Structural Optimization 16, 47–57 © Springer-Verlag 1998

On optimal design of supports in beam and frame structures*

D. Bojczuk
University of Technology, Kielce, Poland

Z. Mróz
Institute of Fundamental Technological Research, Warsaw, Poland

Abstract An algorithm of optimal design of supports including their number , position and stiffness is proposed . The number of supports constitute topological design parameters , their positions correspond to configuration parameters . Both , elastic and rigid supports are considered and the optimization is aimed to minimize the total structure cost . The topology bifurcation points correspond to generation of new supports . The topological sensitivity derivative is used in deriving the optimality conditions . The optimization procedure provides number of supports , their position and stiffness of both supports and beam segments.

1 Introduction

The present paper provides an extension of previous optimal design formulations for beam structures including position and stiffness of supports (Mróz and Rozvany 1975; Rozvany 1975; Szeląg and Mróz 1978; Mróz and Lekszycki 1982; Garstecki and Mróz 1987; Dems and Turant 1997). We shall introduce the topological design parameter, namely the number of supports , increasing with the beam length. Similar to a previous study for trusses (cf. Bojczuk and Mróz 1997), the optimal design path is considered with one size parameter increasing. The topology bifurcation is combined with the usual optimization for optimal support location and cross-sectional stiffness distribution using the optimality conditions and sensitivity gradients (e.g. Mróz and Haftka 1994).

In Section 2, the design parameters are introduced and optimality criteria are derived for elastic supports. The optimality conditions for rigid supports are discussed in Section 3, and illustrative examples are presented in Section 4. The present method can easily be generalized to more complex structures, where topology, configuration, and cross-sectional optimization can be carried out in the uniform way.

2 Optimality conditions for elastic support design

2.1 Design parameters

The design parameters for beam structures can be classified into three classes, namely stiffness, topological, and configuration parameters. The stiffness parameters are represented by beam segment stiffness EI, and support stiffness $k = EA/\ell$ where I and A denote the moment of inertia and cross-sectional area. The topological parameter corresponds to the number of supports which can vary with the beam length. The virtual topology variations are considered in the design process. Figure 1 illustrates the variation of topology in the case of elastic or rigid supports. In Fig. 1a the elastic support is added, in Fig. 1b the existing elastic support is substituted by two elastic supports. Similarly the rigid support addition and substitution is illustrated in Figs. 1c and d. The topology variation in Fig. 1a corresponds to the addition of a support of stiffness k which is regarded as a topological parameter. Similarly, for rigid supports, the support generation is associated with the support reaction R (Fig. 1c). The positions of actually existing supports constitute the configuration parameters.

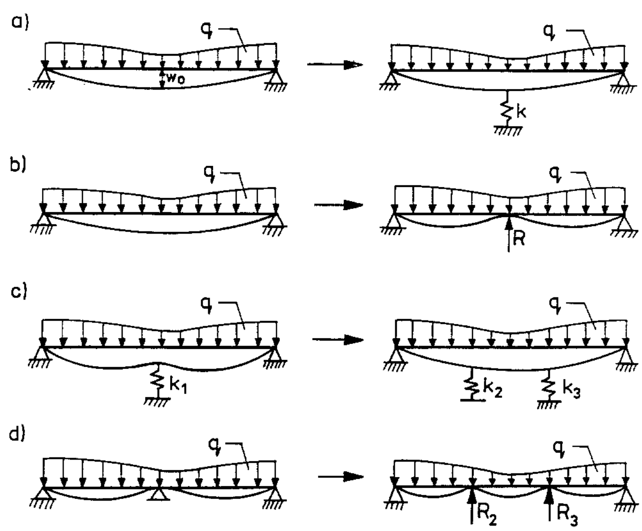

Fig. 1. Variation of support topology: (a) introduction of an elastic support; (b) introduction of a rigid support; (c) substitution of the existing elastic support; (d) substitution of the rigid support

2.2 Optimality conditions for generation of elastic supports

Consider first the case of the generation of new supports. The optimal design is aimed at minimizing the cost of structures for specified global compliance, thus

$$\min_{A(x),s,k} C,$$

* Presented at WCSMO-2, Zakopane, Poland, May 26-30, 1997

subject to

$$\Pi_0 - \Pi \leq 0\,, \tag{1}$$

where $A(x)$ denotes the cross-sectional area of the beam, k is the stiffness of new support, s denotes the parameter specifying the location of the support, Π denotes the potential energy and C is the cost of the structure expressed as follows:

$$C = c\int_0^\ell EA\,\mathrm{d}x + C_S(k)\,, \tag{2}$$

where c is the unit cost of beam material, E denotes Young's modulus and $C_S(k)$ denotes the support cost.

This cost can be assumed as a sum of two partial costs, namely the support material cost $C_S^{(1)}$ and the support installation cost $C_S^{(2)}$. The material cost can be assumed as the linear function of the support stiffness

$$C_S^{(1)} = c_{Sa}k\,, \tag{3}$$

where c_{Sa} is the specific cost. The installation cost is a nonlinear function of support stiffness, for instance

$$C_S^{(2)} = \begin{cases} (c_{Sp} - c_{Sa})k - \frac{c_{Sp}-c_{Sa}}{2k_0}k^2\,, & k < k_0\,, \\ \frac{c_{Sp}-c_{Sa}}{2}k_0\,, & k \geq k_0\,. \end{cases} \tag{4}$$

Alternatively, this cost can be assumed in the form

$$C_S^{(2)} = \alpha \arctan\frac{k}{k_S}\,. \tag{5}$$

The total cost of the elastic support is now expressed as follows (cf. Fig. 2):

$$C_S = C_S^{(1)} + C_S^{(2)} = \begin{cases} c_{Sp}k - \frac{c_{Sp}-c_{Sa}}{2k_0}k^2\,, & k < k_0\,, \\ c_{Sa}k + \frac{c_{Sp}-c_{Sa}}{2}k_0\,, & k \geq k_0\,, \end{cases} \tag{6}$$

or in the form (cf. Fig. 2b)

$$C_S = C_S^{(1)} + C_S^{(2)} = c_{Sa}k + \alpha\arctan\frac{k}{k_S}\,, \tag{7}$$

where c_{Sp}, k_0, k_S and α are positive cost parameters. The derivatives of the cost functions with respect to k are

$$c_S = \frac{\partial C_S}{\partial k} = \begin{cases} c_{Sp} - \frac{c_{Sp}-c_{Sa}}{k_0}k\,, & k < k_0\,, \\ c_{Sa}\,, & k \geq k_0\,, \end{cases} \tag{8}$$

and

$$c_S(k) = \frac{\partial C_S}{\partial k} = c_{Sa} + \frac{\alpha k_S}{k^2 + k_s^2}\,, \tag{9}$$

where by analogy to the first function, we have

$$c_{Sp} = c_S(0) = c_{Sa} + \frac{\alpha}{k_S}\,. \tag{10}$$

Thus the derivatives of two cost functions decrease and tend to the same asymptotic value c_{Sa}.

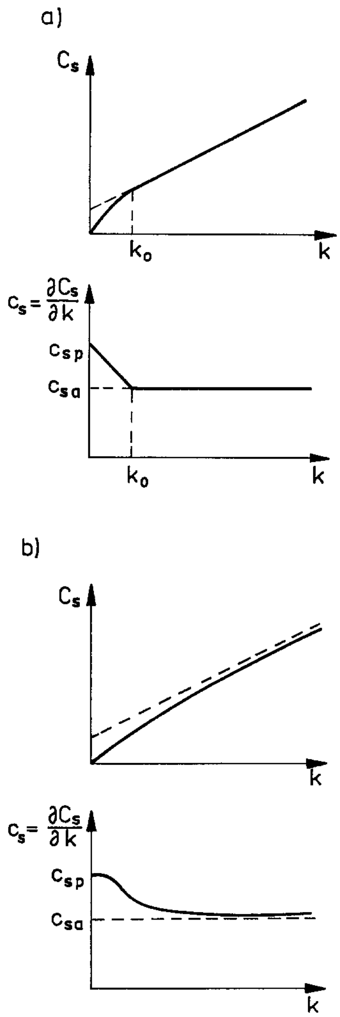

Fig. 2. Variation of support cost and of its derivative for varying stiffness k: (a) cost function specified by (6); (b) cost function specified by (7)

The potential energy of a supported beam is

$$\Pi(w, w_0, A, s, k) = \frac{1}{2}\int_0^\ell EI\kappa^2\,\mathrm{d}x + \frac{1}{2}kw_0^2 - \int_0^\ell qw\,\mathrm{d}x\,, \tag{11}$$

where $w(x)$ is the beam deflection field, w_0 denotes the deflection at the support, κ denotes the beam curvature, $q(x)$ is the distributed transverse loading, and EI is the flexural beam stiffness. The optimal design problem (1) can be formulated as follows:

$$\max_{\lambda}\min_{A,s,k} C^*\,, \tag{12}$$

where

$$C^* = c\int_0^\ell EA\,\mathrm{d}x + C_S(k) +$$

$$\lambda\left[\Pi_0-\left(\frac{1}{2}\int_0^\ell EI\kappa^2\,\mathrm{d}x+\frac{1}{2}kw_0^2-\int_0^\ell qw\,\mathrm{d}x\right)\right], \tag{13}$$

and $\lambda \geq 0$ is the Lagrange multiplier. The optimality conditions with respect to A, s, k and λ are

$$c-\frac{1}{2}\lambda n\kappa^2\frac{I}{A}=0\,,\quad \frac{\partial\Pi}{\partial s}=\frac{\partial C^*}{\partial s}=0\,,$$

$$c_S(k)-\frac{1}{2}\lambda w_0^2=0\,,$$

$$\frac{1}{2}\int_0^\ell EI\kappa^2\,\mathrm{d}x+\frac{1}{2}kw_0^2-\int_0^\ell qw\,\mathrm{d}x=\Pi_0\,, \tag{14}$$

where the following relationships were used:

$$I=\beta A^n\,,\quad \frac{\partial I}{\partial A}=n\beta A^{n-1}=n\frac{I}{A}\,, \tag{15}$$

and n depends on the cross-sectional parameter variation. The first condition specifies the curvature for the optimal design of a varying cross-section. The second condition determines the position of a new elastic support. Following Mróz and Rozvany (1975) and Mróz and Lekszycki (1982), the first derivative of the potential energy with respect to the position, is

$$\left.\frac{\partial\Pi}{\partial s}\right|_{s=s_0}=R\theta\Big|_{x=s_0}\,, \tag{16}$$

where $\theta=-w'$ is the deflection slope. It is seen that at the optimal support there is $\theta=0$. Moreover, R denotes the reaction of support and for the elastic support $R=kw_0$. The third condition provides the optimal support condition. Noting that $c_S(k)$ varies within the interval $[c_{S_a}, c_{S_p}]$, attaining maximum at $k=0$, the condition for introduction of support can be expressed in the form

$$c_{S_p}-\frac{1}{2}\lambda w_p^2=0\,,\quad\text{or}\quad w_p=\sqrt{\frac{2c_{S_p}}{\lambda}}\,, \tag{17}$$

where w_p is now the maximum beam deflection occurring at $k=0$. The value of deflection w_p specified by (17) provides a level (or sensor) line on the deflection diagram. When the deflection line $w=w(x)$ touches the level line $w_p=$ const., a new support is generated at the tangency point. As c_S tends to a steady value c_{S_a}, the second deflection line is specified

$$w_a=\sqrt{\frac{2c_{S_a}}{\lambda}}<w_p\,, \tag{18}$$

which provides the minimal deflection value. The support stiffness is selected so that the actual deflection satisfies the optimality condition $(14)_3$ and for $k>k_0$, the deflection at support is given by (18). Thus except the regions near the rigid end supports, the deflection line between elastic supports lies within domain $w_a \leq w(x) < w_p$. Figure 3 illustrates this concept of optimal design. The line $w=w_p$ plays the role of a "sensor line", generating new supports, and the line $w=w_a$ provides the deflection value at supported points for support stiffness exceeding the critical value k_0.

On the other hand , when $c_S=$ const., the optimal solution, already discussed by Szeląg and Mróz (1978), provides distributed support conditions (cf. Fig. 4). The beam is supported over the segment $-a/2<s<a/2$ with two concentrated supports at the points of the end segments.

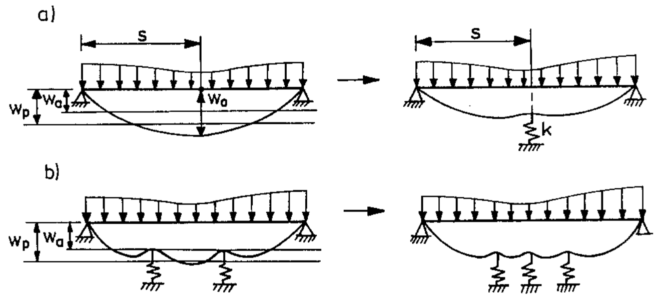

Fig. 3. (a) Introduction of first elastic support; (b) introduction of a consecutive elastic support

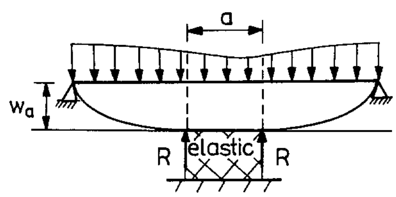

Fig. 4. Optimal support solution for a beam with constant cost derivative c_S, associated with constant deflection $w=w_a$ at the supported portion

For a prismatic beam, $EA=$ const., $EI=$ const., the Lagrangian (13) takes the form

$$C^*=cEA\ell+C_S(k)+$$

$$\lambda\left[\Pi_0-\left(\frac{1}{2}EI\int_0^\ell\kappa^2\,\mathrm{d}x+\frac{1}{2}kw_0^2-\int_0^\ell qw\,\mathrm{d}x\right)\right], \tag{19}$$

and the optimality conditions are

$$c\ell-\frac{1}{2}\lambda n\frac{I}{A}\int_0^\ell\kappa^2\,\mathrm{d}x=0\,,$$

$$\frac{\partial\Pi}{\partial s}=\frac{\partial C^*}{\partial s}=R\theta\Big|_{x=s}=0\,,$$

$$c_S(k)-\frac{1}{2}\lambda w_0^2=0\,,$$

$$\frac{1}{2}EI\int_0^\ell\kappa^2\,\mathrm{d}x+\frac{1}{2}kw_0^2-\int_0^\ell qw\,\mathrm{d}x=\Pi_0\,. \tag{20}$$

The generation of new supports follows the same rules as in the previous case.

The present analysis can be related to the topological derivative concept discussed by Bojczuk and Mróz (1997). Namely, admitting virtual topology variation by introduction of support of vanishing stiffness, the derivative of the Lagrangian with respect to support stiffness equals

$$\left.\frac{\partial C^*}{\partial k}\right|_{k=0} = c_S(0) - \frac{1}{2}\lambda w_0^2 = c_{S_p} - \frac{1}{2}\lambda w_0^2\,, \tag{21}$$

and this derivative becomes nonpositive when $1/2\lambda w_0^2 \geq c_{S_p}$. Thus the support of vanishing stiffness can be introduced when the deflection curve touches the sensor line which is the locus of the vanishing topological derivative.

2.3 Conditions for support substitution

Consider now a different mode of topology variation , by assuming new supports to be generated at the two local maximum deflection points by removing the material from the existing neighbouring support (Fig. 5). Denote by k_1 and w_1 the existing support stiffness and deflection at the support position 1. Similarly, denote by k_2, k_3 and by w_2, w_3 the new support stiffnesses and deflections at the new support positions 2 and 3. Initially, we have $k_1 = k_{10}$, $k = k_2 = k_3 = 0$, $\delta k = \delta k_2 = \delta k_3$, so during the exchange of material between supports 1 and 2, 3 there is

$$k_1 + k_2 + k_3 = k_{10}\,, \quad \delta k_2 + \delta k_3 = -\delta k_1\,,$$

or

$$k_1 + 2k = k_{10}\,, \quad 2\delta k = -\delta k_1\,. \tag{22}$$

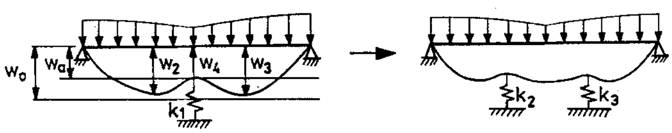

Fig. 5. Substitution of the single support by two supports

The Lagrangian associated with the support substitution process has the form

$$C^* = c\int_0^{\ell} EA\,\mathrm{d}x + C_S^{(1)}(k_1) + C_S^{(1)}(k_2) + C_S^{(1)}(k_3) +$$

$$+ C_S^{(2)}(k_2) + C_S^{(2)}(k_3) + C_S^{(e)}(k_{10}, k_2) + \lambda\Bigg[\Pi_0 -$$

$$\Bigg(\frac{1}{2}\int_0^{\ell} EI\kappa^2\,\mathrm{d}x + \frac{1}{2}k_1 w_1^2 + \frac{1}{2}k_2 w_2^2 +$$

$$\frac{1}{2}k_3 w_3^2 - \int_0^{\ell} qw\,\mathrm{d}x\Bigg)\Bigg]\,, \tag{23}$$

where

$$C_S^{(e)}(k_{10}), k_2) = \begin{cases} C_S^{(2)}(k_{10})\,, & k_2 < k_{10} \\ C_S^{(2)}(k_2)\,, & k_2 \geq k_{10} \end{cases} \tag{24}$$

denotes the cost function of the support installation related to the total exchange of support 1 by support 2 (Fig. 6), and $C_S^{(1)}$ and $C_S^{(2)}$ are defined properly by (3) and (4) or (5). In view of (22) and taking into account that

$$\left.\frac{\partial C_S^{(e)}}{\partial k_2}\right|_{k_2=0} = \left.\frac{\partial C_S^{(e)}}{\partial k}\right|_{k=0} = 0\,, \tag{25}$$

we now obtain, instead of (21), the relationship

$$\left.\frac{\partial C^*}{\partial k}\right|_{k=0} = c_{Sp} - c_{Sa} - \frac{1}{4}\lambda(w_2^2 + w_3^2 - 2w_1^2)\,, \tag{26}$$

representing the condition of the initiation of the support substitution process of the form

$$c_{Sp} - c_{Sa} - \frac{1}{4}\lambda(w_2^2 + w_3^2 - 2w_1^2) \leq 0\,. \tag{27}$$

The optimal positions of supports 2 and 3 are specified by the conditions analogous to the second condition (14).

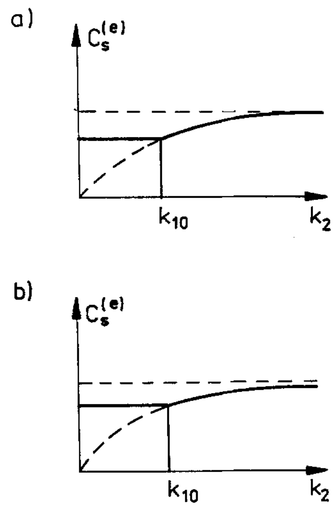

Fig. 6. Cost function of support installation related to total exchange of two supports for: (a) cost function specified by (4); (b) cost function specified by (5)

3 Optimality conditions for rigid supports

The introduction of a rigid support involves finite variation of both static and deflection fields. This case can be studied as a limit transition for an elastic support design by requiring $c_{Sa} = 0$, $w_a = 0$ for $k \to \infty$. However, it is more convenient to reformulate the support cost in terms of its reaction.

The optimization problem has a form similar to previously

$$\min_{A(x), s_k} C\,,$$

subject to

$$\Pi_0 - \Pi \le 0\,, \tag{28}$$

where s_k denotes the position of the k-th support, and the structure cost equals

$$C = c\int_0^\ell EA\,\mathrm{d}x + \sum_{k=1}^{K} C_{RK}\,. \tag{29}$$

The support cost is now expressed in terms of the reaction R as follows (cf. Fig. 7):

$$C_R = C_{R0} + c_{Ra}\mid R\mid\,, \tag{30}$$

where C_{R0} is the initial cost of support installation, and c_{Ra} is the specific cost.

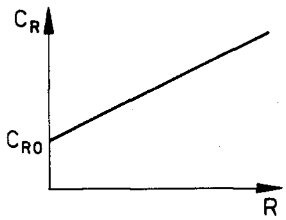

Fig. 7. Cost function for rigid support

The Lagrangian and the optimality conditions are now

$$C^* = c\int_0^\ell EA\,\mathrm{d}x + \sum_{k=1}^{K} C_{Rk} +$$

$$\lambda\left[\Pi_0 - \left(\frac{1}{2}\int_0^\ell EI\kappa^2\,\mathrm{d}x - \int_0^\ell qw\,\mathrm{d}x\right)\right]\,, \tag{31}$$

and

$$c - \frac{1}{2}\lambda n\kappa^2\frac{I}{A} = c - \frac{1}{2}\lambda n\kappa^2\beta A^{n-1} = 0\,,$$

$$(Rw')_k = 0\,,\quad k = 1,\ldots,K\,,$$

$$\frac{1}{2}\int_0^\ell EI\kappa^2\,\mathrm{d}x - \int_0^1 qw\,\mathrm{d}x = \Pi_0\,. \tag{32}$$

The optimality conditions (32) provide optimal beam stiffness distribution and optimal support position. To specify topology variation by introduction of an additional support, we shall consider the final increment of the Lagrangian C^*, namely

$$\Delta C^* = \Delta C - \lambda\Delta\Pi\,, \tag{33}$$

and examine the condition of topology variation $\Delta C^* \le 0$.

Assume that the cross-sectional areas of the beam are fixed; for a design with K existing rigid supports, the $K+1$ support is introduced. The potential energy increase due to the action of this support equals

$$\Delta\Pi = \frac{1}{2}R_{K+1}^{(n)}w_0\,, \tag{34}$$

where $R_{K+1}^{(n)}$ is the reaction of the new support and w_0 denotes the beam deflection at the support point before its introduction (Fig. 8). It is seen that the maximal increase of the potential energy corresponds to maximal displacement w_0 at the support $K+1$. Denoting by R_k the support reactions before introduction of the $(K+1)$-th support, the actual values of reactions are

$$R_k^{(n)} = R_k + \Delta R_k\,,\quad k = 1,2,\ldots,K+1\,, \tag{35}$$

where $R_{K+1} = 0$, $R_{K+1}^{(n)} = \Delta R_{K+1}$. The variation of the cost of the structure now equals

$$\Delta C = C_{R0} + c_{Ra}\sum_{k=1}^{K+1}(\mid R_k + \Delta R_k\mid - \mid R_k\mid)\,, \tag{36}$$

and the variation of the Lagrangian is expressed as follows:

$$\Delta C^* = C_{R0} + c_{Ra}\sum_{k=1}^{K+1}(\mid R_k + \Delta R_k\mid - \mid R_k\mid) -$$

$$\frac{1}{2}\lambda R_{K+1}^{(n)}w_0\,. \tag{37}$$

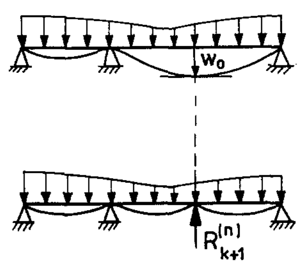

Fig. 8. Variation of the potential energy induced by the introduction of a rigid support

The condition for introduction of new support now takes the form

$$\Delta C^* \le 0\,,$$

or

$$w_0 \ge \frac{2C_{R0} + 2c_{Ra}\sum_{k=1}^{K+1}(\mid R_k + \Delta R_k\mid - \mid R_k\mid)}{\lambda R_{K+1}^{(n)}}\,. \tag{38}$$

Assuming, in a particular case, that all reactions have the same orientation, we have

$$\sum_{k=1}^{K} R_k = \sum_{k=1}^{K+1} R_k^{(n)} = \text{const.}\,, \tag{39}$$

and

$$\sum_{k=1}^{K+1} \Delta R_k = 0 . \tag{40}$$

Then the condition (38) is simplified and can be expressed as follows:

$$C_{R0} - \frac{1}{2}\lambda R^{(n)}_{K+1} w_0 \le 0 ,$$

or

$$w_0 \ge \frac{2C_{R0}}{\lambda R^{(n)}_{K+1}} . \tag{41}$$

The conditions (38) and (41) replace the condition (21) derived in the previous section for the case of the introduction of an elastic support of vanishing stiffness. Now, however, a finite state variation occurs and the optimal support position $x = s$ may differ from the initial position $x = s_0$ at which the condition (38) or (41) was applied.

Let us generate a solution for a beam loaded by the unit reaction force $R = 1$ applied at $x = s_0$. Denote by $w_0(x, s_0)$, $\kappa_0(x, s_0)$, and $M_0(x, s_0)$ the deflection, curvature, and bending moment fields for this solution. The deflection, curvature, and bending moment fields due to applied loading to a beam without the additional support are denoted by $w_i(x)$, $\kappa_i(x)$, and $M_i(x)$. The condition of rigid support at $x = s_0$ is expressed as follows:

$$w(s_0) = w_i(s_0) - R(s_0) w_0(s_0, s_0) = 0 , \tag{42}$$

and provides the value of the reaction

$$R(s_0) = \frac{w_i(s_0)}{w_0(s_0, s_0)} . \tag{43}$$

The deflection, curvature, and bending moment fields for the support position $x = s$ are now

$$w(x, s) = w_i(x) - R(s) w_0(x, s) ,$$

$$M(x, s) = M_i(x) - R(s) M_0(x, s) ,$$

$$\kappa(x, s) = \kappa_i(x) - R(s) \kappa_0(x, s) . \tag{44}$$

The optimal support position corresponds to vanishing deflection and slope at the supports, so we have

$$w(s, s) = w_i(s) - R(s) w_0(s, s) = 0 ,$$

$$w'(s, s) = w_i'(s) - R(s) w_0'(s, s) = 0 . \tag{45}$$

Writing

$$w_i(s) = w_i(s_0) + w_i'(s_0)\Delta s ,$$

$$w_i'(s) = w_i'(s_0) + w_i''(s_0)\Delta s ,$$

$$w_0(s, s) = w_0(s_0, s_0) + \left[w_0'(s_0, s_0) + \frac{\partial w_0}{\partial s}(s_0, s_0) \right] \Delta s ,$$

$$w_0'(s, s) = w_0(s_0, s_0) + \left[w_0''(s_0, s_0) + \frac{\partial w_0'}{\partial s}(s_0, s_0) \right] \Delta s , \tag{46}$$

where $\Delta s = s - s_0$ and $\partial w_0 / \partial s$, $\partial w_0' / \partial s$ denote sensitivity derivatives with respect to the new support position s. From (45) it follows that

$$\Delta s = \left[w_i'(s_0) w_0(s_0, s_0) - w_i(s_0) w_0'(s_0, s_0) \right] \left\{ w_i(s_0) \left[w_0''(s_0, s_0) + \frac{\partial w_0'}{\partial s}(s_0, s_0) \right] - w_i'(s_0) \frac{\partial w_0}{\partial s}(s_0, s_0) - w_i''(s_0) w_0(s_0, s_0) \right\}^{-1} ,$$

$$R(s) = \frac{w_i(s)}{w_0(s, s)} = \frac{w_i(s_0) + w_i'(s_0)\Delta s}{w_0(s_0, s_0) + \left[w'(s_0, s_0) + \frac{\partial w_0}{\partial s}(s_0, s_0) \right] \Delta s} . \tag{47}$$

In order to obtain sensitivity derivatives, we use Betti's principle. For the beams of Fig. 9a and the action of unit force $R = 1$ at two positions $x = s_0$, $x = s$ we can write as follows:

$$1 \cdot w_0(s, s_0) = 1 \cdot w_0(s_0, s) . \tag{48}$$

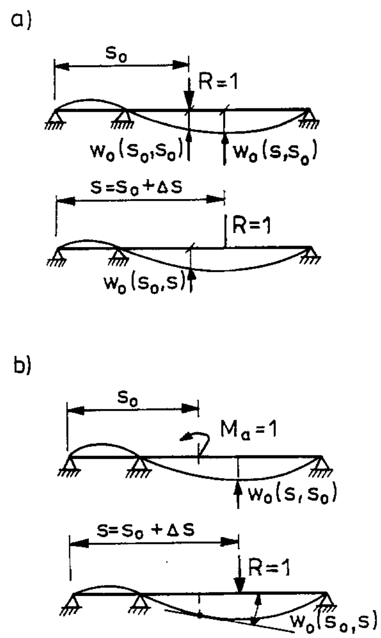

Fig. 9. (a) Sensitivity of displacement with respect to translation of force $R = 1$; (b) Sensitivity of the slope with respect to translation of force $R = 1$

Taking into account $(46)_3$, we have

$$w_0(s, s_0) = w_0(s_0, s_0) + w_0'(s_0, s_0)\Delta s ,$$

$$w_0(s_0, s) = w_0(s_0, s_0) + \frac{\partial w_0}{\partial s}(s_0, s_0)\Delta s\,, \tag{49}$$

and finally the sensitivity derivative equals

$$\frac{\partial w_0}{\partial s}(s_0, s_0) = w_0'(s_0, s_0)\,. \tag{50}$$

Analogously, for the beams of Fig. 9b loaded by the concentrated unit couple $M_a = 1$ at $x = s_0$ and the unit force at $x = s$, we have

$$1 \cdot w_a(s, s_0) = 1 \cdot w_0'(s_0, s)\,, \tag{51}$$

and taking into account that

$$w_a(s, s_0) = w_a(s_0, s_0) + w_a'(s_0, s_0)\Delta s\,,$$

$$w_0'(s_0, s) = w_0'(s_0, s_0) + \frac{\partial w_0'}{\partial s}(s_0, s_0)\Delta s\,, \tag{52}$$

the sensitivity derivative is

$$\frac{\partial w_0'}{\partial s}(s_0, s_0) = w_a'(s_0, s_0)\,, \tag{53}$$

where w_a' is the slope of the beam loaded by the unit couple $M_a = 1$.

The formulae derived provide the transition from the initial support location at $x = s_0$ to the corrected location $x = s$, satisfying the optimality condition.

The conditions (38) or (41) of support generation do not account for variation of the beam design due to introduced support reaction. In fact, by introducing support the beam cost has decreased. Using the first optimality condition (32), the beam cross-sectional area function can be expressed as follows:

$$A(x) = \left[\frac{1}{2\beta c}\lambda n \frac{M^2(x)}{E^2}\right]^{1/(n+1)}\,, \tag{54}$$

where $M = M_i$ for the initial design and $M = M_i - RM_0$ for the modified design with introduced support. The value of the Lagrange multiplier λ for each case, can be determined by substituting (54) into the third optimality condition (32). Denoting the respective designs specified by (54) by $A_i(x)$ and $A(x)$, we have

$$\Delta C_b = c\int_0^\ell E(A - A_i)\,\mathrm{d}x = c\int_0^\ell E\Delta A\,\mathrm{d}x\,, \tag{55}$$

and instead of (33), we can analyse variation of the total cost ΔC. In view of (36) and (55), the topology variation condition is now

$$\Delta C = c_{R0} + \int_0^\ell cE\Delta A\,\mathrm{d}x +$$

$$c_{Ra}\sum_{k=1}^{K+1}(\mid R_k + \Delta R_k \mid - \mid R_k \mid) \le 0\,. \tag{56}$$

This provides the improved assessment of the condition of the new support introduction. The iterative procedure can be developed, for which the support location and beam redesign are carried out consecutively.

For a prismatic beam $EA = \text{const.}$, $EI = \text{const.}$, the first optimality condition (20) provides

$$c\ell - \frac{1}{2}\lambda n \frac{I}{A}\int_0^\ell \kappa^2\,\mathrm{d}x = c\ell - \frac{1}{2}\lambda n \frac{I}{E^2\beta A^{n+1}}\int_0^\ell M^2\,\mathrm{d}x = 0\,, \tag{57}$$

and the cross-sectional areas A_i and A can be calculated from (57) in terms of $M_i(x)$ and $M(x) = M_i(x) - RM_0(x)$. The inequality (56) can then be used in formulating the condition, when a rigid support should be introduced.

The considerations for the second type of topology variation, i.e. substitution of the single rigid support by two new supports, are similar to the case considered previously and are not discussed here.

4 Illustrative examples

In this section we shall present several examples illustrating the general theory of optimal support design including the number of supports as an essential design parameter. The support introduction and support substitution modes will be considered.

4.1 Example 1. Optimal support of an infinite, uniformly loaded beam

Consider an infinite prismatic beam ($EA = \text{const.}$, $EI = \text{const.}$) loaded by the lateral pressure q (Fig. 10). As the boundary conditions have no effect, the optimal support solution will provide the segment length ℓ between uniformly spaced supports.

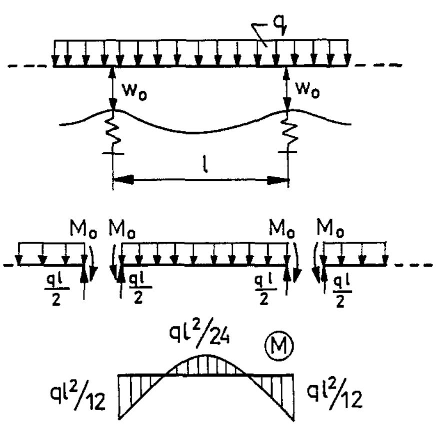

Fig. 10 Static analysis of the single segment of elastically supported beam

4.1.1 Elastic supports. The problem of optimal segment length can be stated as follows:

$$\min_{A,\ell,k} C_d = cEA + \frac{\ell}{\ell}C_s(k)\,,$$

subject to

$$\Pi_0^d - \frac{\Pi}{\ell} \le 0\,, \tag{58}$$

where C_d denotes the beam and support cost per unit length, Π_0^d is the potential energy density per unit length and Π is specified by (11) for the beam segment. The respective Lagrangian is now

$$C_d^* = cEA + \frac{1}{\ell}C_s(k) + \lambda\left(\Pi_0^d - \frac{\Pi}{\ell}\right)\,, \tag{59}$$

and the associated optimality conditions take the form

$$c - \frac{1}{2\ell}n\lambda\frac{I}{A}\int_0^\ell \kappa^2\,\mathrm{d}x = 0\,, \quad c_s(k) - \frac{1}{2}\lambda w_0^2 = 0\,,$$

$$-\frac{1}{\ell^2}C_S(k) + \lambda\frac{1}{\ell^2}\Pi - \lambda\frac{1}{\ell}\frac{\partial \Pi}{\partial \ell} = 0\,,$$

$$\Pi_0^d - \frac{\Pi}{\ell} = 0\,, \tag{60}$$

where w_0 denotes the displacement at the support (Fig. 10). The equilibrium of the segment requires that

$$q\ell - kw_0 = 0\,. \tag{61}$$

As the solution is periodic with respect to the segment length, in view of the symmetry condition the deflection slope θ_0 should vanish at the supported cross-section, thus

$$\theta_0 = 0\,, \tag{62}$$

and the solution for a segment provides the bending moment distribution identical as for a beam clamped at both ends, thus

$$M_0 = -\frac{q\ell^2}{12}\,. \tag{63}$$

The derivative of the potential energy with respect to segment length consists of two terms: the first of the same form as for the translation of a clamped end (cf. Mróz and Rozvany 1975; Szeląg and Mróz 1978; Mróz and Lekszycki 1982), and the second related to deflection of elastic supports, namely

$$\frac{\partial \Pi}{\partial \ell} = -\frac{1}{2}M_0\kappa_0 - kw_0\frac{\partial w_0}{\partial \ell} = -\frac{1}{2}\frac{M_0^2}{EI} - \frac{q^2\ell}{k} = -\frac{q^2\ell^4}{288EI} - \frac{q^2\ell}{k}\,. \tag{64}$$

We also have

$$\int_0^\ell \kappa^2\,\mathrm{d}x = \frac{1}{E^2I^2}\int_0^\ell M^2\,\mathrm{d}x = \frac{q^2\ell^5}{720E^2I^2}\,. \tag{65}$$

Substituting (62)-(65) in the optimality conditions (60), we obtain

$$c - \frac{1}{2}\lambda n\frac{q^2\ell^4}{720\beta E^2A^{n+1}} = 0\,,$$

$$c_s(k) - \frac{1}{2}\lambda\frac{q^2\ell^2}{k^2} = 0\,,$$

$$-\frac{1}{\ell^2}C_S(k) + \frac{1}{\ell}\lambda\Pi_0^d + \lambda\frac{q^2\ell^3}{288\beta EA^n} + \lambda\frac{q^2}{k} = 0\,,$$

$$-\frac{1}{2}\frac{q^2\ell^4}{720\beta EA^n} - \frac{1}{2}\frac{q^2\ell}{k} = \Pi_0^d\,, \tag{66}$$

where $C_S(k)$ and $c_S(k)$ are given by (6) and (8) or (7) and (9). The four unknowns ℓ, A, k and λ can now be specified from (66).

Introduce the nondimensional parameters

$$\eta = C_S^{(2)}/(C_d^0\ell_0)\,,\ \xi_1 = C_d/C_d^0\,,\ \xi_2 = \ell/\ell_0\,,$$

$$\xi_3 = A/A_0\,,\ \xi_4 = k/K_0\,.$$

Assume that $q = 10$ kN/m, $E = 2\cdot 10^5$ MPa, the energy density $\Pi_0^d = -0.625$ J/m, the width of cross-section $b = 0.04$ m, the coefficient of cross-section variation $n = 3$, the unit cost of the beam material $c = 1$ [1/(Nm)], and the coefficients of nondimensional parameters $C_d^0 = 10^8$ [1/m], $\ell_0 = 1$ m, $A_0 = 10^{-2}$ m^2, and $K_0 = 10^8$ N/m. The support cost has the form described by (6), where $c_{Sa} = 1$ m/N, $c_{Sp}/c_{Sa} = 6$. Figure 11 presents values of the nondimensional total cost per unit length ξ_1, and ξ_2, ξ_3, ξ_4 for different values of parameter η characterizing the cost of the support installation. Let us note that for the linear cost function $c_s(k) = c_Sk$, $C_S^{(2)} = 0$, the optimal solution provides $\ell = 0$ and the beam is supported by uniformly distributed linear spring supports.

4.1.2 Rigid supports. The optimal support problem can now be formulated as follows:

$$\min_{A,\ell} C_d\,,$$

subject to

$$\Pi_0^d - \frac{\Pi}{\ell} \le 0\,. \tag{67}$$

The Lagrangian and the optimality conditions are now

$$C_d^* = cEA + \frac{1}{\ell}C_R(R) + \lambda\left(\Pi_0^d - \frac{\Pi}{\ell}\right)\,, \tag{68}$$

and

$$c - \frac{1}{2\ell}n\lambda\frac{I}{A}\int_0^\ell \kappa^2\,\mathrm{d}x = 0\,,$$

$$-\frac{1}{\ell^2}C_{R0} + \lambda\frac{1}{\ell^2}\Pi - \lambda\frac{1}{\ell}\frac{\partial \Pi}{\partial \ell} = 0\,,$$

$$\Pi_0^d - \frac{\Pi}{\ell} = 0\,, \tag{69}$$

where $R = q\ell$ is the support reaction. Following the analysis of the previous case, the set (69) can be presented in the form

$$c - \frac{1}{2}\lambda n\frac{q^2\ell^4}{720\beta E^2A^{n+1}} = 0\,,$$

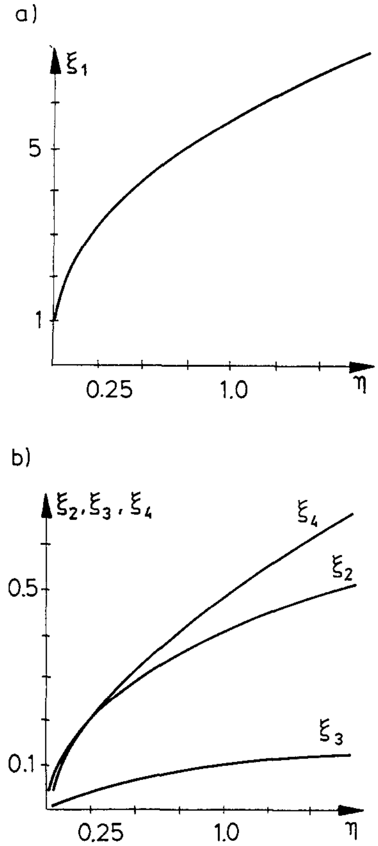

Fig. 11. (a) Optimal nondimensional total cost in function of support installation cost parameter η; (b) optimal nondimensional cross-sectional area, span between supports and support stiffness in function of parameter η

$$-\frac{1}{\ell^2}C_{R0}+\lambda\frac{q^2\ell^3}{360\beta EA^n}=0\,,$$

$$-\frac{1}{2}\frac{q^2\ell^4}{720\beta EA^n}=\Pi_0^d\,, \tag{70}$$

and the solution provides the values of ℓ, A and λ, namely

$$\ell=\left[\frac{1440\beta(-\Pi_0^d)}{E^{n-1}q^2}\left(\frac{nC_{R0}}{4c}\right)^n\right]^{1/(n+4)}, \tag{71}$$

and

$$A=\left[\frac{q^2}{1440\beta E(-\Pi_0^d)}\left(\frac{nC_{R0}}{4cE}\right)^4\right]^{1/(n+4)},$$

$$\lambda=\frac{cE}{n(-\Pi_0^d)}\left[\frac{q^2}{1440\beta E(-\Pi_0^d)}\left(\frac{nC_{R0}}{4cE}\right)^4\right]^{1/(n+4)}. \tag{72}$$

The present example provides the characteristic segment size in the structure, which depends on the cost function, the level of energy density, the value of the loading and the stiffness modulus of the beam.

4.2 Example 2. Optimization of elastic support in a finite beam

Consider the optimal design of a simply supported beam of length ℓ, uniformly loaded over its span. Assume the beam to have a rectangular cross-section of width b and height h, where h is to be determined from the condition of the optimal design for specified global compliance $(-\Pi_0)$. For the cost function (6) the condition of support introduction (17) is satisfied for some value of ℓ. Then the support is introduced at the beam centre. However, for increasing ℓ, the design may be further transformed by the support substitution mode when the condition (27) is satisfied. The central support will be replaced by two supports located symmetrically with respect to the beam centre.

To illustrate such an evolution assume $\ell = 1$ m, $b = 0.04$ m, $h = 0.1$ m, $q = 10$ kN/m, $E = 2{\cdot}10^5$ MPa, $c = 1$ [1/(Nm)], $c_{Sa} = 1$ m/N, $c_{Sp}/c_{Sa} = 6$, $k_0 = 2\cdot 10^7$ N/m.

The cost of the initial design presented in Fig. 12a, without taking into account the cost of rigid supports at the ends, is

$$C^{(0)}=cEA\ell=8\cdot 10^8\,. \tag{73}$$

The sensor line, calculated from (17), equals $w_p = 1.677{\cdot}10^{-4}$ m and the maximum deflection occurring at the centre of the beam can be expressed in the form

$$w_0=\frac{5}{384}\frac{q\ell^4}{EI}=1.953\cdot 10^{-4}\,\text{m}. \tag{74}$$

We have that $w_0 > w_p$ and the condition of topology modification is satisfied. The new elastic support is introduced at the centre of the beam. The optimality conditions (20) provide that the new height and stiffness of the elastic support are $h^{(1)} = 0.0347$ m and $k^{(1)} = 6.991\cdot 10^7$ N/m (Fig. 12b).

The new cost of the structure is

$$C^{(1)}=cEA^{(1)}\ell+c_{Sa}k^{(1)}+\frac{c_{Sp}-c_{Sa}}{2}k_0=3.975\cdot 10^8. \tag{75}$$

The maximum deflections occurring in points located at the distance $x_1 = 0.244$ m from the ends of beam are equal $w_1^{(1)} = 1.774\cdot 10^{-4}$ m and the elastic support deflection is $w_0^{(1)} = 0.877\cdot 10^{-4}$ m. The new sensor line equals $w_p^{(1)} = 2.149\cdot 10^{-4}$ m and now the condition of topology modification by the introduction of an elastic support is not fulfilled, but the condition (27) of the substitution of a single support by two supports is satisfied. Solving for this topology the optimality conditions (20), we have $h^{(2)} = 0.0205$ m, and the new elastic supports of stiffness $k^{(2)} = 3.854\cdot 10^7$ N/m are located at distances $x_1^{(2)} = 0.318$ m from the ends of the beam (Fig. 12c). The cost of the beam is now

$$C^{(2)}=cEA^{(2)}\ell+2\left(c_{Sa}k^{(2)}+\frac{c_{Sp}-c_{Sa}}{2}k_0\right)=$$

$$3.411\cdot 10^8\,. \tag{76}$$

The deflection of the beam centre is $w_0^{(2)} = 1.764\cdot 10^{-4}$ m, the deflections of elastic supports are $w_1^{(2)} = 0.974\cdot 10^{-4}$,

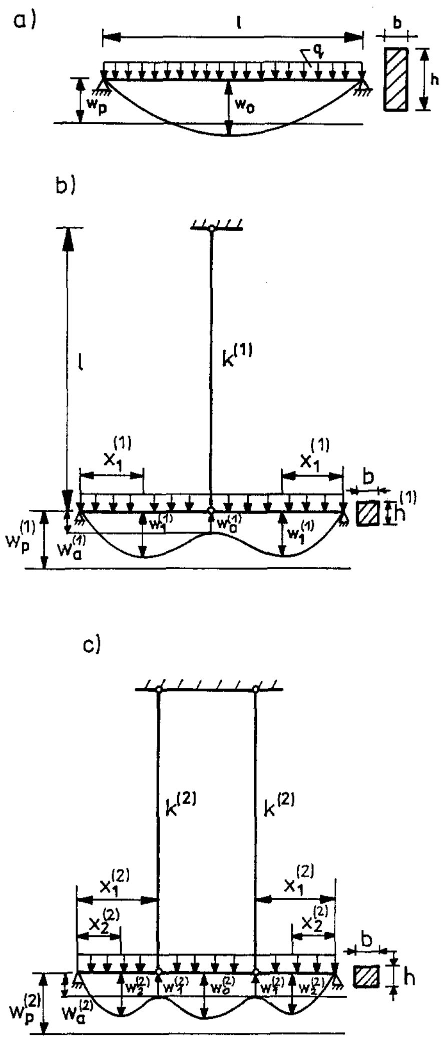

Fig. 12. The evolution of elastic support: (a) the initial design; (b) the beam with the central elastic support; (c) the optimal support of the beam

and the local maximum deflections occurring at points $x_2^{(2)} = 0.163$ m from the beam ends are $w_2^{(2)} = 1.601 \cdot 10^{-4}$ m. The sensor line is $w_p^{(2)} = 2.386 \cdot 10^{-4}$ m, and now no condition of support modification is satisfied. This means that the design shown in Fig. 12c is optimal. The cost of the optimal design is $C^{(0)}/C^{(2)} = 2.346$ times smaller than the cost of the initial design. The distance between elastic supports is $x_S = \ell - 2x_1^{(2)} = 0.374$ m and is slightly bigger than that for the infinite beam of the the same cost, material, and load parameters ($x_S^{(\mathrm{inf})} = 0.290$ m).

4.3 Example 3. Optimization of rigid support in a finite beam

Consider the prismatic beam with the same length, load and boundary conditions as in the initial design in Example 2 (Fig. 13a). Assume the cost of supports in the form described by (30), where $C_{R0} = 0.25 \cdot 10^8$, $c_{Ra} = 104$ [1/N].

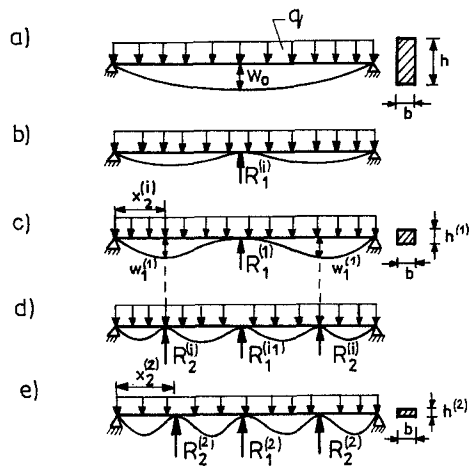

Fig. 13. Optimization of beam with its rigid support: (a) the initial design; (b) test introduction of the support; (c) optimal configuration of structure with one additional support; (d) test introduction of two supports; (e) the optimal structure

The aim is to determine optimal rigid support and height of a rectangular cross-section for the specified global compliance $(-\Pi_0)$, assuming that locations of supports at the ends and width of cross-section are fixed.

The cost of the initial design (Fig. 13a), is

$$C^{(0)} = cEA\ell + 2(C_{R0} + c_{Ra}R) = 9.5 \cdot 10^8 \,, \tag{77}$$

where $R = 1/2q\ell = 5$ kN and the cost of supports is 15.8% of the total cost of the beam. The Lagrange multiplier, calculated from (57) , equals $\lambda = 4.27 \cdot 10^8$ [1/(Nm)], the maximum deflection w_0 is expressed by (74) and the corresponding reaction force (Fig. 13b), is $R_1^{(i)} = 6.25$ Kn. Now we have

$$w_0 > \frac{2C_{R0}}{\lambda R_1^{(i)}} \,, \tag{78}$$

and the condition (41) of introduction of new rigid support is satisfied. The optimality conditions (57), $(32)_2$ and $(32)_3$ provide the structure shown in Fig. 13c, where $h^{(1)} = 0.0286$ m, $R_1^{(1)} = 6.25$ kN; the actual reactions at the ends are $R^{(1)} = 1.875$ kN and the Lagrange multiplier is $\lambda^{(1)} = 1.22 \cdot 10^8$ [1/(Nm)]. The cost of structure is now

$$C^{(1)} = cEA^{(1)}\ell + 3C_{R0} + c_{Ra}(2R^{(1)} + R_1^{(1)}) = 4.039 \cdot 10^8 \,. \tag{79}$$

The points of maximum deflections $w_1^{(1)} = 2.164 \cdot 10^{-4}$ m are located symmetrically at the distance $x_2^{(i)} = 0.211$ m from the beam ends and corresponding reaction forces are $R_2^{(i)} = 2.76$ kN (Fig. 13d). The condition of the simultaneous introduction of two rigid supports arising from (32), which has the form

$$w_1^{(1)} \geq \frac{C_{R0}}{\lambda^{(1)} R_2^{(i)}} \,, \tag{80}$$

is satisfied. The optimality conditions (57), $(32)_2$, $(32)_3$ provide $h^{(2)} = 0.0099$ m, $\lambda^{(2)} = 0.42 \cdot 10^8$ [1/(Nm)] and the optimal positions of additional rigid supports specified by the distance $x_2^{(2)} = 0.225$ m from the beam ends (Fig. 13e). The reactions of supports are $R^{(2)} = 0.86$ kN, $R_1^{(2)} = 2.75$ kN, $R_2^{(2)} = 2.76$ kN, and the cost of the structure is

$$C^{(2)} = cEA^{(2)}\ell + 5C_{R0} + c_{Ra}(2R^{(2)} + R_1^{(2)} + 2R_2^{(2)}) = 3.038 \cdot 10^8 . \tag{81}$$

Now, the condition of topology modification (41), or the more precise condition (56), and conditions of support substitution are not satisfied. This means that the design shown in Fig. 13e is optimal. The cost of the optimal design is $C^{(0)}/C^{(2)} = 3.13$ times smaller than the cost of the initial design. The length between internal rigid supports is $x_s = 0.5(\ell - 2x_2^{(2)}) = 0.275$ m and it is slightly bigger than for the infinite beam of the same cost, material and load parameters ($x_S^{(\text{inf})} = 0.259$ m).

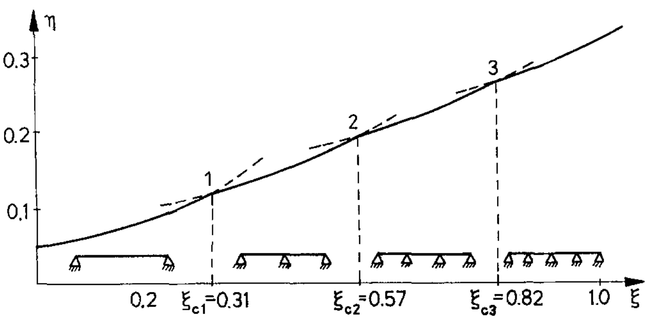

Fig. 14. Evolution of the optimal design for increasing values of the length parameter

Figure 14 presents the nondimensional cost $\eta = C/C^{(0)}$ variation for increasing values of the length parameter $\xi = \ell/\ell_0$, where $\ell_0 = 1$ m. It is seen, that for $\xi = \xi_{c\ell} = 0.31$ the introduction of a new support at the middle of the beam provides the designs associated with the lower cost curve. The states at which a new topology of support are generated will be called the topology bifurcation points. It is natural to expect that for increasing ξ, the consecutive bifurcation points will appear. In our case these bifurcations occur for $\xi_{c2} = 0.58$, $\xi_{c3} = 0.82$, etc., and they correspond to the beam with two and three additional supports. It is important to note that despite of discontinuity in state fields and design variables induced by rigid support introduction, the cost function is continuous and only its topological derivative exhibits discontinuity

Received Sept. 9, 1997

5 Concluding remarks

The method of the simultaneous optimization of topology, configuration, and cross-sectional dimensions of beams, including support design in order to minimize the cost function, was presented in this paper. The cost of the structure is assumed as the sum of the beam material cost and the cost of supports, including the cost of support installation. New supports are introduced in the optimal position and the conditions of support generation are formulated.

The proposed optimization procedure may not correspond to the global minimum. However, it provides new possibilities to generate more effective optimal designs and can easily be implemented with the use of any structural analysis method and of any optimization code. The approach presented here may also be used for maximum stiffness design with cost constraints, as well as for other types of optimization criteria and constraints, and can easily be generalized to more complex structures.

Acknowledgement

The present work was partially supported through grant No. 3P40400207 by the Committee of Scientific Research (KBN), established by the Polish Government.

References

Bojczuk, D.; Mróz, Z. 1997: Optimal design of trusses with account for topology variation. *Mech. Struct. Mach.* (submitted)

Dems, K.; Turant, J. 1997: Sensitivity analysis and optimal design of elastic hinges and supports in beam and frame structures. *Mech. Struct. Mach.* (in print)

Garstecki, A.; Mróz, Z. 1987: Optimal design of supports of elastic structures subject to loads and initial distortions. *Mech. Struct. Mach.* **15**, 47-68

Mróz, Z.; Haftka, R.T. 1994: Design sensitivity of non-linear structures in regular and critical states. *Int. J. Solids Struct.* **31**, 2071-2098

Mróz, Z.; Lekszycki, T. 1982: Optimal support reaction in elastic frame structures. *Comp. Struct.* **14**, 179-185

Mróz, Z.; Rozvany, G.I.N. 1975: Optimal design of structures with variable support positions. *J. Opt. Theory & Appl.* **15**, 85-101

Rozvany, G.I.N. 1975: Analytical treatment of some extended problems in structural optimization. Parts I-II. *J. Struct. Mech.* **3**, 359–385, 387–402

Szeląg, D.; Mróz, Z. 1978: Optimal design of elastic beams with unspecified support positions. *ZAMM* **58**, 501-510