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Quadrature Formulae for H^p Functions

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Recently the problem of obtaining finite quadrature formulae for $\int_{-1}^{1} f(x)\,dx$, where $f(x)$ ranges over the Hardy class H^p, $p>1$, has received much attention. Indeed, improving over earlier estimates due to Bojanov, Wilf et al., see e.g. [1] and [5], Loeb and Werner, [3] discovered the remarkable fact that n-term quadrature formulae exist which approximate $\int_{-1}^{1} f(x)\,dx$ to within the order of $e^{-c\sqrt{n}}$! (We refer the reader to their paper for the precise statements of the problem and the results.)

They leave open the question of whether this marvelous proximity can be improved even further, and it is that question to which we address ourselves in this paper. Our answer is both *yes* and *no*. Thus we show that for each fixed $p>1$ their answer, $e^{-c\sqrt{n}}$, is the correct one, but that their estimate for c as a function of p is definitely improvable. The bound that Loeb and Werner obtain is that c is of the size $\frac{1}{q}$ $\left(\text{where here, as throughout, we write } q \text{ for the conjugate index to } p\text{, i.e. } q=\frac{p}{p-1}\right)$. We will show that c can be chosen of the size $\frac{1}{\sqrt{q}}$ and that this is then the correct, unimprovable answer.

To state our result precisely, we refer back to [3] where a duality argument is given which allows us to define our *proximity index* as

$$(1)\quad \mathscr{E}_{n,p}=\inf_B \sup_f \left|\int_{-1}^{1} f(x)\,B(x)\,dx\right|$$

where $B(z)$ ranges over all n-th degree Blaschke products and $f(z)$ ranges over all H^p functions of norm 1.

Our result is

Theorem. *With $\mathscr{E}_{n,p}$ given by* (1), *we have*

$$\tfrac{2}{3}\, e^{-6\sqrt{\frac{n}{q}}} \leqq \mathscr{E}_{n,p} \leqq 11\, e^{-\frac{1}{2}\sqrt{\frac{n}{q}}}.$$

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(In particular for $p=1$, no quadrature formula "works".)

The Upper Bound

By Hölder's inequality, we have

$$(2)\quad \left|\int_{-1}^{1} f(x)\,B(x)\,dx\right| \leqq \left(\int_{-1}^{1} |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_{-1}^{1} |B(x)|^q\,dx\right)^{\frac{1}{q}}$$

and it is well known [2], page 48, that

$$(3)\quad \left(\int_{-1}^{1} |f(x)|^p\,dx\right)^{\frac{1}{p}} \leqq \pi^{\frac{1}{p}}\,\|f(z)\|_p.$$

Therefore, by (1) we obtain

$$(4)\quad \mathscr{E}_{n,p} \leqq \pi^{\frac{1}{p}} \left(\int_{-1}^{1} |B(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Our job, then, is to construct a $B(x)$ so as to make this right side as small as we can.

We proceed in our construction of $B(x)$ very much as Loeb and Werner did. The only difference is that whereas they used the exact construction given in [4], we find it preferable to modify it somewhat.

Namely it is shown in [4] that

$$(5)\quad \text{With } r(t)=\frac{p(-t)}{p(t)}, \qquad p(t)=\prod_{k=0}^{N-1}(t+\xi^k), \qquad \xi=e^{-\frac{1}{\sqrt{N}}},$$

we have $|r(t)| \leqq e^{-\sqrt{N}}$ for $e^{-\sqrt{N}} \leqq t \leqq 1$.

We now define $R(t)$ by

$$(6)\quad R(t)=r(t)\,r(tM)\,r(tM^2)\ldots r(tM^{Q-1}) \quad \text{where} \quad M=e^{\sqrt{N}}.$$

and we observe that, for all t in $[e^{-Q\sqrt{N}}, 1]$, one of these factors, by (5), is bounded by $e^{-\sqrt{N}}$ while all of the others are bounded by 1. Thus we have

$$(7)\quad |R(t)| \leqq e^{-\sqrt{N}} \qquad \text{for } e^{-Q\sqrt{N}} \leqq t \leqq 1.$$

We now define

$$(8)\quad Q=[q], \qquad N=\left[\frac{n}{2Q}\right], \qquad B(x)=R\left(\frac{1-x^2}{1+x^2}\right),$$

and we note that this $B(x)$ *is* a Blaschke product, and that it has degree $2\cdot Q\cdot N \leqq n$. Furthermore, by (7), we have $|B(x)| \leqq e^{-\sqrt{N}}$ for $\frac{1-x^2}{1+x^2} \geqq e^{-Q\sqrt{N}}$, i.e. for $x^2 \leqq \frac{1-e^{-Q\sqrt{N}}}{1+e^{-Q\sqrt{N}}}$, which is surely the case if $|x| \leqq 1-e^{-Q\sqrt{N}}$. In short we have

$$(9)\quad |B(x)| \leqq e^{-\sqrt{N}} \qquad \text{for } |x| \leqq 1-e^{-Q\sqrt{N}}.$$

We may now estimate $\int_{-1}^{1} |B(x)|^q\,dx$ by using (9) for $|x| \leqq 1 - e^{-Q\sqrt{N}}$ and $|B(x)| \leqq 1$ on the rest of the interval. This gives

$$(10)\quad \int_{-1}^{1} |B(x)|^q\,dx \leqq 2\,e^{-q\sqrt{N}} + 2\,e^{-Q\sqrt{N}} \leqq 4\,e^{-Q\sqrt{N}}.$$

Next it is seen from (8) that $\sqrt{N} \geqq \sqrt{\frac{n}{2Q}} - 1$ so that $Q\sqrt{N} \geqq \sqrt{\frac{nQ}{2}} - Q$ and since $q \geqq 1$ we see that $q \geqq Q \geqq \frac{q}{2}$ so that $Q\sqrt{N} \geqq \frac{1}{2}\sqrt{nq} - q$ and thus (10) becomes

$$(11)\quad \left(\int_{-1}^{1} |B(x)|^q\,dx\right)^{\frac{1}{q}} \leqq 4^{\frac{1}{q}}\,e \cdot e^{-\frac{1}{2}\sqrt{\frac{n}{q}}}.$$

Finally, by (4), we conclude that

$$(12)\quad \mathscr{E}_{n,p} \leqq \pi^{\frac{1}{p}}\,4^{\frac{1}{q}}\,e \cdot e^{-\frac{1}{2}\sqrt{\frac{n}{q}}} \leqq 11\,e^{-\frac{1}{2}\sqrt{\frac{n}{q}}}$$

as asserted.

The Lower Bound

Let $B(z)$ be an arbitrary n-th degree Blaschke product. We wish to pick $f(z) \in H^p$, $\|f(z)\|_p = 1$, so as to make $\int_{-1}^{1} f(x)\,B(x)\,dx$ as large as we can. Our choice will be

$$(13)\quad f(z) = \frac{c}{1 - \rho z^2}\,\overline{B(\bar{z})},\ \rho = 1 - e^{-\sqrt{nq}},\ \text{and } c \text{ chosen so that } \|f\|_p = 1$$

$$\left(\text{i.e. } c = \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1 - \rho\,e^{i\theta}|^p}\right)^{\frac{-1}{p}}\right).$$

We thereby obtain

$$(14)\quad \int_{-1}^{1} f(x)\,B(x)\,dx = c\int_{-1}^{1} \frac{|B(x)|^2}{1 - \rho x^2}\,dx \geqq c\int_{-\rho}^{\rho} \frac{|B(x)|^2}{1 - \rho x^2}\,dx$$
$$\geqq \frac{2}{3}\,c\int_{-\rho}^{\rho} \frac{|B(x)|^2}{1 - x^2}\,dx,$$

$\left(\text{this last since } \frac{1}{1 - \rho x^2} \geqq \frac{2}{3}\frac{1}{1 - x^2} \text{ throughout } [-\rho, \rho]\right)$.

If we now define

$$(15)\quad a = \int_{-\rho}^{\rho} \frac{dx}{1 - x^2}\ \left(= \log\frac{1+\rho}{1-\rho} \geqq \sqrt{nq}\right)$$

then $\frac{1}{a}\frac{1}{1-x^2}dx$ is a probability measure on $[-\rho,\rho]$ and we may apply the Arithmetico-Geometric inequality to conclude that

$$(16)\quad \frac{1}{a}\int_{-\rho}^{\rho}\frac{|B(x)|^2}{1-x^2}dx \geqq \exp\left(\frac{1}{a}\int_{-\rho}^{\rho}\frac{\log|B(x)|^2}{1-x^2}dx\right).$$

Now let us examine $\int_{-1}^{1}\log\left|\frac{x-\alpha}{1-\bar{\alpha}x}\right|\frac{dx}{1-x^2}=\int_{-1}^{1}\log\left|\frac{x-\alpha}{1-\alpha x}\right|\frac{dx}{1-x^2}$. This is a bounded harmonic function of α in the half-disc $|\alpha|<1$, $\operatorname{Im}\alpha>0$. It is seen to vanish everywhere on the circular boundary, $|\alpha|=1$. As for the flat boundary, $-1<\alpha<1$, we observe that the measure $\frac{dx}{1-x^2}$ is invariant under the maps $t=\frac{x-\alpha}{1-\alpha x}$, $-1<\alpha<1$, so that we have there $\int_{-1}^{1}\log\left|\frac{x-\alpha}{1-\alpha x}\right|\frac{dx}{1-x^2}=\int_{-1}^{1}\log|t|\frac{dt}{1-t^2}=-\frac{\pi^2}{4}$. By the minimum principle then, we conclude that, throughout our half disc,

$$(17)\quad \int_{-1}^{1}\log\left|\frac{x-\alpha}{1-\alpha x}\right|\frac{dx}{1-x^2}\geqq -\frac{\pi^2}{4},$$

and exactly the same argument holds for the lower half disc. From this inequality we of course derive the fact that

$$(18)\quad \int_{-\rho}^{\rho}\frac{\log|B(x)|}{1-x^2}dx \geqq \int_{-1}^{1}\frac{\log|B(x)|}{1-x^2}dx \geqq -n\frac{\pi^2}{4}$$

so that by (1), (14), and (16) we may conclude that

$$(19)\quad \mathscr{E}_{n,p}\geqq\frac{2}{3}c\,a\exp\left(\frac{-n\pi^2}{2a}\right).$$

We finally need an estimate of c. To this end we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\rho e^{i\theta}|^p}\leqq\frac{1}{(1-\rho)^{p-1}}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\rho e^{i\theta}|}$$

and since

$$\frac{d}{d\rho}\frac{\rho}{|1-\rho e^{i\theta}|}=\frac{1-\rho\cos\theta}{|1-\rho e^{i\theta}|^3}\leqq\frac{1}{|1-\rho e^{i\theta}|^2}$$

we get

$$\frac{d}{d\rho}\left(\rho\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\rho e^{i\theta}|}\right)\leqq\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\rho e^{i\theta}|^2}=\frac{1}{1-\rho^2}$$

which by integration gives $\rho\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\rho e^{i\theta}|}\leqq\frac{1}{2}\log\frac{1+\rho}{1-\rho}$. Hence

$$(20)\quad c\geqq(1-\rho)^{\frac{1}{q}}\cdot\frac{2\rho}{\log\frac{1+\rho}{1-\rho}}=e^{-\sqrt{\frac{n}{q}}}\cdot\frac{2\rho}{a}\geqq e^{-\sqrt{\frac{n}{q}}}\cdot\frac{1}{a},$$

(this last from (13) and (15)).

If we now insert this in (19) and recall (15) we find that

$$(21)\quad \mathscr{E}_{n,p}\geqq\frac{2}{3}e^{-\sqrt{\frac{n}{q}}}\exp\left(-\frac{\pi^2}{2}\sqrt{\frac{n}{q}}\right)\geqq\frac{2}{3}e^{-6\sqrt{\frac{n}{q}}}.$$

The proof is complete.

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