. Consequently, $L(P)$ itself is uniquely determined up to unique isomorphism over P. Looking at one realization of $L(P)$ and all intermediate sets E,

$$P \subset E \subset L(P),$$

we are actually looking at all non-isomorphic join-extensions of P (cf. Bruns [6], J. Schmidt [22]).

$L(P)$ is actually a join-completion – and it suffices to postulate (0.6) for join-completions only. (For an inner characterization and a universal property of $L(P)$, cf. J. Schmidt [22].) In particular, $L(P)$ is a meet-semilattice. By Corollary 0.3 then, the join-extensions E of P which are meet-semilattices are, up to isomorphism, the (meet-)subsemilattices of $L(P)$ containing P, and the least such join-extension is the subsemilattice of $L(P)$ generated by P. We may here always throw in the largest element, e, of $L(P)$ (Corollary 0.2), so that we wind up with the least join-extension of P, $M(P)$, which is a meet-semilattice with identity e. Likewise, the join-completions of P are, up to isomorphism, the closure retracts of $L(P)$ containing P (Banaschewski [3]), and the least join-completion of P is the closure retract of $L(P)$ generated by P, i.e., the set of all meets (in $L(P)$) of subsets $M \subset P$. It is known as *the normal* or *Dedekind-MacNeille completion of* P, denoted by $N(P)$. P is join- and meet-dense in P. Actually, $N(P)$ is the largest join- and meet-extension of P and the only join- and meet-completion of P (Banaschewski [3]). By Lemma 0.1, P is completely meet- and join-faithful in $N(P)$. However, this frequently stated observation, together with completeness, fails to characterize $N(P)$ in general.

1. Relative Pseudo-Complements in Join-Extensions

Let P be a meet-semilattice and a, $z \in P$. We denote by $a \to z$ (or $a \xrightarrow[P]{} z$ if necessary) *the relative pseudo-complement of a with respect to z*,

(1.1) $a \to z = \max\{x \in P \mid a \wedge x \leqq z\}$.

Lemma 0.1, concerning meets, has now an exact analogue for "arrows":

Lemma 1.1. *Let P be a meet-semilattice and E a join-extension of P, also a meet-semilattice. Then for a, z∈P:*

(1.2) *if* $a \xrightarrow[P]{} z$ *exists, then* $a \xrightarrow[E]{} z$ *exists, and* $a \xrightarrow[E]{} z = a \xrightarrow[P]{} z$.

Consequently, $a \xrightarrow[P]{} z$ *exists iff* $a \xrightarrow[E]{} z$ *exists and belongs to P.*

Proof. $a \wedge (a \xrightarrow[P]{} z) \leqq z$, holding in P, holds in E too. Let now $x \in E$ be such that $a \wedge x \leqq z$. By join-density, $x = \sup_E M$ for some $M \subset P$. For $y \in M$, we get $a \wedge y \leqq a \wedge x \leqq z$, whence $y \leqq a \xrightarrow[P]{} z$. This being true for each $y \in M$, $x \leqq a \xrightarrow[P]{} z$.

A *Brouwerian (relatively pseudo-complemented, pseudo-Boolean, Heyting, implicative) semilattice* is a meet-semilattice – necessarily with largest element e – in which all relative pseudo-complements $a \to z$ exist. It is well-known that relative pseudo-complementation can be axiomatized by equations either for the unary operators $a \to$ (cf., e.g., Rasiowa-Sikorski [18]) or the unary operators $\to z$

(Katriňák-Mitschke [13]). Anyway, Brouwerian semilattices form an equational class of algebras $(E, \wedge, e, \rightarrow)$, where $(E, \wedge, e)$ is a meet-semilattice with identity and $\rightarrow$ satisfies those aforementioned equations. A *Brouwerian subalgebra* is a subsemilattice containing e and closed under $\rightarrow$. It is again a Brouwerian semilattice by itself.

Here is an exact analogue of Corollaries 0.2 and 0.3:

Corollary 1.2. *Let P be a partially ordered set, E a Brouwerian join-extension (a join-extension which is a Brouwerian semilattice). Then P is a Brouwerian semilattice iff P is a Brouwerian subalgebra of E.*

Proof. Corollaries 0.2, 0.3, Lemma 1.1.

Note that $E = L(P)$ is a Brouwerian (semi)lattice since $L(P)$ is even complete and has the appropriate infinite distributivity (cf. Birkhoff [4], Ch.V, Theorem 24). Hence the Brouwerian join-extensions of P are exactly the Brouwerian subalgebras of $L(P)$ containing P. In particular, there is *the least Brouwerian join-extension of P*, $B(P)$, first observed by Ch. Dial, namely the Brouwerian subalgebra of $L(P)$ generated by P. Note that this Brouwerian subalgebra is obtained from P by first closing under $\rightarrow$, then under finite meets (including $e = \inf \varnothing$). This is indeed true in any Brouwerian semilattice $E \supset P$, join-density of P in E being not required here. So in our case $E = L(P)$, we pass from P to its closure under $\rightarrow$, call it $A(P)$. We may write, cum grano salis,

$$(1.3) \quad B(P) = M(A(P)).$$

(Recall that $M(P)$ was the subsemilattice with e generated by P in the given model of $L(P)$.) In particular:

Proposition 1.3. *Let P be a partially ordered set closed in $L(P)$ under $\rightarrow$, $P = A(P)$. Then $M(P)$ is the least Brouwerian join-extension of P, $B(P) = M(P)$.*

A partially ordered set P such that $P = A(P)$ might be called a *Brouwerian partially ordered set*. Note that the total restriction to P of the operation $\rightarrow$ in $L(P)$ can be described completely within P itself:

$$(1.4) \quad a \rightarrow z = \max \{x \in P \mid (a] \cap (x] \subset (z]\},$$

where $(x]$ denotes the principal ideal of P generated by x. This extended notion of the relative-pseudo-complement not even involving meets seems to have been used first by Katriňák [12]. All statements of this section can be extended, mutatis mutandis, from our Brouwerian semilattices to Brouwerian partially ordered sets. The term "Brouwerian semilattice" would, of course, become ambiguous now. As a matter of fact, Katriňák loc.cit. considers Brouwerian join-semilattices.

Proposition 1.3 dealing with finite meets, let us consider the infinitary analogue. By virtue of Corollaries 0.3 and 1.2, the Brouwerian join-completions of P are exactly the closure retracts of $L(P)$ closed under $\rightarrow$ and containing P. Again, there is *the least Brouwerian join-completion of P*. Again, it is obtained by first closing under $\rightarrow$, then under arbitrary meets. So it may be written

$N(A(P))$,

which, by virtue of (1.3), coincides with $N(B(P))$. However, in contrast to (1.3), join-density plays a role here. Indeed, we have to show closure of $N(A(P))$ under $\rightarrow$. The proof of that is different from, and even simpler than, the proof of (1.3), and we actually get somewhat more, namely $a \rightarrow z \in N(A(P))$ for each $z \in N(A(P))$ and any $a \in L(P)$. Indeed, represent a as the join in $L(P)$ of elements $a_i \in A(P)$, z as the meet in $L(P)$ of elements $z_j \in A(P)$. It follows that $a_i \rightarrow z_j \in A(P)$ for each i and j, so that

$$a \rightarrow z = \bigwedge_{i,j} (a_i \rightarrow z_j)$$

is in $N(A(P))$. As an immediate consequence, we have

Theorem 1.4. *Let P be a partially ordered set such that $P = A(P)$. Then the Dedekind-MacNeille completion $N(P)$ is Brouwerian.*

Theorem 1.4 was first stated and proven by Rasiowa ([17], Theorem 3.8) for lattices with zero. Actually, her long-winded method of proof (cf. also Rasiowa-Sikorski [18], Ch. IV, 9.1) does not apply to Brouwerian semilattices, not to speak of Brouwerian partially ordered sets. For she uses the analogous Glivenko-Stone result for Boolean lattices (Theorem 2.3 below) and McKinsey-Tarski's theory of the representation of Brouwerian lattices with zero as lattices of open elements of closure algebras (Boolean algebras with topological closure operators in them; cf. [15]). D. Smith [27] seems to have been the first to publish a direct proof of Rasiowa's result (without mentioning the latter), now for Brouwerian lattices with or without zero. Meanwhile, the result has become an exercise in Balbes-Dwinger's book [1] (p. 238, ex. 11), again for Brouwerian lattices with zero. Nothing in their book seems to indicate the present line of thought.

We proved above that $N(A(P))$ is somewhat more than just a Brouwerian subalgebra of $L(P)$. We may say it is even a "subact" of $L(P)$. For some of the fundamental rules of a Brouwerian semilattice L – like $L(P)$ above – can be interpreted to the effect that the semilattice L acts on itself, with the unary operators $a \rightarrow$ ($a \in L$) as the individual actions (cf. J. Schmidt [25]). A *Brouwerian subact* then, in general, should be a meet-subsemilattice $N \subset L$ containing the identity of L and closed under those unary operators $a \rightarrow$ ($a \in L$). This makes N more than just a Brouwerian subalgebra. Note that each filter $F \subset L$ is a Brouwerian subact. That $N(A(P))$ was a Brouwerian subact of $L(P)$ does not come as an accident. The proof above actually shows:

Lemma 1.5. *Any Brouwerian closure retract N of a Brouwerian join-completion L is a Brouwerian subact of L.*

It was mentioned in § 0 that in general $N(P)$ is the only join- and meet-completion of P so that P is completely meet- and join-faithful in $N(P)$. However, there may be other completions featuring this double faithfulness. For instance, there is the largest join-completion in which P is completely join-faithful, denoted by $I_\infty(P)$ (cf. J. Schmidt [23, 24]). In general, $N(P) \subset I_\infty(P)$, but equality may not take place. However, if P is a Brouwerian meet-semilattice, we are in a better position, due to

Proposition 1.6. *Let P be a meet-semilattice, let B be any Brouwerian join-completion of P and E any join-extension of P in which P is completely join-faithful. Suppose $B \subset E$. Then actually $B = E$.*

Proof. Suppose $a \in E$. We can write a as the join in E of elements $a_i \in P$. By the completeness of B, the join in B, b, of these elements a_i exists. Certainly $a \leqq b$. On the other hand, b is the join in E of elements $b_j \in P$. For every j, the infinite distributivity of B yields

$$b_j = b \wedge b_j = {}^B\bigvee_i (a_i \wedge b_j).$$

Since P is a meet-semilattice, $a_i \wedge b_j \in P$. Since $b_j \in P$ too,

$$b_j = {}^P\bigvee_i (a_i \wedge b_j).$$

Since P is completely join-faithful in E,

$$b_j = {}^E\bigvee_i (a_i \wedge b_j) \leqq a.$$

This being true for each j, $b \leqq a$, whence $a = b \in B$ and $E = B$.

Corollary 1.7. *If P is a Brouwerian semilattice, $N(P)$ is the only completely join-faithful join-completion of P.*

(For a closely related characterization of $N(P)$ for an arbitrary partially ordered set P, cf. J. Schmidt [21].)

2. Applications to Pseudo-Complements

Throughout this section, the partially ordered set P will have a zero, o. Then each join-extension E of P has a zero. The latter, however, may be different from the zero of P, i.e., E may have a new zero, a "tail", e.g., $E = L(P)$. Throughout this section, we will only consider join-extensions having the same zero as P. Note that $M(P)$, $N(P)$, $I_\infty(P)$, $A(P)$, $B(P)$, $N(A(P))$ are among them. The largest among them is $L^*(P)$, i.e., $L(P)$ with its "tail" removed. It is the largest join-completion of P with the same zero.

The *pseudo-complement of $a \in P$* in P is the element

(2.1) $\neg a = a \to o = \max\{x \in P \mid a \wedge x = o\}$.

If necessary, we will write $\neg_P a$. We can now reformulate Lemma 1.1:

Lemma 2.1. *Let P be a meet-semilattice with zero and E a join-extension of P, also a meet-semilattice, with the same zero as P. Then for $a \in P$:*

(2.2) *if* $_{P}\neg a$ *exists, then* $_{E}\neg a$ *exists and* $_{E}\neg a = {}_{P}\neg a$.

Consequently, $_{P}\neg a$ *exists iff* $_{E}\neg a$ *exists and belongs to P.*

A *pseudo-complemented semilattice* is a meet-semilattice with zero in which all pseudo-complements $\neg a$ exist. We can look at them as algebraic systems $(E, \wedge, e, \neg)$ characterized by equations (cf. Balbes-Horn [2]) which are special cases of the aforementioned equations for the operator $\rightarrow z$ (Katriňák-Mitschke loc.cit.). We may call these algebraic systems, for lack of an established word, PSC-*algebras.* A PSC-*subalgebra* then is a subsemilattice closed under $\neg$ and containing the largest element e (hence the least element $o = \neg e$). Again, the PSC-subalgebras of $L^*(P)$ containing P are exactly the join-extensions with the same zero which are pseudo-complemented semilattices. Among them is certainly a least one, the PSC-subalgebra generated by P. Note, however, that there is no relatively simple description of the latter comparable to (1.3). On the other hand, the least pseudo-complemented join-completion (with the same zero) will be

$$N(P \cup \neg P \cup \neg\neg P),$$

where $\neg P = \{\neg p \mid p \in P\}$. Actually, $P \cup \neg P \cup \neg\neg P$ is the closure in $L^*(P)$ of P under $\neg$. More generally, the pseudo-complemented join-completions of P are exactly those join-completions (with the same zero), E, containing even $P \cup \neg P$. Indeed, represent $a \in E$ as the join in $L^*(P)$ of elements $a_i \in P$. Then $\neg a$ is the meet of the pseudo-complements $\neg a_i$, hence belongs to E. With that, we have a strong analogue of Rasiowa's result, Theorem 1.4:

Proposition 2.2. *Let P be a partially ordered set with zero such that* $\neg P \subset P$. *Then every join-completion of P (with the same zero) is pseudo-complemented.*

We might even have omitted "with the same zero" here since a complete lattice with a "tail" is trivially pseudo-complemented. Also, there is an extension of Proposition 2.2 to certain join-extensions. We leave that to the reader.

Let us finish this section with the aforementioned Glivenko-Stone result (Glivenko [9], Stone [28]).

Theorem 2.3. *Let P be a Boolean lattice. Then N(P) is the only Boolean join-completion – with the same zero! – of P. N(P) is even the largest Boolean join-extension E – with the same zero – of P: Each such E – P included – is a Boolean subalgebra of N(P).*

Note that the addition "with the same zero" is essential here unless we exclude the one-element Boolean lattice P.

Rather than basing the proof of the strengthened Rasiowa result, Theorem 1.4, on this classical predecessor, we use Theorem 1.4 here. So $N(P)$ is certainly Brouwerian, hence distributive. The Boolean complementation of P is the restriction to P of the pseudo-complementation in $N(P)$. Let $x \in N(P)$. We want to show that $\neg\neg x = x$. Let $x = \bigwedge_i x_i$, where $x_i \in P$, and consider the join in $N(P)$, y, of the elements $\neg x_i$. So

$$\neg y = \bigwedge_i \neg\neg x_i = \bigwedge_i x_i = x.$$

With that, $\neg\neg x = \neg\neg\neg y = \neg y = x$. Trivially, a distributive pseudo-complemented lattice with involutionary pseudo-complementation is Boolean. (By virtue of Huntington's theorem [11], §2, one can omit distributivity here; but this is no longer trivial. We got distributivity free of charge anyway.)

Let now E be any Boolean join-extension of P, with the same zero. In this case, the join-density of P in E implies meet-density. So $E \subset N(P)$. An E is a PSC-subalgebra, here: a Boolean subalgebra, of $N(P)$.

Here is some converse:

Proposition 2.4. *Let P be a pseudo-complemented semilattice which has a Boolean join-extension, with the same zero, E. Then P is Boolean.*

Proof. P is a PSC-subalgebra of E. Hence pseudo-complementation of P is still involutionary, making P a lattice, even a sublattice of E. So P is again distributive, completing the proof.

3. Relative Pseudo-Complements and Meet-Retractions

A *meet-retraction* of a meet-semilattice E is an idempotent (meet-)endomorphism.

Lemma 3.1. *Let γ be a meet-retraction of the meet-semilattice E. Let $P = \operatorname{im}\gamma$ be the corresponding meet-retract. Let $a, z \in P$ be such that $a \xrightarrow[E]{} z$ exists. Then $a \xrightarrow[P]{} z$ exists, and*

(3.1) $\quad a \xrightarrow[P]{} z = \gamma(a \xrightarrow[E]{} z)$.

Proof. One has $a \wedge \gamma(a \xrightarrow[E]{} z) = \gamma(a) \wedge \gamma(a \xrightarrow[E]{} z) = \gamma(a \wedge (a \xrightarrow[E]{} z)) \leqq \gamma(z) = z$. Next, let $x \in P$ be such that $a \wedge x \leqq z$. So $x \leqq a \xrightarrow[E]{} z$, whence $x = \gamma(x) \leqq \gamma(a \xrightarrow[E]{} z)$.

Combining this with Lemma 1.1, we get

Corollary 3.2. *Let P be a meet-retract of the meet-semilattice E and P join-dense in E. Let $a, z \in P$. Then*

(3.2) $\quad a \xrightarrow[P]{} z = a \xrightarrow[E]{} z$,

where the left hand side exists iff the right hand side exists.

An immediate consequence of Lemma 3.1 alone:

Corollary 3.3. *Every meet-retract P of a Brouwerian semilattice E is a Brouwerian semilattice.*

For the extreme special case $E = \mathfrak{P}(X)$ and P the family of open sets of some topology on X, this is a very old observation. McKinsey and Tarski [15] replaced $\mathfrak{P}(X)$ by an arbitrary Boolean lattice E and the topological interior operator by an arbitrary kernel operator still satisfying the Kuratowski axioms (actually, their setup is dual to this paper). They also showed that each Brouwerian lattice with zero can be obtained that way. Siemion Fajtlowicz observed that (3.1) holds also for meet-preserving closure operators. Actually, for the latter we have a somewhat stronger observation:

Lemma 3.4. *Let γ be a meet-preserving closure operator of the meet-semilattice E and $P = \operatorname{im}\gamma$. Let $a \in E$ and $z \in P$. Then*

$$(3.3)\quad \gamma(a) \xrightarrow[P]{} z = a \xrightarrow[E]{} z,$$

where the left hand side exists iff the right hand side exists.

Corollary 3.5. *If, in addition, E is Brouwerian, then $P = \operatorname{im}\gamma$ is a Brouwerian subalgebra, even a subact, of E.*

Here is some converse:

Lemma 3.6. *Let E be a meet-semilattice and γ a closure operator of E onto a Brouwerian subact $P = \operatorname{im}\gamma$. Then γ is meet-preserving.*

Since E is not assumed Brouwerian by itself, we redefine the notion of a *Brouwerian subact P*. P should still be a meet-subsemilattice of E (which is, of course, true once P is a closure retract of E). Moreover, for each $z \in P$ and $a \in E$, $a \xrightarrow[E]{} z$ is assumed to exist in E and to belong to P. Note that a Brouwerian subact will certainly be a Brouwerian semilattice by itself. For if both a and z are in P, then $a \xrightarrow[E]{} z$, being also in P, is $a \xrightarrow[P]{} z$ indeed. The identity of P, $z \xrightarrow[P]{} z = z \xrightarrow[E]{} z$, will then be the identity of E.

The proof of Lemma 3.6 is very simple. Let $a, b \in E$. Then $a \wedge b \leqq \gamma(a \wedge b)$. But $\gamma(a \wedge b) \in P$, so $b \rightarrow \gamma(a \wedge b) \in P$. Hence $\gamma(a) \leqq b \rightarrow \gamma(a \wedge b)$ and $\gamma(a) \wedge b \leqq \gamma(a \wedge b)$. Repeating this switching, we get $\gamma(a) \wedge \gamma(b) \leqq \gamma(a \wedge b)$.

Combining this with Corollary 3.5, we get

Corollary 3.7. *Let E be a Brouwerian semilattice and γ a closure operator in E. Then $P = \operatorname{im}\gamma$ is a Brouwerian subact* iff *γ if is meet-preserving.*

Let us collect several results about join-completions:

Corollary 3.8. *Let P the a partially ordered set, E a join-completion of P, γ the corresponding closure operator from L(P) onto E. Then the following are equivalent:*

(i) *γ is meet-preserving;*

(ii) *E is a meet-retract of L(P);*

(iii) *E is Brouwerian;*

(iv) *E is a Brouwerian subalgebra of L(P);*

(v) *E is a Brouwerian subact of L(P).*

Proof. (v) ⇒ (iv) ⇒ (iii) being trivial, (iii) ⇒ (v) by Lemma 1.5 ((iii) ⇒ (iv) already by Corollary 1.2). (v) ⇒ (i) by Lemma 3.6 (and the converse holds by Corollary 3.5). (i) ⇒ (ii) is trivial. (ii) ⇒ (iii) by Corollary 3.3.

4. Around Glivenko's Theorem

The proof of Glivenko-Stone's Theorem 2.3 is usually based on the famous Glivenko result which, loosely speaking, ties Brouwerian and Boolean lattices together. In this section, we are first going to extend Glivenko's result somewhat and to arrive at some converse.

Theorem 4.1 (Glivenko-Frink). *Given a meet-semilattice P and an element $z \in P$ such that $a \to z$ exists for each $a \in P$. Then γ_z, defined by*

(4.1) $\gamma_z(a) = (a \to z) \to z \quad (a \in P)$,

is a meet-preserving closure retraction, and the associated closure retract,

(4.2) $B_z = \operatorname{im} \gamma_z$,

is a Boolean lattice with least element z.

Proof. γ_z is a closure operator in P, so that B_z is a subsemilattice of P. B_z is even a Brouwerian subact of P: Let $a \in P$ and $x \in B_z$, then x is of the form $x = b \to z$ for some $b \in P$. Certainly, $a \wedge b \to z \in B_z$. But in P, $a \wedge b \to z = a \to (b \to z)$, whence $a \to x \in B_z$. This makes B_z a Brouwerian semilattice by itself. Since z is the least element of B_z, B_z is pseudo-complemented, with pseudo-complementation

(4.3) ${}_{B_z}\neg\, a = a \underset{B_z}{\longrightarrow} z = a \underset{P}{\longrightarrow} z \quad (a \in B_z)$.

This pseudo-complementation being involutionary, B_z is actually a lattice. Being Brouwerian, it is distributive, hence Boolean. Lemma 3.6 makes γ_z meet-preserving:

(4.4) $\gamma_z(a \wedge b) = \gamma_z(a) \wedge \gamma_z(b)$.

This classical result due for (Brouwerian) lattices to Glivenko [9] and extended to semilattices by Frink [8], Grätzer and E.T. Schmidt (cf. [10], p. 58, Remark; [19], Lemmas 4 and 5) seems to have been formulated only for $z = o$ (least element, whose existence we do not require here). Rarely, on the other hand, is the important equation (4.4) included in the result.

As in the proof of Glivenko-Stone's Theorem 2.3, we got distributivity free of charge here. Neither did we have to calculate, nor did we have to use one of those clever axiomatizations of Boolean lattices not mentioning distributivity at all. On the contrary, some of them (cf. Huntington [11], §2; also Birkhoff [4], Ch. II, Theorem 17) are more or less immediate consequences of Theorem 4.1 (applied to $z = 0$), requiring very little, if any, additional calculation (for details, cf. J. Schmidt [25], § 8).

We now prove the following converse:

Proposition 4.2. *Let P be a meet-semilattice and γ a meet-preserving closure operator whose closure retract $B = \operatorname{im} \gamma$ is Boolean, with least element z. Then $a \to z$ exists for each $a \in P$, and $B = B_z$.*

Proof. Let $a \in P$. Since the Boolean complement $\neg \gamma(a) = \gamma(a) \underset{B}{\longrightarrow} z$ exists in B, $a \underset{P}{\longrightarrow} z$ exists in P and $a \underset{P}{\longrightarrow} z = \neg \gamma(a)$ by virtue of Lemma 3.4–(3.3) from left to right. Hence

$$\gamma_z(a) = (a \underset{P}{\longrightarrow} z) \underset{P}{\longrightarrow} z = \neg \gamma(a) \underset{P}{\longrightarrow} z = \neg\neg \gamma(a) = \gamma(a),$$

whence $\gamma_z = \gamma$ and $B_z = B$.

Summarizing Theorem 4.1 and Proposition 4.2, we have established a one-to-one correspondence between the Boolean closure retracts B with meet-preserving closure operators γ and those elements $t\in P$ admitting all relative pseudo-complements $a\to z$. The shortest description of this correspondence does not even mention the closure operator γ:

(4.5) $B=\mathrm{im}(\to z), \quad z=\min B.$

Note that these *right Brouwerian elements* z form a subsemilattice, $b_r(P)$, of P (as opposed to our former least Brouwerian join-extension $B(P)$). Incidentally, $b_r(P)$ is empty unless P has an identity e. We then have $e\in b_r(P)$ and $B_e=\{e\}$. Trivially, each Brouwerian subact is contained in $b_r(P)$. In particular, $B_z\subset b_r(P)$ for each $z\in b_r(P)$, so that actually

(4.6) $b_r(P)=\bigcup_{z\in b_r(P)} B_z,$

making $b_r(P)$ itself *the largest Brouwerian subact of* P. Note that $b_r(P)$ becomes a closure retract of P once P is complete, so that $b_r(P)$ itself will be complete. Also, the Boolean closure retracts B_z ($z\in b_r(P)$) will be complete, and the closure operators γ_z: $P\to B_z$, completely join-preserving anyway (cf. (0.4)), will preserve finite joins and meets (Eq. (4.4)), hence be lattice homomorphisms. (For more details about this "right" one-to-one correspondence and a "left" counterpart, cf. J. Schmidt [25]; cf. also E.T. Schmidt [19], [20]; Varlet [31].)

We finish with a generalization of Glivenko-Stone's Theorem 2.3:

Theorem 4.3. *Let P be a meet-semilattice. Let E be a join-completion of P. Then*

(4.7) $b_r(P)\subset b_r(E)$

and, for each $z\in b_r(P)$,

(4.8) $B_z^E=N(B_z^P),$

where the upper indices refer to the semilattice we are in.

Proof. Let $z\in b_r(P)$. We actually show that $a\to z\in E$ for each $a\in L(P)$. Represent a as the join in $L(P)$ of elements $a_i\in P$. So $a_i\to z\in P$, so that $a\to z$, the meet of those elements $a_i\to z$, is in E, and $z\in b_r(E)$, proving (4.7). Evidently, $B_z^P\subset B_z^E$ for every $z\in b_r(P)$. Any $x\in B_z^E$ can be written in the form $x=a\to z$, for some $a\in E$. Represent a as the join in E of elements $a_i\in P$. Again, x is the meet of the elements $a_i\to z\in B_z^P$. With that, B_z^P is meet-dense in B_z^E. However, by Glivenko-Frink's Theorem 4.1, both B_z^P and B_z^E are Boolean lattices. Actually, B_z^P is a Boolean subalgebra of B_z^E. So B_z^P, being meet-dense in B_z^E, is also join-dense in B_z^E. But B_z^E, being a closure retract of the complete lattice E, is complete itself. With that, B_z^E is the Dedekind-MacNeille completion of B_z^P.

For $z=0$, we have the following

Corollary 4.4. *Let P be a pseudo-complemented semilattice, E any join-completion — with the same zero — of P. Then*

(4.9) $B_o^E=N(B_o^P).$

If P is now Boolean, then P will coincide with B_o^P, so that (4.9) runs

(4.10) $B_o^E = N(P)$.

Another application of Glivenko-Frink's Theorem 4.1 makes $N(P)$ Boolean: Glivenko-Stone's Theorem 2.3. In particular, $\gamma_o^E: E \to N(P)$ is a lattice homomorphism (even completely join-preserving, remember). For all this, E may still be any join-completion of P. In the literature, E has always been assumed to be the ideal completion.

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