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Integral Equations and Operator Theory

NON-SELFADJOINT PERIODIC DIRAC OPERATORS WITH FINITE-BAND SPECTRA

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We prove that skew-symmetric potential matrices generating Dirac operators with finite-band spectra are dense in the space of all skew-symmetric matrices.

1. Statement of the problem and result

We consider Dirac operators

$$L = J\frac{d}{dx} + Q(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where J and $Q(x)$ are 2×2 matrices, $J = \text{const}$, with

$$J^2 = -I, \quad JQ(x) + Q(x)J = 0, \quad Q(x+\pi) = Q(x), \tag{1.2}$$

I being the unit matrix. For 2-vectors $F = \text{col}\{f_1, f_2\} \in \mathbb{C}^2$ and 2×2 matrix W, let $\|F\| = (|f_1|^2 + |f_2|^2)^{1/2}$ and

$$\|W\| = \sup_{\|F\| \le 1} \|WF\|.$$

Denote by $\mathcal{L}_2^2(0,x)$ and $\mathcal{L}_{2,2}^2(0,x)$, respectively, the spaces of 2-coordinate vector functions $F(t) = \text{col}\{f_1(t), f_2(t)\}$ and 2×2 matrix functions $W(t)$ with finite norms

$$\|F\|_{\mathcal{L}_2^2(0,x)} = \left(\int_0^x \|F(t)\|^2\, dt\right)^{1/2}, \quad \|W\|_{\mathcal{L}_{2,2}^2(0,x)} = \left(\int_0^x \|W(t)\|^2\, dt\right)^{1/2}.$$

We denote by $\mathcal{D}$ the class of all operators (1.1) satisfying (1.2) and such that $Q \in \mathcal{L}_{2,2}^2(0,\pi)$.

Let $U(x,\lambda)$ be the solution of the Cauchy problem

$$\begin{cases} JU'(x,\lambda)dx + Q(x)U(x,\lambda) = \lambda U(x,\lambda), \\ U(0,\lambda) = I, \end{cases} \tag{1.3}$$

and let $U(\lambda) = U(\pi,\lambda)$ be the monodromy matrix of operator $L \in \mathcal{D}$. It is well known that the spectrum $\sigma(L)$ of L in the space $L^2(\mathbb{R})$ is described by the relation

$$\sigma(L) = \{\lambda \in \mathbb{C} : \Delta(\lambda) \in [-1,1]\}$$

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where $\Delta(\lambda) = 1/2 \text{ Tr } U(\lambda)$ is the Hill discriminant of L. Since $\Delta(\lambda)$ is an entire function, the spectrum is a union of analytic arcs with end-points at λ's such that $\Delta(\lambda) = \pm 1$ and which intersect at λ's such that $\Delta(\lambda) \in [-1, 1], \Delta'(\lambda) = 0$.

Let $\mathcal{D}_+$ be the subclass of $\mathcal{D}$ formed by all operators with Hermitian symmetric potential matrices $Q(x) = Q^*(x)$. Every operator L from $\mathcal{D}_+$ is selfadjoint in $L^2(\mathbb{R})$ and its spectrum is real. Moreover, it is well known [1] that the spectrum has a band structure:

$$\sigma(L) = \bigcup_{n=-\infty}^{\infty} [\mu_n^+, \mu_{n+1}^-]$$

where

$$\cdots \leq \mu_{n-1}^+ < \mu_n^- \leq \mu_n^+ < \mu_{n+1}^- \leq \cdots,$$

and the Hill discriminant $\Delta(\lambda)$ is a monotonic function on intervals $[\mu_n^+, \mu_{n+1}^-]$, which takes on values ± 1 at their end-points. The adjacent intervals (μ_n^-, μ_n^+) are called spectral gaps, and if they collapse, except finitely many of them, then the spectrum is finite-band: it is composed of finite number of intervals and two infinite rays. Using the method due to Marchenko and Ostrovskii [2], Misyura [3] proved that potential matrices generating Hill operators from $\mathcal{D}_+$ with finite-band spectra are dense in the subspace of all potential matrices generating $\mathcal{D}_+$ with respect to the norm of $\mathcal{L}^2_{2,2}(0, \pi)$.

The aim of the present paper is to prove a density theorem for the subclass $\mathcal{D}_-$ of $\mathcal{D}$ formed by all operators with skew-symmetric potential matrices, i.-e., matrices satisfying

$$Q(x) = -Q^*(x). \tag{1.4}$$

Dirac operators of class $\mathcal{D}_-$ became a subject of special interest (cf. [4], [5], [6]) since Zaharov and Shabat [8] found that they are L parts of the Lax $L - A$ pairs for non-linear Schrödinger equation in the focusing case.

Since the spectrum of a general non-selfadjoint Dirac operator does not lie on a line, the notion of finite-band spectrum has no straight-forward geometric meaning, and to clarify it we first state a proposition from [9] describing all monodromy matrices of operators belonging to $\mathcal{D}_-$. In what follows we choose a basis in $\mathbb{C}^2$ such that

$$J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Theorem 1.1. *For a 2×2- matrix $U(\lambda)$ to be the monodromy matrix of some operator $L \in \mathcal{D}_-$ it is necessary and sufficient that it has the form*

$$U(\lambda) = \begin{Vmatrix} c(\lambda) & -s^*(\lambda) \\ s(\lambda) & c^*(\lambda) \end{Vmatrix} \tag{1.5}$$

where $c(\lambda)$ and $s(\lambda)$ are entire functions of exponential type π, $c^*(\lambda) = \overline{c(\bar{\lambda})}$, $s^*(\lambda) = \overline{s(\bar{\lambda})}$, and the following conditions are satisfied:

i) the representations

$$c(\lambda) = \cos \lambda\pi + f(\lambda), \quad s(\lambda) = \sin \lambda\pi + g(\lambda) \tag{1.6}$$

are valid with f, $g \in \mathcal{PW}_\pi$ where $\mathcal{PW}_\pi$ is the space of all entire functions of exponential type not exceeding π, endowed with the norm

$$\|f\|_{\mathcal{PW}} = \|f\|_{L^2(\mathbb{R})};$$

ii) the identity holds

$$c(\lambda)c^*(\lambda) + s(\lambda)s^*(\lambda) \equiv 1; \tag{1.7}$$

iii) for each $x \in [0,\pi]$, the integral equations

$$K^T(t) + \int_0^x K^T(s)F(s,t)\,ds = 0, \quad 0 \le t \le x,$$

and

$$K^T(t) + \int_0^x K^T(s)G(s,t)\,ds = 0, \quad 0 \le t \le x,$$

with kernels

$$F(x,t) = \sum_k \left\{ \operatorname*{res}_{s^*(\lambda_k)=0} \left(\frac{c(\lambda)}{s^*(\lambda)} Y(x,\lambda)Y^T(t,\lambda) \right) - \frac{1}{\pi} Y(x,k)Y^T(t,k) \right\},$$

and

$$G(x,t) = -\sum_k \left\{ \operatorname*{res}_{c(\lambda_k)=0} \left(\frac{s_*(\lambda)}{c(\lambda)} Z(x,\lambda)Z^T(t,\lambda) \right) - \frac{1}{\pi} Z(x,k+\frac{1}{2})Z^T(t,k+\frac{1}{2}) \right\},$$

and $X(t) = \mathrm{col}\{\cos\lambda t, \sin\lambda t\}$, $Z(t) = \mathrm{col}\{-\sin\lambda t, \cos\lambda t\}$, have only the trivial solution in $\mathcal{L}_2^2(0,x)$.

It follows from (1.7) that $|c(\lambda)| \le 1$ and $\Delta(\lambda) \in [-1,1]$ for real λ's. Therefore, $\mathbb{R} \subset \sigma(L)$, and it is easy to deduce from Theorem 1.1 that $\mathbb{R} = \sigma(L)$ if and only if $Q(x) \equiv 0$. For $Q(x) \not\equiv 0$ the spectrum contains non-real points. These may be either "spines" symmetric with respect to the real axis and intersecting it at points λ such that $\Delta'(\lambda) = 0$, or finite analytic arcs not connected with the real axis.

We shall say that $L \in \mathcal{D}_-$ is an operator with the *finite-band spectrum* if for all λ's, except finitely many of them, $\Delta'(\lambda) = 0$ implies $\Delta(\lambda) = \pm 1$. According to our definition, $L \in \mathcal{D}_-$ is an operator with the finite-band spectrum if and only if the part of its spectrum outside some disc does not contain either spectral spines or isolated arcs, and hence is reduced to two rays lying on the real axis. Another definition of finite-band spectrum for non-selfadjoint operators was introduced and investigated by Gesztezy and Weikard (cf., [10]-[13]); in the present situation both definitions coincide.

Our aim is to prove that the set of potential matrices generating Dirac operators from $\mathcal{D}_-$ with finite-band spectra is dense with respect to the norm of $\mathcal{L}_{2,2}^2(0,\pi)$ in the subspace of all potential matrices generating $\mathcal{D}_-$. More precisely, we prove the following statement.

Theorem 1.2. *Given an operator $L_0 \in \mathcal{D}_-$ with a potential matrix $Q_0 \in \mathcal{L}_{2,2}^2(0,\pi)$ and an arbitrary number $\epsilon > 0$, there exists a matrix $Q_\epsilon \in \mathcal{L}_{2,2}^2(0,\pi)$ generating an operator $L_\epsilon \in \mathcal{D}_-$ with finite-band spectrum and such that $\|Q_0 - Q_\epsilon\|_{\mathcal{L}_{2,2}^2(0,\pi)} \le \epsilon$.*

Instead of operators $L \in \mathcal{D}_-$, Theorem 1.1 permits us to consider their monodromy matrices. Given an arbitrary operator $L_0 \in \mathcal{D}_-$ with a potential matrix $Q_0 \in \mathcal{L}^2_{2,2}(0,\pi)$, we make small perturbations of its monodromy matrix $U_0(\lambda)$ to arrive at a matrix $U_\epsilon(\lambda)$ corresponding to some operator $L_\epsilon \in \mathcal{D}_-$ with a potential matrix $Q_\epsilon \in \mathcal{L}^2_{2,2}(0,\pi)$ and finite-band spectrum. To perform these perturbations we use analytic techniques developed in [14], [15]. The control of difference between potential matrices $Q_\epsilon(x)$ and $Q_0(x)$ is based on the following proposition that is also proved in [9].

Theorem 1.3. *If L_1 and L_2 are two operators from class $\mathcal{D}_-$ with potential matrices $Q_1(x)$ and $Q_2(x)$, and if $U_1(\lambda)$ and $U_2(\lambda)$ are their monodromy matrices, respectively, then*

$$||U_1 - U_2||_{\mathcal{PW}} \leq K||Q_1 - Q_2||_{\mathcal{L}^2_{2,2}(0,\pi)} \exp(K||q||_{\mathcal{L}^2(0,\pi)})$$

where $||U||_{\mathcal{PW}}$ is the maximal $\mathcal{PW}$-norm of elements of $U(\lambda)$, $q(t) = \max\{||Q_1(t)||, ||Q_2(t)||\}$ and K is independent of $L_i, Q_i, i = 1, 2$.

If, on the other hand, $U(\lambda)$ is an entire matrix function of the form (1.5) *with elements $c(\lambda)$ and $s(\lambda)$ satisfying conditions i)–iii) of Theorem 1.1, then there exists a number K such that for each sufficiently small number $\epsilon > 0$ every entire matrix function $V(\lambda)$ of the same form* (1.5)*, satisfying ii) and $||U - V||_{\mathcal{PW}} \leq \epsilon$ also satisfies conditions i) and iii), and if $Q_U(x)$ and $Q_V(x)$ are potential matrices of corresponding operators from $\mathcal{D}_-$, then*

$$||Q_U - Q_V||_{\mathcal{L}^2_{2,2}(0,\pi)} \leq K||U - V||_{\mathcal{PW}}. \tag{1.8}$$

The main difficulty in proving Theorem 1.2 is that the parametrization of operator $L \in \mathcal{D}_-$ given by Theorem 1.1 uses functions $c(\lambda)$ and $s(\lambda)$, while the notion of finite-band spectrum is related to the Hill discriminant $\Delta(\lambda) \equiv (c(\lambda) + c^*(\lambda))/2$. If $c(\lambda) = c^*(\lambda)$, then $\Delta(\lambda) = c(\lambda)$ and, given an arbitrary $\epsilon > 0$, we can construct a function $c_\epsilon(\lambda) = c^*_\epsilon(\lambda)$ with all critical values equal to ± 1, except finitely many of them, with $||c_\epsilon - c||_{\mathcal{PW}} \leq \epsilon$ and $1 - c^2_\epsilon(\lambda) \geq 0, \lambda \in \mathbb{R}$. It follows now that a factorization $1 - c^2_\epsilon(\lambda) = s_\epsilon(\lambda)s^*_\epsilon(\lambda)$ is possible with $||s_\epsilon - s||_{\mathcal{PW}} \leq K\epsilon$. The pair $c_\epsilon(\lambda), s_\epsilon(\lambda)$ generates the Dirac operator $L_\epsilon \in \mathcal{D}_-$ with a potential matrix $Q_\epsilon(x)$ such that $||Q_\epsilon - Q||_{\mathcal{L}^2_{2,2}(0,\pi)} \leq K\epsilon$. The Hill discriminant of L_ϵ coincides with $c_\epsilon(\lambda)$ and therefore the spectrum of L_ϵ is finite-band.

Unfortunately, the case $c(\lambda) = c^*(\lambda)$ is not generic: it is easy to derive from Theorem 1.3 that the set of matrices corresponding to operators with $c(\lambda) \neq c^*(\lambda)$ is open with respect to the norm of $\mathcal{L}^2_{2,2}(0,\pi)$ in the space of all matrices generating $\mathcal{D}_-$. To prove Theorem 1.2 in a general case, we construct in Sec. 3 an auxiliary operator L_ϵ which is a "spoiled" version of initial operator L_0. The spectrum of L_ϵ is not finite-band. Moreover, all its spectral spines do not degenerate, but since the elements of its monodromy matrix have a well-controlled asymptotic behavior, we are able to make an additional small perturbation to obtain an operator with finite-band spectrum.

2. A class of entire functions of exponential type π

Denote by $\mathcal{H}$ the class of all entire functions $u(\lambda)$ of exponential type π, which are real on the real line, satisfy the condition $u^2(\lambda) \leq 1$ for real values of λ, and which may be represented in the form

$$u(\lambda) = \cos\lambda\pi + f(\lambda) \tag{2.1}$$

with $f \in \mathcal{PW}_\pi$ and $\{f(n)\}_{n=-\infty}^{\infty} \in \ell^1$.

Lemma 2.1. *If $f \in \mathcal{PW}_\pi$, then*

i) for every $H > 0$ the relation

$$\lim_{\lambda \to \infty, |\Im\lambda| \le H} f(\lambda) = 0$$

holds;

ii) for every sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ with $\lambda_n - n = o(1)$ as $|n| \to \infty$ and every $R > 0$ the condition

$$\sum_{n=-\infty}^{\infty} \max_{|t-\lambda_n| \le R} |f(t)|^2 < \infty$$

is fulfilled. In particular,

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2 < \infty.$$

Proof. Without loss of generality we assume $H > 2R$ and use the well-known "subharmonic" arguments (cf., [7]). Since $f \in \mathcal{PW}_\pi$, the functions $f_\pm(\lambda) = e^{\pm i\pi\lambda} f(\lambda)$ belong to the Hardy spaces in $\{\lambda = x + iy : \pm y > 0\}$, and if $P_H = \{\lambda : |\Im\lambda| \le H\}$ then

$$\int_{P_{2H}} |f(x+iy)|^2 dxdy \le e^{2\pi H} \int_{\mathbb{R}} \left(\int_0^{2H} |f_+(x+iy)|^2 \, dy + \int_{-2H}^{0} |f_-(x+iy)|^2 \, dy \right) dx < \infty.$$

On the other hand, $|f(\lambda)|^2$ is a subharmonic function in $\mathbb{C}$ and hence

$$\int_{d(\lambda,R)} |f(x+iy)|^2 dxdy = \int_0^R \tau d\tau \int_0^{2\pi} |f(\lambda + \tau e^{i\theta})|^2 d\theta \ge \pi R^2 \, |f(\lambda)|^2.$$

For every $\lambda \in P_H$ the disc $d(\lambda, R) = \{\mu : |\mu - \lambda| \le R\}$ is located inside the strip P_{2H} and the integral on the left-hand side here vanishes as $|\lambda| \to \infty$ proving part *i*) of Lemma 2.1.

If $t \in d(\lambda_n, R)$ and $|n|$ is sufficiently big, then $d(t, R) \subset d(\lambda_n, 2R) \subset P_{2H}$ and

$$\max_{t \in d(\lambda_n, R)} |f(t)|^2 \le (\pi R^2)^{-1} \int_{d(\lambda_n, 2R)} |f(x+iy)|^2 dxdy.$$

Since $\lambda_n - n = o(1)$ as $|n| \to \infty$, there exists a number $N > 0$ such that every point of the strip P_{2H} is covered by not more than N discs $d(\lambda_n, 2R)$. Therefore

$$\sum_{n=-\infty}^{\infty} \max_{t \in d(\lambda_n, R)} |f(t)|^2 \le N(\pi R^2)^{-1} \int_{P_{2H}} |f(x+iy)|^2 dxdy < \infty,$$

which completes the proof of Lemma 2.1.

Lemma 2.2. *If $s(\lambda) = \sin \lambda\pi + f(\lambda)$ with $f \in \mathcal{PW}_\pi$ and $\{\lambda_n\}_{n=-\infty}^{\infty}$ is the zero sequence of $s(\lambda)$, then $\{\lambda_n - n\}_{n=-\infty}^{\infty} \in \ell^2$.*

Proof. First we note that outside the exceptional set

$$E = \bigcup_{n=-\infty}^{\infty} d(n, 10^{-1})$$

an estimate $|\sin\lambda\pi| \geq c\exp(\pi|\Im\lambda|)$ is valid with some $c > 0$. Hence

$$\lim_{|\lambda|\to\infty, \lambda\notin E} \left|\frac{f(\lambda)}{\sin\lambda\pi}\right| = 0,$$

and by the Rouché Theorem zeros λ_n of $s(\lambda)$ with big $|n|$ are inside E. For every such n we have

$$|\lambda_n - n| \leq |f(\lambda_n)| \max_{|t|\leq 10^{-1}} \left|\frac{t}{\sin\pi t}\right| \leq C|f(\lambda_n)|,$$

and the statement of Lemma 2.2 follows from Lemma 2.1.

Lemma 2.3. *The Hill discriminant of operator $L \in \mathcal{D}_-$ belongs to class $\mathcal{H}$.*

Proof. Let L be a Dirac operator from class $\mathcal{D}_-$ and let $U(\lambda)$ be its monodromy matrix (1.5) with elements $c(\lambda)$ and $s(\lambda)$ and with properties described in Theorem 1.1. The Hill discriminant $\Delta(\lambda) = (c(\lambda) + c^*(\lambda))/2$ has the form $\Delta(\lambda) = \cos\lambda\pi + h(\lambda)$ where $h \in \mathcal{PW}_\pi$. If we set $v(\lambda) = (c(\lambda) - c^*(\lambda))/2i$, then $v \in \mathcal{PW}_\pi$, both $\Delta(\lambda)$ and $v(\lambda)$ are real for real λ's, $c(\lambda) = \Delta(\lambda) + iv(\lambda), c^*(\lambda) = \Delta(\lambda) - iv(\lambda)$, and equation (1.7) yields

$$1 - \Delta^2(\lambda) - v^2(\lambda) - s(\lambda)s^*(\lambda) = 0 \tag{2.2}$$

implying, in particular, $0 \leq \Delta^2(\lambda) \leq 1, \lambda \in \mathbb{R}$.

Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ be the zero set of $s(\lambda)$ with account taken of multiplicities. It follows from Lemma 2.2 that $\{\lambda_n - n\}_{n=-\infty}^{\infty} \in \ell^2$. If we substitute $\lambda = \lambda_n$ in (2.2) and apply Lemma 2.1 to the function $v(\lambda)$, we will find $\{1 - \Delta^2(\lambda_n)\}_{n=-\infty}^{\infty} = \{v^2(\lambda_n)\}_{n=-\infty}^{\infty} \in \ell^1$. Since $\Delta(\lambda_n)$ is asymptotic to $(-1)^n$, we have

$$|h(\lambda_n)| \leq K\left|\frac{1 - \Delta^2(\lambda_n)}{(-1)^n + \Delta(\lambda_n)} - ((-1)^n - \cos\lambda_n\pi)\right| \leq K(|v(\lambda_n)|^2 + |\lambda_n - n|^2),$$

and $\{h(\lambda_n)\}_{n=-\infty}^{\infty} \in \ell^1$. According to the Taylor formula

$$h(\lambda_n) - h(n) = h'(n)(\lambda_n - n) + \int_n^{\lambda_n} h''(s)(\lambda_n - s)ds.$$

Since $h' \in \mathcal{PW}_\pi$, Lemma 2.2 yields $\{h'(n)\}_{n=-\infty}^{\infty} \in \ell^2$. In addition, $h''(\lambda)$ is bounded in the strip P_1 and hence $|h(\lambda_n) - h(n)| \leq |h'(n)(\lambda_n - n)| + K|\lambda_n - n|^2$. We conclude $\{h(n)\}_{n=-\infty}^{\infty} \in \ell^1$ completing the proof of Lemma 2.3.

In this section we study small deformations of functions belonging to $\mathcal{H}$.

Let $\mathcal{M}(u) = \{\mu_n\}_{n=-\infty}^{\infty}$ be the sequence of all critical points of $u \in \mathcal{H}$, i.e., the sequence of all solutions of equation $u'(\lambda) = 0$ with account taken of multiplicities. It follows from (2.1) that $-\pi^{-1}u'(\lambda) = \sin\lambda\pi + g(\lambda)$ with $g \in \mathcal{PW}_\pi$ and hence by Lemma 2.2 the representation

$$\mu_n = n + \delta_n, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{2.3}$$

is valid with $\{\delta_n\}_{n=-\infty}^{\infty} \in l^2$. It is easy to see, using (2.1), that if m_n is the multiplicity of $\mu_n \in \mathcal{M}(u)$ as a zero of $u(\lambda)-u(\mu_n)$, then $m_n = 2$ for all n's with possible exception of finitely many of them. Substituting $\lambda = n$ into (2.1) we find $\{u(n)-(-1)^n\}_{n=-\infty}^{\infty} = \{f(n)\}_{n=-\infty}^{\infty} \in \ell^1$ and again using the Taylor formula to estimate $u(\mu_n) - u(n)$ we obtain

$$\sum_{n=-\infty}^{\infty} |(-1)^n - u(\mu_n)|^{2/m_n} < \infty.$$

Let $\mathcal{CR}(u)$ be the sequence of all critical values of $u \in \mathcal{H}$, i.e.,

$$\mathcal{CR}(u) = \{\gamma_n : \gamma_n = u(\mu_n), \quad \mu_n \in \mathcal{M}(u)\}.$$

It is evident that if $\mu_n = \mu_k, n \neq k$, then $\gamma_n = \gamma_k$.

Denote by $\mathcal{R}(u)$ the subset of all integers such that critical points $\mu_n \in \mathcal{M}(u)$ with $n \in \mathcal{R}(u)$ are real. Since $u(\lambda)$ is a real function for real λ's, for each integer $n \notin \mathcal{R}(u)$ we have $\overline{\mu}_n \in \mathcal{M}(u)$ and $u(\overline{\mu}_n) = \overline{u(\mu_n)}$. With sets $\mathcal{M}(u)$ and $\mathcal{R}(u)$ being defined, let us introduce the set $\mathcal{G}(u)$ of sequences $\{\gamma_n\}_{n=-\infty}^{\infty}$ of complex numbers with the following properties:

1. If $\mu_n = \mu_k, n \neq k$, then $\gamma_n = \gamma_k$;
2. If $n \in \mathcal{R}(u)$, then $\Im\gamma_n = 0$ and $\gamma_n^2 \leq 1$;
3. For every $n \notin \mathcal{R}(u)$, there exists $p = p(n)$ such that $\overline{\mu}_n = \mu_{p(n)}, \overline{\gamma}_n = \gamma_{p(n)}$ and $p(p(n)) = n$.

Theorem 2.1. *Given a function $u_0 \in \mathcal{H}$ with $\mathcal{M}(u_0) = \{\mu_{n,0}\}_{n=-\infty}^{\infty}$ and $\mathcal{CR}(u_0) = \{\gamma_{n,0}\}_{n=-\infty}^{\infty}$, there exists a number $K > 0$ such that for every sufficiently small $\epsilon > 0$ and every sequence of complex numbers $\{\gamma_n\}_{n=-\infty}^{\infty} \in \mathcal{G}(u_0)$ such that*

$$\left(\sum_{n=-\infty}^{\infty} |\gamma_n - \gamma_{n,0}|^{2/m_n}\right)^{1/2} \leq \epsilon \tag{2.4}$$

there exists a function $u_\epsilon \in \mathcal{H}$, with the set of critical values $\mathcal{CR}(u_\epsilon) = \{\gamma_n\}_{n=-\infty}^{\infty}$ and such that

$$\left(\int_{-\infty}^{\infty} |u_0(\lambda) - u_\epsilon(\lambda)|^2 \, d\lambda\right)^{1/2} \leq K\epsilon, \qquad \sum_{n=-\infty}^{\infty} |u_0(n) - u_\epsilon(n)| \leq K\epsilon. \tag{2.5}$$

Proof. Theorem itself and its proof are variations on the themes discussed in [14] and [15]. Here we will describe construction of $u_\epsilon(\lambda)$, skip details repeating [14] and [15] and give arguments requested by the present situation.

Let $\mathcal{N}$ be the set of all integers n for which the numbers μ_n are pairwise different and $\gamma_n \neq \gamma_{n,0}$, and containing both n and $p(n)$ if $n \notin \mathcal{R}(u_0)$. For $n \in \mathcal{N}$, we set $\sigma_n = \gamma_n - \gamma_{n,0}$, choose a small number ρ_n, $\rho_n \geq 2\sigma_n$, and denote by ℓ_n the closed curve

$$\ell_n = \{\lambda : |u_0(\lambda) - \gamma_{n,0}| = \rho_n\}. \tag{2.6}$$

We assume that ℓ_n is positively oriented with respect to its interior ω_n^+, and that $\mu_{n,0}$ lies inside ω_n^+.

It is evident that for each $n \in \mathcal{N}$ and sufficiently small ϵ in (2.4) every function $v_n(\lambda) = (\pm(u_0(\lambda) - \gamma_{n,0}))^{1/m_n}$ is a shift function in ω_n^+. For $n \in \mathcal{R}(u_0) \bigcap \mathcal{N}$ the numbers $\mu_{n,0}$ and $\gamma_{n,0}$ are real, the curve ℓ_n is symmetric with respect to the real axis and, defining $v_n(\lambda)$, we choose the sign + or − such that $v_n(\lambda)$ is real for real $\lambda \in \omega_n^+$. If, on the other hand, $n \in \mathcal{N} \backslash \mathcal{R}(u_0)$, then $\Im \mu_n \neq 0$, $\overline{\mu}_n = \mu_{p(n)} \in \mathcal{M}(u_0)$, and we can assume that $\overline{\ell}_n = \ell_{p(n)}$ and $\overline{v_n(\lambda)} = v_{p(n)}(\overline{\lambda}), \lambda \in \omega_n^+$.

With $\epsilon > 0$ sufficiently small, the curves ℓ_n do not intersect, and we can assume

$$\begin{aligned} K^{-1}\rho_n^{1/m_n} \leq |\ell_n| \leq K\rho_n^{1/m_n} \\ |t-z| \leq \left|\int_z^t ds\right| \leq K|t-z|, \quad t,\ z \in \ell_n, \end{aligned} \tag{2.7}$$

where $|\ell_n|$ is the length of ℓ_n and the integral is taken over the shortest arc linking t and z in ℓ_n. Besides,

$$\inf\{|t-z|,\ t \in \ell_n, z \in \ell_m, n \neq m\} \geq K^{-1}|n-m|. \tag{2.8}$$

Here and in what follows K denotes some number determined by the function $u_0(\lambda)$ and not depending on λ, n, σ_n, ρ_n.

The function $v_n(\lambda)$ is analytic in the domain ω_n^+, real for real $\lambda \in \omega_n^+$ and maps ω_n^+ one-to-one onto the disc $d_n = \{w : |w - \gamma_{n,0}| \leq \rho_n^{1/m_n}\}$. We define in d_n the function $b_n(w)$ inverse to $v_n(\lambda)$ and set

$$\alpha_n(\lambda) = b_n\left(v_n(\lambda)\left(\frac{1 \mp \sigma_n v_n^{-m_n}(\lambda)}{1 \mp \overline{\sigma}_n \rho_n^{-2} v_n^{m_n}(\lambda)}\right)^{1/m_n}\right), \quad \lambda \in \ell_n, \tag{2.9}$$

with the sign opposite to that in the definition of $v_n(\lambda)$ and with positive values of the root for real $\lambda \in \omega_n^+$.

Since $|v_n(\lambda)|^{m_n} = \rho_n$ for $\lambda \in \ell_n$ and $|\sigma_n\rho_n^{-1}| \leq 1/2$, the function $\alpha_n(\lambda)$ is an analytic diffeomorphism of ℓ_n onto itself satisfying the relations

$$\alpha_n(\overline{\lambda}) = \overline{\alpha_n(\lambda)}, \quad \lambda \in \ell_n, \quad n \in \mathcal{R}(u_0), \tag{2.10}$$

or

$$\alpha_n(\overline{\lambda}) = \overline{\alpha_{p(n)}(\lambda)}, \quad \lambda \in \ell_n, \quad n \notin \mathcal{R}(u_0), \tag{2.11}$$

and, according to (2.9),

$$u_0(\alpha_n(\lambda)) = \frac{(u_0(\lambda) - \gamma_{n,0}) - \sigma_n}{1 - \overline{\sigma}_n\rho_n^{-2}(u_0(\lambda) - \gamma_{n,0})} + \gamma_{n,0}, \quad \lambda \in \ell_n.$$

Using (2.9) and the identity $b_n(v_n(\lambda)) \equiv \lambda$, $\lambda \in \ell_n$, we find that there exists a constant K depending only on the function $u_0(\lambda)$ and independent of σ_n and ρ_n such that the estimates

$$\left.\begin{aligned} |\alpha_n(\lambda) - \lambda| &\leq K|\sigma_n|\rho_n^{1/m_n - 1} \\ |\alpha_n'(\lambda) - 1| &\leq K|\sigma_n|\rho_n^{-1} \\ |\alpha_n''(\lambda)| &\leq K|\sigma_n|\rho_n^{-1/m_n - 1} \end{aligned}\right. \quad \lambda \in \ell_n, \tag{2.12}$$

hold. If $\tau_n(\lambda)$ is the diffeomorphism inverse to $\alpha_n(\lambda)$, then

$$\left.\begin{aligned} |\tau_n(\lambda) - \lambda| &\leq K|\sigma_n|\rho_n^{-1/m_n} \\ |\tau_n'(\lambda) - 1| &\leq K|\sigma_n|\rho_n^{-1} \\ |\tau_n''(\lambda)| &\leq K|\sigma_n|\rho_n^{-1/m_n - 1} \end{aligned}\right. \quad \lambda \in \ell_n. \tag{2.13}$$

Let us set now

$$U(\lambda)=\begin{cases}\dfrac{(u_0(\lambda)-\gamma_{n,0})+\sigma_n}{1+\bar\sigma_n\rho_n^{-2}(u_0(\lambda)-\gamma_{n,0})}+\gamma_{n,0} & \lambda\in\omega_n^+\\ u_0(\lambda) & \lambda\in\omega^-\end{cases}\tag{2.14}$$

where

$$\omega^-=\mathbb{C}\setminus\omega^+,\qquad \omega^+=\operatorname{clos}\bigcup_{n\in\mathcal{N}}\omega_n^+.$$

The function $U(\lambda)$ is analytic outside $\partial\omega^-$ and takes on the critical value γ_n at the point $\mu_{n,0}\in\omega_n^+$. The boundary values

$$U_-(\lambda)=\lim_{\mu\to\lambda,\mu\in\omega^-}U(\mu),\;\; U_+(\lambda)=\lim_{\mu\to\lambda,\mu\in\omega_n^+}U(\mu),\;\; \lambda\in\partial\omega_n^+,$$

satisfy the relation

$$U_+(\alpha_n(\lambda))=U_-(\lambda),\;\; \lambda\in\partial\omega_n^+.$$

To transform $U(\lambda)$ in an entire function, let us construct two shift functions $\Phi^+(z)$ and $\Phi^-(z)$ analytic in ω^+ and ω^-, respectively, satisfying the glueing condition

$$\Phi^+(\alpha_n(z))=\Phi^-(z),\quad z\in\partial\omega_n^+.\tag{2.15}$$

Suppose that these are functions of the form

$$\Phi^+(z)=z+\varphi^+(z),\quad \Phi^-(z)=z+\varphi^-(z)\tag{2.16}$$

where $\varphi^+(z)$ and $\varphi^-(z)$ are representable by their Cauchy integrals

$$\varphi^+(z)=\sum_{n\in\mathcal{N}}\frac{1}{2\pi i}\int_{\ell_n}\frac{\varphi_n^+(t)}{t-z}\,dt,\quad z\in\omega^+,\tag{2.17}$$

$$\varphi^-(z)=-\sum_{n\in\mathcal{N}}\frac{1}{2\pi i}\int_{\ell_n}\frac{\varphi_n^-(t)}{t-z}\,dt,\quad z\in\omega^-.\tag{2.18}$$

It follows from (2.15) and (2.16), that

$$\varphi_n^+(\alpha_n(z))-\varphi_n^-(z)=\beta_n(z)\tag{2.19}$$

with

$$\beta_n(z)=z-\alpha_n(z).\tag{2.20}$$

Denote by $\mathcal{L}_n^2, n\in\mathcal{N}$, the Hilbert space of all complex-valued functions on ℓ_n with the norm

$$\|\varphi\|_n=\left(\frac{1}{|\ell_n|}\int_{\ell_n}|\varphi(t)|^2|dt|\right)^{1/2},$$

where, as above, $|\ell_n|$ is the length of ℓ_n, and introduce the Hilbert space $\mathbb{L}^2$ of all functional sequences $\Phi=\{\varphi_n(z)\}_{n\in\mathcal{N}},\varphi_n\in\mathcal{L}_n^2$, with the norm

$$\|\Phi\|=\left(\sum_{n\in\mathcal{N}}\|\varphi_n\|^2\right)^{1/2}.$$

According to (2.12), $|\beta_n(z)| \le K|\sigma_n|\rho_n^{1/m_n-1} \le K|\sigma_n|^{1/m_n}$, and if $B = \{\beta_n(z)\}_{n\in\mathcal{N}}$, then

$$\|B\| \le K\left(\sum_{n\in\mathcal{N}} |\sigma_n|^{2/m_n}\right)^{1/2}.$$

Using the Plemelj formula, we obtain

$$\frac{1}{2}\varphi_n^-(z) + \frac{1}{2\pi i}\int\limits_{\ell_n} \frac{\varphi_n^-(t)}{t-z}\,dt + {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int\limits_{\ell_n} \frac{\varphi_k^-(t)}{t-z}\,dt = 0, \quad z \in \ell_n, \tag{2.21}$$

and since

$$\sum_{n\in\mathcal{N}} \frac{1}{2\pi i}\int\limits_{\ell_n} \frac{\varphi_n^+(t)}{t-z}\,dt = 0, \quad z \in \omega^-,$$

then

$$\frac{1}{2}\varphi_n^+(z) - \frac{1}{2\pi i}\int\limits_{\ell_n} \frac{\varphi_n^+(t)}{t-z}\,dt - {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int\limits_{\ell_k} \frac{\varphi_k^+(t)}{t-z}\,dt = 0, \quad z \in \ell_n. \tag{2.22}$$

Here and in what follows the prime means that the n-th term is omitted from the sum. We replace z in (2.22) by $\alpha_n(z)$, set $t = \alpha_k(s)$ for the k-th integrand, add the resulting identity to (2.21) and use (2.19) to find that $\Phi^- = \{\varphi_n^-(z)\}_{n\in\mathcal{N}}$ is a solution of equation

$$\varphi_n(z) - \int\limits_{l_n} K_n(z,t)\varphi_n(t)\,dt - \mathbb{R}[\Phi]_n(z) = \gamma_n(z), \quad z \in \ell_n. \tag{2.23}$$

Here the kernel $K_n(z,t)$ and the operator $\mathbb{R}$ are defined by the formulas

$$K_n(z,t) = \frac{1}{2\pi i}\left(\frac{\alpha_n'(t)}{\alpha_n(t)-\alpha_n(z)} - \frac{1}{t-z}\right), \quad t, z \in \ell_n, \tag{2.24}$$

$$\mathbb{R}[\Phi]_n(z) = {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int\limits_{\ell_k} \left(\frac{\alpha_k'(t)}{\alpha_k(t)-\alpha_n(z)} - \frac{1}{t-z}\right)\varphi_k(t)\,dt, \quad z \in \ell_n, \tag{2.25}$$

and

$$\gamma_n(z) = -\frac{1}{2}\beta_n(z) + \frac{1}{2\pi i}\int\limits_{\ell_n} \frac{\beta_n(t)}{t-z}\,dt + \int\limits_{\ell_n} K_n(z,t)\beta_n(t)\,dt + \mathbb{P}[B]_n(z) \tag{2.26}$$

with

$$\mathbb{P}[B]_n(z) = {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int\limits_{\ell_k} \frac{\alpha_k'(t)}{\alpha_k(t)-\alpha_n(z)}\beta_k(t)\,dt, \quad z \in \ell_n. \tag{2.27}$$

Let us prove that $\Gamma = \{\gamma_n(z)\}_{n\in\mathcal{N}} \in \mathbb{L}^2$ and obtain an estimate for $\|\Gamma\|$. To this end we prove that there exists a number K determined by the function $u_0 \in \mathcal{H}$ such that

$$\left|\int\limits_{\ell_n} K_n(z,t)\varphi_n(t)\,dt\right| \le K|\sigma_n|\rho_n|^{-1}\|\varphi_n\|_n, \quad z \in \ell_n, \tag{2.28}$$

$$\|\mathbb{P}\| \le K\epsilon, \tag{2.29}$$

$$\|\mathbb{R}\| \leq K\epsilon, \tag{2.30}$$

where

$$\epsilon = \left(\sum_{n\in\mathcal{N}} |\sigma_n|^{2/m_n}\right)^{1/2} = \left(\sum_{n\in\mathcal{N}} |\gamma_n - \gamma_{n,0}|^{2/m_n}\right)^{1/2}.$$

We start with representing $K_n(z,t)$ in the form

$$K_n(z,t) = \left(-\int_z^t \alpha_n''(s)(t-s)\,ds\right)\left((t-z)\int_z^t \alpha_n'(s)\,ds\right)^{-1}, \quad z,t\in\ell_n,$$

with the integrations over the shortest arc of ℓ_n joining z and t. Relations (2.7) and (2.12) yield the estimate

$$\left|\int_{\ell_n} K_n(z,t)\varphi_n(t)\,dt\right| \leq K|l_n||\sigma_n|\rho_n^{-1/m_n-1}\|\varphi_n\|_n \leq K|\sigma_n|\rho_n^{-1}\|\varphi_n\|_n$$

proving (2.28). To prove (2.29), let us introduce the operator

$$\mathbb{P}_0[\Phi]_n(z) = {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int_{\ell_k} \frac{\varphi_k(t)}{\alpha_k(t)-\alpha_n(z)}\,dt, \quad z\in\ell_n, \quad \Phi = \{\varphi_k(t)\}_{k\in\mathcal{N}} \in \mathbb{L}^2,$$

and represent it in the form

$$\mathbb{P}_0[\Phi]_n(z) = {\sum_{k\in\mathcal{N}}}' \frac{1}{k-n}\frac{1}{2\pi i}\int_{\ell_k} \varphi_k(t)\,dt + p_n(z), \tag{2.31}$$

where

$$p_n(z) = {\sum_{k\in\mathcal{N}}}' \frac{1}{2\pi i}\int_{\ell_k} \frac{(\alpha_n(z)-n)-(\alpha_k(t)-k)}{(\alpha_n(z)-\alpha_k(t))(n-k)}\,\varphi_k(t)dt.$$

The sum in (2.31) is the n-th coordinate of the discrete Hilbert transform of the sequence

$$s_k = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\ell_k} \varphi_k(t)\,dt & k\in\mathcal{N}, \\ 0 & k\notin\mathcal{N}, \end{cases} \tag{2.32}$$

and since $|s_k| \leq K|\ell_k|\|\varphi_k\|_k$, its $\mathbb{L}^2$-norm is bounded by $K\epsilon\|\Phi\|$. To estimate the latter sum we note that if $z\in\ell_n$, $t\in\ell_k$, then $|\alpha_n(z)-\mu_{n,0}| \leq K\rho_n^{1/m_n}$, $|\alpha_k(t)-\mu_{k,0}| \leq K\rho_k^{1/m_k}$ which, being combined with (2.3), yields $|\alpha_n(z)-n| \leq K\rho_n^{1/m_n}+\delta_n$, $|\alpha_k(t)-k| \leq K\rho_k^{1/m_k}+\delta_k$ with $\{\delta_n\}_{n=-\infty}^{\infty} \in \ell^2$. Using (2.8), we obtain

$$|p_n(z)| \leq K(\rho_n^{1/m_n}+\delta_n){\sum_{k\in\mathcal{N}}}'|\ell_k|\|\varphi_k\|_k + K{\sum_{k\in\mathcal{N}}}' \frac{(\rho_k^{1/m_k}+\delta_k)\ell_k}{|k-n|^2}\|\varphi_k\|_k. \tag{2.33}$$

Therefore

$$\sum_{n\in\mathcal{N}} \|p_n\|_n^2 \leq K^2\epsilon^2\|\Phi\|^2$$

proving $\|\mathbb{P}_0[\Phi]\| \leq K\epsilon\|\Phi\|$. Applying this estimate to $\Phi = \{\alpha_k'(t)\varphi_k(t)\}_{k\in\mathcal{N}}$ we arrive at (2.29).

To prove (2.30) we use the representation

$$\mathbb{R}[\Phi]_n(z) = \mathbb{P}_0[\Psi]_n(z) + q_n(z)$$

where $\Psi = \{(\alpha_k'(t)-1)\varphi_k(t)\}_{k\in\mathcal{N}}$ and

$$q_n(z) = \sum_{k\in\mathcal{N}}{}' \frac{1}{2\pi i}\int_{\ell_k} \frac{(\alpha_n(z)-z)-(\alpha_k(t)-t)}{(\alpha_k(t)-\alpha_n(z))(t-z)}\,\varphi_k(t)\,dt.$$

Similar to (2.33) we have

$$|q_n(z)| \leq K\left(\sigma_n\rho_n^{1/m_n-1}\sum_{k\in\mathcal{N}}|\ell_k|\|\varphi_k\|_k + \sum_{k\in\mathcal{N}}{}' \frac{|\sigma_k|\rho_k^{1/m_k-1}|\ell_k|\|\varphi_k\|_k}{|n-k|^2}\right)$$

which yields (2.30).

For every $n\in\mathcal{N}$ the Hilbert operator is bounded in space $\mathcal{L}_n^2$ by a constant K not depending on n, i.e.,

$$\left\|\frac{1}{2\pi i}\int_{\ell_n}\frac{\varphi_n(t)}{t-z}\,dt\right\|_n \leq K\|\varphi_n\|_n.$$

Combining these estimates with (2.28) and (2.29) we find that if $B = \{\beta_n(z)\}_{n\in\mathcal{N}}$ is defined by (2.20) and $\Gamma = \{\gamma_n(z)\}_{n\in\mathcal{N}}$ is defined by (2.24), (2.26) and (2.27), then $\|\Gamma\| \leq K\|B\| \leq K\epsilon$. Therefore operator $\mathbb{K}$ defined by the relation

$$\mathbb{K}[\Phi]_n(z) = \int_{\ell_n} K_n(z,t)\varphi_n(t)\,dt + \mathbb{R}[\Phi]_n(z), \quad z\in\ell_n,$$

is bounded in $\mathbb{L}^2$ and

$$\|\mathbb{K}\| \leq K(\epsilon + \sup_{n\in\mathcal{N}}|\sigma_n|\rho_n^{-1}). \tag{2.34}$$

Since the number K in (2.34) does not depend on σ_n and ρ_n, we can assume that $K|\sigma_n|\rho_n^{-1} < 1/4$ and choose ϵ in (2.4) so small that $K\epsilon < 1/4$. Now we are ready to construct functions $\Phi^+(z)$ and $\Phi^-(z)$ satisfying (2.15).

First, given a function $u_0 \in \mathcal{H}$ and the number K being fixed as described above, we define $B = \{\beta_n(z)\}_{n\in\mathcal{N}}$ by relations (2.20) and $\Gamma = \{\gamma_n(z)\}_{n\in\mathcal{N}}$ by (2.24), (2.26) and (2.27), and according to (2.23) consider the equation

$$(\mathbb{I}-\mathbb{K})\,\Phi = \Gamma. \tag{2.35}$$

Since $\|\mathbb{K}\| \leq 1/2$, this equation has the unique solution $\Phi = \{\varphi_n(z)\}_{n\in\mathcal{N}} \in \mathbb{L}^2$ and the estimate $\|\Phi\| \leq 2\|\Gamma\| \leq K\epsilon$ is valid. Repeating arguments from [15] we prove that the functions $\Phi^+(z)$ and $\Phi^-(z)$ defined by (2.16) with

$$\varphi^-(z) = -\sum_{k\in\mathcal{N}}\frac{1}{2\pi i}\int_{\ell_k}\frac{\varphi_k(t)}{t-z}\,dt, \quad z\in\omega^-, \tag{2.36}$$

and

$$\varphi^+(z) = \sum_{k\in\mathcal{N}} \frac{1}{2\pi i} \int_{\ell_k} \frac{\zeta_k(t)}{t-z}\,dt, \quad z\in\omega^+, \quad \zeta_k(t) = \varphi_k(\tau_k(t)) + \beta_k(\tau_k(t)), \tag{2.37}$$

are the shift functions in ω^- and ω^+, respectively, satisfying (2.15).

Denote by $\tilde\omega^-$ and $\tilde\omega^+$ the images of ω^- and ω^+ with respect to $\Phi^-(z)$ and $\Phi^+(z)$, and by $\Omega^-(z)$ and $\Omega^+(z)$ the inverse maps. Let

$$u_\epsilon(\lambda) = \begin{cases} U(\Omega^-(\lambda)) & \lambda\in\tilde\omega^- \\ U(\Omega^+(\lambda)) & \lambda\in\tilde\omega^+ \end{cases} \tag{2.38}$$

where $U(\lambda)$ is defined by (2.14). If $\lambda\in\partial\tilde\omega^-$, then there exists $n\in\mathcal{N}$ such that $z=\Omega^-(\lambda)\in\partial\omega_n^+$, and

$$\lim_{\mu\in\tilde\omega^-,\mu\to\lambda} u_\epsilon(\mu) = U(z) = u_0(z).$$

On the other hand, according to (2.15), $\Omega^+(\lambda) = \alpha_n(z)$. Looking now at λ as a point of $\partial\tilde\omega^+$ and using (2.14) we find

$$\lim_{\mu\in\tilde\omega^+,\mu\to\lambda} u_\epsilon(\mu) = U(\alpha_n(z)) = u_0(z).$$

Therefore $u_\epsilon(\lambda)$ is an entire function. Let us prove that it satisfies conditions (2.5).

To this end we choose $a>2$ such that the set ω^+ is located inside the strip $\{\lambda : |\mathrm{Im}\lambda| \le a\}$, and find for $z=t+is$, $|s|\ge 2a$, $t\in\mathbb{R}$, an entire n such that $n-1/2\le t\le n+1/2$. If $n\in\mathcal{N}$, we write (2.36) in the form

$$\varphi^-(z) = -\frac{1}{2\pi i}\int_{\ell_n}\frac{\varphi_n(t)}{t-z}\,dt - \sum_{k=-\infty}^{\infty}{}' \frac{s_k}{k-n} + \psi_n(z) \tag{2.39}$$

where s_k is defined by (2.32) and

$$\psi_n(z) = -\sum_{k\in\mathcal{N}}{}' \frac{1}{2\pi i}\int_{\ell_k} \frac{(k-t)+(z-n)}{(k-n)(t-z)}\,\varphi_k(t)\,dt.$$

If $n\notin\mathcal{N}$, the integral in (2.39) is redundant. Since $|s_k|\le K|\ell_k|\|\varphi_k\|_k$, and the discrete Hilbert transform is a bounded operator in ℓ^2, we have

$$\sum_{n=-\infty}^{\infty}\left|\sum_{k\in\mathcal{N}}{}'\frac{s_k}{k-n}\right|^2 \le K\sum_{k\in\mathcal{N}}|s_k|^2 \le K\sup_{k\in\mathcal{N}}|\ell_k|^2\|\Phi\|^2 \le K\epsilon^2\|\Phi\|^2.$$

Since $|\Im z| = |s|\ge 2a$, then $|t-z|\ge 1$ for $t\in\ell_n$ and $|t-z|\ge K|k-n|$ for $t\in\ell_k, k\ne n$. Therefore

$$\left\|\frac{1}{2\pi i}\int_{\ell_n}\frac{\varphi_n(t)}{t-z}\,dt\right\|_n \le K|\ell_n|\|\varphi_n\|_n, \quad |\psi_n(z)|\le K\sum_{k\in\mathcal{N}}{}'\frac{(|\ell_k|+|\ell_n|)\|\varphi_k\|_k}{|k-n|^2}, \tag{2.40}$$

and we conclude that

$$|\varphi^-(z)|\le\chi_n, \quad n-1/2\le\Re z\le n+1/2, \quad |\Im z|\ge 2a, \tag{2.41}$$

with $\|\{\chi_n\}_{n=-\infty}^{\infty}\|_{l^2} \le K\epsilon\|\Phi\|$. Since $\|\Phi\| \le 2\|\Gamma\| \le K\epsilon$, we obtain

$$\max_{|\Im z|\ge 2a} |\varphi^-(z)| \le K\epsilon\|\Phi\| \le K^2\epsilon^2 \tag{2.42}$$

and

$$\int_{\mathbb{R}} |\varphi^-(t+2ai)|^2\, dt \le K^2\|\Phi\|^2 \le K^2\epsilon^2.$$

For sufficiently small ϵ we have $K\epsilon < 1 < a$, and hence $\{z : |\mathrm{Im} z| \ge 3a\} \subset \tilde{\omega}^-$. It means that for $z = t + is$, $s = 3a$, $t \in [n-1/2,\ n+1/2]$, there exists $\lambda = \tau + i\delta$, $|\delta| \ge 2a$, such that $z = \Phi^-(\lambda) = \lambda + \varphi^-(\lambda)$, and hence $\lambda = \Omega^-(z)$. It follows from (2.42) that $|\varphi^-(\lambda)| \le 1$ and hence $\Re\lambda \in [n-3/2,\ n+3/2]$. It permits us to use (2.41) to obtain the estimates $|\Omega^-(z) - z| = |\lambda - \Phi^-(\lambda)| = |\varphi^-(\lambda)| \le \chi_{n-1} + \chi_n + \chi_{n+1}$ and

$$|u_\epsilon(z)| = |u_0(\Omega^-(z))| \le C\exp\pi|\Omega^-(z)| \le C\exp\pi|z|.$$

Therefore $u_\epsilon(z)$ is an entire function with exponential type not exceeding π, and

$$\int_{\mathbb{R}} |u_\epsilon(t+3ai) - u_0(t+3ai)|^2\, dt = \int_{\mathbb{R}} |u_0(\Omega^-(t+3ai)) - u_0(t+3ai)|^2\, dt \le$$

$$\le K \max_{|\Im s|\le 4a} |u_0'(s)|^2 \sum_{n=-\infty}^{\infty} |\chi_{n-1} + \chi_n + \chi_{n+1}|^2 \le K\|\Phi\|^2 \le K^2\epsilon^2$$

which proves the first estimate in (2.5).

To prove the second estimate in (2.5), we assume that the number ϵ is small enough for the $2\rho_n^{1/m_n}$ - neighborhoods of domains $\omega_n^+, n \in \mathcal{N}$, do not intersect. For $n \in \mathcal{N}$ let us choose an arbitrary number $\lambda_n \in \omega^-$ such that $\mathrm{dist}(\lambda_n, \omega_n^+) = \rho_n^{1/m_n}$. If $n \notin \mathcal{N}$, we set $\lambda_n = \mu_{n,0}$ and note that $\|\{\lambda_n - n\}_{n=-\infty}^{\infty}\|_{l^2} \le K$, and $|\lambda_n - t| \ge K|k-n|, t \in \ell_k, n \ne k$. With λ_n being fixed, we set $z_n = \Phi^-(\lambda_n)$ and, according to (2.38), obtain $u_\epsilon(z_n) = u_0(\Omega^-(z_n)) = u_0(\lambda_n)$. Since the function $u_0(\lambda)$ belongs to class $\mathcal{H}$, it has the form $u_0(\lambda) = \cos\pi\lambda + f_0(\lambda), f_0 \in \mathcal{PW}_\pi, \{f_0(n)\}_{n=-\infty}^{\infty} \in \ell^1$. Hence $u_\epsilon(z_n) = \cos\pi\lambda_n + f_0(\lambda_n)$ and $u_\epsilon(z_n) - u_0(z_n) = u_0(\lambda_n) - u_0(z_n) = (\cos\pi\lambda_n - \cos\pi z_n) + (f_0(\lambda_n) - f_0(z_n))$ which yields an estimate

$$|u_\epsilon(z_n) - u_0(z_n)| \le K\left(|\lambda_n - z_n||\lambda_n - n| + |\lambda_n - z_n|^2 + |\lambda_n - z_n| \max_{|t-\mu_{n,0}|\le a} |f_0'(t)|\right).$$

Instead of the first inequality in (2.40) now we have

$$\left\| \frac{1}{2\pi i} \int_{\ell_n} \frac{\varphi_n(t)}{t-\lambda_n}\, dt \right\|_n \le K|\ell_n|\rho_n^{-1/m_n}\|\varphi_n\|_n \le K\|\varphi_n\|_n.$$

Using again (2.36) and the second inequality from (2.40) which is valid for $z = \lambda_n$ we obtain $|\lambda_n - z_n| = |\varphi^-(\lambda_n)| \le K\|\varphi_n\|_n + \chi_n, n \in \mathcal{N}$, and $|\lambda_n - z_n| = \chi_n, n \notin \mathcal{N}$ with $\|\{\chi_n\}_{n=-\infty}^{\infty}\|_{l^2} \le K\epsilon$. These inequalities together with Lemma 2.1 applied to the function $f_0' \in \mathcal{PW}_\pi$ imply an estimate

$$\sum_{n=-\infty}^{\infty} |u_\epsilon(z_n) - u_0(z_n)| \le K\epsilon. \tag{2.43}$$

Besides, due to the choice of λ_n, we have $\|\{z_n - n\}_{n=-\infty}^{\infty}\|_{l^2} \leq K$ with K not depending on ϵ. The second inequality (2.5) now follows Lemma 2.1 applied to $u_\epsilon'(\lambda) - u_0'(\lambda)$ and from (2.43). We arrive at the representation $u_\epsilon(\lambda) = \cos\pi\lambda + f_\epsilon(\lambda), f_\epsilon(\lambda) = f_0(\lambda) + (u_\epsilon(\lambda) - u_0(\lambda))$ which shows that $u_\epsilon \in \mathcal{H}$.

Let us now check that $u_\epsilon(\lambda)$ takes on real values for real λ's.

First we note that if $n \in \mathcal{R}(u_0)\bigcap\mathcal{N}$ and $\lambda \in \ell_n$, then (2.10) holds and according to (2.24) we have $\overline{K_n(z,t)} = -K_n(\bar{z},\bar{t})$. On the other hand, if $n \in \mathcal{N}\backslash\mathcal{R}(u_0)$ and $z,t \in \ell_n$, then $\bar{z},\bar{t} \in \ell_{p(n)}$, $\overline{\alpha_n(\lambda)} = \alpha_{p(n)}(\bar{\lambda})$, and $\overline{K_n(z,t)} = -K_{p(n)}(\bar{z},\bar{t})$. Since $p(p(n)) = n$ and the complex conjugation inverts the orientation on ℓ_n, we find $\overline{\mathbb{R}[\Phi]_n(z)} = \mathbb{R}[\Phi^*]_n(\bar{z})$, $\overline{\mathbb{P}[\Phi]_n(z)} = \mathbb{P}[\Phi^*]_n(\bar{z})$ where $\Phi^*(z) = \overline{\Phi(\bar{z})}$, and $\mathbb{R}$ and $\mathbb{P}$ are defined by (2.25) and (2.27), respectively. Since $\beta^*(z) = \beta(z)$, we obtain from (2.26) $\gamma_n^*(z) = \gamma_n(z)$, and hence the unique solution Φ of (2.35) has the property $\Phi^* = \Phi$. It follows now from (2.36) and (2.37) that $(\Phi^+(z))^* = \Phi^+(z), (\Phi^-(z))^* = \Phi^-(z)$, which means that $\Phi^+(z)$ and $\Phi^-(z)$ are real for real λ's. Therefore $u_\epsilon(\lambda)$ is also real for real λ's.

Let $\tilde{\mu}_n$ be a critical point of $u_\epsilon(\lambda)$. According to (2.14) and (2.38), either $u_0'(\Omega^-(\tilde{\mu}_n)) = 0$ or $u_0'(\Omega^+(\tilde{\mu}_n)) = 0$, depending on either $\tilde{\mu}_n \in \tilde{\omega}^-$ or $\tilde{\mu}_n \in \tilde{\omega}^+$, respectively. In the former case $u_\epsilon(\tilde{\mu}_n) = u_0(\mu_{n,0}) = \gamma_{n,0}$, while in the latter case $u_\epsilon(\tilde{\mu}_n) = \gamma_{n,0} + \sigma_n = \gamma_n$, which proves that the sequence $\mathcal{CR}(u_\epsilon)$ of critical values of $u_\epsilon(\lambda)$ coincides with $\{\gamma_n\}_{n=-\infty}^{\infty}$.

According to our previous assumption, domains ω_n^+ are disjoint. In particular, it implies that if $n \notin \mathcal{R}(u_0)$, i.e., $\mu_{n,0}$ is not real, then $\operatorname{dist}(\omega_n^+, \mathbb{R}) > 0$. There exists only finite number of such $\mu_{n,0}$ and for all of them and all sufficiently small $\epsilon > 0$ we have $\Im(\Phi^\pm(\mu_{n,0})) \neq 0$. Therefore, if $\tilde{\mu}_n$ is a real critical point of $u_\epsilon(\lambda)$, then $n \in \mathcal{R}(u_0), \tilde{\mu}_n = \Phi^\pm(\mu_{n,0})$ with real $\mu_{n,0}$, and $|u_\epsilon(\tilde{\mu}_n)| = |\gamma_n| \leq 1$. Hence $u_\epsilon^2(\lambda) \leq 1$ for $\lambda \in \mathbb{R}$ which completes the proof of Theorem 2.1.

Remark 1. Let $\{\mu_{n,0}^\pm\}_{n=-\infty}^{\infty}$ and $\{\mu_n^\pm\}_{n=-\infty}^{\infty}$ be the zero sequences of $1 - u_0^2(\lambda)$ and $1 - u^2(\lambda)$, respectively. Later, in the proof of Theorem 1.2, we will need an estimate

$$\|\{\mu_{n,0}^\pm - \mu_n^\pm\}_{n=-\infty}^{\infty}\|_{l^2} \leq K\epsilon \tag{2.44}$$

for a particular case of the sequence $\{\gamma_n\}_{n=-\infty}^{\infty}$ with

$$\gamma_n = \begin{cases} \gamma_{n,0} & |n| \leq M \\ (-1)^n & |n| > M. \end{cases} \tag{2.45}$$

To prove it we note that $\mu_n^\pm = \Phi^-(\mu_{n,0}^\pm)$ if $n \notin \mathcal{N}$, and $\mu_n^+ = \mu_n^- = \mu_n = \Phi^+(\mu_{n,0})$ if $n \in \mathcal{N}$. Since $|\mu_{n,0}^\pm - \mu_{n,0}|^2 \leq K|(-1)^n - \gamma_{n,0}|$, we have for $n \in \mathcal{N}$ an estimate

$$|\mu_n^\pm - \mu_{n,0}^\pm| \leq |\mu_n^\pm - \mu_{n,0}| + |\mu_{n,0} - \mu_{n,0}^\pm| \leq K(|\varphi^+(\mu_{n,0})| + |\sigma_n|^{1/2}).$$

For $t \in \ell_k, k \notin \mathcal{N}$, we have $|t - \mu_{n,0}| \geq K|k-n|$ and, similar to (2.41), $|\varphi^-(\mu_{n,0}^\pm)| \leq \tau_n$ for $n \notin \mathcal{N}$, and $|\varphi^+(\mu_{n,0})| \leq K\|\zeta_n\|_n + \tau_n$ for $n \in \mathcal{N}$ with $\|\{\tau_n\}_{n=-\infty}^{\infty}\|_{l^2} \leq K\epsilon$. It follows from (2.13), (2.20) and (2.37)

$$\left(\sum_{n=-\infty}^{\infty} \|\zeta_n\|_n^2\right)^{1/2} \leq K\epsilon$$

which yields (2.44).

Remark 2. Since $\mu_{n,0} - n \to 0$ as $|n| \to \infty$, for all sufficiently large M and $|n| > M$ every disc $\{\lambda : |\lambda - n| \le 10^{-4}\}$ contains domain ω_n^+ . If for such n we have $|\lambda - n| = 10^{-3}$, then $\operatorname{dist}(\lambda, \ell_n) \ge 9 \times 10^{-4}$, and similar to (2.42) an estimate holds

$$|\varphi^-(\lambda)| \le K\epsilon, \qquad |\lambda - n| = 10^{-3}. \tag{2.46}$$

3. An auxiliary operator

Let an operator $L_0 \in \mathcal{D}_-$ be given, and let $c_0(\lambda) = \cos\lambda\pi + f_0(\lambda)$ and $s_0(\lambda) = \sin\lambda\pi + g_0(\lambda)$ be the entries of its monodromy matrix. According to (1.7), the identity $c_0(\lambda)c_0^*(\lambda) + s_0(\lambda)s_0^*(\lambda) \equiv 1$ holds. If $\theta_0(\lambda) = c_0(\lambda)c_0^*(\lambda)$, and $u_0(\lambda) = 2\theta_0(\lambda/2) - 1$, then $0 \le \theta_0(\lambda) \le 1, |u(\lambda)| \le 1$ for real λ's and $u_0(\lambda) = \cos\lambda\pi + F_0(\lambda)$ where

$$F_0(\lambda) = 2(f_0(\lambda/2) + f_0^*(\lambda/2))\cos\lambda\pi/2 + 2f_0(\lambda/2)f_0^*(\lambda/2).$$

The function $\theta_0(\lambda)$ takes on real values at real λ's, and it follows from part $i)$ of Theorem 1.1 that $F_0 \in \mathcal{PW}_\pi$. For every integer n we have

$$F_0(2n+1) = 2f_0(n+1/2)f_0^*(n+1/2), \quad F_0(2n) = 2(-1)^n(f_0(n) + f_0^*(n)) + 2f_0(n)f_0^*(n).$$

According to Lemma 2.3, $\Delta_0(\lambda) \equiv (c_0(\lambda)+c_0^*(\lambda))/2 \in \mathcal{H}$ and therefore $\{f_0(n)+f_0^*(n)\}_{n=-\infty}^{\infty} \in \ell^1$. In addition, Lemma 2.1 applied to the function $f_0(\lambda/2)$ shows that $\{f_0(n+1/2)f_0^*(n+1/2)\}_{n=-\infty}^{\infty} \in \ell^1$, $\{f_0(n/2)f_0^*(n/2)\}_{n=-\infty}^{\infty} \in \ell^1$ and we conclude that $u_0 \in \mathcal{H}$.

Let, as before, $\mathcal{CR}(u_0) = \{\gamma_{n,0}\}_{n=-\infty}^{\infty}$ be the sequence of all critical values of $u_0(\lambda)$. Given a number $\epsilon > 0$, let us choose $M = M(\epsilon)$ such that

$$\left(\sum_{|n|\ge M} |(-1)^n - \gamma_{n,0}|\right)^{1/2} \le \epsilon, \tag{3.1}$$

and define the sequence $\Gamma = \{\gamma_n\}_{n=-\infty}^{\infty} \in \mathcal{G}(u_0)$ by (2.45). According to (3.1), the estimate (2.4) holds and by Theorem 2.1 there exists a function $u_\epsilon \in \mathcal{H}$ for which Γ is the sequence of critical values and (2.5) is satisfied.

According to (3.1), all critical points of $u_\epsilon(\lambda)$, with possible exception of finitely many of them, are real and simple, and the corresponding critical values are ± 1. It means that $h(\lambda) = (u_\epsilon'(\lambda))^2(1 - u_\epsilon^2(\lambda))^{-1}$ is a meromorphic function with real values on the real line. Moreover, (2.1) shows that $h(\lambda)$ is a rational function and $\lim_{|\lambda|\to\infty} h(\lambda) = \pi^2$. The function $\sqrt{h(\lambda)}$ is continued as a single-valued analytic function outside some disc $\{\lambda : |\lambda| \le r\}$, and the representation holds

$$\sqrt{h(\lambda)} = \pi + \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \qquad |\lambda| > r,$$

with real numbers $a_k, k \ge 1, |a_k| \le r^k$. On the other hand, for real λ's we have

$$\sqrt{h(\lambda)} = \left| \frac{u_\epsilon'(\lambda)}{\sqrt{1 - u_\epsilon^2(\lambda)}} \right|$$

and

$$\int_{\lambda_0}^{\lambda} \sqrt{h(t)}dt = \begin{cases} 2k\pi + c_0 + \arccos u_\epsilon(\lambda) & \mu_{2k} \le \lambda \le \mu_{2k+1} \\ (2k+2)\pi + c_0 - \arccos u_\epsilon(\lambda) & \mu_{2k+1} \le \lambda \le \mu_{2k+2} \end{cases}$$

where $\{\mu_n\}_{n=-\infty}^{\infty}$ is the sequence of critical points of $u_\epsilon(\lambda)$, and $\arccos t \in [0, \pi]$. Hence

$$u_\epsilon(\lambda) = \cos \pi \left(\lambda + a_1 \ln \lambda + b_0 + \sum_{n=1}^{\infty} \frac{b_k}{\lambda^k} \right), \qquad \lambda > r. \tag{3.2}$$

Since $u_\epsilon(\lambda)$ is an entire function, we conclude that $a_1 = 0$, and comparing (3.2) to (2.1) we find $\cos \pi b_0 = 1$. Therefore $u_\epsilon(\lambda) = \cos \pi(\lambda + S(\lambda))$ where $S(\lambda)$ is analytic in $\{\lambda : |\lambda| > r\}$, real for real λ's, and $\lim_{\lambda\to\infty} S(\lambda) = 0$.

Let us now set

$$\theta_\epsilon(\lambda) = \frac{1 + u_\epsilon(2\lambda)}{2}, \qquad \psi_\epsilon(\lambda) = \frac{1 - u_\epsilon(2\lambda)}{2}. \tag{3.3}$$

The zero set $\{\mu_n^\pm\}_{n=-\infty}^{\infty}$ of $\theta_\epsilon(\lambda)$ coincides with the set of points at which $u_\epsilon(2\lambda)$ takes on value -1. It means that $\{2\mu_n^\pm\}_{n=-\infty}^{\infty}$ is the set of points at which $u_\epsilon(\lambda)$ takes on the value -1.

According to the construction of the function $u_\epsilon(\lambda)$ and Remark 1 to Theorem 2.1 there exists a factorization $\theta_\epsilon(\lambda) = c_\epsilon(\lambda)c_\epsilon^*(\lambda)$ with $c_\epsilon(\lambda) = \cos \pi\lambda + f_\epsilon(\lambda)$ such that

$$\left(\int_{-\infty}^{\infty} |c_0(\lambda) - c_\epsilon(\lambda)|^2 \, d\lambda \right)^{1/2} \le K\epsilon. \tag{3.4}$$

Similar arguments yield factorization $\psi_\epsilon(\lambda) = s_\epsilon(\lambda)s_\epsilon^*(\lambda)$ with $s_\epsilon(\lambda) = \sin \pi\lambda + g_\epsilon(\lambda)$ such that

$$\left(\int_{-\infty}^{\infty} |s_0(\lambda) - s_\epsilon(\lambda)|^2 \, d\lambda \right)^{1/2} \le K\epsilon, \tag{3.5}$$

and we obtain the matrix $U_\epsilon(\lambda)$ of the form (1.5) generated by the functions $c(\lambda) = c_\epsilon(\lambda)$ and $s(\lambda) = s_\epsilon(\lambda)$. It follows now

$$\theta_\epsilon(\lambda) = \cos^2 \pi(\lambda + S(\lambda)), \qquad \psi_\epsilon(\lambda) = \sin^2 \pi(\lambda + S(\lambda)), \qquad |\lambda| > r,$$

with $R(\lambda) = S(2\lambda)/2$. Hence

$$c_\epsilon(\lambda) = G(\lambda) \cos \pi(\lambda + S(\lambda)), \qquad c_\epsilon^*(\lambda) = G^*(\lambda) \cos \pi(\lambda + S(\lambda)),$$

where $G(\lambda)$ is analytic in $\{\lambda : |\lambda| > r\}$, $G(\lambda)G^*(\lambda) \equiv 1$ and $\lim_{\lambda\to\infty}(G(\lambda) - 1) = 0$. Therefore, $G(\lambda) = 1 + ic_1/\lambda + c_2/\lambda^2 + \ldots$ with some real number c_1. Now we set $\Delta_\epsilon(\lambda) = (c_\epsilon(\lambda) + c_\epsilon^*(\lambda))/2, v_\epsilon(\lambda) = (c_\epsilon(\lambda) - c_\epsilon^*(\lambda))/2i$ and and since $\Delta_\epsilon^2(\lambda) \le 1$ for $\lambda \in \mathbb{R}$, we obtain

$$\Delta_\epsilon(\lambda) = \frac{G(\lambda) + G^*(\lambda)}{2} \cos \pi(\lambda + S(\lambda)) = (1 - \frac{C_1^2}{2\lambda^2} + \ldots) \cos \pi(\lambda + S(\lambda)) \tag{3.6}$$

$$v_\epsilon(\lambda) = \frac{G(\lambda) - G^*(\lambda)}{2i} \cos \pi(\lambda + S(\lambda)) = (\frac{c_1}{\lambda} + \ldots) \cos \pi(\lambda + S(\lambda)) \tag{3.7}$$

with a real C_1.

Assume now that $c_\epsilon(\lambda) \neq c_\epsilon^*(\lambda)$. In this case there exists a non-real zero $\nu = x+iy, y \neq 0$, of $c_\epsilon(\lambda)$. It follows from the representation

$$c_\epsilon(\lambda) = \text{const} \prod_{n=-\infty}^{\infty} \frac{\lambda_n - \lambda}{n+1/2}$$

that for any real $y' \neq 0, |y'| < |y|$, the function $c(\lambda) = c_\epsilon(\lambda)(\lambda-\nu')(\lambda-\nu)^{-1}, \nu = x+iy'$, satisfies $|c(\lambda)| < 1, |c(\lambda)| \leq |c_\epsilon(\lambda)|$, and hence $1 - c(\lambda)c^*(\lambda) \geq 1 - c_\epsilon(\lambda)c_\epsilon^*(\lambda) \geq 0$ for real λ's. We conclude that there exists an entire function $s(\lambda)$ such that $c(\lambda)c^*(\lambda) + s(\lambda)s^*(\lambda) \equiv 1$, and for all sufficiently small values of $|y-y'|$ we have $\|c - c_0\|_{\mathcal{PW}} \leq K\epsilon$, $\|s - s_0\|_{\mathcal{PW}} \leq K\epsilon$. Moreover, since $\Re(\nu - \nu') = 0$, we have

$$c(\lambda) = G(\lambda)\frac{\lambda - \nu'}{\lambda - \nu}\cos\pi(\lambda + S(\lambda)) = (1 + \frac{ic}{\lambda} + \ldots)\cos\pi(\lambda + S(\lambda))$$

where $c = c_1 + 2\Im(\nu - \nu')$, and with a proper choice of ν' we can assume that $c \neq 0$. Similar to (3.6) and (3.7), we set

$$\Delta(\lambda) \equiv \frac{c(\lambda) + c^*(\lambda)}{2} = (1 - \frac{C^2}{2\lambda^2} + \ldots)\cos\pi(\lambda + S(\lambda)) \tag{3.8}$$

$$v(\lambda) = \frac{c(\lambda) - c^*(\lambda)}{2i} = (\frac{c}{\lambda} + \ldots)\cos\pi(\lambda + S(\lambda)), \tag{3.9}$$

and for real λ's we obtain

$$0 \leq 1 - \Delta^2(\lambda) - v^2(\lambda) = 1 - \left(1 - \frac{C^2}{\lambda^2} + \ldots\right)\cos^2\pi(\lambda + S(\lambda)) - \left(\frac{c^2}{\lambda^2} + \ldots\right)\cos^2\pi(\lambda + S(\lambda))$$

which implies $C^2 \geq c^2 > 0$.

The pair of functions $c(\lambda)$ and $s(\lambda)$ defines, according to Theorem 1.3, the operator $L \in \mathcal{D}_-$ with the Hill discriminant $\Delta(\lambda)$. Since $|c(\lambda)| < 1$ for real λ's, we have $|\Delta(\lambda)| < 1$ for such λ's, which means that all possible spines in the spectrum of L do not degenerate.

4. Proof of Theorem 1.2

The most simple is to prove Theorem 1.2 if the function $c_0(\lambda) \equiv c(\lambda)$ in (1.5) is real for real λ's. In this case $\Delta_0(\lambda) = c_0(\lambda) = c_0^*(\lambda), v_0(\lambda) = 0$, and $\Delta_0^2(\lambda) + s_0(\lambda)s_0^*(\lambda) \equiv 1$. Given a fixed sufficiently small $\epsilon > 0$, we choose sufficiently big integer $M > 0$ to satisfy inequality (3.1) where $\Gamma = \{\gamma_{n,0}\}_{n=-\infty}^{\infty}$ is the sequence of critical values of $\Delta_0(\lambda)$, and using Theorem 2.1 find a function $\Delta_\epsilon \in \mathcal{H}, \|\Delta_\epsilon - \Delta_0\|_{\mathcal{PW}} \leq K\epsilon$, with all critical values equal to ± 1 except finitely many of them. Using Remark 1 to Theorem 2.1 we obtain representation $1 - \Delta_\epsilon^2(\lambda) = s_\epsilon(\lambda)s_\epsilon^*(\lambda)$ with $\|s_\epsilon - s_0\|_{\mathcal{PW}} \leq K\epsilon$. According to Theorem 1.3, the pair $c_\epsilon(\lambda), s_\epsilon(\lambda)$ with $c_\epsilon(\lambda) \equiv \Delta_\epsilon(\lambda)$ generates the operator $L_\epsilon \in \mathcal{D}_-$ with the finite-band spectrum and potential matrix $Q_\epsilon(x)$ satisfying $\|Q_0 - Q_\epsilon\|_{\mathcal{L}^2_{2,2}(0,\pi)} \leq K\epsilon$.

Let now $c_0(\lambda) \neq c_0^*(\lambda)$. Then there exists at least one non-real zero of $c_0(\lambda)$ and according to Sec. 3 we can assume that for an initial operator $L_0 \in \mathcal{D}_-$ the representations

$$\Delta_0(\lambda) \equiv \frac{c_0(\lambda) + c_0^*(\lambda)}{2} = (1 - \frac{C^2}{2\lambda^2} + \ldots)\cos\pi(\lambda + S(\lambda)) \tag{4.1}$$

$$v_0(\lambda) = \frac{c_0(\lambda) - c_0^*(\lambda)}{2i} = (\frac{c}{\lambda} + \ldots)\cos\pi(\lambda + S(\lambda)) \tag{4.2}$$

are valid with $C^2 \geq c^2 > 0$. In addition, $|\Delta_0(\lambda)| < 1$ for real λ's.

In what follows we will use the following proposition.

Lemma 4.1. *Let $v(\lambda)$ be an entire function of exponential type π, bounded on the real line, whose zeros $\{\nu_{n,0}\}_{n=-\infty}^{\infty}$ are real, except finitely many of them, and satisfy relation $\|\nu_{n,0} - n - 1/2\|_{l^2} < \infty$, and let $\{\mu_{n,0}\}_{n=-\infty}^{\infty}$ be a sequence of numbers which are real, except finitely many of them, and satisfy $\|\mu_{n,0} - n\|_{l^2} < \infty$. Denote by M_0 a number such that*

$$\sum_{|n|>M_0} |\mu_{n,0} - n|^2 < 10^{-6}$$

and define

$$v_M(\lambda) = v(\lambda) \prod_{|n|>M} \frac{\mu_n - \lambda}{\nu_{n,0} - \lambda},$$

where M is an integer, $M > M_0$, the numbers μ_n are real, and

$$\sum_{|n|>M_0} |\mu_n - \mu_{n,0}|^2 < 10^{-6}.$$

Then $|v_M(\lambda)| \leq K$ for all real λ's with K not depending either on M or the set $\{\mu_n\}_{|n|\geq M}$.

Proof. Without loss of generality we can assume that M_0 is sufficiently large integer and $\mu_n < \nu_{n,0} < \mu_{n+1}, \nu_{-n,0} < \mu_{-n} < \nu_{-n+1,0}, n \geq M_0$. If U is the 10^{-3}-neighborhood of the zero set of $v(\lambda)$, then there exists a constant $K > 0$ such that

$$K^{-1}e^{\pi|\Im\lambda|} \leq |v(\lambda)| \leq Ke^{\pi|\Im\lambda|}, \qquad \lambda \notin U.$$

Since $|\mu_n - n| < 2 \times 10^{-3}$, the lines $\{\lambda : |\Re\lambda| = M + 1/4\}$ do not intersect U, and since $|\mu_n - \lambda| < |\nu_{n,0} - \lambda|$ for $\{\lambda : |\Re\lambda| = M + 1/4\}$ and $|n| > M$, we obtain

$$|v_M(\lambda)| \leq Ke^{\pi|\Im\lambda|} \prod_{|n|>M} \left|\frac{\mu_n - \lambda}{\nu_{n,0} - \lambda}\right| \leq Ke^{\pi|\Im\lambda|}, \qquad |\Re\lambda| = M + 1/4.$$

By the Phragmen-Lindelöf Theorem we obtain

$$|v_M(\lambda)| \leq Ke^{\pi|\Im\lambda|}, \qquad |\Re\lambda| \leq M + 1/4.$$

Denote $s_M(\lambda)$ the entire function

$$s_M(\lambda) = \lambda \prod_{0<|n|\leq M} \frac{n - \lambda}{n} \prod_{|n|>M} \frac{\mu_n - \lambda}{n}.$$

Since $|\mu_n - n| \leq 2 \times 10^{-3}$ there exists a constant K not depending on either M or $\{\mu_n\}_{n=-\infty}^{\infty}$ such that

$$K^{-1}e^{\pi|\Im\lambda|} \leq |s_M(\lambda)| \leq Ke^{\pi|\Im\lambda|}, \qquad \lambda \notin V,$$

where V is the 10^{-2}-neighborhood of the zero set of $s_M(\lambda)$. Now we have $|v_M(\lambda)s_M^{-1}(\lambda)| \leq K$, $|\Re\lambda| = M + 1/4$. Since $v_M(\lambda)s_M^{-1}(\lambda)$ is analytic and bounded in $\{\lambda : |\Re\lambda| \geq M + 1/4\}$, the same Phragmen-Lindelöf Theorem shows that estimate $|v_M(\lambda)| \le K$ is valid for all real λ's, which proves Lemma 4.1.

As soon as an operator $L_0 \in \mathcal{D}_-$ is given, the functions $\Delta_0(\lambda), v_0(\lambda)$ and $s_0(\lambda)$ are fixed, as well as some numbers are fixed related to them. Namely, if λ is real, then $\Delta_0(\lambda) = \cos\lambda\pi + o(1), \Delta_0''(\lambda) = -\pi^2\cos\lambda\pi + o(1)$ as $|\lambda| \to \infty$, and since the elementary estimates $|\sin\lambda\pi| \ge \pi 20^{-1}(1 - \pi^2 2^{-1}20^{-2}), 20^{-1} \le |\lambda| \le 2^{-1}; |\cos\lambda\pi| \ge (1 - \pi^2 2^{-1}10^{-2}), |\lambda| \le 10^{-1}$ are valid, there exists N_0 such that for $|\lambda| \ge N_0$ the estimates

$$1 - \Delta_0^2(\lambda) \ge \pi^2 20^{-2}(1 - \pi^2 20^{-2}), \quad 2^{-1} \ge |\lambda - n| \ge 20^{-1}, \quad |n| > N_0, \tag{4.3}$$

$$|\Delta_0''(\lambda)| \ge \pi^2(1 - 3\pi^2 2^{-2}10^{-2}), \quad |\lambda - n| \le 10^{-1}, \quad |n| > N_0, \tag{4.4}$$

are fulfilled. Furthermore, if $\{\mu_{n,0}\}_{n=-\infty}^{\infty}$ is the sequence of all critical points of $\Delta_0(\lambda)$, and $\{\mu_{n,0}^{\pm}\}_{n=-\infty}^{\infty}$ is the sequence of all zeros of $1 - \Delta_0^2(\lambda)$, then we can assume $|\mu_{n,0} - n| < 10^{-4}, |\mu_{n,0}^{\pm} - n| < 10^{-4}$, $|n| \ge N_0$. In addition, fixed are all numbers K from Theorem 2.1 with $u_0(\lambda) = \Delta_0(\lambda)$, the number K from (2.44) and (2.46), the numbers M_0 and K from Lemma 4.1 with $v(\lambda) = v_0(\lambda)$, $\{\nu_{n,0}\}_{n=-\infty}^{\infty}$ being the zero sequence of $v_0(\lambda)$ and $\{\mu_{n,0}\}_{n=-\infty}^{\infty}$ being the sequence of critical points of $1-\Delta_0^2(\lambda)$. The first inequality in (2.5) implies $|\Delta_\epsilon(\lambda) - \Delta_0(\lambda)| \le K\epsilon$ and (4.2) implies $|(\lambda + i)v_0(\lambda)| \le K$ for all real λ's. Of course we can assume that all these K's are equal and $N_0 > 10^2\pi K$.

From now on we fix sufficiently small $\epsilon > 0$ such that $\pi^2 K\epsilon < 10^{-4}, \epsilon^{-2} > M_0 + N_0$, the inequalities (2.5) are fulfilled with $u_0(\lambda) = \Delta_0(\lambda)$, and using (4.1) and (4.2) find an integer $m > \epsilon^{-2} + 10^3 K$ such that

$$\min_{\Im\lambda=0, |\lambda-n|\le 1/2} (1 - \Delta_0^2(\lambda)) \ge \frac{C^2(1-\epsilon)}{n^2}, \qquad |n| \ge m, \tag{4.5}$$

while

$$\max_{\Im\lambda=0, |\lambda-n|\le 1/2} v_0^2(\lambda) \le \frac{c^2(1+\epsilon)}{n^2}, \qquad |n| \ge m. \tag{4.6}$$

If necessary, we can increase m in such a way that $1 - \Delta_0^2(\lambda)$ takes its minimal value on the interval $I_n = \{\lambda : |\lambda - n| \le 1/2\}, |n| \ge m$, at a point $\chi_{n,0}$ with $|\chi_{n,0} - n| < 10^{-1}$. Finally we assume

$$\left(\int_{|\lambda|\ge m} |v_0(\lambda)|^2 \, d\lambda \right)^{1/2} \le \epsilon. \tag{4.7}$$

With m being fixed, we choose an arbitrary integer $M > \epsilon^{-1}m$ satisfying (3.1), define $\{\gamma_n\}_{n=-\infty}^{\infty}$ according to (2.45) and use Theorem 2.1 with $u_0(\lambda) = \Delta_0(\lambda)$ to obtain the function $\Delta_\epsilon \in \mathcal{H}$, $\|\Delta_\epsilon - \Delta_0\|_{\mathcal{PW}} \le K\epsilon$, with all critical values $\gamma_n = (-1)^n, |n| > M$. Let us now construct a function $v_\epsilon(\lambda)$, real for real λ's, satisfying $\|v_\epsilon - v_0\|_{\mathcal{PW}} \le K\epsilon$ and such that

$$1 - \Delta_\epsilon^2(\lambda) - v_\epsilon^2(\lambda) \ge 0, \quad \Im\lambda = 0. \tag{4.8}$$

To this end, let us set $w_\epsilon(\lambda) = (1 - 2\epsilon)v_0(\lambda)$. Since $\Delta_0^2(\lambda) < 1$ and $1 - \Delta_0^2(\lambda) - v_0^2(\lambda) \ge 0$ for all real λ's, the strong inequality $1 - \Delta_0^2(\lambda) - w_\epsilon^2(\lambda) > 0$ holds for all $\epsilon > 0$. As $M \to +\infty$ the sequence of functions $\Delta_\epsilon(\lambda)$ converges to $\Delta_0(\lambda)$ uniformly on compact sets, and we fix sufficiently large M for the inequality $1-\Delta_\epsilon^2(\lambda)-w_\epsilon^2(\lambda)>0$ to be valid for $\lambda\in[-m,m]$. Let us prove the same inequality for $\lambda\in[-M-1+10^{-1},M+1-10^{-1}]$.

Assume that $|\lambda-n|=10^{-3},|n|>M$. Since $K\epsilon\le 10^{-4}$, then according to Remark 2 to Theorem 2.1 we have (in notations of Sec.2) $|\varphi^-(\lambda)|\le 10^{-4}$. Hence $|\Phi^-(\lambda)-n|\le|\lambda-n|+|\phi^-(\lambda)|\le 10^{-2}$, and if now $|\lambda-n|=10^{-1}$, then $\lambda\in\tilde\omega^-$. We conclude that the set

$$\mathbb{C}\setminus\bigcup_{|n|>M}\{\lambda:|\lambda-n|<10^{-1}\} \tag{4.9}$$

is contained inside the set $\tilde\omega^-$. According to (2.38), if λ belongs to this set, then $\Delta_\epsilon(\lambda)=\Delta_0(\Omega_-(\lambda))$, which implies that $1-\Delta_\epsilon^2(\lambda)$ takes its minimal value $1-\Delta_0^2(\chi_{n,0})$ on an interval $I_n,m\le|n|\le M$ at its inner point $\chi_n=\Phi^-(\chi_{n,0})$, and its minimal value on $\{\lambda:10^{-1}\le\lambda+M+1\le 2^{-1}\}$ and $\{\lambda:-2^{-1}\le\lambda-M-1\le-10^{-1}\}$ is not less than $1-\Delta_0^2(\chi_{-M-1,0})$ and $1-\Delta_0^2(\chi_{M+1,0})$, respectively. It follows now from (4.5) that if either $\lambda\in I_n,m\le|n|\le M$, or $\lambda\in[-M-1+10^{-1},-M-2^{-1}]\cap I_{-M-1}$ or $\lambda\in[M+2^{-1},M+1-10^{-1}]\cap I_{M+1}$, then $1-\Delta_\epsilon^2(\lambda)\ge C^2(1-\epsilon)n^{-2}$. We compare this estimate to (4.6) and since $C^2(1-\epsilon)\ge c^2(1+\epsilon)(1-2\epsilon)^2$, we find that the inequality

$$1-\Delta_\epsilon^2(\lambda)-w_\epsilon^2(\lambda)\ge 0,\qquad \lambda\in[-M-1+10^{-1},M+1-10^{-1}], \tag{4.10}$$

is valid.

As a matter of fact, this inequality cannot hold for all λ's with $|\lambda|\ge M$ as requested by (4.8). Indeed, let $\{\mu_n^\pm\}_{n=-\infty}^\infty$ be the zero sequence of $1-\Delta_\epsilon^2(\lambda)$, and let again $\{\nu_{n,0}\}_{n=-\infty}^\infty$ be the zero sequence of both $v_0(\lambda)$ and $w_\epsilon(\lambda)$. According to (2.45), $\mu_n^+=\mu_n^-,|n|>M$, and it follows from (3.8) and (3.9) that $\mu_n^\pm=n+o(1)$, $\nu_{n,0}=n+1/2+o(1)$. Hence

$$\nu_{-n,0}<\mu_{-n+1}^+=\mu_{-n}^-<\nu_{-n,0},\qquad \nu_{n-1,0}<\mu_n^+=\mu_n^-<\nu_{n,0},\qquad n\ge M,$$

and $1-\Delta_\epsilon^2(\lambda)=0$ at points close to $\lambda=n$, while zeros of $w_\epsilon(\lambda)$ are close to $\lambda=n\pm 1/2$. Therefore (4.10) cannot be true for all sufficiently big real λ's.

To obtain (4.8) we introduce instead of $w_\epsilon(\lambda)$ the function

$$v_\epsilon(\lambda)=w_\epsilon(\lambda)\prod_{|n|>M}\frac{\lambda-\mu_n^+}{\lambda-\nu_{n,0}}\,. \tag{4.11}$$

In other words, to define $v_\epsilon(\lambda)$, we move all zeros of $w_\epsilon(\lambda)$ located in $\{\lambda:|\lambda|>M\}$ along the real axis in the direction of $\lambda=0$ until they meet their next neighbor from the zero set of $1-\Delta_\epsilon^2(\lambda)$. Let us now prove that (4.8) holds. From now on we assume that λ is real.

If $-M-1\le\lambda\le M+1$ and $|n|\ge M+1$, then $|\lambda-\mu_n^+||\lambda-\nu_{n,0}|^{-1}<1$, and for such λ we have $|v_\epsilon(\lambda)|\le|w_\epsilon(\lambda)|$. Using (4.10) we obtain

$$1-\Delta_\epsilon^2(\lambda)-v_\epsilon^2(\lambda)\ge 1-\Delta_\epsilon^2(\lambda)-w_\epsilon^2(\lambda)\ge 0,\qquad \lambda\in[-M-1+10^{-1},M+1-10^{-1}].$$

For $|n|\ge M+1$ we have $1-\Delta_\epsilon^2(\mu_n^+)=v_\epsilon^2(\mu_n^+)=0,\Delta_\epsilon(\mu_n^+)\Delta_\epsilon'(\mu_n^+)=v_\epsilon(\mu_n^+)v_\epsilon'(\mu_n^+)=0,(1-\Delta_\epsilon^2(\mu_n^+))''=-2\Delta_\epsilon(\mu_n^+)\Delta_\epsilon''(\mu_n^+)$. Since $\mu_n^+=\Phi^+(\mu_{n,0})$, we find using (2.44) that $|\mu_n^+-n|\le|\mu_{n,0}^+-\mu_n^+|+|\mu_{n,0}^+-n|\le K\epsilon+10^{-4}\le 10^{-3}$ which permits us to use (4.4) with $\lambda=\mu_n^+$. By virtue of the S. Bernstein theorem [7] we have $|\Delta_\epsilon''(\lambda)-\Delta_0''(\lambda)|\le\pi^2K\epsilon\le 10^{-4}$, and using (4.4) we obtain the estimate $|\Delta_\epsilon''(\mu_n^+)| \geq |\Delta_0''(\mu_n^+)| - |\Delta_\epsilon''(\mu_n^+) - \Delta_0''(\mu_n^+)| \geq \pi^2(1 - 3\pi^2 2^{-1}10^{-2}), |n| \geq m$. On the other hand, by the same S. Bernstein theorem we have $|(1-\Delta_\epsilon^2(\lambda))'''| \leq 8\pi^3$ for all real λ's. Using the Taylor formula, we obtain

$$1 - \Delta_\epsilon^2(\lambda) = \left| -\Delta_\epsilon(\mu_n^+)\Delta_\epsilon''(\mu_n^+)(\lambda - \mu_n^+)^2 + \frac{1}{2}\int_{\mu_n^+}^{\lambda} (1 - \Delta_\epsilon^2(t))'''(\lambda - t)^2\, dt \right| \geq 5^{-1}\pi^2|\lambda - \mu_n^+|^2$$
$$|\lambda - \mu_n^+| \leq 10^{-1}, \ |n| > m.$$

Since the function $(\lambda + i)w_\epsilon(\lambda)$ and the sequences $\{\nu_{n,0}\}_{n=-\infty}^{\infty}$, $\{\mu_{n,0}\}_{n=-\infty}^{\infty}$, and $\{\mu_n^+\}_{n=-\infty}^{\infty}$ (the latter being in the capacity of $\{\mu_n\}_{n=-\infty}^{\infty}$) satisfy conditions of Lemma 4.1, and since $v_\epsilon(\lambda)$ is defined by (4.11), we have $|(\lambda + i)v_\epsilon(\lambda)| \leq K$. Hence $|v_\epsilon''(\lambda)| \leq 4\pi^2 K(|\lambda| + 1)^{-1}$ for real λ's and

$$v_\epsilon^2(\lambda) \leq \frac{(2K\pi^2)^2}{m^2}|\lambda - \mu_n^+|^2 \leq \pi^2 10^{-1}|\lambda - \mu_n^+|^2 \leq 1 - \Delta_\epsilon^2(\lambda), \qquad |\lambda - \mu_n^+| \leq 10^{-1}, \ |n| > M.$$

If $|\lambda - n| \leq 20^{-1}$ and $|n| > M$, then $|\lambda - \mu_n^+| \leq |\lambda - n| + |\mu_n^+ - n| \leq 20^{-1} + 2 \times 10^{-3} < 10^{-1}$, and according to the previous inequality we obtain $1 - \Delta_\epsilon^2(\lambda) - v_\epsilon^2(\lambda) \geq 0$. On the other hand, if $2^{-1} \geq |\lambda - n| \geq 20^{-1}, |n| > M$, then according to (4.3) we have $1 - \Delta_\epsilon^2(\lambda) \geq |1 - \Delta_0^2(\lambda)| - |\Delta_\epsilon^2(\lambda) - \Delta_0^2(\lambda)| \geq \pi^2 20^{-2}(1 - \pi^2 20^{-2}) - 2K\epsilon \geq \pi^2 2^{-2} 20^{-2} > 10^{-4}, |v_\epsilon(\lambda)|^2 \leq K^2(|\lambda| + 1)^{-2} \leq K^2 m^{-2} \leq 10^{-4} \leq 1 - \Delta_\epsilon^2(\lambda)$, which completes the proof of (4.8) for all $\lambda \in \mathbb{R}$.

Let us now prove an estimate

$$\|v_\epsilon - v_0\|_{PW} \leq K\epsilon. \tag{4.12}$$

It is sufficient to prove an inequality $\|v_\epsilon - w_\epsilon\|_{PW} \leq K\epsilon$. For $\lambda \in [-m, m]$ we have

$$|(\lambda + i)(v_\epsilon(\lambda) - w_\epsilon(\lambda))| = \left| (\lambda + i)w_\epsilon(\lambda)\left(1 - \exp \sum_{|n|>M} \ln\left(1 + \frac{\nu_{n,0} - \mu_n^+}{\lambda - \nu_{n,0}}\right)\right)\right|.$$

Since, for such λ's and $|n| > M$,

$$\left|\frac{\nu_{n,0} - \mu_n^+}{\lambda - \nu_{n,0}}\right| \leq \frac{2}{M - m} < \frac{1}{4},$$

and since $mM^{-1} \leq \epsilon, m \geq \epsilon^{-2}$, we obtain

$$|(\lambda + i)(v_\epsilon(\lambda) - w_\epsilon(\lambda))| \leq K \sum_{|n|>M}\left|\frac{\nu_{n,0} - \mu_n^+}{\lambda - \nu_{n,0}}\right|$$

$$\leq K\left(\left|\sum_{|n|>M}\frac{1}{2n}\right| + \sum_{|n|>M}\frac{|\nu_{n,0} - n - 1/2| + |\mu_n^+ - n|}{|n|} + \frac{|\lambda| + |\nu_{n,0} - n|}{n^2}\right) \leq K\epsilon$$

where K does not depend on M. Therefore

$$\left(\int_{|\lambda|\leq m} |v_\epsilon(\lambda) - w_\epsilon(\lambda)|^2\, d\lambda\right)^{1/2} \leq K\epsilon. \tag{4.13}$$

To estimate the "tails" of integrals we use (4.7) and the inequality $|v_\epsilon(\lambda)| \leq K|\lambda + i|^{-1}$ and obtain

$$\int\limits_{|\lambda|\geq m} |v_\epsilon(\lambda) - w_\epsilon(\lambda)|^2\, d\lambda \leq 2 \int\limits_{|\lambda|\geq m} |v_\epsilon(\lambda)|^2\, d\lambda + 2 \int\limits_{|\lambda|\geq m} |v_0(\lambda)|^2\, d\lambda \leq 2(K^2 m^{-1} + \epsilon^2) \leq 4K^2\epsilon^2$$

which together with (4.13) proves (4.12).

We define now $c_\epsilon(\lambda) = \Delta_\epsilon(\lambda) + iv_\epsilon(\lambda)$ and obtain $1 - \Delta_\epsilon^2(\lambda) - v_\epsilon^2(\lambda) = 1 - c_\epsilon(\lambda)c_\epsilon^*(\lambda) \geq 0$. Since $\|\Delta_\epsilon - \Delta_0\|_{\mathcal{PW}} \leq K\epsilon$ and (4.12) is fulfilled, we have $\|c_\epsilon - c_0\|_{\mathcal{PW}} \leq K\epsilon$, and there exists a factorization $1 - c_\epsilon(\lambda)c_\epsilon^*(\lambda) = s_\epsilon(\lambda)s_\epsilon^*(\lambda)$ with $\|s_\epsilon - s_0\|_{\mathcal{PW}} \leq K\epsilon$. The pair of functions $c_\epsilon(\lambda), s_\epsilon(\lambda)$ generates the operator $L_\epsilon \in \mathcal{D}_-$ with the finite-band spectrum and potential matrix $Q_0(x)$ satisfying $\|Q_\epsilon - Q_0\|_{\mathcal{L}^2_{2,2}(0,\pi)} \leq K\epsilon$.

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