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On the Existence of Solutions of Certain Asymptotically Homogeneous Problems

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In this paper we study the equation

$$Lu = g(u) - f. \tag{1}$$

Our interest is in deciding whether (1) has a solution for every $f \in L^2(\Omega)$. Here L is an unbounded self-adjoint operator on $L^2(\Omega)$ with compact resolvent (i.e. $(L - iI)^{-1}$ is compact) and $g: L^2(\Omega) \to L^2(\Omega)$ is defined by $(g(u))(x) = \bar{g}(u(x))$ where $\bar{g}: \mathbf{R} \to \mathbf{R}$ is continuous and $y^{-1}\bar{g}(y) \to \mu(\nu)$ as $y \to \infty(-\infty)$. We always assume that μ and ν are finite and that Ω has finite measure. (Usually, we will not distinguish between g and $\bar{g}$. This will cause no confusion.)

It turns out (cp. [7, § 3]) that, *in many cases*, (1) has a solution for every $f \in L^2(\Omega)$ if and only if the simpler equation

$$Lu = \mu u^+ + \nu u^- - f \tag{2}$$

has a solution for every $f \in L^2(\Omega)$. Here $u^+(x) = \sup\{u(x), 0\}$ and $u^-(x) = \inf\{u(x), 0\}$. (Note that this differs from the notation in [7].) One only expects (cp. [7] and [8]) (1) and (2) to behave similarly if the particular case of (2)

$$Lu = \mu u^+ + \nu u^- \tag{3}$$

has only trivial solutions. Thus it is of interest to determine

$$A_0 \equiv \{(\mu, \nu) \in \mathbf{R}^2 \colon \text{(3) has a non-trivial solution}\}.$$

It seems difficult to determine A_0 even in simple cases (for example, even if L is the Laplacian with Dirichlet boundary condition and Ω is a sphere). Thus it becomes of interest to determine properties of A_0. In § 1, we prove that if τ is an eigenvalue of L, then the component of A_0 containing (τ, τ) is unbounded. This is a bifurcation type theorem. We use this result to partially answer a question of Berestycki [5].

In § 2, we prove that, if a certain homotopy index for Eq. (3) is non-zero, then Eq. (1) is solvable for every $f \in L^2(\Omega)$. This improves a result in [7]. (The homotopy index is defined in [6].)

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In § 3, we prove that, if $(\mu, v) \notin A_0$, if $[\mu, v]$ contains precisely one eigenvalue of L and if this eigenvalue is simple, then the solvability of (2) does determine the solvability of (1). Finally, in § 4, we discuss some open problems.

Many of our results could be abstracted further. In particular, $g(u)$ could be replaced by more general operators in a number of the results. For example, we could replace $g(u)$ by $g(x, u)$, replace μ by $\mu f_1(x)$ and replace v by $v f_2(x)$ (provided that f_1 and f_2 are bounded and positive on Ω and we place a uniformity condition on how $y^{-1} g(x, y)$ tends to $\mu f_1(x)$ $(v f_2(x))$ as $y \to \infty(-\infty)$. In this case, our results still hold if appropriately modified. (For most, but not all, of the results, we need to assume that $f_1(x) = f_2(x)$ a.e. on Ω.)

The author has been considerably influenced by the recent work of Amann [2] and Amann and Zehnder [3]. I should like to thank them for sending me preprints of their work.

§ 1. The Bifurcation Theorem

Let $A_0^+ = \{(\mu, v) \in A_0 \colon \mu \geqq v\}$. Note that $(\mu, v) \in A_0$ if and only if $(v, \mu) \in A_0$ and that $(\mu, \mu) \in A_0^+$ if and only if μ is an eigenvalue of L. In this section, we prove the following theorem. We assume that the assumptions of the introduction hold.

Theorem 1. *If τ is an eigenvalue of L, then the component C_τ of A_0^+ containing (τ, τ) is unbounded.*

Before proving Theorem 1, we need some preliminaries. Note that, by Dunford and Schwartz [13, Lemma 7.6.13], $\sigma(L)$ consists of isolated points. If C_τ is bounded, choose $K > 0$ such that $K \geqq \sup\{|\mu| + |v| + 1 \colon (\mu, v) \in C_\tau\}$. If $a, b \in \sigma(L) \cup \{\infty\} \cup \{-\infty\}$ and $a < b$, let $Q_{a,b}$ be the orthogonal projection onto the subspace $N_{a,b}$ of $L^2(\Omega)$ spanned by the eigenvectors of L corresponding to eigenvalues in (a, b). Define $P_{a,b} = I - Q_{a,b}$ and $W_{a,b} = N_{a,b}^{\perp}$. Now Eq. (3) is equivalent to the pair of equations

$$L w = P_{a,b}(\mu(n+w)^+ + v(n+w)^-), \tag{4}$$

$$L n = Q_{a,b}(\mu(n+w)^+ + v(n+w)^-). \tag{5}$$

(Remember that L is self-adjoint.) Note that $N_{a,b} \subseteq \mathscr{D}(L)$.

Lemma 1. *If $a < v < \mu < b$, Eq. (4) has a unique solution $w = S_{\mu,v}(n)$ in $\mathscr{D}(L) \cap W_{a,b}$ for each $n \in N_{a,b}$. Moreover, $S_{\mu,v}$ is positive homogeneous and continuous as a map of $N_{a,b}$ into $\mathscr{D}(L)$ with the graph norm and there is a $K > 0$ such that $\|S_{\mu,v}(n)\|' \leqq K\|n\|$ for $n \in N_{a,b}$, where $\| \ \|'$ denotes the graph norm.*

Remark. $S_{\mu,v}$ is easily seen to be Lipschitz continuous.

Proof. This is a rather standard application of the contraction mapping theorem. Hence we only sketch the argument. For simplicity assume that $-\infty < a$ and $b < \infty$. Let $\gamma = \frac{1}{2}(a+b)$. Then (4) is equivalent to

$$w = Z P_{a,b}(\mu_1(n+w)^+ + v_1(n+w)^-), \tag{6}$$

where Z is the inverse of $(L - \gamma I)|_{W_{a,b}}$, $\mu_1 = \mu - \gamma$ and $v_1 = v - \gamma$. Since L is self-adjoint, $\|Z\| = \sup\left\{\frac{1}{b-\gamma}, \frac{1}{\gamma-a}\right\} = 2(b-a)^{-1}$. Since $P_{a,b}$ is orthogonal, $\|P_{a,b}\| = 1$. Moreover, the map $y \to \mu_1 y^+ + \nu y^-$ (of $\mathbf{R}$ to $\mathbf{R}$) is Lipschitz continuous with Lipschitz constant $\sup\{|\mu_1|, |\nu_1|\}$. Thus the same result holds for the map $u \to \mu_1 u^+ + \nu_1 u^-$ (as a map of $L^2(\Omega)$ into itself). Using these results, we find by a simple calculation that for each $n \in N_{a,b}$ the right-hand side of (6) defines a contraction mapping of $W_{a,b}$ into itself. Hence most of the result follow. Note that $S_{\mu,\nu}$ will be positive homogeneous because each term of (6) is. The contraction mapping theorem implies that $S_{\mu,\nu}$ is continuous as a map of $N_{a,b}$ into $L^2(\Omega)$. Equation (4) now implies that $LS_{\mu,\nu}(n)$ is continuous in n and so the required continuity follows. The estimate for $S_{\mu,\nu}$ can be easily proved directly. Alternatively, it follows from the continuity and positive homogeneity of $S_{\mu,\nu}$.

Thus, we see that $(\mu, \nu) \in A_0^+$ if and only if there is a non-trivial solution (in $N_{a,b}$) of

$$0 = F_{\mu,\nu}(n) \equiv Ln - Q_{a,b}(\mu(n + S_{\mu,\nu}(n))^+ + \nu(n + S_{\mu,\nu}(n))^-). \tag{7}$$

(Remember that $S_{\mu,\nu}(0) = 0$.) Note that the map $n \to F_{\mu,\nu}(n)$ is still positive homogeneous. We assume that $\sigma(L) \cap (a, b)$ is finite (and thus $N_{a,b}$ is finite-dimensional. Moreover, F is also a gradient mapping. It is the gradient of the mapping

$$f_{\mu,\nu}(n) \equiv \tfrac{1}{2}(L(n + S_{\mu,\nu}(n)), n + S_{\mu,\nu}(n)) - G(n + S_{\mu,\nu}(n)),$$

where $G(u) = \frac{1}{2}\int_\Omega (\mu(u^+)^2 + \nu(u^-)^2)$. There are a number of ways of proving this. The shortest way is to apply Theorem 2.3 in Amann [2] (as in §3 there). Note that his saddle point reduction is equivalent to ours and that his function f agrees with ours except for a linear isomorphism of $N_{a,b}$. Alternatively, it can be proved by (i) an approximation argument (noting that the map $u \to g(u)$ is Fréchet differentiable as a map of $\mathcal{D}(L)$ into $L^2(\Omega)$ if g is continuously differentiable and g' is bounded on $\mathbf{R}$), or (ii) by a careful direct differentiation of $f_{\mu,\nu}$ (and the use of the equation satisfied by $S_{\mu,\nu}(n)$). (The latter proof requires the use of the definition of the gradient since the chain rule is inapplicable and the use of the Lipschitz property of $S_{\mu,\nu}(n)$. Some care must also be exercised because L is unbounded.)

We now wish to apply the homotopy index in the sense of Conley [6]. We assume some knowledge of its basic properties. The idea of using the homotopy index in a similar situation is due to Amann and Zehnder [3]. *Assume that* $(\mu, \nu) \notin A_0$. Since $F_{\mu,\nu}$ is a gradient mapping and $F_{\mu,\nu}(n) \neq 0$ if $n \in N_{a,b} \setminus \{0\}$, zero is the only bounded solution of $n'(t) = F_{\mu,\nu}(n(t))$ (cp. [3], proof of Theorem 9.1). Thus the homotopy index of the isolated invariant set zero is defined. We use $h(0, F_{\mu,\nu})$ to denote this homotopy index.

Lemma 2. *If $\mu \notin \sigma(L)$ and $a < \mu < b$, $h(0, F_{\mu,\mu})$ is defined and is the homotopy type of pointed m_μ-sphere where m_μ is the number of eigenvalues (counting multiplicities) of L in (μ, b).*

Proof. If $\mu = \nu$, (4) becomes

$$Lw = P_{a,b}(\mu(n + w)) = \mu w.$$

Thus, since $\mu \notin \sigma(L)$, $S_{\mu,\mu}(n)=0$. Hence $F_{\mu,\mu}(n)=Ln-\mu n$. The result now follows from the statement in § 1.4.3 in [6].

In particular, if $\alpha, \beta \in \mathbf{R} \backslash \sigma(L)$ and $[\alpha, \beta] \cap \sigma(L) \neq \varnothing$, then $m_\alpha \neq m_\beta$ and thus $h(0, F_{\alpha,\alpha}) \neq h(0, F_{\beta,\beta})$ (since pointed spheres S^k and S^l do not have the same homotopy type if $k \neq l$).

We need one more technical lemma.

Lemma 3. *Let D denote a closed half disc in $\mathbf{R}^2$ (e.g. $\{(x,y) \in R^2: x^2+y^2 \leqq a, y \geqq 0\}$) with centre A and let E and H denote the curved and straight parts of ∂D respectively (where E and H are closed). Assume that K_1 and K_2 are two compact subsets of D such that $K_1 \cap K_2 = \varnothing$, $A \in K_1$ and $K_1 \cap E = \varnothing$. Then there is an arc T in $D \backslash (K_1 \cup K_2 \cup E)$ such that the ends of T lie in distinct components of $H \backslash (\{A\} \cup E)$.*

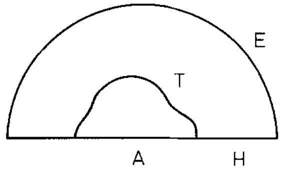

Fig. 1

Proof. The result is well-known but it is difficult to find an explicit reference. It follows easily from the Alexander lemma (Newman [17], Theorem 5.8.1). Essentially the same result is proved in part of the proof of Corollary 2.3 in Turner [22]. The result can also be proved by using the Mayer-Vietoris sequence of Čech cohomology.

Proof of Theorem 1. If C_τ is bounded we can choose a half disc D in $\{(\mu, \nu) \in \mathbf{R}^2: \mu \geqq \nu\}$ with centre (τ, τ) such that the curved part E of ∂D does not interest C_τ. (Thus the straight part H of ∂D lies in the line $\mu = \nu$.) Moreover, by our earlier choice of K, we can choose D such that $|\mu|, |\nu| < K$ for $(\mu, \nu) \in D$. Since C_τ is a component of $D \cap A_0^+$, a standard argument (cp. [22], proof of Corollary 2.3) shows that $D \cap A_0^+ = K_1 \cup K_2$, where K_1 and K_2 are compact, $K_1 \cap K_2 = \varnothing$, $C_\tau \subseteq K_1$ and K_1 does not intersect the curved part of ∂D. By Lemma 3, there is an arc $T \subseteq D \backslash (K_1 \cup K_2) \subseteq D \backslash A_0^+$ such that the two ends (α, α) and (β, β) of T lie in different components of $\{(\mu, \mu) \in D: \mu \neq \tau\}$. In our earlier construction of $F_{\mu,\nu}$, we take $a < -K$ and $b > K$ such that $(a, b) \cap \sigma(L)$ is finite. (Thus $F_{\mu,\nu}$ is defined for $(\mu, \nu) \in D$.) Since $T \cap A_0^+ = \varnothing$, $F_{\mu,\nu}(n) \neq 0$ if $(\mu, \nu) \in T$ and $n \in N_{a,b} \backslash \{0\}$. Thus, by our earlier comments, the homotopy index $h(0, F_{\mu,\nu})$ is defined for $(\mu, \nu) \in T$ and, by homotopy invariance (cp. Conley [6], Theorem 4.1.4), it is constant on T. In particular, $h(0, F_{\alpha,\alpha}) = h(0, F_{\beta,\beta})$. However, by the construction of T, $\tau \in [\alpha, \beta]$ and thus, by Lemma 2 and the comments following it, $h(0, F_{\alpha,\alpha}) \neq h(0, F_{\beta,\beta})$. Hence we have a contradiction and C_τ must be unbounded.

Remarks. 1. The author's original proof avoided the use of the saddle point reduction work of Amann and used results in Gromoll and Meyer [15] instead of the homotopy index. However, Amann's work shortened the proof while the homotopy index seems more flexible to use than the results in [15].

2. If L is not self-adjoint, degree theory could be used to prove a rather weaker result. Our method could also be used to obtain a weaker result in some cases where L does not have compact resolvent.

3. Note that we do *not* claim to prove that there is an unbounded connected subset of $\{(\mu, \nu) \in A_0 \colon \mu > \nu\}$ with (τ, τ) in its closure. We do not know whether this is true or false.

4. Theorem 1 can be improved slightly. However, it is convenient to defer the discussion of this till the end of § 2.

Finally, for this section, we indicate how Theorem 1 can be used to give a partial answer to a question of Berestycki [5, p. 390]. If L is a second order elliptic partial differential operator with appropriate boundary condition and $K \in \mathbf{R}$ he asks whether the equation

$$Lu = \mu u^+ + (\mu - K) u^- \tag{8}$$

has a non-trivial solution for an infinite number of μ's. For simplicity, let us assume that L is bounded below and let $\lambda_1 < \lambda_2 < \dots$ denote the distinct eigenvalues of L.

Proposition 1. *Let* $e = \limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n)$. *If* $0 < |K| < \frac{1}{2} e$, (8) *has an infinite number of solutions.*

Proof. As usual, it suffices to assume that $K > 0$. Choose e_1 such that $K < \frac{1}{2} e_1 < \frac{1}{2} e$. By the proof of Theorem 4 in Dancer [17], $(\mu, \nu) \notin A_0$ if $\lambda_n < \nu \leqq \mu < \lambda_{n+1}$. Thus, if $\lambda_{n+1} - \lambda_n > e_1$, $(\mu, \nu) \notin A_0$ whenever $\mu + \nu = \frac{1}{2}(\lambda_n + \lambda_{n+1})$ and $0 \leqq \mu - \nu \leqq \frac{1}{2} e_1$. Choose $\tau \in \sigma(L)$ such that $\alpha_n < \alpha_{n+1} < \dots \tau \dots < \beta_n < \beta_{n+1}$ where α_n and α_{n+1} (β_n and β_{n+1}) are successive eigenvalues of L, where $\alpha_{n+1} - \alpha_n > e_1$ and where $\beta_{n+1} - \beta_n > e_1$. Then C_τ must intersect $\mu - \nu = K$ at a point (μ, ν) with $\alpha_{n+1} + \alpha_n \leqq \mu + \nu \leqq \beta_n + \beta_{n+1}$. (This follows because (i) C_τ is unbounded and thus C_τ must leave the box $0 \leqq \mu - \nu \leqq K$, $\alpha_n + \alpha_{n+1} \leqq \mu + \nu \leqq \beta_n + \beta_{n+1}$ and because (ii) we have shown that A_0 (and hence C_τ) does not intersect the ends on which $\mu + \nu = \alpha_n + \alpha_{n+1}$ or $\beta_n + \beta_{n+1}$.) Since we can choose the α's and β's arbitrarily large (because $e_1 < e$), the result follows.

Remark. Let $N(\lambda)$ denote the number of eigenvalues of L less than or equal to λ (counting multiplicity). If $N(\lambda) \sim c\lambda$ as $\lambda \to \infty$, then it is easy to see that $e \geqq 1/c$ while, if $\lambda^{-1} N(\lambda) \to 0$ as $\lambda \to \infty$ then $e = \infty$. Since there are known estimates for $N(\lambda)$ for many elliptic partial differential operators (cp. Agmon [2] Theorem 14.6), we obtain estimates for e in some cases. For example, it turns out that e tends to be infinite if the order of the operator is higher than the dimension of Ω while we obtain an estimate for e if equality holds. For example, if $\Omega \subseteq \mathbf{R}^2$ is bounded and smooth and if L is the Laplacian with Dirichlet boundary condition then $e \geqq 4\pi (m(\Omega))^{-1}$, where $m(\Omega)$ denotes the measure of Ω.

2. On the symmetric domain, one can often obtain better results by restricting to a subspace of functions with appropriate symmetries (because, in the subspace, the corresponding e may be larger).

§ 2. On the Solvability of $Lu = g(u) - f$

In this section, we show how the homotopy index can be used to improve a result of [7] on the solvability for all $f \in L^2(\Omega)$ of the equation

$$Lu = g(u) - f. \tag{9}$$

We assume that all the conditions of the introduction hold. Our main result is the following.

Theorem 2. *Assume that* $(\mu, \nu) \notin A_0$, *and that* $h(0, F_{\mu,\nu})$ *is non-zero. Then* (9) *has a solution for every* $f \in L^2(\Omega)$.

Remarks. 1. Whether the homotopy index is non-zero apriori depends upon the choice of a and b (with the notation of § 1). We will prove, however, that it is independent of a and that whether it is zero is independent of b. Here a and b are restricted by the requirement that $\sigma(L) \cap (a, b)$ is finite.

2. By using the homotopy invariance of the homotopy index, we see that it suffices to check that $h(0, F_{\tilde{\mu},\tilde{\nu}}) \neq 0$ for some point $(\tilde{\mu}, \tilde{\nu})$ of the component of $\mathbf{R}^2 \setminus A_0$ containing (μ, ν). It is also easy to show that $h(0, F_{\mu,\nu}) = h(0, F_{\nu,\mu})$.

Before proving Theorem 2, we need some preliminaries. In particular, we need to justify Remark 1. For the moment, we write $F_{\mu,\nu,a,b}$ to indicate the explicit dependence upon a and b. For simplicity, we assume that 0 is not a local maximum of $F_{\mu,\nu,a,b}$. (The arguments in this case are similar but easier.) Since $F_{\mu,\nu,a,b}$ is positive homogeneous, we can argue as in Conley [6, p. 55] to show that

$$h(0, F_{\mu,\nu,a,b}) = \Sigma^1 \wedge [F^+_{\mu,\nu,a,b}/x^+] \tag{10}$$

where $\Sigma^1 \wedge$ denotes a suspension,

$$F^+_{\mu,\nu,a,b} = \{x \in N_{a,b} \colon \|x\| = 1 \text{ and } f^+_{\mu,\nu,a,b}(x) \geqq 0\},$$

x^+ is any point of $F^+_{\mu,\nu,a,b}$ and [ ] denotes homotopy type. Suppose now that $\tilde{a} \in \sigma(L) \cup \{-\infty\}$ such that $(\tilde{a}, a]$ contains exactly one distinct eigenvalue λ_i of L (possibly multiple). Now $N_{\tilde{a},b} = N_{a,b} \oplus K_i$, where $K_i = \{x \in L^2(\Omega) \colon Lx = \lambda_i x\}$. It follows easily from our construcation in § 1 that

$$f_{\mu,\nu,a,b}(n) = f_{\mu,\nu,\tilde{a},b}(n + \tilde{S}(n))$$

where $\tilde{S}(n)$ is the unique solution in K_i of the equation

$$Lw = P_i F_{\mu,\nu,\tilde{a},b}(n + w),$$

where $n \in N_{a,b}$, $w \in K_i$ and P_i is the orthogonal projection of $N_{\tilde{a},b}$ onto K_i. Now it follows from the work of Amann (cp. [2], Lemma 3.3) (and is also easily seen directly) that $f_{\mu,\nu,\tilde{a},b}(n+z)$ (where $n \in N_{a,b}$, $z \in K_i$) is concave in z and has its maximum at $z = \tilde{S}(n)$. Thus $T_n \equiv \{z \in K_i \colon f_{\mu,\nu,\tilde{a},b}(n+z) \geqq 0\}$ is either empty or is a closed convex set containing $\tilde{S}(n)$. Since $f_{\mu,\nu,\tilde{a},b}(0+z) < 0$ if $z \neq 0$ (as is easily seen), it follows that

$$Z_{\tilde{a}} \equiv \{u \in N_{\tilde{a},b} \setminus \{0\} \colon f_{\mu,\nu,\tilde{a},b}(u) \geqq 0\} = \cup (\{n\} + T_n), \tag{11}$$

where the union is over $\{n \in N_{a,b} \setminus \{0\} \colon f_{\mu,\nu,\tilde{a},b}(n + \tilde{S}(n)) \geqq 0\}$, i.e., over $Z_a \equiv \{n \in N_{a,b} \setminus \{0\} \colon f_{\mu,\nu,a,b}(n) \geqq 0\}$. Since T_n is convex, it can be contracted down to $\{\tilde{S}(n)\}$ by the obvious homotopy. Thus, by (11), $Z_{\tilde{a}}$ and Z_a have the same homotopy type. Since $f_{\lambda,\mu,\tilde{a},b}(\lambda x) = \lambda^2 f_{\lambda,\mu,\tilde{a},b}(x)$ if $\lambda \geqq 0$ (because $F_{\lambda,\mu,\tilde{a},b}$ is positive homogeneous), we can contract Z_a radially and deduce that $Z_{\tilde{a}}$ and $F^+_{\mu,\nu,\tilde{a},b}$ have the same homotopy type. Moreover, a similar result holds for Z_a. Hence

$F^+_{\mu,\nu,\tilde{a},b}$ and $F^+_{\mu,\nu,a,b}$ have the same homotopy type and thus, by (10), $h(0, F^+_{\mu,\nu,a,b}) = h(0, F_{\mu,\nu,\tilde{a},b})$, as required. (Note that we may ignore base points because our sets are absolute neighbourhood retracts as we see below.)

We now need to consider the increasing b to $\tilde{b} \in \sigma(L) \cup \{\infty\}$ such that $(b, \tilde{b}) \cap \sigma(L) = \varnothing$. In this case, we prove a weaker result which suffices for our purposes. Adding an eigenvalue of L greater than b in the construction of $F_{\mu,\nu,a,b}$ is the same as adding an eigenvalue of $-L$ less than $-b$ in the construction of $-F_{\mu,\nu,a,b}$. Thus, by our arguments above,

$$\tilde{Z}_b \equiv \{x \in N_{a,b}:\ \|x\| = 1,\ -f_{\mu,\nu,a,b}(x) \geqq 0\}$$

and

$$\tilde{Z}_{\tilde{b}} \equiv \{x \in N_{a,\tilde{b}}:\ \|x\| = 1,\ -f_{\mu,\nu,a,\tilde{b}}(x) \geqq 0\}$$

have the same homotopy type and thus the same cohomology. Hence, by Alexander's duality (cp. Dold [12, p. 301]),

$$\tilde{H}_n(F^+_{\mu,\nu,a,b}) = \tilde{H}_{n+k}(\tilde{F}^+_{\mu,\nu,a,\tilde{b}}),$$

where $k = \dim N_{a,\tilde{b}} - \dim N_{a,b}$,

$$\tilde{F}^+_{\mu,\nu,a,b} = \{x \in N_{a,b}:\ \|n\| = 1,\ f^+_{\mu,\nu,a,b}(n) > 0\},$$

$\tilde{F}^+_{\mu,\nu,a,\tilde{b}}$ is defined similarly and $\tilde{H}_n$ denotes reduced homology. (Here we have used that $N_{a,b} \backslash \tilde{Z}_b = \tilde{F}^+_{\mu,\nu,a,b}$ and the corresponding result for $N_{a,\tilde{b}}$. Thus, by Dold [12, p. 51], $\Sigma^1 \wedge \Sigma^k \wedge \tilde{F}^+_{\mu,\nu,a,b}$ and $\Sigma^1 \wedge \tilde{F}^+_{\mu,\nu,a,\tilde{b}}$ have the same homology. To complete the proof, we need to make a few observations. Firstly, implicit in the proof of the result of [6, p. 55] quoted earlier is that $F^+_{\mu,\nu,a,b}$ is a strong deformation retract of a neighbourhood of itself in the sphere. (This is proved by using the tangential component of the differential equation and deforming along flow lines of this equation on the sphere.) Thus $F^+_{\mu,\nu,a,b}$ is an absolute neighbourhood retract (in the sense of Spanier [21, p. 56]). A similar argument shows that $\tilde{F}^+_{\mu,\nu,a,b}$ and $F^+_{\mu,\nu,a,b}$ have the same homotopy type and thus $\Sigma^1 \wedge \Sigma^k \wedge F^+_{\mu,\nu,a,b}$ and $\Sigma^1 \wedge F^+_{\mu,\nu,a,\tilde{b}}$ have the same homology. Since each of these spaces is simply connected (by Spanier [21, Corollary 8.5.3]) and since they are absolute neighbourhood retracts (because $F^+_{\mu,\nu,a,b}$ and $F^+_{\mu,\nu,a,\tilde{b}}$ are), Corollary 7.8.5 in Hu [16] implies that either space is contractible if and only if all of its homology groups are zero. Thus, since they have the same homology, if one is contractible, so is the other. Hence, if $h(0, F_{\mu,\nu,a,b}) \neq 0$, then

$$h(0, F_{\mu,\nu,a,\tilde{b}}) \neq 0.$$

Remark. A rather more elaborate argument (using, for example, Lemma 1 in Dancer [11]) implies that

$$h(0, F_{\mu,\nu,a,\tilde{b}}) = \Sigma^k \wedge h(0, F_{\mu,\nu,a,b}).$$

We have shown that adding an eigenvalue below a or above b does not make the homotopy index zero. Thus the assumption of Theorem 2 is well defined. Note that, we have also shown that it suffices to use homology (or cohomology) to decide whether the homotopy index is non-zero.

We now return to the proof of Theorem 2.

Step 1. We prove that (9) is solvable for every $f \in L^2(\Omega)$ if g is continuously differentiable on **R** *and there is a $K>0$ such that $|g'(y)| \leqq K$ on* **R**.

If $a < -K$ and $b > K$ we can repeat the proof of Lemma 1 to reduce Eq. (9) to the equation

$$Ln = Q_{a,b}\, g(n+S_1(n)) - Q_{a,b} f \tag{12}$$

on $N_{a,b}$. Here $S_1(n)$ is the unique solution in $W_{a,b}$ of

$$Lw = P_{a,b}\, g(n+w) - P_{a,b} f. \tag{13}$$

A simple estimation shows that

$$\|S_1(n)\| \leqq K_1 \|n\| + K_2. \tag{14}$$

Moreover, as before, (12) is a gradient system. Now since $y^{-1}(g(y) - \mu y^+ - \nu y^-) \to 0$ as $|y| \to \infty$ and since g is continuous, it is easy to see (cp. [7, Lemma 2]) that

$$\|u\|^{-1}(g(u) - \mu u^+ - \nu u^-) \to 0 \tag{15}$$

in $L^2(\Omega)$ as $\|u\| \to \infty$. Since $S_1(n)$ is a solution of (13) and since $S(n)$ is a solution of (6), a simple estimation shows that $\|n\|^{-1} \|S(n) - S_1(n)\| \to 0$ as $n \to \infty$. (Here we use (14) and (15).) From this inequality and (15), we find that

$$Ln - Q_{a,b}\, g(n+S_1(n)) = F_{\mu,\nu}(n) + r(n), \tag{16}$$

where $\|n\|^{-1} \|r(n)\| \to 0$ as $\|n\| \to \infty$. By similar arguments, r is bounded on bounded sets.

Choose a fixed bounded neighbourhood T of zero. Then T is an isolating neighbourhood for $F_{\mu,\nu}$. Suppose we look for a solution of (12) of the form $n = st$, where $t \in T$ and s is large. By (16) and the positive homogeneity of $F_{\mu,\nu}$, (12) becomes

$$F_{\mu,\nu}(t) + s^{-1} r(st) - s^{-1} Q_{a,b} f = 0. \tag{17}$$

If s is large, our earlier estimates for r imply that the last two terms are small on all of T. Thus, by homotopy invariance,

$$h(I, F_1) = h(0, F_{\mu,\nu}) \neq 0,$$

if s is sufficiently large. (Here F_1 denotes the mapping on the left-hand side of (17) and I denotes the set of bounded solutions of $n'(t) = F_1(n(t))$ in T.) Since the left-hand side of (17) is still a gradient equation (because (12) is), it follows easily (cp. [3, proof of Theorem 9.1]) that (17) has a solution in T and hence (9) has a solution, as required.

Step 2. We remove the extra condition on g.

Fix $f \in L^2(\Omega)$. Now, since $(\mu,\nu) \notin A_0$, there is a $\tilde{K} > 0$ such that

$$\|Lu - \mu u^+ - \nu u^-\| \geqq \tilde{K} \|u\| \tag{18}$$

for $u \in L^2(\Omega) \cap \mathscr{D}(L)$. If $|g_1(y)| \leq \frac{1}{2}\tilde{K}|y| + M$ on R (where $g(y) = \mu y^+ + \nu y^- + g_1(y)$), it is easy to see that

$$\|g_1(u)\| \leq \tfrac{1}{2}\tilde{K}\,\|u\| + M_1 \tag{19}$$

on $L^2(\Omega)$, where M_1 depends only on $\tilde{K}$, M and the measure of Ω. It follows from (18) and (19) that, if u is a solution of (9), then $\|u\| \leq M_2$, where M_2 depends only on $\tilde{K}$, M and the measure of Ω.

It is easy, but tedious, to construct continuously differentiable g_n: $\mathbf{R} \to \mathbf{R}$ such that (i) $g_n \to g$ uniformly on compact subsets of $\mathbf{R}$, (ii) $|g_n'(y)| \leq K_n$ on $\mathbf{R}$ and (iii) for every $\varepsilon > 0$, there is an $M_\varepsilon > 0$ with

$$|g_n(y) - \mu y^+ - \nu y^-| \leq \varepsilon |y| + M_\varepsilon \tag{20}$$

on $\mathbf{R}$ for all n. By Step 1, there is a solution u_n of $Lu = g_n(u) - f$ and, by the argument in the previous paragraph, $\|u_n\| \leq M_2'$ for all n. Hence by (20) and the equation satisfied by u_n, we find that $\{Lu_n\}$ is bounded in $L^2(\Omega)$. Since L has compact resolvent, it follows that $\{u_n\}$ is compact in $L^2(\Omega)$. Thus, by choosing a subsequence if necessary, we may assume that $u_n \to v$ in $L^2(\Omega)$ as $n \to \infty$. Since L is closed, the required result that $Lv = g(v) - f$ will follow if we show that $g_n(u_n) \to g(v)$ in $L^2(\Omega)$ as $n \to \infty$. By Egorov's theorem and the construction of the g_n, $g_n(u_n) \to g(v)$ uniformly except on a set of small measure. Hence the result will follow if we show that $\int g_n(u_n)^2$ and $\int g(v)^2$ are small over sets of small measure. This follows easily from (20) since $\{u_n^2\}$ are equi-integrable (because $\{u_n\}$ is pre-compact in $L^2(\Omega)$). This proves Theorem 2.

Remarks. 1. Note that our assumption that L has compact resolvent is used essentially in the proof of Step 2 of Theorem 2.

2. It would be preferable to have a type of homotopy index in infinite dimensions that could be used directly.

3. Our methods (and especially Step 1) could be applied to many other asymptotically small gradient perturbations. For example, the methods should be applicable to some Hamiltonian systems of ordinary differential equations. The technique in Step 2 should be useful in other situations. It could be used to improve some of the results in [3].

4. The homotopy index condition in Theorem 2 is more general than the corresponding condition on the degree (as in § 3 of [7]). (It can be shown that, except for sign, the appropriate degree is independent of a and b and equals the infinite-dimensional Leray-Schauder degree.) The homotopy index condition is more general because a slight generalization of the result in Rothe [20] and Alexander duality imply that the degree is an alternating sum of the ranks of the reduced homology groups $\tilde{H}_n(F_{\mu,\nu}^+)$. (Remember that the homotopy index is zero if and only if $\tilde{H}_n(F_{\mu,\nu}^+) = 0$ for all n.) Moreover the homotopy index condition is sometimes more convenient to check. For example, the homotopy index is non-zero if $F_{\mu,\nu}^+$ is not connected. (This is used in § 3.) However, if $[\mu, \nu]$ contains at most, two eigenvalues of L (counting multiplicity), the homotopy index is zero precisely when the degree is zero. It is possible to construct an example where $N_{a,b}$ is three-dimensional, the degree is zero but the homotopy index is non-zero.

We now consider an important special case where our assumptions are satisfied. Assume $\tau \in \sigma(L)$ and $(s, y) \in \mathbf{R}^2$ with $s^2 + y^2 = 1$ and $s \geqq y$. Let Q_τ denote the orthogonal projection onto K_τ, the kernel of $L - \tau I$, and define $k_{s,y}$: $K_\tau \to K_\tau$ by $k_{s,y}(n) = -Q_\tau(s n^+ + y n^-)$. We easily see (cp. [7, §3]) that, if $k_{s,y}(n) \neq 0$ for $n \in K_\tau \setminus \{0\}$ and if t is sufficiently small and non-zero, then $(\tau + t s, \tau + t y) \notin A_0$. Note that $k_{s,y}$ is positive homogeneous and is a gradient mapping. (It is the gradient of $-\int_\Omega [s(n^+)^2 + y(n^-)^2]$.)

Theorem 3. *Assume that $k_{s,y}(n) \neq 0$ for $n \in K_\tau \setminus \{0\}$ and $a < \tau < b$. If t is sufficiently small and positive, $h(0, F_{\mu(t), v(t), a, b}) = \Sigma^k \wedge h(0, k_{s,y})$, where $\mu(t) = \tau + t s$, $v(t) = \tau + t y$ and k is the number of eigenvalues of L (counting multiplicity) in (τ, b). In particular, if $h(0, k_{s,y}) \neq 0$, if g: $R \to R$ is continuous with $y^{-1} g(y) \to \mu(v)$ as $y \to \infty$ $(-\infty)$, and if (μ, v) lies in the component of $\mathbf{R}^2 \setminus A_0$ containing $(\mu(t), v(t))$ (where t is small and positive), then Eq. (9) has a solution for all $f \in L^2(\Omega)$.*

Proof. The second statement follows from the first, Theorem 2 and Remark 2 after Theorem 2 if we note that, as earlier, whether the homotopy index is non-zero is determined by homology and thus cannot be made zero by suspensions. We now prove the first statement. With our earlier notation (except we write $S_t(n)$ to show the variation of $S(n)$ with t),

$$L S_t(n) = P_{a,b}[\mu(t)(n + S_t(n))^+ + v(t)(n + S_t(n))^-]$$

i.e.

$$(L - \tau I) S_t(n) = t P_{a,b}[s(n + S_t(n))^+ + y(n + S_t(n))^-].$$

Since $(L - \tau I)|_{W_{a,b}}$ is invertible, it follows easily that there is a $K > 0$ such that $\|S_t(n)\| \leqq t K \|n\|$ if t is small (and positive) and $n \in N_{a,b}$. Now we must consider the homotopy index of zero of

$$\begin{aligned} F_{\mu(t), v(t)}(n) = L n - \tilde{P}[\mu(t)(n + S_t(n))^+ + v(t)(n + S_t(n))^-] \\ - Q_\tau[\mu(t)(n + S_t(n)^+ + v(t)(n + S_t(n))^-], \end{aligned}$$

where $\tilde{P} = P_{a,b} - Q_\tau$, i.e., of

$$\begin{aligned} (L - \tau I) \tilde{P} n - t \tilde{P}[s(n + S_t(n))^+ + y(n + S_t(n))^-] \\ - t Q_\tau[s(n + S_t(n)^+ + y(n + S_t n)^-]. \end{aligned}$$

Since multiplying by the positive self adjoint $\tilde{P} + \frac{1}{t} Q_\tau$ does not affect the homotopy index of zero (cp. [6], p. 55), we have to evaluate the homotopy index of 0 for

$$\begin{aligned} (L - \tau I) \tilde{P} n - t \tilde{P}[s(n + S_t(n)^+ + y(n + S_t(n))^-] \\ - Q_\tau[s(n + S_t(n))^+ + y(n + S_t(n))^-]. \end{aligned}$$

Since $\|S_t(n)\| \leqq t K \|n\|$, we see by letting t tend to zero, that we must evaluate the homotopy index of

$$(L - \tau I) \tilde{P} n - Q_\tau[s n^+ + y n^-].$$

Now, by the homotopy $(L-\tau I)\tilde{P}n - Q_\tau[s(tn+(1-t)Q_\tau n)^+ + y(tn+(1-t)Q_\tau n)^-]$ our homotopy index reduces to the homotopy index of $(L-\tau I)\tilde{P}n - Q_\tau[s(Q_\tau n)^+ + y(Q_\tau n)^-]$. (In the last homotopy, it is easy to check directly that, for any bounded solution $n(r)$ of the corresponding differential equation, $\tilde{P}n(r) = 0$ for all $r \in \mathbf{R}$.) The result now follows for the product theorem and the formula for the homotopy index of zero of a linear mapping (cp. [6], § 3.6D and § 1.4.3).

Remarks. 1. The proof could be shortened a little if we used the remark just before the proof of Theorem 2.

2. A similar result holds for $t<0$. ($k_{s,y}$ is replaced by $-k_{s,y}$). It can be deduced from this and Remark 2 after the statement of Theorem 2 that, if t is small and positive, then $(\tau - ty, \tau - ts) \notin A_0$ provided that $k_{s,y}(n) \neq 0$ on $N_{a,b} \setminus \{0\}$. Moreover the corresponding homotopy index is non-zero if and only if $h(0, k_{s,y}) \neq 0$ (and thus if and only if the corresponding homotopy index for $\mu(t)$, $\nu(t)$ is non-zero).

3. Theorem 2 improves a result in [7]. (Our result is equivalent to the one there if $\dim K_\tau \leqq 2$.) As before, there is an example with $\dim K_\tau = 3$, with the homotopy index non-zero and with the degree zero.

Finally we explain how Theorem 1 can be improved. Assume that $\tau \in \sigma(L)$ and that $K_1, \ldots, K_p$ are closed "intervals" on

$$B = \{(s, y) \in \mathbf{R}^2 \colon\ s \geqq y,\ s^2 + y^2 = 2\}$$

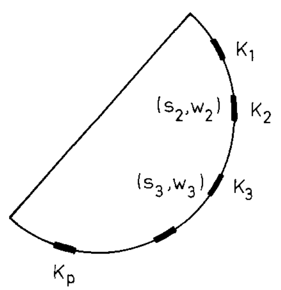

Fig. 2

such that $k_{s,y}(n) \neq 0$ if $n \in K_\tau \setminus \{0\}$ and $(s, y) \in B \setminus \bigcup_{i=1}^{p} \operatorname{int} K_i$ and such that, for each i, $k_{s,y}(n_0) = 0$ for some $n_0 \in K_\tau \setminus \{0\}$ and some $(s, y) \in K_i$. Let m_i denote $h(0, k_{s_i, w_i})$, where (s_i, w_i) is the end point of K_i nearest $(1, 1)$ and let $m_{p+1} = h(0, k_{-1,-1})$. It is easy to show (cp. [7]), that if α is sufficiently small, if $(\mu, \nu) \in A_0^+$ and if $\|(\mu, \nu) - (\tau, \tau)\|_2 \leqq \alpha$ (where $\|\ \|_2$ denote the L^2 norm on $\mathbf{R}^2$), then $(\mu, \nu) - (\tau, \tau) = rw$ where $w \in \operatorname{int} K_i$ for some i. Let $\tilde{K}_i = \{(\mu, \nu) \colon (\mu, \nu) - (\tau, \tau) = rw$, where $w \in K_i$, $0 < r < \frac{1}{2}\alpha\}$. We say that K_i and K_j are strongly related if there is a connected set $T \subseteq A_0^+ \setminus \{(\tau, \tau)\}$ such that T contains points arbitrarily close to (τ, τ) in both $\tilde{K}_i$ and $\tilde{K}_j$. (Intuitively, A_0^+ contains a "loop",)

We say that K_i and K_j are related if $i = j$ or if $i_0, i_1, \ldots, i_s$ are such that $i_0 = i$, $i_s = j$ and K_{i_t} is strongly related to $K_{i_{t+1}}$ for $0 \leqq t \leqq s-1$. (Intuitively, there is a succession of "loops".) Theorem 1 can be improved as follows.

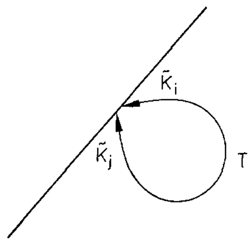

Fig. 3

Proposition 2. *Suppose that $1 \leqq i \leqq p$. Let i_1 be the smallest integer such that $i_1 \leqq i$ and K_{i_1} is related to K_i and let i_2 be the largest integer such that $i_2 \geqq i$ and K_{i_2} is related to K_i. Then either* (i) *there is a j such that K_i and K_j are related and an unbounded connected subset A of $A_0^+ \setminus (\tau, \tau)$ containing points arbitrarily close to (τ, τ) in $\tilde{K}_j$ or* (ii) *m_{i_1} and m_{i_2+1} have the same stable homotopy type.*

(Two sets X and Y have the same stable homotopy type if there is a non-negative integer p such that the suspensions $S^p X$ and $S^p Y$ have the same homotopy type. If $m_i = [X_i]$, for each i, when we say that m_i and m_j have different stable homotopy type, we really mean X_i and X_j have different stable homotopy types. Note that we need only consider a finite number of p's by (Spanier [21, Theorem 8.5.11]) and that, if X_i and X_j have different homology, they have different stable homotopy type.) This proposition improves Theorem 1 if $p \geqq 2$ because it is not difficult to use homology to prove that m_j and m_1 have different stable homotopy types if $j > 1$ and that m_j and m_{p+1} have different stable homotopy types if $j < p+1$. Proposition 2 is proved by using Theorem 3 and properties of connected sets (as in § 2 of [9]) and by more carefully using the ideas in the proof of Theorem 1. We omit the proof. Note that stable homotopy occurs because of the suspension in Theorem 3. (If L is bounded above or below, we often need only consider fewer suspensions.) There is one case when the statement of our (rather complicated) Proposition 2 can be simplified. Assume that, for each i, $A_0^+ \cap \tilde{K}_i$ is a simple curve (or is empty). Then, in possibility (i) of Proposition 2, we may take $i = j$ and strongly related is equivalent to related. This assumption is true in many cases because if (i) $k_{s_0, y_0}(n_0) = 0$ where $\|n_0\| = 1$; if (ii) $\{x \in \Omega: n_0(x) = 0\}$ has measure zero and if (iii) the non-radial part of k_{s_0, y_0} has locally a Lipschitz continuous inverse near n_0 (as a map of the sphere in K_τ into the tangent bundle to the sphere), then a rather tedious application of the contraction mapping theorem implies that $\{(\mu, \nu, u) \in \mathbf{R}^2 \times L^2(\Omega): \|u\| = 1, u$ is near n_0, $(\mu, \nu) - (\tau, \tau)$ is small and has direction near (s_0, y_0), $Lu = \mu u^+ + \nu u^-\}$ is a curve parameterized by t, where $t = \|(\mu, \nu) - (\tau, \tau)\|_2$. (In addition, in this case, $k_{s,y}(n) \neq 0$ if n is near n_0 on the sphere, if $n \neq n_0$ and if (s, y) is near (s_0, y_0).)

Finally, the homotopy index considerably restricts the manner in which a connected subset A of A_0^+ intersecting $\tilde{K}_i$ arbitrarily close to (τ, τ) can meet (γ, γ) where $\gamma \in \sigma(L) \setminus \{\tau\}$.

§ 3. On a Simple Special Case

If $(\mu, \nu) \notin A_0$, if the range of $Lu - \mu u^+ - \nu u^-$ is not equal to $L^2(\Omega)$ and if $g: \mathbf{R} \to \mathbf{R}$ is continuous such that $y^{-1} g(y) \to \mu(\nu)$ as $y \to \infty(-\infty)$, then

$$\mathscr{R} \equiv \{f \in L^2(\Omega): (9) \text{ has a solution}\}$$

is a closed proper subset of $L^2(\Omega)$. (This is proved in [7].) On the other hand, Theorem 3 implies that, if a certain homotopy index is non-zero, then $\mathscr{R}=L^2(\Omega)$. It is natural to ask whether these two possibilities exhaust $\mathbf{R}^2\backslash A_0$. Thus we make the rather optimistic conjecture.

Conjecture A. If $(\mu, \nu)\notin A_0$ and the range of $Lu-\mu u^+-\nu u^-$ is equal to $L^2(\Omega)$, then $h(0, F_{\mu,\nu})\neq 0$.

It would be of interest to prove a result of this type under additional assumptions on L or on (μ, ν). Proposition 1 of [7] shows that Conjecture A is true if L is a second order ordinary differential operator with separated boundary conditions. We will prove a result of the second type in a moment. There are numerous weaker variants of Conjecture A. The weakest is probably the following.

Conjecture A′. Suppose that $(\mu, \nu)\notin A_0$ and that there is a continuous g: $\mathbf{R}\to\mathbf{R}$ such that $y^{-1}g(y)\to\mu(\nu)$ as $y\to\infty(-\infty)$ and such that $\mathscr{R}\neq L^2(\Omega)$. Then, for every continuous g with $y^{-1}g(y)\to\mu(\nu)$ as $y\to\infty(-\infty)$, $\mathscr{R}\neq L^2(\Omega)$.

We only prove one very simple result.

Proposition 3. *Assume that $\lambda_{i-1}<\lambda_i<\lambda_{i+1}$ are distinct successive eigenvalues of L and λ_i is simple. If $\lambda_{i-1}<\mu$, $\nu<\lambda_{i+1}$ and $(\mu, \nu)\notin A_0$, then either* (i) *the range of $Lu-\mu u^+-\nu u^-$ does not equal $L^2(\Omega)$ or* (ii) $h(0, F_{\mu,\nu})\neq 0$. *(If $i=1$, we let $\lambda_0=-\infty$.)*

Proof. Let $a=\lambda_{i-1}$ and $b=\lambda_{i+1}$ in the reduction method of § 1. If $f\in N_{a,b}$, and thus $P_{a,b}f=0$, it is easy to see that the equation $Lu=\mu u^+ +\nu u^- -f$ has a solution if and only if the equation $F_{\mu,\nu}(n)=f$ has a solution (in the notation of § 1). By our assumptions, $N_{a,b}$ is one-dimensional. It is easy to see that a positive homogeneous mapping g on $\mathbf{R}$ with $g(1)\neq 0$ and $g(-1)\neq 0$ satisfies either (i) $h(0, g)\neq 0$ or (ii) the range of g is $[0,\infty)$ or $(-\infty, 0]$. Thus $h(0, F_{\mu,\nu})\neq 0$ or the range of $F_{\mu,\nu}$ is not equal to $N_{a,b}$, i.e., there is an f in $N_{a,b}$ such that $Lu=\mu u^+ +\nu u^- -f$ has no solution. This completes the proof.

Remarks. 1. In fact, if $h(0, F_{\mu,\nu})\neq 0$, the corresponding degree is also non-zero. Essentially, our proof is valid because every positive homogeneous mapping of $\mathbf{R}$ to $\mathbf{R}$ satisfies the analogue of Conjecture A. This result does not extend to maps of $\mathbf{R}^2$ (even for maps g which are gradient mappings and such that $G(z)+G(-z)=k\|z\|^2$, where g is the gradient of G and $z\in\mathbf{R}^2$.

2. There is one variant of Proposition 3. We delete the assumption that λ_i is simple but assume that there is a group G of unitary operators on $L^2(\Omega)$ commuting with L and with the map $u\to\mu u^+ +\nu u^-$ such that K_{λ_i}, the kernel of $L-\lambda_i I$, is generated by one element under the action of the group. Then it is easy to show that $f_{\mu,\nu}$ is invariant under the action of the group G of unitaries and thus $f_{\mu,\nu}(n)=C\|n\|^2$. It follows easily that Proposition 3 holds in this case and, in fact $h(0, F_{\mu,\nu})\neq 0$ if $(\mu, \nu)\notin A_0$ and $\lambda_{i-1}<\mu$, $\nu<\lambda_{i+1}$. Note that such a group of unitaries is usually a reflection of symmetries of the domain Ω. The result is useful because, in many cases, a multiple eigenvalue is a consequence of such a group of symmetries.

3. In fact, the method in Remark 2 can be used to obtain results if we weaken the assumption there that K_{λ_i} is generated by one element under the action of the group. For example, assume that Ω is a square in $\mathbf{R}^2$, that L is the Laplacian with Dirichlet boundary conditions and that $i=2$. Then either $(\mu, \nu)\in A_0$ or the corresponding homotopy index is non-zero (where $\lambda_1<\mu$, $\nu<\lambda_3$). This

follows because, by the symmetries, $F_{\mu,\nu}^{+}$ is $\{x \in N_{a,b}: \|x\|=1\}$, or it is empty or it has at least four components. (In fact, a more difficult result of Nussbaum [18] implies that, if $(\mu, \nu) \notin A_0$, then the corresponding degree is non-zero.) Symmetries can be used to simplify the calculation of the homotopy index and of the degree in many other cases. For example, assume that (i) Ω is the disc in $\mathbf{R}^2$, (ii) L is the Laplacian with Dirichlet boundary conditions, (iii) $(\mu, \nu) \notin A_0$ and (iv) (9) has a radially symmetric solution for every radially symmetric $f \in L^2(\Omega)$. Then it can be shown that (9) has a solution for all $f \in L^2(\Omega)$. Note that, by a variant of Proposition 1 in [7], condition (iv) reduces to a simpler condition on the radially symmetric solutions of eqn (3). The proof will appear elsewhere. In addition, we will show that there are cases where the symmetries enable one to quickly decide whether the homotopy index is non-zero but it is difficult to decide whether the degree is non-zero.

4. We mentioned earlier that there is an example with $N_{a,b}$ three-dimensional, the homotopy index non-zero and the degree zero. This example can be chosen such that Ω is the unit disc in $\mathbf{R}^2$ and such that a circle group of symmetries acts on $N_{a,b}$. The example can be used to construct an example where (i) Ω is the unit disc in $\mathbf{R}^2$; (ii) L is invariant under the circle group of symmetries generated by the rotations and (iii) there is a (relatively) open subset W of the radially symmetric functions in $L^2(\Omega)$ such that (9) has a solution for all $f \in L^2(\Omega)$ but no radially symmetric solution for all $f \in W$. It would be of interest to decide whether similar behaviour can occur when L is the Laplacian.

§ 4. Additional Open Problems

In this section, we want to mention some other problems. The first problem is to try to understand better the structure of A_0. In particular, does A_0 contain an open set and does every component of A_0 contain an element (μ, ν) with $\mu = \nu$? Little is known of A_0 even in such simple cases as (*a*) Ω is the disc in $\mathbf{R}^n$ and L is the Laplacian with Dirichlet boundary conditions or (*b*) $\Omega = [-1, 1]$ and $Ly = y^4$ with the boundary conditions $y(-1) = y^1(-1) = y(1) = y^1(1) = 0$. (The latter case is a problem raised by Fučik.) A few comments can be made. For some partial differential operators, a few elements of A_0 can be found by a separation of variables. Some results can be proved for problem (b) above. Firstly, the analogues of all the results of § 2 in [7] hold (though some of the proofs there need to be modified). Secondly, it can be shown that, if $u \neq 0$, if $Lu = \mu u^+ + \nu u^-$ and if (μ, ν) belongs to the component C_i of A_0 containing (λ_i, λ_i) (where λ_i is the ith eigenvalue of L), then u has exactly $(i-1)$ zeros in $(0, 1)$ and each of these is simple. It follows that $C_j \cap C_i = \emptyset$ if $j \neq i$. Thirdly, some of our comments in § 2 and the easily proved result that $\|h^+\|_2 \neq \|h^-\|_2$ for each eigenfunction of L with an even number of zeros can be used to obtain a rather complete understanding of A_0 near the eigenvalues of L.

We mention two more problems. Firstly, *if* $(\mu, \nu) \in A_0$, *prove that the range* $\mathscr{R}_1$ *of* $Lu - \mu u^+ - \nu u^-$ *is not dense in* $L^2(\Omega)$. This is probably a rather optimistic conjecture and it would be of interest to prove it under additional assumptions on L or on (μ, ν). The reason it is of interest is that, if $(\mu, \nu) \in A_0$, if $\bar{\mathscr{R}}_1 \neq L^2(\Omega)$, if $g: \mathbf{R} \to \mathbf{R}$ is continuous and if $g(y) - \mu y^+ - \nu y^-$ is bounded on $\mathbf{R}$, it follows that $\bar{\mathscr{R}} \neq L^2(\Omega)$ (cp. [8]). The conjecture is known to be true for some second order ordinary differential operators (cp. [8]). (The class includes all second order operators with Dirichlet boundary conditions.) It is not difficult to prove that, if the range of $F_{\mu,\nu}$ is not dense in $N_{a,b'}$ then $\bar{\mathscr{R}}_1 \neq L^2(\Omega)$. It follows that the conjecture is true when $\lambda_{i-1} < \mu$, $\nu < \lambda_{i+1}$ and λ_i is simple (by a similar argument to that in the proof of Proposition 3). As in Remark 2 after Proposition 3, the conjecture is still true when we replace "λ_i is simple" by an appropriate symmetry condition.

Finally, if g is convex, if $y^{-1}g(y) \to \mu(\nu)$ as $y \to \infty(-\infty)$, $(\mu, \nu) \notin A_0$ and $\lambda_{i-1} < \mu < \lambda_i < \nu < \lambda_{i+1}$ it would be of interest to understand the structure of $\mathscr{R}$ and the number of solutions of (9) for each $f \in \mathscr{R}$. Here we take $\lambda_0 = -\infty$. Partial results are known if μ and ν are near λ_i (e.g. Podolak [19]). If λ_i is simple and if $\mathscr{R} \neq L^2(\Omega)$, it is easy to use some of our earlier ideas to show that there is a continuous function $\tilde{g}: K_i^\perp \to \mathbf{R}$ such that $\mathscr{R} = \{\alpha h_i + v: \alpha \geqq \tilde{g}(v)\}$ (or $\alpha \leqq \tilde{g}(v)$). Here h_i spans K_i. Moreover, if $f \in \operatorname{int} \mathscr{R}$, the equation has at least two solutions. In fact, the convexity could be weakened. In some cases where h_1 is positive on Ω a much more precise result is known (cp. [4] or [10]). (This result is also true for $i = 1$ in case (b) above provided that $\mu \geqq 0$.) Unfortunately, if h_i changes sign in Ω, it can be shown under fairly very weak hypotheses on L that such a precise result is no longer true. In particular, there is an f in $L^2(\Omega)$ for which the equation has at least 4 solutions.

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