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How to Conjugate C^1-Close Group Actions

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The aim of this paper is to prove the following stability theorem for group actions.

Theorem A. *Let G be a compact Lie group and let M be a connected compact differentiable manifold. Then any two C^1-close G-actions on M are conjugate by an explicitly defined diffeomorphism isotopic to the identity map of M.*

The existence of a map conjugating two C^1-close G-actions has already been proved by Palais [5]. Palais' proof relies essentially on the fact that there exists a representation of G in an orthogonal group $O(n)$ and an equivariant embedding of M in $\mathbb{R}^n$. The main tool in our approach is to define a "*center of mass*" for almost constant maps. This enables us to define a specific map conjugating the two group actions if they are C^1-close.

Using this notion of center we prove in the last paragraph a differential geometric version of the theorem:

If M carries a Riemannian metric and one of the actions is by isometries, then we say in terms of curvature bounds for M, independent of the dimension of M, how C^1-close the actions have to be.

We hope that this will be useful in e.g. pinching problems, compare the special case $G=\mathbb{Z}_2$ in Grove and Karcher [3] and also Shiohama [6].

A key point in our work was to find a useful "*center of mass*" of a map. We thank D. Burghelea for stimulating discussions on that. He arrived at such a definition for maps with connected compact domain using properties of the heat equation and harmonic maps, — we believe that his center can also be used to prove Theorem A for connected compact Lie groups.

1. Center of almost Constant Maps

Let N and M be compact Riemannian manifolds and denote by $C^0(N, M)$ the Banach manifold of continuous maps from N to M; the component of the constant maps is modelled on $C^0(N, \mathbb{R}^n)$. The

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tangent bundle $TC^0(N,M)$ is naturally equivalent to $C^0(N,TM)$ and $C^0(N,M)$ carries a Finsler structure given by

$$(1.1)\qquad \|X\|_{C^0}:=\max_{p\in N}\|X(p)\|_{f(p)}\qquad \text{for all } X\in T_f\,C^0(N,M)$$

see e.g. Eliasson [1].

There is a natural embedding $\hat{e}$: $M\to C^0(N,M)$ defined by $\hat{e}(m)(p):=m$ for all $p\in N$ and all $m\in M$. Put $\hat{e}(m):=\hat{m}$ and $\hat{e}(M):=\hat{M}$. Then we have the Banach bundle $TC^0(N,M)|_{\hat{M}}$ over $\hat{M}$ containing the tangent bundle $T\hat{M}$ of $\hat{M}$ as a finite dimensional subbundle. We define now a complementary bundle to $T\hat{M}$ in $TC^0(N,M)|_{\hat{M}}$. Let P: $TC^0(N,M)|_{\hat{M}}\to TC^0(N,M)|_{\hat{M}}$ be given by

$$(1.2)\qquad P(X)(p)=\int_N X,\qquad (\text{normalization: vol}(N)=1)$$

for all $X\in TC^0(N,M)|_{\hat{M}}$ and all $p\in N$. P is a differentiable bundle map with $P^2=P$ i.e. a bundle projection. The image of P equals $T\hat{M}$, so $E:=\ker P\subseteq TC^0(N,M)|_{\hat{M}}$ is a bundle with $E\oplus T\hat{M}=TC^0(N,M)|_{\hat{M}}$.

The connection on M induces a connection on $C^0(N,M)$ such that the exponential map of $C^0(N,M)$ is given by $C^0(\exp)$: $TC^0(N,M)\to C^0(N,M)$ i.e. for $X\in T_f\,C^0(N,M)$ we have $C^0(\exp)(X)(p)=\exp_{f(p)}X(p)$ for all $p\in N$, – see Eliasson [1]. Thus from the bundle projection π: $E\to\hat{M}$ we get a differentiable deformation retraktion of a tubular neighborhood $U_\varepsilon:=C^0(\exp)(E_\varepsilon)$ of $\hat{M}$ in $C^0(N,M)$ to $\hat{M}$, – denote this by $\tilde{\pi}:=\pi\circ C^0(\exp)^{-1}|_{U_\varepsilon}$.

Now if $f\in C^0(N,M)$ is sufficiently C^0-close to a constant map $\hat{m}\in C^0(N,M)$ we have $f\in U_\varepsilon$. We shall say that f is "*almost constant*". We define the "*center of mass*" $\mathscr{C}(f)$ of f by

$$(1.3)\qquad \mathscr{C}(f):=\tilde{\pi}\circ f\in\hat{M}.$$

Then for any isometry A: $M\to M$ on M we have

$$(1.4)\qquad A\circ\mathscr{C}(f)=\mathscr{C}(A\circ f).$$

To prove this we just note that for the constant map $\hat{c}=\mathscr{C}(f)$ we have $\int_N \exp_c^{-1}\circ f=0$ and $A_*\circ\exp_c^{-1}\circ f=\exp_{A(c)}^{-1}\circ A\circ f$, where A_* is the differential of the isometry A.

2. The Conjugation Map

Let now N be a compact Lie group G with right invariant metric and corresponding volume, and consider two differentiable G-actions on M μ_i: $G\times M\to M$, $i=1,2$. Choose a riemannian metric on M such that μ_1 operates by isometries. Let R_h denote right translation on G

by $h \in G$; we have for every $f \in C^0(G, M)$ which has a well defined center $\mathscr{C}(f)$ (i.e. $f \in U_\varepsilon$), that

$$\mathscr{C}(f) = \mathscr{C}(f \circ R_h). \tag{2.1}$$

Define now the map $\eta\colon G \times M \to M$ by $\eta(g, m) = \mu_1(g^{-1}, \mu_2(g, m))$ for all $(g, m) \in G \times M$ and let $\hat{\eta}\colon M \to C^1(G, M) \subset C^0(G, M)$ be the corresponding map $\hat{\eta}(m) := \eta(m, \cdot)$. It is easy to see that $\hat{\eta}\colon M \to C^0(G, M)$ is an embedding and if μ_2 is C^1-close to μ_1, then $\hat{\eta}$ is C^1-close to $\hat{e}\colon M \to C^0(G,M)$ ($e\colon G \times M \to M$ is the trivial action). Since $\hat{\eta}$ is of course C^0-close to $\hat{e}$ we have that $\hat{\eta}(M) \subset U_\varepsilon$, so

$$S := \tilde{\pi} \circ \hat{\eta}\colon M \to \hat{M} \equiv M \tag{2.2}$$

is a well defined differentiable map. Since π is a bundle projection and $\hat{\eta}$ is C^1-close to $\hat{e}$ we have $S = \tilde{\pi} \circ \hat{\eta}$ is C^1-close to $1_M = \tilde{\pi} \circ \hat{e}$, i.e. S is a diffeomorphism on M isotopic to 1_M.

To complete the proof of Theorem A we show that S conjugates μ_1 and μ_2. To this end, note that $S(m) = \mathscr{C}(\hat{\eta}(m))$ for all $m \in M$, so using (1.4) and (2.1) we get for each $h \in G$

$$\begin{aligned} \mu_1(h, S(m)) &= \mu_1(h, \mathscr{C}(\hat{\eta}(m))) \\ &= \mathscr{C}(\mu_1(h, \cdot) \circ \hat{\eta}(m)) \\ &= \mathscr{C}(\mu_1(h, \cdot) \circ \hat{\eta}(m) \circ R_h) \\ &= \mathscr{C}(\hat{\eta}(\mu_2(h, m))) \\ &= S(\mu_2(h, m)). \quad \text{Q.E.D.} \end{aligned} \tag{2.3}$$

Corollary 2.4. *Let $\mu_1, \mu_2\colon H \to G$ be two homomorphisms of the compact Lie group H into the Lie group G with biinvariant metric. We can view μ_1 and μ_2 as actions of H on G by isometries, namely left translations $\mu_i(h, g) := \mu_i(h) \cdot g$. If μ_1 and μ_2 are C^0-close then the map S(2.2) turns out to be a left translation. In other words the subgroups $\mu_1(H)$, $\mu_2(H)$ are conjugate in G.*

Proof. By assumption the map $\hat{\eta}_g\colon H \to G$, $\hat{\eta}_g(h) := \mu_1(h^{-1}, \mu_2(h, g)) = \mu_1(h^{-1}) \cdot \mu_2(h) \cdot g$ is almost constant so that the center $\mathscr{C}(\hat{\eta}_g) =: S(g)$ is defined. Now (1.4) and $\hat{\eta}_g = R_g \circ \hat{\eta}_e$ imply $\mathscr{C}(\hat{\eta}_g) = \mathscr{C}(\hat{\eta}_e) \cdot g$ or $S(g) = S(e) \cdot g$. The left translation S is a diffeomorphism of G which conjugates the actions μ_1, μ_2 and $S(e) \in G$ conjugates the subgroups $\mu_1(H)$, $\mu_2(H)$.

Remark. As example consider $O(n)$ with scalar product $\langle A, B \rangle := \frac{1}{2} \cdot \operatorname{trace} AB^*$. The metric is so normalized that the shortest closed geodesics — e.g. $\exp t\cdot A$, $0\leqq t\leqq 2\pi$,

$$A=\begin{pmatrix} 0 & 1 & \\ -1 & 0 & 0 \\ & 0 & 0 \end{pmatrix}^{-}$$

have length 2π. The cutlocus distance is π and for the sectional curvatures K holds $0\leqq K\leqq\frac{1}{2}$. It follows from (3.1) and (3.3) that a map into $O(n)$ has a well defined center if its image is contained in a ball of radius $\pi/4$ (actually $\pi/3$ with a refinement of (3.3)). Two subgroups $\mu_1(H)$, $\mu_2(H)$ are therefore conjugate, if $\max_H d(\mu_1(h),\mu_2(h))\leqq\frac{\pi}{3}$.

Corollary 2.5. *Let M be compact, The identity $1_M\in\mathrm{Diff}(M)$ has a neighborhood U_ε in the C^1-topology, such that any homomorphism τ: $H\to U_\varepsilon\subset\mathrm{Diff}(M)$ of a compact Lie group H is trivial.*

Proof. By Theorem A τ is conjugate to the trivial action of H on M.

Remark. The notion of almost constant map with a well defined center can be extended to more general situations. Consider a map $f\colon X\to C^0(N,M)$, X of finite volume, N compact. We call f almost constant, if for each $n\in N$ the map $f_n\colon X\to M$, $f_n(x)=f(x)(n)\in M$ is almost constant. We define the center of f to be the map $\mathscr{C}(f)\in C^0(N,M)$ given by $\mathscr{C}(f)(n)=\mathscr{C}(f_n)$. The interpretation of the center is given by the following: For every $n\in N$ holds $\int_X \exp^{-1}_{\mathscr{C}(f)(n)} f(x)(n)\,dx=0$. For example an almost constant $f\colon X\to\mathrm{Diff}(M)$ has a center in the space of maps from M to M; according to the proof of Theorem A the center is actually in $\mathrm{Diff}(M)$ if f is almost constant in the C^1-sense.

3. A Differential Geometric Version of the Previous Construction

In this section we assume that M is a Riemannian manifold and that the action μ_1 of the compact Lie group G is by isometries. We describe in terms of the Riemannian metric how close another action μ_2 (by diffeomorphisms) has to be to μ_1, so that the center-map is a well defined diffeomorphism conjugating μ_1 and μ_2.

Let δ be the minimum and Δ the maximum of the sectional curvatures K of M, and let D be the minimum of the cut-locus distance $d(p,C(p))$ (a continuous function on M). Put $\rho:=\frac{1}{2}D$ if $\Delta\leqq 0$ or $\rho:=\mathrm{Min}(\frac{1}{2}D,\frac{1}{2}\pi\Delta^{-\frac{1}{2}})$ if $\Delta>0$. Then every metric ball B of radius $<\rho$ is strongly convex, i.e. for any points $p,q\in B$ there is a unique minimizing geodesic from p to q in M and this geodesic segment lies in B, see e.g. Karcher [4].

Proposition 3.1. *Let X be a measure space of volume 1 and $f\colon X\to M$ a measurable map such that the image $f(X)$ of f in M is contained in a* *metric ball B of radius $\leqq \rho'$ determined below, Then f has in B a well defined center $\mathscr{C}(f) \in B$.*

Proof. For every $p \in B$ we define a tangent vector v_p at p by

$$v_p := \int_X \exp_p^{-1} f(x)\, dx. \tag{3.2}$$

We claim that the vector field $p \to v_p$ is differentiable and vanishes only at one point of B. Therefore in B the center $\mathscr{C}(f)$ is well defined. To prove the claim we need the following:

Lemma 3.3. *For $q \in M$ let c: $[0,1] \to M$ be a geodesic with $\|\dot{c}\| = \beta < 2\rho$ and $c(0) = q$. Let J be a Jacobi field along c with $J(0) = 0$ (i.e. $J'' + R(J, \dot{c})\dot{c} = 0$ where R is the curvature tensor and "$'$" denotes covariant derivative). Then for $\delta > 0$ we have,*

$$\|J(t) - t\,J'(t)\| \leqq \left(\frac{\Delta}{\delta}\right)^{\frac{3}{2}} \cdot \frac{\sin \delta^{\frac{1}{2}} \beta\, t - \delta^{\frac{1}{2}} \beta\, t \cos \delta^{\frac{1}{2}} \beta\, t}{\sin \Delta^{\frac{1}{2}} \beta\, t} \cdot \|J(t)\|$$

$$\leqq k_1(\Delta, \delta, \beta) \cdot \|J(t)\| \qquad \left(\text{e.g. } k_1\left(\frac{1}{2}, 0, \frac{\pi}{2}\right) \leqq \frac{\pi^3}{48}\right)$$

and similar formulas for other combinations of signs of δ and Δ (as will be seen from the proof).

Proof. $\frac{d}{dt} \langle J - t J', J - t J' \rangle = 2 \langle -t J'', J - t J' \rangle$ implies that

$$\|J(t) - t\,J'(t)\| \leqq \int_0^t \|t \cdot J''\|\, dt \leqq \max |K| \cdot \beta^2 \cdot \int_0^t t \cdot \|J\|\, dt. \tag{3.4}$$

For $\|J\|$ we have the Rauch-estimates, see e.g. Gromoll, Klingenberg, Meyer [2],

$$\|J(t)\| \leqq \begin{cases} \|J'(0)\| \cdot \delta^{-\frac{1}{2}} \beta^{-1} \sin \delta^{\frac{1}{2}} \beta\, t & \text{if } \delta > 0 \\ \|J'(0)\| \cdot t & \text{if } \delta = 0 \\ \|J'(0)\| \cdot (-\delta)^{-\frac{1}{2}} \beta^{-1} \sinh(-\delta)^{\frac{1}{2}} \beta\, t & \text{if } \delta < 0 \end{cases}$$

and similar lower bounds with δ replaced by Δ. Insert these estimates in (3.4) and integrate:

$$\|J(t) - t\,J'(t)\| \leqq \|J'(0)\| \max |K| \cdot \beta^2 (\delta^{\frac{1}{2}} \beta)^{-3} \cdot (\sin \delta^{\frac{1}{2}} \beta\, t - \delta^{\frac{1}{2}} \beta\, t \cos \delta^{\frac{1}{2}} \beta\, t),$$

(resp. if $\delta < 0$: $\leqq \|J'(0)\| \max |K| \beta^2 ((-\delta)^{\frac{1}{2}} \beta)^{-3}$

$\cdot ((-\delta)^{\frac{1}{2}} \beta\, t \cosh(-\delta)^{\frac{1}{2}} \beta\, t - \sinh(-\delta)^{\frac{1}{2}} \beta\, t)$ etc.).

Now use the lower Rauch-estimates to prove the lemma:

$$\|J(t)-tJ'(t)\| \leqq \|J(t)\| \cdot \max |K| \, \beta^2 (\delta^{\frac{1}{2}} \beta)^{-3}$$
$$\cdot (\sin \delta^{\frac{1}{2}} \beta t - \delta^{\frac{1}{2}} \beta t \cdot \cos \delta^{\frac{1}{2}} \beta t) \cdot \Delta^{\frac{1}{2}} \beta (\sin \Delta^{\frac{1}{2}} \beta t)^{-1}$$

and similarly for $\Delta = 0$ or $\Delta < 0$.

Remark. In pinching situations or if $\Delta < 0$ the lemma can be improved by estimating $\|\cos kt \cdot J - k^{-1} \sin kt \cdot J'\|$ with e.g. $2k^2 = (\Delta + \delta) \beta^2$.

To continue the proof of Proposition 3.1 we choose $\rho' < \rho$ such that $\beta \leqq 2\rho'$ in Lemma 3.3 implies that $\|J(t) - tJ'(t)\| \leqq q \cdot \|J(t)\|$ for some $q < 1$. We consider now the vector field v along a geodesic γ defined by $\gamma(\varepsilon) = \exp_p \varepsilon \cdot A$ with $A \in T_p M$ and $\|A\| = 1$. For each $x \in X$ we get the contribution $\exp_{\gamma(\varepsilon)}^{-1} f(x)$ to the integral $v_{\gamma(\varepsilon)}$. Therefore define the family of geodesics $c(\varepsilon, t) = \exp_{\gamma(\varepsilon)}(1-t) \exp_{\gamma(\varepsilon)}^{-1} f(x)$ which join $c(\varepsilon, 0) = f(x)$ to $c(\varepsilon, 1) = \gamma(\varepsilon)$. Consider then the Jacobi fields $J_{\varepsilon_0}(t) = \dfrac{d}{d\varepsilon} c(\varepsilon_0, t)$ along the geodesics $c(\varepsilon_0, .)$, obviously $J_{\varepsilon_0}(0) = 0$ and $J_\varepsilon(1) = \dot{\gamma}(\varepsilon)$. We get

$$-\frac{D}{d\varepsilon} \exp_{\gamma(\varepsilon)}^{-1} f(x) = \frac{D}{d\varepsilon} \frac{d}{dt} c(\varepsilon, t)\Big|_{t=1} = J'_\varepsilon(1) = J'_\varepsilon(1; x).$$

All these derivatives exist uniformly in B, therefore we have

$$\frac{D}{d\varepsilon} v_{\gamma(\varepsilon)} = \int_X -J'_\varepsilon(1; x)\, dx. \tag{3.5}$$

From $J_\varepsilon(1; x) = \dot{\gamma}(\varepsilon)$ together with Lemma 3.3 we obtain

$$\left\| \dot{\gamma}(\varepsilon) + \frac{D}{d\varepsilon} v_{\gamma(\varepsilon)} \right\| \leqq q \cdot \|\dot{\gamma}(\varepsilon)\|. \tag{3.6}$$

Therefore v is a differentiable vector field with isolated singularities. The index of the vector field $-v$ is $+1$, since at the boundary of the convex ball B v is an average over inward pointing vectors (and B is contractible). From this we have that v has at least one singularity. On the other hand (3.6) implies that if γ is a geodesic leaving an isolated singularity of v, then the component of v in the direction $-\dot{\gamma}$ is strictly increasing, so the index of each singularity of $-v$ is $+1$, – thus v has exactly one singularity in B, the center $\mathscr{C}(f)$. (3.6) implies also $d(p, \mathscr{C}(f)) \leqq \|v_p\| \cdot (1-q)^{-1}$.

Let now X be a compact Lie group G with right-invariant metric of volume 1. Let μ_1 be an action on M by isometries and μ_2 an action by diffeomorphisms. As in Section 2 we consider the map η: $G \times M \to M$, and we denote now $\hat{\eta}(m)$: $G \to M$ by η_m and similarly $\eta(g, \cdot)$: $M \to M$ by η^g. We assume that the actions are so close in the C^0-topology, that for each

$m \in M$ we have that $\eta_m(G)$ is contained in some convex ball $B_{\rho'}$ as described in Proposition 3.1. We therefore have a unique *center of mass* of η_m *in* $B_{\rho'}$. Assuming now that for all $m \in M$ $\eta_m(G)$ lies in a ball of radius $\frac{1}{2}\rho'$, then there is in M a unique center $S(m) := \mathscr{C}(\eta_m)$ with the property

$$\|\exp_{S(m)}^{-1} \circ \eta_m\|_{C^0} < \rho',$$

i.e. $C^0(\exp)$ is a diffeomorphism on $E_{\rho'}$, see Section 1, thus $S: M \to M$ is differentiable. (In the notation of Section 2 $S = \tilde{\pi} \circ \hat{\eta}$, where $\tilde{\pi}$ is defined on $U_{\rho'}$.) As we have seen in Section 2 $\mu_1(g, S(m)) = S(\mu_2(g, m))$ for all $(g, m) \in G \times M$.

Proposition 3.7. *There is an explicit condition* (3.15) *for the* C^1-*distance of* μ_1 *and* μ_2 *which guarantess that S is a diffeomorphism, and a slightly sharper condition so that S is even isotopic to* 1_M.

Proof. We fix $m \in M$ and a unit tangent vector $A \in T_m M$. Put $\eta^g(m) =: m_g$, $\eta^g_*(A) =: A_g \in T_{m_g} M$ and $S(m) =: c$. Let γ_g: $[0, 1] \to M$ be the geodesic from c to m_g i.e. $\gamma_g(0) = c$ and $\gamma_g(1) = m_g$, put $\|\dot{\gamma}_g\| =: \beta_g$. Next consider $c(\varepsilon) := S(\exp_m \varepsilon \cdot A)$ and the geodesics $\gamma_{g\varepsilon}$ from $c(\varepsilon)$ to $m_{g\varepsilon} := \eta^g(\exp_m \varepsilon \cdot A)$. By that we get the Jacobi fields $J_g(t) = \frac{d}{d\varepsilon} \gamma_{g\varepsilon}(t)\Big|_{\varepsilon=0}$ with $J_g(0) = S_*(A)$ and $J_g(1) = A_g$. Moreover, by definition of $c(\varepsilon)$ we have $\int_G \dot{\gamma}_{g\varepsilon}(0)\, dg = 0$ and thus

$$\text{(3.8)} \qquad \frac{D}{d\varepsilon} \int_G \dot{\gamma}_{g\varepsilon}(0)\, dg = \int_G J_g'(0) = 0.$$

We shall now approximate $S_*(A)$ by an average of the A_g parallel translated to $c(0) = S(m)$ and thereby derive $S_*(A) \neq 0$. Let us split J_g in $J_g = Y_g + Z_g$, where Y_g and Z_g are Jacobi fields along γ_g with

$$\text{(3.9)} \qquad Y_g(0) = 0, \qquad Y_g(1) = A_g$$

and

$$\text{(3.10)} \qquad Z_g(0) = S_*(A), \qquad Z_g(1) = 0.$$

Lemma 3.3 gives us

$$\text{(3.11)} \qquad \|Z_g(0) + Z_g'(0)\| \leqq k_1(\Delta, \delta, \beta_g) \cdot \|Z_g(0)\|.$$

Similarly we have to compare $Y_g'(0)$ and B_g = parallel translate of A_g along γ_g to $\gamma_g(0) = c$. Therefore

Lemma 3.12. *If J is a Jacobi field along* γ *with* $J(0) = 0$ *and* P_t *denotes parallel translation along* γ, *then, if* $\delta > 0$,

$$\|J(t) - t \cdot P_t J'(0)\| \leqq \max|K| \cdot \delta^{-\frac{3}{2}} \cdot (\delta^{\frac{1}{2}} \beta t - \sin \delta^{\frac{1}{2}} \beta t) \cdot \Delta^{\frac{1}{2}} \cdot (\sin \Delta^{\frac{1}{2}} \beta t)^{-1} \cdot \|J(t)\|$$

and similar estimates for other signs of δ *and* Δ.

Proof. Put $W = J - t \cdot P_t J'(0)$, hence $W'' = J'' = -R(J, \dot{c})\dot{c}$. Let Q be the unit parallel field such that $Q(\tau) = \|W(\tau)\|^{-1} W(\tau)$ for a fixed τ. Then

$$\|W(\tau)\| = \langle W(\tau), Q(\tau)\rangle = \int_0^\tau \int_0 \langle W'', Q\rangle \leqq \int_0^\tau \int_0 \max |K| \, \beta^2 \, \|J\|,$$

we insert the upper Rauch estimate to get

$$\|W(\tau)\| \leqq \|J'(0)\| \cdot \max |K| \cdot \delta^{-\frac{3}{2}} \beta^{-1} (\delta^{\frac{1}{2}} \beta \tau - \sin \delta^{\frac{1}{2}} \beta \tau),$$

and the lower Rauch estimate to complete the proof.

Remark. Also this lemma can be improved for pinching situations.

Continuing the proof of Proposition 3.7 we get from Lemma 3.12

$$\|B_g - Y_g'(0)\| \leqq k_2(\Delta, \delta, \beta_g) \cdot \|A_g\|. \tag{3.13}$$

We approximate now $S_*(A) = \int_G Z_g(0)\, dg$ by $\int_G B_g\, dg$:

$$\int_G Z_g(0)\, dg - \int_G B_g\, dg \underset{(3.8)}{=} \int_G (Z_g(0) + Z_g'(0))\, dg - \int_G (B_g - Y_g'(0))\, dg,$$

hence by (3.11) and (3.13)

$$\|S_*(A) - \int_G B_g\, dg\| \leqq k_1(\Delta, \delta, \beta) \cdot \|S_*(A)\| + \int_G k_2(\Delta, \delta, \beta_g) \cdot \|A_g\|\, dg. \tag{3.14}$$

This inequality contains the desired information. We make now the *assumption* that the two actions of G are so close in the C^1-topology, that parallel translation of A_g along any once broken geodesic form m_g to m in $B_{\rho'}$ gives a vector PA_g such that $\measuredangle(A, PA_g) \leqq \Phi_g$. Then (3.14) implies that if

$$k_1(\Delta, \delta, \beta) < 1, \qquad k_2(\Delta, \delta, \beta_g) < \cos \Phi_g \tag{3.15}$$

we have

$$(1 + k_1(\Delta, \delta, \beta)) \|S_*(A)\| \geqq \int_G (\cos \Phi_g - k_2(\Delta, \delta, \beta_g)) \cdot \|A_g\|\, dg > 0.$$

So (3.15) *is the condition for S to be of maximal rank and since S is also homotopic to 1_M it is a diffeomorphism.*

Moreover if $\varphi := \measuredangle(S_*(A), \int_G B_g\, dg)$ *we have from* (3.14) *and* (3.15) *also*

$$\sin \varphi \leqq k_1 + k_2 \frac{1 + k_1}{\cos \Phi_g - k_2}. \tag{3.16}$$

Therefore we have also an explicit condition for S to satisfy the assumptions of the diffeotopy Theorem 3.1 *in Grove and Karcher* [3], *so S is then isotopic to* 1_M.

Example. In the case of M being the standard sphere i.e. $\delta = \Delta = 1$ we can, knowing the structure of Jacobifields on the sphere (compare (3.9) and (3.10)), obtain more explicit estimates. With X^{*g} denoting the reflection of the vector $X \in T_{S(m)} M$ at the tangent hyperplane orthogonal to γ_g we have,

$$\int_G (1-\alpha_g)\, S_*(A)\, dg - \int_G (1+\varepsilon_g)\, PA_g\, dg = \int_G \alpha_g\, S_*(A)^{*g}\, dg - \int_G \varepsilon_g\, PA_g^{*g}\, dg \tag{3.17}$$

where

$$\alpha_g := \frac{1}{2}\left(1 - \frac{\beta_g}{\tan \beta_g}\right) \quad \text{and} \quad \varepsilon_g := \frac{1}{2}\left(\frac{\beta_g}{\sin \beta_g} - 1\right). \tag{3.18}$$

Except in the case $G = \mathbb{Z}_2$ all the reflections "$*g$" are different; therefore it is not obvious how to get a reasonably sharp sufficient condition from (3.17). A rough one is: If for all $m \in S^n$ $\eta_m(G)$ lies in a ball of radius $\leqq \frac{\pi}{4}$ and $\Phi_g \leqq 76°$ for all $g \in G$ then μ_1 and μ_2 are conjugate.

Since a spherical triangle in a ball of radius $r = 45°$ ($=36°$, $=30°$) has area $\leqq 55.5°$ ($\leqq 33.1°$, $\leqq 22.2°$), we can guarantee $\Phi_g \leqq 76°$ if the angle Ψ_g between $A \in T_m S^n$ and $A_g \in T_{m_g} S^n$ parallel translated from m_g to m along the unique shortest geodesic is $\leqq \Phi_g$-area, i.e. $\leqq 20.5°$ ($\leqq 42.9°$, $\leqq 53.8°$).

Added in Proof. § 3 gives a proof of Theorem A without using Banach manifolds except for the differentiability argument for S on top of p. 17. To give a completely selfcontained differential geometric proof of Theorem A we prove the differentiability of the conjugating map S as follows:

On the neighborhood $U = \{(m, n) \in M \times M \mid d(m, n) < \rho'\}$ of the diagonal of $M \times M$ we define the map

$$v\colon U \to TM \quad \text{by} \quad v(m, n) := \int_G \exp_n^{-1}(\eta_m(g))\, dg \in T_n M.$$

We know $v(m, n) = 0 \Rightarrow n = \mathscr{C}(\eta_m) = S(m)$; therefore graph $S = v^{-1}(Z)$, where Z is the zero-section of TM. Now we prove that already the partial map $v(m, \cdot)\colon B_{\rho'}(m) \to TM$ is transversal to Z. Then graph S is (i) a differentiable submanifold of U and (ii) has no "vertical" tangent (= tangent to the second factor of $M \times M$), hence S is a differentiable map.

To prove the transversality assume $v(m, n) = 0$, note that $v(m, \cdot)$ is a vector field on $B_{\rho'}$, consider geodesics γ with $\gamma(0) = n$ and their image curves under $v(m, \cdot)$ in TM: $X(t) = v(m, \gamma(t))$. We identify $(\pi_*, K)\colon TTM \to$

$TM \oplus TM$, G. K. M. [2] p. 45, where π_* is the differential of $\pi\colon TM \to M$ and K is the connection map. Under this identification $\dot{X}(0) = (\dot{\gamma}, D_{\dot{\gamma}} X)(0)$, while tangent vectors to the zero-section are represented by $(\dot{\gamma}, 0)$. Now (3.6) proves the transversality.

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