Structural Optimization 16, 68–75 © Springer-Verlag 1998

Review Article

Numerical instabilities in topology optimization: A survey on procedures dealing with checkerboards, mesh-dependencies and local minima

O. Sigmund

Department of Solid Mechanics, Technical University of Denmark, DK-2800 Lyngby, Denmark

J. Petersson

Department of Mechanical Engineering, University of Linköping, S-58183 Linköping, Sweden

Abstract In this paper we seek to summarize the current knowledge about numerical instabilities such as checkerboards, mesh-dependence and local minima occuring in applications of the topology optimization method. The checkerboard problem refers to the formation of regions of alternating solid and void elements ordered in a checkerboard-like fashion. The mesh-dependence problem refers to obtaining qualitatively different solutions for different mesh-sizes or discretizations. Local minima refers to the problem of obtaining different solutions to the same discretized problem when choosing different algorithmic parameters. We review the current knowledge on why and when these problems appear, and we list the methods with which they can be avoided and discuss their advantages and disadvantages.

1 Introduction

The topology optimization method for continuum structures (Bendsøe and Kikuchi 1988; or Bendsøe 1995, for an overview) has reached a level of maturity where it is being applied to many industrial problems and it has widespread academic use, not only for structural optimization problems but also in material, mechanism, electromagnetics and other coupled field design problems.

Despite its level of matureness, there still exist a number of problems concerning convergence, checkerboards and mesh-dependence which are subject to debate in the topology optimization community. In this paper, we seek to summarize the current knowledge on these problems and to discuss methods with which they can be avoided.

The topology optimization problem consists in finding the subdomain Ω_s with limited volume V^*, included in a predetermined design domain Ω, that optimizes a given objective function f (e.g. compliance). Finding the optimal topology corresponds to finding the connectedness, shape and number of holes such that the objective function is extremized. Introducing a density function ρ defined on Ω, taking the value 1 in Ω_s and 0 elsewhere, the nested version of the problem can be written as

$$\left.\begin{aligned} &\min_{\rho} \quad f(\rho)\,, \\ &\text{s.t.} \quad \int_{\Omega} \rho \,\mathrm{d}x \le V^*\,, \\ &\qquad \rho(\mathbf{x}) = 0 \text{ or } 1, \quad \forall\, \mathbf{x} \in \Omega\,, \end{aligned}\right\} \tag{1}$$

i.e. wherever the displacement function $\mathbf{u}$ (or in general, the state function) appears, it has been eliminated through the implicit dependence defined by the equilibrium equation.

Typically the topology optimization problem is treated by *discretizing* (1) by dividing Ω into N finite elements. Usually one approximates the density as element-wise constant and thereby the discretized density can be represented by the N-vector $\boldsymbol{\rho}$. Taking ρ to be constant in each element is practical since integrations over elements can be performed with ρ outside the integral sign, and consequently one can operate with simple scalings of the usual element stiffness matrices. The discretized 0-1 topology optimization problem becomes

$$\left.\begin{aligned} &\min_{\rho} \quad f(\rho)\,, \\ &\text{s.t.} \quad V = \sum_{i=1}^{N} \rho_i\, v_i \le V^*\,, \\ &\qquad \rho_i = 0 \text{ or } 1, \quad i = 1,\dots,N\,, \end{aligned}\right\} \tag{2}$$

where ρ_i and v_i are element densities and volumes, respectively, and V^* is the total volume bound.

1.1 *Nonexistence*

It is well-known that the 0-1 topology optimization problem (1) lacks solutions in general. The reason is that given one design the introduction of more holes, without changing the structural volume, will generally increase the efficiency measure [decrease the function value f in (1)], i.e. there is a lack of closedness of the set of feasible designs. The type of numerical instability where a larger number of holes appear for larger N, which hence refers to *nonexistence* of solutions, is indeed found in numerical solutions of the topology optimization problem (2) and it is often termed *mesh-dependence.*

Given the fact that problem (1) generally is not solvable, there are two possibilities of computing something which is close to what an engineer wants.

a Modify problem (1) in such a way that the new version possesses a solution. Then discretize this new problem version, analyse the discretization procedure to make sure that the discrete problem will give solutions that are close to the exact ones, and, finally, develop algorithms to solve the discretized problem.

b Discretize problem (1). The discrete version of problem (1) generally *has* solutions since it is posed in finite dimension. Develop methods to solve this discrete problem and use heuristic rules to avoid unwanted effects such as e.g. checkerboard patterns.

Most often, methods belonging to **b** produce results with comparatively little computational effort. However, this methodology is dubious in the sense that by using it, one simply produces pictures that people want to see, without knowing which continuum problem is solved (because probably there is none). Choosing methodology **a**, there are two possible ways to overcome the nonexistence problem, namely by *relaxation* or *restriction* of the design problem. The former approach was used in the original works on topology optimization and will be briefly described in the following. The latter has become increasingly popular in recent years and will be discussed also in Section 3.

1.2 Relaxation

In principle relaxation means an enlargement of the design set to achieve existence.

In order to encompass composites, i.e. structures with fine microstructure, Bendsøe and Kikuchi (1988) introduced a microscale through the use of the so-called *homogenization approach to topology optimization.* Today we know that the ill-posedness of the 0-1 problem [for $f(\rho)$ being compliance], can be overcome by using so-called ranked layered materials (see Allaire and Kohn 1993 and references therein). Using this approach results in structures with large regions of perforated microstructure or composite materials, i.e. the resulting density ρ has "grey regions" ($0 < \rho < 1$).

Another approach to obtain a well-posed problem for a broad range of problems is to allow all materials with a symmetric and positive semidefinite elasticity tensor to compete in the problem. This converts the problem to one where the energy depends linearly on ρ (Bendsøe *et al.* 1994). Here, the variable ρ is allowed to attain all values between 0 and 1. This linear problem in one sense provides the "most relaxed" problem, and gives a useful bound on the maximum structural efficiency. It also models the *variable thickness sheet problem*, where ρ is interpreted as the thickness function of a two-dimensional sheet. Henceforth we will use this term for any topology optimization problem where the energy depends linearly on ρ.

1.3 Restriction

Restriction methods seek to find a set of designs which is smaller than the original one in (1), but which possesses sufficient closedness and thereby solutions. This is achieved by introducing extra constraints that bound the maximum allowed oscillation of ρ, i.e. the maximum degree of perforation of the structure. Examples, which will be explained in more detail in Section 3.2, are upper bounds on total variation, local or global gradients of ρ. The first corresponds to the perimeter of the set Ω_s. The last two make sense only if ρ attains intermediate values.

1.4 Penalization of intermediate densities

There are at least two different reasons for allowing but penalizing intermediate densities:

1. post-processing of solutions to relaxed problems,
2. to avoid integer programming techniques when solving (2) (with some perimeter-type constraint).

If one has chosen a homogenization approach, but has realized that in many cases manufacturing or other considerations exclude perforated or intermediate density regions and call for macroscopic 0-1 solutions, then one could add a penalization to the method, i.e. **1**. Attacking directly a restricted version of (2), optimization methods for discrete 0-1 variables often fail to solve large-scale problems. Therefore, for computational reasons, one has suggested to allow continuous variables, but penalize the intermediate ρ-values to somehow approach a black/white design, i.e. **2**. Doing this, one has perturbed the problem and a justification is needed, i.e. to show that solutions of the new problem approach those of the original discrete-valued one, as the penalization is made stronger. That this is indeed possible, at least with the first technique given below and perimeter control, and is proven by Petersson (1997b). If the order of problem manipulations is reversed, i.e. if the penalization and intermediate values are introduced in the problem *before* the design set restriction (which yields existence of solutions in both cases), then the restricting constraint can be expressed in terms of a continuous variable, cf. the gradient constraints in Sections 3.2.2 and 3.2.3.

In conclusion, for problems that exclude microstructures, allowing intermediate density values *and* simultaneously including penalization is in itself neither a relaxation nor a design set restriction* but rather an attempt to make classical optimization schemes applicable while leaving the solution set nearly unchanged (in contrast to restriction and relaxation).

Several different penalization techniques have been suggested. One example is to replace $f(\rho)$ by $f(\rho)+c\int\rho(1-\rho)\mathrm{d}x$ (Allaire and Francfort 1993; Allaire and Kohn 1993; Haber *et al.* 1996). Here c is the penalization parameter determining the degree of penalization. Another probably more popular method is the so-called SIMP [Simple Isotropic Material with Penalization (Zhou and Rozvany 1991)] approach (cf. Bendsøe 1989). Using the SIMP approach the stiffness tensor of an intermediate density material is $\mathbf{C}_{ijk\ell}(\rho) = \mathbf{C}^0_{ijk\ell}\rho^p$, where $\mathbf{C}^0_{ijk\ell}$ is the stiffness tensor of solid material and p

* If one excludes the penalization, then this procedure is indeed a relaxation since the design set is enlarged by allowing intermediate values and, consequently, from (1) one can arrive at the variable thickness sheet problem (which has solutions)

is the penalization factor which ensures that the continuous design variables are forced towards a black and white (0/1) solution. The influence of the penalty parameter can be explained as follows. By specifying a value of p higher than one, the local stiffness for $\rho < 1$ is lowered, thus making it "uneconomical" to have intermediate densities in the optimal design. The discretized SIMP optimization problem is written as

$$\left.\begin{aligned} &\min_{\rho} \quad f(\rho)\,, \\ &\text{s.t.} \quad V = \sum_{i=1}^{N} \rho_i\, v_i \leq V^*\,, \\ &\qquad 0 < \rho_{\min} \leq \rho \leq 1\,, \end{aligned}\right\} \tag{3}$$

where the design variables ρ are continuous variables and $\rho_{\min}$ are lower bounds on densities, introduced to prevent singularity of the equilibrium problem. If the objective is compliance, then $f(\rho) = \mathbf{F}^T\mathbf{K}^{-1}(\rho)\mathbf{F}$ where $\mathbf{F}$ is the load vector,

$$\mathbf{K}(\rho) = \sum_{i=1}^{N} \rho_i^p \mathbf{K}_i\,, \tag{4}$$

and $\mathbf{K}_i$ denote (global level) element stiffness matrices. When $p = 1$ the optimization problem corresponds to the variable thickness sheet problem.

1.5 Numerical problems

We have divided the common numerical problems appearing in topology optimization into three categories:

- *Checkerboards* refer to the problem of formation of regions of alternating solid and void elements ordered in a checkerboard like fashion.
- *Mesh dependence* refers to the problem of not obtaining qualitatively the same solution for different mesh-sizes or discretizations.
- *Local minima* refers to the problem of obtaining different solutions to the same discretized problem when choosing different algorithmic parameters.*

The definitions, appearances, mathematical and physical explanations and techniques to prevent the three problems are shown schematically in Table 1 and graphically in Fig. 1 and are discussed in detail in the following sections.

2 Checkerboards

The checkerboard problem is illustrated in Fig. 1b and consists of regions in the "optimal topology" consisting of alternating solid and void elements. It was earlier believed that these regions represented some sort of optimal microstructure, but papers by Díaz and Sigmund (1995) and Jog and Haber (1996) have shown that the checkerboard patterns are due to bad numerical modelling of the stiffness of checkerboards.

* E.g. different starting solutions

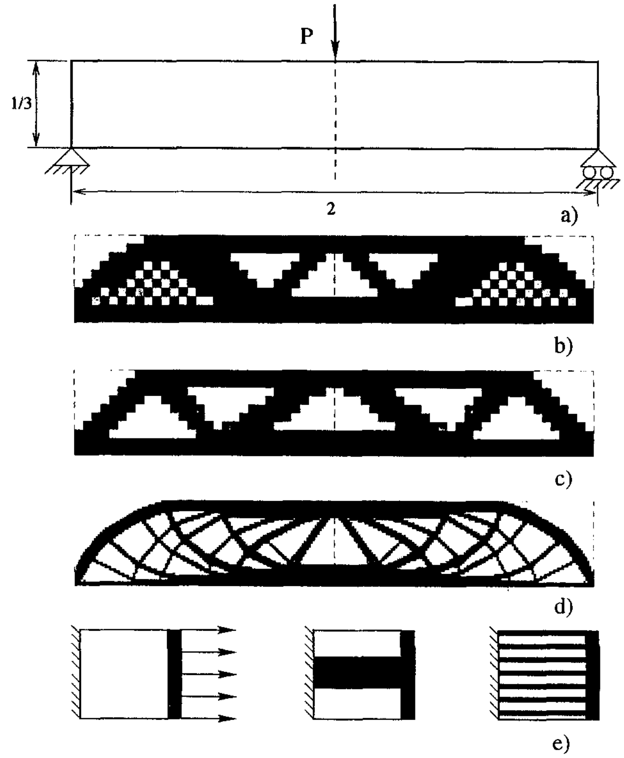

Fig. 1. (a) Design problem, (b) example of checkerboards, (c) solution for 600 element discretization, (d) solution for 5400 element discretization and (e) nonuniqueness example

Assume that the problem (1) has been adjusted so that solutions exist. If (2) is a good approximation of (1), the solutions of (2) will approach those of (1) as N is increased. This – henceforth referred to as *FE-convergence* – usually requires a careful study. A typical example of nonconvergence is the formation of checkerboards, and a guarantee of not obtaining such anomalies is one of the important byproducts of a FE-convergence proof.

Díaz and Sigmund (1995) compared the stiffness of checkerboard configurations in a discretized setting to the stiffness of uniformly distributed materials and concluded that the checkerboard structure has artificially high stiffness. Jog and Haber (1996) presented a theoretical framework based on a linearized, incremental form of the problem and a patch test was proposed. They argued that spurious thickness modes can be detected by investigating the nonuniqueness of solutions to the discretized incremental equation system. Both works provide useful guidelines regarding choice of stable elements and they show that checkerboard patterns are prone to appear in both the homogenization and the SIMP approach.

Theoretical studies of the appearance of checkerboards in three-dimensional problems have not yet been carried out. However numerical experience shows that checkerboards also appear for this case (e.g. Beckers 1997b).

Agreeing that the checkerboard problem is explained by

Table 1. Definition of problems found in discretized topology optimization. An "∃" indicates that existence of solutions has been proved

Numerical experience	Mathematical problem	Physical explanation	Prevention techniques
Checkerboards	Nonconvergence of FE-solutions	Erroneous FE-modelling of checkerboards	Higher order finite elements Patches Filtering Restriction methods below
Mesh dependence (a) Necessarily finer and finer structure	(a) Nonexistence	(a) "convergence" to microstructure	(a) Relaxation (∃) Perimeter (∃) Global/local gradient constraint (∃) Mesh-independent filtering
(b) Possibly finer and finer structure	(b) Nonuniqueness	(b) Ex.: uniaxial stress	(b) Nothing (maybe manufacturing preferences)
Local minima	Nonconvergence of algorithm	Nonconvexity Flatness	Continuation methods

bad numerics, several papers have suggested methods to prevent them. Some of these prevention schemes, which are almost all based on heuristics, are listed in the following:

2.1 Smoothing

Having obtained the "optimal solution" (with checkerboards) the output picture is smoothed with image processing. This method ignores the underlying problem and should be avoided. Many commercial post-processing codes automatically use smoothing of the output images, so here precautions should be taken.

2.2 Higher-order finite elements

Many papers suggest the use of higher-order finite elements for the displacement function **u** to avoid the checkerboard problem. Díaz and Sigmund (1995) and Jog and Haber (1996) show that checkerboards are mostly prevented when using 8 or 9-node finite elements for the homogenization approach. For the SIMP approach, however, checkerboards are only prevented using 8 or 9-node elements if the penalization power is small enough [i.e. p should be smaller than 2.29 for a specific example in the paper by Díaz and Sigmund (1995)]. This is also a byproduct of the FE-convergence proof in the paper of Petersson (1997a). In this work it is shown that, under a biaxial stress assumption, the mathematical analyses of mixed FEM for the Stokes' flow problem can be extended to prove strong FE-convergence and uniqueness of solutions in the variable thickness sheet problem.

A drawback of using higher-order finite elements is the substantial increase in cpu-time.

2.3 Patches

To save cpu-time but still obtain checkerboard free designs, Bendsøe *et al.* (1993) suggested the use of a patch technique inspired by similar problems in Stokes flow (Johnson and Pitkäranta 1982). This technique effectively introduces a kind of superelement to the finite element formulation and has in a practical test shown to damp the appearance of checkerboards. However in topology optimization it does not remove them entirely.

2.4 Filter

Based on filtering techniques from image processing, Sigmund (1994) suggested a checkerboard prevention filter implying modification of the design sensitivities used in each iteration of the algorithm solving the discretized problem. The filter makes the design sensitivity of a specific element depend on a weighted average over the element itself and its eight direct neighbours and is very efficient in removing checkerboards. However, the procedure belongs to methodology **b** described in Section 1, and therefore suffers from the principal difficulties characteristic for such methods. An extension of the method to ensure mesh independence is described in more detail in the next section.

2.5 Other methods

Furthermore, most of the restriction methods described in the next section on mesh-dependence will also reduce checkerboarding. The reason for this is that when one enforces a constraint, strong enough to make the set compact so that solutions exist, any sequence of admissible designs, such as FE-solutions, has convergent subsequences due to this compactness. The FE-convergence question for problems that have solutions without this compactification, e.g. Stokes' problem and the variable thickness sheet problem, is much more difficult since the sequence of FE-solutions cannot immediately be shown to converge. It is then necessary to make sure that the finite dimensional space for displacements **u** is sufficiently large compared to the space for ρ, i.e. as mentioned above, given that ρ is approximated as constant in each element, one needs a sufficiently large number of displacement nodes in each finite element. Checkerboards appear when this is not the case. In the restricted problems, however, this effect will be visible only for coarse meshes. If the mesh is made finer it follows from the compactness and (strong) convergence mentioned above that checkerboards are made arbitrarily weak. This was illustrated in the paper by Petersson and Sigmund (1998).

3 Mesh-dependence

The mesh-dependence problem is illustrated in Figs. 1c and d. Figure 1c shows the optimal topology for the so-called MBB-problem discretized by 600 finite elements (optimized using the SIMP approach). Solving the same problem but now with a 5400 finite element discretization, results in a much more detailed structure (Fig. 1d). Ideally mesh-refinement should result in a better finite element modelling of the same optimal structure and a better description of boundaries – not in a more detailed and qualitatively different structure.

As seen in Table 1, mesh-dependence problems can be divided into two categories, namely (a) the problem of (necessarily) obtaining finer and finer structure with mesh-refinement, which is due to to the previously discussed problem of nonexistence of solutions, and (b) problems with many optima, i.e. *nonunique solutions*. An example of the latter is the design of a structure in uni-axial tension. Here a structure consisting of one thick bar will be just as good as a structure made up of several thin bars with the same overall area (see Fig. 1e in which Poisson's ratio is zero and the vertical black strip at the points of force application is assumed rigid). In category (a) the refinement into a finer structure follows necessarily since it gives a strictly better value of the objective function. In (b) a finer structure is always possible but not necessary. The oscillation in ρ perpendicular to the direction of uniaxial stress is arbitrary and not bounded. If the stress is biaxial, however, it is proven by Petersson (1997a) that the optimal ρ is unique.

Naturally, one cannot set up schemes that remove the nonuniqueness problem, but by introducing manufacturing constraints such as a minimum area constraint a less oscillating solution can be determined. Schemes to prevent the nonexistence problems will be discussed in the following.

3.1 Relaxation

As discussed in the Introduction, the 0-1 topology optimization problem can be made well-posed by relaxation using the homogenization or free material approaches. Unless one is interested in resulting design with composite areas, this approach does in general not result in easily manufacturable solutions (i.e. macroscopic 0-1 designs).

3.2 Restriction methods

To obtain macroscopic 0-1 solutions some sort of global or local restriction on the variation of density must be imposed on the original topology optimization problem. So far, four different restriction schemes have been proposed, namely, perimeter control, global or local gradient constraint and mesh-independent filtering. The four methods, for which existence of solutions have been proved for the first three and the latter is based on heuristics, are discussed in the following.

3.2.1 Perimeter control. The *perimeter* of Ω_s is, vaguely speaking, the sum of the circumferences of all holes and outer boundaries. Existence of solutions to the perimeter controlled topology optimization problem was proved by Ambrosio and Buttazzo (1993). The first numerical implementation of the scheme was done by Haber *et al.* (1996). The scheme introduces intermediate density values with penalization in order to use some "usual" gradient-based algorithm. An upper bound on the *total variation* is used, $\mathrm{TV}(\rho) \leq P^*$. This makes sense since the total variation of ρ coincides with the perimeter of Ω_s when ρ is 1 in Ω_s and 0 elsewhere. The difference, hence, is that the total variation is defined even when ρ takes intermediate values. In case the function ρ is smooth,

$$\mathrm{TV}(\rho) = \int_\Omega |\nabla \rho| \, \mathrm{d}x. \tag{5}$$

For the *discretized* density, the total variation is in the implementation calculated as

$$P = \sum_{k=1}^{K} \ell_k \left(\sqrt{\langle \rho \rangle_k^2 + \varepsilon^2} - \varepsilon \right), \tag{6}$$

where $\langle \rho \rangle_k$ is the jump of material density through element interface k of length ℓ_k, $K \approx 2N$ is the number of element interfaces. The parameter ε is a small positive number which guarantees the differentiability of the perimeter. This expression is exactly the total variation of the element-wise constant density when $\varepsilon = 0$.

In the first implementation of the perimeter method, Haber *et al.* (1996) used an interior penalty method to impose the constraint and reported that some experiments with algorithm parameters were required to make the algorithm converge to mesh-independent designs. Using a mathematical programming method to solve the optimization problem, Duysinx (1997) reported the perimeter constraint to be quite difficult to approximate resulting in fluctuations in the design variables. Arguing that the evaluation of the perimeter is computationally cheap compared to the finite element solution, Duysinx (1997) proposed an internal loop procedure for perimeter approximation and reported very good convergence behaviour.

Perimeter control has been implemented in continuous variable settings in the references mentioned above and in a discrete variable setting in the report by Beckers (1996).

The following two gradient constraints presuppose that ρ is sufficiently smooth and defined for intermediate values. This is the case e.g. when the SIMP approach has been applied to the ill-posed problem (1). After this has been done, one can restrict the problem with (7) or (8) to ensure existence of solutions. It is less clear what these constraints correspond to as ρ approaches a discrete-valued function. [The measure in (5) converges to the perimeter].

3.2.2 Global gradient constraint. By a "global gradient constraint" we here simply mean the norm of the function ρ in the Sobolov space $H^1(\Omega)$,

$$||\rho||_{H^1} = \left(\int_{\Omega} \left(\rho^2 + |\nabla \rho|^2 \right) \, \mathrm{d}x \right)^{\frac{1}{2}} \leq M. \tag{7}$$

Proof of existence when including this bound was given by Bendsøe (1995). The existence proof holds also if the term ρ^2 is removed in (7). The "only" difference between global gradient and total variation is then the exponent 2 in (7).

To our knowledge neither a FE-convergence proof nor numerical experiments with global gradients in the setting of topology optimization have been carried out. Such work is expected to be similar in content to perimeter control, but less interesting due to the ambiguity of (7) as one passes to the limit in the penalization (of intermediate ρ-values).

3.2.3 Local gradient constraint. Introduction of a local gradient constraint on thickness variation of plates was first done by Niordson (1983). Proof of existence, FE-convergence and numerical implementation of a scheme introducing local gradient constraint on density variation was given by Petersson and Sigmund (1998). The constraint on the local density variation is written as the following pointwise constraint on the derivatives of the function ρ:

$$\left| \frac{\partial \rho}{\partial x_i} \right| \leq c \quad (i = 1, 2). \tag{8}$$

The convergence proof implies that checkerboards and other numerical anomalies will be eliminated, or at least, they can be made arbitrarily weak using this scheme. Implementation of the scheme results in up to $2N$ extra constraints in the optimization problem and the method must therefore be considered to be too slow for practical design problems.

3.2.4 Mesh independent filtering. This filter proposed by Sigmund (1994, 1997) is an extension of the checkerboard filter mentioned earlier. The filter modifies the design sensitivity of a specific element based on a weighted average of the element sensitivities in a fixed neighborhood. It must be emphasized that this filter is purely heuristic but it produces results very similar to local gradient constrained results (Petersson and Sigmund 1998), requires little extra cpu-time and is very simple to implement compared to the other approaches. Similar ideas of weighted averages have been used to ensure mesh-independence in bone-mechanics simulation (Mullender *et al.* 1994) and in plastic softening materials (e.g. Leblond *et al.* 1994).

The mesh-independence scheme works by modifying the element sensitivities as follows:

$$\widehat{\frac{\partial f}{\partial \rho_k}} = (\rho_k)^{-1} \frac{1}{\sum_{i=1}^{N} \hat{H}_i} \sum_{i=1}^{N} \hat{H}_i \, \rho_i \, \frac{\partial f}{\partial \rho_i}. \tag{9}$$

The convolution operator (weight factor) $\hat{H}_i$ is written as

$$\hat{H}_i = r_{\min} - \mathrm{dist}(k, i), \quad \{ i \in N \mid \mathrm{dist}(k, i) \leq r_{\min} \} ,$$
$$k = 1, \ldots, N, \tag{10}$$

where the operator dist (k, i) is defined as the distance between the centre of element k and the centre of element i. The convolution operator $\hat{H}_i$ is zero outside the filter area. The convolution operator for element i is seen to decay linearly with the distance from element k.

This means that instead of using the real sensitivities (e.g. $\partial f / \partial \rho_k$), the filtered sensitivities (9) are used. It is worthwhile noting that the sensitivity (9) converges to the original sensitivity when $r_{\min}$ approaches zero and that all sensitivities will be equal (resulting in an even distribution of material) when $r_{\min}$ approaches infinity.

The existence issue for the mesh-independence filter has yet to be proved, but applications in several papers on material design and mechanism design by the first author shows that the method in practice produces mesh-independent designs.

3.2.5 Comparison of methods. The perimeter, local gradient and mesh-independence filter methods produce very similar designs, but there are some differences.

The perimeter control scheme is a global constraint and will allow the formation of locally very thin bars. The local gradient and filtering schemes are local constraints and will generally remove thin bars.

Predicting the value of the perimeter constraint for a new design problem must be determined by experiments, since there is no direct relation between local scale in the structure and the perimeter bound. If the perimeter bound is too tight, there may be no solution to the optimization problem. This problem is particularly difficult for three-dimensional problem. In contrast, the gradient and filtering schemes define a local length scale under which structural variation is filtered out. This local length scale corresponds to a lower limit on bar/beam widths and can easily be defined when machining constraints are taken into consideration.

Another important difference is the implementation aspect. The perimeter control scheme requires an extra constraint added to the optimization problem. Although the addition of one extra constraint to the optimization problem should not be a problem for advanced large scale mathematical programming algorithms, practice has shown that implementation of the constraint can give some convergence problems. The local gradient constraint scheme is considered impractical due to the addition of $2N$ extra constraints to the optimization problem. The big advantage of the filtering scheme is that it requires no extra constraints in the optimization problem. Furthermore it is very easy to implement, and experience shows that its implementation even stabilizes convergence. The disadvantage of the filtering method is that it is based on heuristics.

The perimeter and mesh-independence filter methods have both been applied to three-dimensional problems with success (Beckers 1997b; Sigmund *et al.* 1998, respectively). The comparisons above also hold for three-dimensional problems.

Finally, it should be mentioned that there is a nonisotropy inherent in the implementation of the discretized perimeter measure. E.g. a straight edge angled 45 degrees to the edges of the finite element mesh will be approximated by a perimeter which is $\sqrt{2}$ times the true value because of the jagged edge, i.e. (6) favours structural edges parallel to those of the FEs. In fact, it has been proved by Petersson (1997b) that the perimeter control used to date, (6), is actually the proper discretization of

$$\mathrm{TV}_{\mathrm{new}}(\rho) = \int_{\Omega} \left(|\frac{\partial \rho}{\partial x_1}| + |\frac{\partial \rho}{\partial x_2}| \right) \, \mathrm{d}x \tag{11}$$

instead of (5), (assuming the FE-edges are parallel to the x_i-axes). This means that the numerical results will approach solutions of an original problem statement including a new perimeter that equals the sum of the structures edge lengths projected onto the coordinate axes. This type of modification was earlier introduced in the field of image segmentation, see the paper by Chambolle (1995) in which an analogous "FE"-convergence proof is also given.

4 Local minima

Considering the many differing solutions to, for instance the MBB-beam problem, having appeared in literature, it is clear that topology optimization problems have extremely many local minima. To a large extent local minima appear for the nonrestricted 0-1 topology optimization problems. The schemes producing well-posed problems discussed in the previous section tend to convexify the problems and produce reproducible designs. Nevertheless, small variations in initial parameters such as move limits, geometry of design domains, number of elements, perimeter constraint value or filter parameter, etc., can result in drastical changes in the "optimal design". These problems are partly due to flatness of the objective function, but probably more importantly, due to the numerical optimization procedures used to solve the problems. Convergence proofs of algorithms producing iterates to solve convex programs are common, while for nonconvex programs the statements usually only ensure the algorithm iterates' convergence to a nearby stationary point (which certainly need not be similar to a global solution). Most global optimization methods seem to be unable to handle problems of the size of a typical topology optimization problem. Based on experience, it seems that *continuation* methods must be applied because, by construction, they take also "global" information into account and are thus more likely to ensure "global" convergence (or at least convergence to better designs).

The idea of continuation methods is to gradually change the optimization problem from an (artificial) convex problem to the original (nonconvex) design problem in a number of steps. In each step a gradient-based optimization algorithm is used until convergence. Different continuation procedures have been suggested.

Allaire and Francfort (1993) and Allaire and Kohn (1993a) suggested a continuation method were the structure is first optimized allowing grey or perforated regions. After convergence, the penalization scheme discussed earlier is gradually introduced to obtain a 0/1 design.

For the perimeter constraint, Haber *et al.* (1996) suggested a gradual raise in the penalization factor. For a low value of the penalization factor, the design problem resembles the variable thickness sheet problem which is convex. For increasing penalization factor the problem is expected to gradually converge to the desired 0-1 design.

For the mesh-independence filter Sigmund (1997) and Sigmund and Torquato (1997) suggest starting with a large value of the filter size $r_{\min}$ ensuring a convex solution and gradually to decrease it, to end up with a 0-1 design.

Recently, Guedes and Taylor (1997) suggested a continuation approach where costs of intermediate density elements are gradually increased by adjusting a weight function ω in the resource constraint $\int \omega\rho \le V^*$.

5 Conclusions

Checkerboards, mesh dependence and local minima appearing in structural topology optimization problems have been discussed. It is concluded that precautions must be taken to avoid them. At present the two most promising approaches seem to be the perimeter control and mesh-independent filtering.

The perimeter scheme corresponds to problems for which there are both existence and FE-convergence proofs, and it can be generalized to all topology optimization problems. The method, however, can be a little difficult to make robust, but results can be improved by combining it with filtering techniques (e.g. Beckers 1997a).

The mesh-independence filter which, despite of its lack of theoretical justification (at the present time), produces good results, does not introduce extra constraints in the optimization problem and is very simple to implement.

Independent of which approach one uses, any single optimization formulation that will produce (close to) $0-1$ designs, will be inherently nonconvex. To obtain close to global optima, different types of ad-hoc continuation methods have been used so far. The concepts of continuation in topology optimization are not very coherent, and more research is needed to obtain general stable methods, maybe in combination with (other) global optimization methods.

Acknowledgements

The authors are grateful to Martin P. Bendsøe for valuable discussions on the subject. They also acknowledge the valuable comments of two reviewers of an earlier version of the paper. The work presented in this paper received support from Denmark's Technical Research Council (Programme of Research on Computer-Aided Design), (OS), and the Swedish Research Council for Engineering Sciences and the National Network in Applied Mathematics, (JP).

References

Allaire, G.; Francfort, G.A. 1993: A numerical algorithm for topology and shape optimization. In: Bendsøe, M.P.; Mota Soares, C.A. (eds.) *Topology design of structures*, pp. 239–248. Dordrecht: Kluwer

Allaire, G.; Kohn, R.V. 1993: Topology optimization and optimal shape design using homogenization. *In*: Bendsøe, M.P.; Mota Soares, C.A. (eds) *Topology design of structures*, pp. 207–218. Dordrecht: Kluwer

Ambrosio, L.; Buttazzo, G. 1993: An optimal design problem with perimeter penalization. *Calc. Var.* **1**, 55–69

Beckers, M. 1996: Optimisation topologique de structures continues en variable discretes. *Technical Report OF-38*, LTAS, University of Liège

Beckers, M. 1997a: Methodes du perimetre et des filtres pour l'optimisation topologique en variable discretes. *Technical Report OF-45*, LTAS, University of Liège

Beckers, M. 1997b: Optimisation topologique de structures tridimensionelles en variable discretes. *Technical Report OF-44*, LTAS, University of Liège

Bendsøe, M.P. 1989: Optimal shape design as a material distribution problem. *Struct. Optim.* **1**, 193–202

Bendsøe, M.P. 1995: *Optimization of structural topology, shape and material.* Berlin, Heidelberg, New York: Springer

Bendsøe, M.P.; Guedes, J.M.; Haber, R.B.; Pedersen, P.; Taylor, J.E. 1994: An analytical model to predict optimal material properties in the context of optimal structural design. *Trans. ASME, J. Appl. Mech.* **61**, 930–937

Bendsøe, M.P.; Kikuchi, N. 1988: Generating optimal topologies in optimal design using a homogenization method. *Comp. Meth. Appl. Mech. Engng.* **71**, 197–224

Bendsøe, M.P.; Kikuchi, N.; Diaz, A.R. 1993: Topology and generalized layout optimization of elastic structures. In: Bendsøe, M.P.; Mota Soares, C.A. (eds.) *Topology design of structures*, pp. 159–205. Dordrecht: Kluwer

Chambolle, A. 1995: Image segmentation by variational methods: Mumford and shah functional and the discrete approximations. *SIAM J. Appl. Math.* **55**, 827–863

Díaz, A.R.; Sigmund, O. 1995: Checkerboard patterns in layout optimization. *Struct. Optim.* **10**, 40–45

Duysinx, P. 1997: Layout optimization: A mathematical programming approach. (Submitted). Also: *DCAMM Report No. 540*, Technical University of Denmark

Guedes, J.M.; Taylor, J.E. 1997: On the prediction of material properties and topology for optimal continuum structures. *Struct. Optim.* **14**, 183–192

Haber, R.B.; Bendsøe, M.P.; Jog, C. 1996: A new approach to variable-topology shape design using a constraint on the perimeter. *Struct. Optim.* **11**, 1–12

Jog, C.S.; Haber, R.B. 1996: Stability of finite element models for distributed-parameter optimization and topology design. *Comp. Meth. Appl. Mech. Engng.* **130**, 203–226

Johnson, C.; Pitkäranta, J. 1982: Analysis of some mixed finite element methods related to reduced integration. *Mathematics of Computations* **38**, 375–400

Leblond, J.B.; Perrin, G.; Deveaux, J. 1994: Bifurcation effects in ductile metals with damage delocalization. *J. Appl. Mech.* **61**, 236–242

Mullender, M.G.; Huiskes, R.; Weihnans, H. 1994: A physiological approach to the simulation of bone remodelling as a self-organizational controll process. *J. Biomech.* **11**, 1389–1394

Niordson, F.I. 1983: Optimal design of plates with a constraint on the slope of the thickness function. *Int. J. Solids & Struct.* **19**, 141–151

Petersson, J. 1997a: A finite element analysis of optimal variable thickness sheets. (Submitted). Also: *Mat. Report No. 1996-4*, December 1996, Technical University of Denmark

Petersson, J. 1997b: Some convergence results in perimeter-controlled topology optimization. (Submitted). Also: *Report LiTH-IKP-R-995*, Linköping University, Sweden

Petersson, J.; Sigmund, O. 1998: Slope constrained topology optimization. *Int. J. Numer. Meth. Engng.* **41**, 1417–1434

Sigmund, O. 1994: *Design of material structures using topology optimization.* Ph.D. Thesis, Department of Solid Mechanics, Technical University of Denmark

Sigmund, O. 1997: On the design of compliant mechanisms using topology optimization. *Mech. Struct. Mach.* **25**, 495–526

Sigmund, O.; Torquato, S. 1997: Design of materials with extreme thermal expansion using a three-phase topology optimization method. *J. Mech. Phys. Solids* **45**, 1037–1067

Sigmund, O.; Torquato, S.; Aksay, I.A. 1998: On the design of 1–3 piezocomposites using topology optimization. *J. Mat. Res.* **13** (4)

Zhou, M.; Rozvany, G.I.N. 1991: The COC algorithm, Part II: Topological, geometry and generalized shape optimization. *Comp. Meth. Appl. Mech. Engng.* **89**, 197–224

Received Oct. 20, 1997
Revised manuscript received March 9, 1998