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Blow-Up Theorems for Nonlinear Wave Equations

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0. Introduction

Consider semi-linear wave equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) \qquad \left(\Delta = \text{Laplacian} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right). \tag{$*$}$$

The basic question we ask is: under what conditions on the data and the nonlinear function f can there be solutions of $(*)$ which blow-up in finite time?

Global Existence for solutions to the Cauchy Problem when $n=3$ was first demonstrated by Jörgens in [9]; his basic assumptions are that $G(u)=\int_0^u f(s)\,ds$ is nonpositive, that the data is of finite energy, and that f satisfies a growth restriction at infinity. The Cauchy problem then has a unique classical solution existing for all time. (No growth restriction is necessary when $n=1$, as Jörgens has shown in [8] and [10].) A multiplication of $(*)$ by $\partial u/\partial t$ and an integration over all space shows that the energy, E, is a constant independent of time, if the Cauchy data is sufficiently small at infinity:

$$E = \int_{R^n} \left[\tfrac{1}{2} u_t^2 + \tfrac{1}{2} |\nabla u|^2 - G(u)\right] dx.$$

Thus automatic *a priori* bounds are obtained, provided $G(u) \leqq 0$. If, however, $G(u)$ is unbounded above, existence for all time becomes suspect. Indeed, Keller in [12] has shown, by comparing the solution u of $(*)$ with the solution $\phi(t)$ of the ordinary differential equation $\ddot{\phi} = f(\phi)$, that $(*)$ has solutions which blow-up in finite time, if $G(u) \to \infty$ at a sufficiently rapid rate as $u \to \infty$. In three dimensions, Keller's comparison theorem requires that one of the datum functions remain constant on a bounded set; it cannot be extended to dimension $n>3$ because the Riemann function is no longer positive. Using an extension of Keller's idea, Jörgens in [10] has proved a nonexistence theorem for $(*)$. Levine [13] has recently obtained some abstract blow-up theorems; his results are described at the end of Section 4. However, Berger in [2] and [3] has demonstrated that even when $G(u)\to\infty$, (*) may have certain periodic solutions ("stationary states"); see the remarks following Theorem 3.1.

Similar nonexistence theorems have been given for parabolic equations of the form

$$\frac{\partial u}{\partial t}-\Delta u=f(u)$$

by Kaplan [11] for a bounded domain, and by Fujita ([5] and [6]) for the Cauchy problem. In addition, Tsutsumi [18] has established a blow-up theorem for the equation

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+u^{1+\alpha}$$

on a bounded domain when $p<2+\alpha$.

We first show that Kaplan's method can be applied to (*) on a bounded domain. Next we show that, for $n\leqq 3$ and for a positive, convex function $f(u)$, there is a large class of initial data for which solutions to the Cauchy problem for (*) blow-up in finite time. Consideration of the spherical means of a solution then shows that this theorem can be extended to any dimension $n>3$. Finally, Section 5 is devoted to the Cauchy problem for the "accretive" equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=f(u_t);$$

the effect of a positive convex function f is seen to cause blow-up at a rate greater than or equal to that found in Section 3.

The positive integer n denotes the number of space dimensions. We shall write

$$\nabla u=\operatorname{grad}_x u=\left(\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n}\right)$$

so that

$$\Delta u=\text{Laplacian } u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}.$$

ω denotes a unit vector in $\mathbb{R}^n$; $d\omega$ = element of surface measure on the unit sphere in $\mathbb{R}^n$; ω_n = area of the unit sphere in $\mathbb{R}^n$.

1. Preliminaries

In Sections 2–4 we consider the equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=f(u).$$

We shall always assume enough smoothness on the data and on the nonlinearity $f(u)$ to assure the existence of a unique local solution to the Cauchy Problem (or, in Section 2, to the mixed problem). In this regard, see [7-9, 15, 16], and [17].

In order to simplify the exposition, we list two assumptions which will be made repeatedly. Let $S \subseteq \mathbb{R}^n$ and let λ, α, β denote nonnegative constants. The first hypothesis (H_1) concerns the Cauchy data:

$$(\mathrm{H}_1)\quad u(x,0) \geqq \alpha, \qquad \frac{\partial u}{\partial t}(x,0) \geqq \beta \qquad \text{for all } x \in S.$$

Secondly, we specify the nature of the nonlinearity:

(H_2) $f(s)$ is bounded below by a locally Lipschitzian, convex function $g(s)$ satisfying

i) $g(s) - \lambda s$ is a nonnegative, nondecreasing function for $s \geqq \alpha$;

ii) $g(s)$ grows fast enough as $s \to +\infty$ so that the integral

$$T_0 = \int_\alpha^\infty \left[\lambda \alpha^2 + \beta^2 - \lambda s^2 + 2\int_\alpha^s g(\xi)\, d\xi\right]^{-\frac{1}{2}} ds \tag{1.1}$$

converges.

Hypotheses (H_1) and (H_2) will appear in slightly weaker or stronger forms throughout Sections 2–5.

Before proceeding, we prepare a simple lemma on an ordinary differential inequality, which will be much used in the sequel:

Lemma 1.1. *Let $\phi(t) \in C^2$ satisfy*

$$\ddot{\phi} \geqq h(\phi) \qquad (t \geqq 0)$$

with $\phi(0) = \alpha > 0$, $\dot{\phi}(0) = \beta > 0$. Suppose that $h(s) \geqq 0$ for all $s \geqq \alpha$. Then

a) *$\dot{\phi}(t) > 0$ wherever $\phi(t)$ exists; and*

b) *the inequality*

$$t \leqq \int_\alpha^{\phi(t)} \left[\beta^2 + 2\int_\alpha^s h(\xi)\, d\xi\right]^{-\frac{1}{2}} ds$$

obtains.

Proof. If a) is false, let $t = t_1$ be the first point where $\dot{\phi}(t_1) = 0$. Then integrating the differential inequality we obtain

$$\dot{\phi}(t) \geqq \dot{\phi}(0) + \int_0^t h(\phi(s))\, ds$$

so that $0 = \dot{\phi}(t_1) \geqq \beta + \int_0^{t_1} h(\phi(s))\, ds$.

By the definition of t_1, $\phi(s)\geqq\alpha$ for $0\leqq s\leqq t_1$. Thus the integral term above is nonnegative, and the resulting contradiction proves a). To prove b) we use a) and multiply the differential inequality by $\dot\phi(t)$ to get

$$\dot\phi\,\ddot\phi\geqq\dot\phi\,h(\phi),$$

or

$$\frac{d}{dt}\left(\tfrac{1}{2}\dot\phi^2-\int_\alpha^\phi h(\xi)\,d\xi\right)\geqq 0.$$

Thus

$$(\dot\phi(t))^2\geqq\beta^2+2\int_\alpha^{\phi(t)} h(\xi)\,d\xi$$

and, since $\dot\phi(t)>0$, we may separate variables and integrate to obtain b).

2. Bounded Domain

The blow-up problem on a bounded domain is the easiest mathematically, for the method of proof, which closely follows that of Kaplan ([11]), is independent of both the spatial dimension and the Riemann function of the wave operator.

Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the mixed problem

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}-\Delta u=f(u)\qquad(x\in\Omega,\ t>0)\\&u(x,t)=0\qquad\text{for }\ x\in\partial\Omega,\ t\geqq 0,\end{aligned}\tag{2.1}$$

with given smooth Cauchy data. Let $\psi(x)$ denote the first eigenfunction for the problem

$$\Delta\psi+\mu\psi=0\qquad(x\in\Omega)\tag{2.2}$$

under the Dirichlet condition $\psi=0$ on $\partial\Omega$, and let $\mu=\mu_1$ be the corresponding first eigenvalue. By a classical theorem (see [4], Vol. I, pp. 451–455) we may suppose that $\psi(x)>0$ in Ω.

We assume that

i) $u(x,0)\geqq 0$, $u_t(x,0)\geqq 0$ for all $x\in\Omega$; there exist $x_0, x_1\in\Omega$ such that $u(x_0,0)>0$, $u_t(x_1,0)>0$.

ii) (H_2) holds with $\lambda=\mu$,

$$\alpha=\int_\Omega\psi(x)\,u(x,0)\,dx;\qquad\beta=\int_\Omega\psi(x)\,u_t(x,0)\,dx.$$

(Note that both α and β are positive by hypothesis.)

We may now prove

Theorem 2.1. *Let* $u(x,t)$ *be a* C^2 *solution of* (2.1) *for which* i) *and* ii) *are satisfied. Then*

$$\lim_{t\to t_0^-}\sup_{x\in\Omega}|u(x,t)|=+\infty$$

for some finite time $t_0 \leqq T_0$, *where* T_0 *is given by* (1.1).

Proof. Let $\psi(x)$ be as defined by (2.2). Without loss of generality, we may assume that ψ is normalized:

$$\int_\Omega \psi(x)\,dx=1.$$

Let

$$\phi(t)=\int_\Omega \psi(x)\,u(x,t)\,dx,$$

multiply (2.1) by ψ and integrate over Ω. Since $u\in C^2$, we obtain

$$\int_\Omega \psi\,u_{tt}\,dx=\ddot{\phi}=\int_\Omega \psi\,\Delta u\,dx+\int_\Omega \psi\,f(u)\,dx.$$

By Jensen's inequality and ii), we have

$$\int_\Omega \psi\,f(u)\,dx\geqq\int_\Omega \psi\,g(u)\,dx\geqq g\Big(\int_\Omega \psi\,u\,dx\Big)=g(\phi)$$

since ψ is normalized. Now

$$\psi\,\Delta u=\nabla\cdot(\psi\,\nabla u)-\nabla\cdot(u\,\nabla\psi)+u\,\Delta\psi;$$

using this and the boundary conditions satisfied by u and ψ, we see that

$$\int_\Omega \psi\,\Delta u\,dx=\int_\Omega u\,\Delta\psi\,dx=-\mu\int_\Omega u\psi\,dx=-\mu\,\phi(t).$$

Thus we arrive at

$$\ddot{\phi}+\mu\,\phi\geqq g(\phi)$$

with

$$\phi(0)=\int_\Omega \psi(x)\,u(x,0)\,dx=\alpha>0;\qquad \dot{\phi}(0)=\int_\Omega \psi(x)\,u_t(x,0)\,dx=\beta>0.$$

Hypothesis ii) implies that Lemma 1.1 is applicable with $h(s)=g(s)-\mu\,s$; therefore

$$t\leqq\int_\alpha^{\phi(t)}\left[\mu\,\alpha^2+\beta^2-\mu\,s^2+2\int_\alpha^s g(\xi)\,d\xi\right]^{-\frac{1}{2}}ds,$$

and thus $\phi(t)$ develops a singularity in a finite time $t_0\leqq T_0$, where

$$T_0=\int_\alpha^\infty\left[\mu\,\alpha^2+\beta^2-\mu\,s^2+2\int_\alpha^s g(\xi)\,d\xi\right]^{-\frac{1}{2}}ds.$$

Finally, since $\phi(t)>0$, we have

$$\begin{aligned}\phi(t)=|\phi(t)|&=\Big|\int_\Omega \psi(x)\,u(x,t)\,dx\Big|\\&\leqq \sup_{x\in\bar\Omega}|u(x,t)|\int_\Omega \psi(x)\,dx\\&=\sup_{x\in\bar\Omega}|u(x,t)|,\end{aligned}$$

which proves the theorem.

Corollary. *For each* p, $1\leqq p\leqq\infty$,

$$\|u(t)\|_{L_p(\Omega)}=\Big(\int_\Omega |u(x,t)|^p\,dx\Big)^{1/p}$$

blows up in finite time.

The proof of the corollary is simply to apply Hölder's inequality to the term

$$\Big|\int_\Omega \psi(x)\,u(x,t)\,dx\Big|.$$

Remarks on the Proof. 1. As Kaplan has noted ([11]), Δ may be replaced by any uniformly elliptic self-adjoint second-order operator

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

with smooth coefficients $a_{ij}(x)$.

2. The same result holds if the boundary condition is of the form

$$u(x,t)=\Phi(x,t)\qquad\text{for } x\in\partial\Omega,\ t\geqq 0$$

provided $\Phi(x,t)\geqq 0$ for all $x\in\partial\Omega$, $t\geqq 0$. To demonstrate this, we need only show that the term

$$-\int_\Omega \nabla\cdot(u\,\nabla\psi)\,dx$$

is nonnegative. Now ψ obeys the maximum (minimum) principle, and assumes its minimum value (zero) on $\partial\Omega$. It follows that $\nabla\psi$ is directed toward the interior of Ω, so that

$$\frac{\partial\psi}{\partial\nu}\leqq 0\quad\text{on}\quad\partial\Omega,$$

where $\nu=$outer normal to $\partial\Omega$. Then clearly

$$-\int_\Omega \nabla\cdot(u\,\nabla\psi)\,dx=-\int_{\partial\Omega}\Phi(x,t)\,\frac{\partial\psi}{\partial\nu}(x)\,dS_x\geqq 0.$$

3. The corresponding problem with general linear homogeneous boundary conditions on u can be treated similarly. We define $\psi(x)$ as the first eigenfunction of $\Delta\psi+\mu\psi=0$ in Ω, satisfying the same boundary conditions as u on $\partial\Omega$.

For example, if the boundary condition is

$$\frac{\partial u}{\partial \nu}=0 \qquad (x\in\partial\Omega),$$

we choose

$$\psi(x)=\text{const}=(\text{measure}(\Omega))^{-1}\equiv(m(\Omega))^{-1}$$

and accordingly

$$\phi(t)=\int_{\Omega}\psi(x)\,u(x,t)\,dx=\frac{1}{m(\Omega)}\int_{\Omega}u(x,t)\,dx.$$

We then easily obtain $\ddot{\phi}\geqq g(\phi)$, and proceed as above.

3. Cauchy Problem, $n\leqq 3$

We consider now the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=f(u) \qquad (x\in\mathbb{R}^n,\ t>0) \tag{3.1}$$

for $n\leqq 3$. Only the case $n=3$ will be analyzed; the method is similar when $n=1$ or 2.

The proof in Section 2 cannot be extended to the Cauchy problem by integrating over all space, since no such positive eigenfunction exists. However, Fujita in [5] and [6] was successful in modifying this method for a parabolic equation of the form

$$\frac{\partial u}{\partial t}-\Delta u=f(u)$$

by exploiting the fact that the Green's function for the "heat" operator is positive in any dimension. Only for $n\leqq 3$ is the Riemann function for (3.1) positive. In Section 3 we shall show that a combination of the methods of Kaplan and Keller provides a nonexistence theorem for (3.1).

For any $R>0$, define

$$\psi(x)=\frac{c}{r}\sin\frac{\pi r}{R} \qquad \text{for } |x|=r\leqq R, \tag{3.2}$$

where $c>0$ is chosen so that $\int_{|x|\leqq R}\psi(x)\,dx=1$. Let $\mu=\pi^2/R^2$; we assume that

i) (H_1) holds for arbitrary $\alpha>0$, $\beta>0$, with $S=\{x\in\mathbb{R}^3\colon |x|\leqq R+2T_0\}$, where T_0 is given by (1.1) with $\lambda=\mu$;

ii) $\Delta u(x,0) \geqq 0$ for all $x \in S$;

iii) (H_2) holds with $\lambda = \mu$ in the following weakened form: the function $g(u)$ is assumed convex only for $u \geqq \alpha$.

Under these conditions we shall prove

Theorem 3.1. *Let $u(x,t)$ be a C^2 solution of* (3.1) *for which* i)–iii) *hold. Then*

$$\lim_{t \to t_0^-} \sup_{|x| \leqq R} |u(x,t)| = +\infty$$

for some finite time $t_0 \leqq T_0$.

Remark. When $n=3$, Keller (cf. [12], p. 528) assumes that on some set $|x-x_0| \leqq T$, the data satisfies

$$u(x,0) = \alpha = \text{const.}; \qquad u_t(x,0) \geqq \beta = \text{const.}$$

which, when $\beta > 0$, is a special case of (H_1).

Proof. The solution $u(x,t)$ of (3.1) satisfies the following nonlinear integral equation:

$$u(x,t) = u_0(x,t) + \frac{1}{4\pi} \int_0^t \frac{1}{t-\tau} \int_{|y-x|=t-\tau} f(u(y,\tau))\, dS_y\, d\tau$$

where $u_0(x,t)$ is the solution of the linear equation with the same data as that of u when $t=0$. Thus

$$u_0(x,t) = \frac{t}{4\pi} \int_{|\omega|=1} \frac{\partial u}{\partial t}(x+\omega t, 0)\, d\omega + \frac{1}{4\pi} \int_{|\omega|=1} u(x+\omega t, 0)\, d\omega$$
$$+ \frac{1}{4\pi t} \int_{|y-x| \leqq t} \Delta u(y,0)\, dy.$$

From i) we clearly have $u_0(x,t) \geqq \alpha + \beta t$ for $|x| \leqq R + T_0$, $0 \leqq t \leqq T_0$.

We now claim that $u(x,t) \geqq \alpha$ for $|x| \leqq R$, $0 \leqq t \leqq T_0$. Let $C(x_0, T_0)$ be any backward characteristic cone, the x-coordinate of whose vertex x_0 satisfies $|x_0| \leqq R$. We shall show that $u(x,t) \geqq \alpha$ in $C(x_0, T_0)$ using Keller's method. Then, since x_0 was an arbitrary point in $|x| \leqq R$, we will have proved the claim. Suppose the assertion $u(x,t) \geqq \alpha$ in $C(x_0, T_0)$ is false. Let

$$t_1 = \inf\{t\colon u(x,t) < \alpha \text{ in } C(x_0, T_0)\},$$

and let $(x_1, t_1 + \varepsilon)$, with sufficiently small ε, be a point in $C(x_0, T_0)$ where $u(x_1, t_1 + \varepsilon) < \alpha$. Then from the integral equation we have

$$u(x_1, t_1+\varepsilon) - \alpha = u_0(x_1, t_1+\varepsilon) - \alpha$$
$$+ \frac{1}{4\pi} \int_0^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau} \int_{|y-x_1| = t_1+\varepsilon-\tau} f(u(y,\tau))\, dS_y\, d\tau.$$

Now $|x_1| \leqq R+T_0$, so that $u_0(x_1, t_1+\varepsilon)-\alpha \geqq \beta t_1$ by the above. Using (H_2) and the definition of the point t_1, we have

$$u(x_1, t_1+\varepsilon)-\alpha$$
$$\geqq \beta t_1 + \frac{1}{4\pi} \int_{t_1}^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau} \int_{|y-x_1|=t_1+\varepsilon-\tau} g(u(y,\tau))\, dS_y\, d\tau$$
$$\geqq C + \frac{1}{4\pi} \int_{t_1}^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau} \int_{|y-x_1|=t_1+\varepsilon-\tau} [g(u(y,\tau))-g(\alpha)]\, dS_y\, d\tau$$

where $C>0$ depends on β, t_1, ε, and the (positive) value of $g(\alpha)$. We now split this integral into two components, one over the set $u \geqq \alpha$; the other over the complement of this set. Whenever $u \geqq \alpha$, $g(u)-g(\alpha)$ is non-negative, so we may restrict our attention to the region $u<\alpha$. There, since g is Lipschitzian, we have

$$u(x_1, t_1+\varepsilon)-\alpha \geqq \varepsilon K(u-\alpha)_{\min} + C,$$

where $(u-\alpha)_{\min}$ is the least value of $(u-\alpha)$ in the backward characteristic cone $C(x_1, t_1+\varepsilon)$ for $t \geqq t_1$, and where K is proportional to the Lipschitz constant for g. Taking $\varepsilon<1/K$ and applying the above to the point in $C(x_1, t_1+\varepsilon)$ where $(u-\alpha)$ assumes its minimum, we get

$$(u-\alpha)_{\min} \geqq C + \varepsilon K(u-\alpha)_{\min};$$

thus

$$(u-\alpha)_{\min} \geqq \frac{C}{1-\varepsilon K} > 0$$

which is impossible. Hence no such point t_1 exists, and the claim is proved.

Now let $\psi(x)$ and μ be as defined above. Then

$$\Delta\psi + \mu\psi = 0 \quad \text{in} \quad |x|<R;$$

ψ vanishes on $|x|=R$, and

$$\left.\frac{\partial\psi}{\partial r}\right|_{|x|=R} < 0.$$

We multiply (3.1) by $\psi(x)$ and integrate over $|x| \leqq R$; with

$$\phi(t) = \int_{|x| \leqq R} \psi(x)\, u(x,t)\, dx$$

we obtain

$$\ddot{\phi} = \int_{|x| \leqq R} \psi \Delta u\, dx + \int_{|x| \leqq R} \psi f(u)\, dx.$$

Now $u(x,t) \geqq \alpha$ for $|x| \leqq R$, $0 \leqq t \leqq T_0$; thus

$$\int_{|x| \leqq R} \psi f(u)\, dx \geqq \int_{|x| \leqq R} \psi\, g(u)\, dx \geqq g(\phi)$$

by the convexity of g and Jensen's inequality. Using the properties of ψ and the fact that u is a positive solution, we get

$$\int_{|x|\leqq R} \psi\, \Delta u\, dx = \int_{|x|\leqq R} [\nabla\cdot(\psi \nabla u) - \nabla\cdot(u \nabla \psi) + u\, \Delta\psi]\, dx \geqq -\mu\, \phi(t).$$

Therefore $\ddot{\phi} + \mu\, \phi \geqq g(\phi)$ with

$$\phi(0) = \int_{|x|\leqq R} \psi(x)\, u(x,0)\, dx \equiv \alpha_1 \geqq \alpha; \quad \dot{\phi}(0) = \int_{|x|\leqq R} \psi(x)\, u_t(x,0)\, dx \equiv \beta_1 \geqq \beta.$$

Lemma 1.1 now applies with $h(s) = g(s) - \mu\, s$; we find that $\phi(t)$ blows up in a finite time $t_0 \leqq T_1$, where

$$T_1 = \int_{\alpha_1}^{\infty} \left[\mu\, \alpha_1^2 + \beta_1^2 - \mu\, s^2 + 2\int_{\alpha_1}^{s} g(\xi)\, d\xi\right]^{-\frac{1}{2}} ds.$$

It remains only to show that $T_1 \leqq T_0$. For this purpose, set

$$\begin{aligned} T^* = T^*(\alpha,\beta) &= \int_{\alpha}^{\infty} \left[\mu\, \alpha^2 + \beta^2 - \mu\, s^2 + 2\int_{\alpha}^{s} g(\xi)\, d\xi\right]^{-\frac{1}{2}} ds \\ &= \int_{\alpha}^{\infty} \left[\beta^2 + 2\int_{\alpha}^{s} (g(\xi) - \mu\, \xi)\, d\xi\right]^{-\frac{1}{2}} ds. \end{aligned}$$

Then from iii) we have that $T^*(\alpha,\beta)$ decreases as α, β increase. Hence since $\alpha_1 \geqq \alpha$, $\beta_1 \geqq \beta$, we find that

$$T_1 = T^*(\alpha_1, \beta_1) \leqq T^*(\alpha, \beta) = T_0.$$

Thus $\phi(t)$ blows up in a finite time $t_0 \leqq T_0$. Then

$$\phi(t) = |\phi(t)| \leqq \sup_{|x|\leqq R} |u(x,t)| \int_{|x|\leqq R} \psi(x)\, dx = \sup_{|x|\leqq R} |u(x,t)|$$

which completes the proof.

Corollary. *For each p, $1 \leqq p \leqq \infty$, the expressions*

$$\Big(\int_{|x|\leqq R} |u(x,t)|^p\, dx\Big)^{1/p}$$

blow-up in finite time.

Two comments on this result are now in order. First, note that the integral defining T_0 converges if $g(s) \geqq s^{1+\varepsilon}$ as $s \to \infty$ for arbitrary $\varepsilon > 0$. Secondly, a brief comparison of this result and the work of Berger in [2] and [3] should be made. Berger considers (3.1) with

$$f(u) = -m^2 u + u\, |u|^{\sigma} \qquad (m>0, \sigma>0)$$

and seeks stationary solutions; that is, solutions of the form

$$u(x,t) = e^{i\lambda t}\, v(x)$$

where λ is real and $v(x) \to 0$ as $|x| \to \infty$. His results show that, for $n=3$, stationary solutions exist with $v(x) \to 0$ exponentially as $|x| \to \infty$ if and only if $|\lambda| < m$ and $0 < \sigma < 4$. Clearly hypothesis iii) is satisfied for such an $f(u)$ if $u^\sigma \geqq m^2 + \mu$ for $u \geqq \alpha$ (i.e. if α is sufficiently large). However, due to the special form of stationary solutions, hypothesis i) on the data cannot be satisfied by the function $v(x)$, since for such solutions $u(x, 0) = v(x)$; $u_t(x, 0) = i\lambda v(x)$.

4. Cauchy Problem – Extension to Dimension $n \geqq 3$

We again examine the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) \qquad (x \in \mathbb{R}^n, t > 0) \tag{4.1}$$

where now $n \geqq 3$. As noted above, the Riemann function for the wave operator is no longer positive for $n > 3$, so the proof in Section 3 cannot be extended as it stands. In Section 4 we shall show that consideration of the "spherical means" of a solution circumvents this difficulty, and furnishes a blow-up theorem for (4.1). When $n=3$, Theorem 3.1 is stronger than Theorem 4.1, since in the latter we require that the function $g(s)$ be everywhere convex.

To begin, let $x_0 \in \mathbb{R}^n$ and let $r \geqq 0$. The *spherical mean* of u about x_0 is defined by

$$\bar{u}(r, t) = \frac{1}{\omega_n} \int_{|\omega|=1} u(x_0 + \omega r, t)\, d\omega. \tag{4.2}$$

Let a, b (with $0 < a < b < \infty$) denote constants which will be specified below. Define $\psi(r)$ to be the first eigenfunction, and μ, the corresponding first eigenvalue, of the ordinary differential boundary-value problem:

$$\psi'' - \frac{n-1}{r}\psi' + \frac{n-1}{r^2}\psi + \mu\psi = 0 \qquad (a < r < b) \tag{4.3}$$

$$\psi(a) = \psi(b) = 0.$$

(The choice of ψ will be explained later.) Since the theorem below is our major result, we list our assumptions explicitly:

i) a) for arbitrary $\alpha > 0$, $\beta > 0$,

$$u(x, 0) \geqq \alpha, u_t(x, 0) \geqq \beta \qquad \text{for } x \in S = \{x : a - T_0 \leqq |x| \leqq b + T_0\},$$

where

$$T_0 = \int_\alpha^\infty \left[\beta^2 + \int_\alpha^s f(\xi)\, d\xi\right]^{-\frac{1}{2}} ds$$

b) the data are subharmonic on S; i.e.

$$\Delta u(x, 0) \geqq 0, \qquad \Delta u_t(x, 0) \geqq 0 \quad \text{for all } x \in S$$

ii) $f(s)$ is bounded below by a locally Lipschitzian convex function $g(s)$ satisfying

a) $g(s)-2\mu s$ is a nonnegative, nondecreasing function for $s \geqq \alpha$;

b) $f(s)$ grows fast enough at infinity so that $T_0 < \infty$.

Our goal is to prove

Theorem 4.1. *Let $u(x,t)$ be a C^2 solution of* (4.1). *There exists a positive constant r_0, depending only on n, with the property that: if* i) *and* ii) *are satisfied for any constants a, b with $b > a \geqq T_0(r_0+2)$, then*

$$\lim_{t \to t_0^-} \sup_{a \leqq |x| \leqq b} |u(x,t)| = +\infty$$

for some finite time $t_0 \leqq T_0$.

Before proceeding, we shall establish several lemmas in order to simplify the proof. First we show that (4.1) can be reduced to a one-dimensional problem; this is the content of

Lemma 4.1. *Let $\bar{u}(r,t)$ be defined by* (4.2) *with $x_0=0$. Then $\bar{u}$ satisfies*

$$\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{(n-1)}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\partial^2 \bar{u}}{\partial r^2} = \frac{1}{\omega_n} \int_{|\omega|=1} f(u(\omega r,t))\, d\omega.$$

Proof. From direct computation, we obtain

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \frac{1}{\omega_n} \int_{|\omega|=1} u_{tt}(\omega r,t)\, d\omega;$$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial r} &= \frac{1}{\omega_n} \int_{|\omega|=1} \omega \cdot \nabla u(\omega r,t)\, d\omega = \frac{1}{\omega_n r^{n-1}} \int_{|y|=r} \nu \cdot \nabla u(y,t)\, dS_y \\ &= \frac{1}{\omega_n r^{n-1}} \int_{|y| \leqq r} \Delta u(y,t)\, dy; \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \bar{u}}{\partial r^2} &= \frac{-(n-1)}{\omega_n r^n} \int_{|y| \leqq r} \Delta u(y,t)\, dy + \frac{1}{\omega_n r^{n-1}} \int_{|y|=r} \Delta u(y,t)\, dS_y \\ &= \frac{-(n-1)}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{\omega_n} \int_{|\omega|=1} \Delta u(\omega r,t)\, d\omega, \end{aligned}$$

hence the result.

Corollary. *The partial differential inequality*

$$\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{(n-1)}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\partial^2 \bar{u}}{\partial r^2} \geqq g(\bar{u})$$

obtains.

The corollary is an immediate consequence of hypothesis ii) and Jensen's inequality.

The Riemann function for the operator on the lefthand side of (4.4) is well-known and yields

Lemma 4.2. *$\bar{u}(r,t)$ satisfies the following nonlinear integral equation:*

$$\bar{u}(r,t)=\bar{u}(r,0)+t\,\bar{u}_t(r,0)$$
$$+\int_0^t\int_{r-(t-\tau)}^{r+(t-\tau)} R(\xi,\tau;r,t)\left[\frac{1}{\omega_n}\int_{|\omega|=1}\{\Delta u(\omega\xi,0)+\tau\,\Delta u_t(\omega\xi,0)\right.$$
$$\left.+f(u(\omega\xi,\tau))\}\,d\omega\right]d\xi\,d\tau$$

where the kernel R, the Riemann function, is given by

$$R(\xi,\tau;r,t)=\frac{2^{n-3}\xi^{n-1}}{[(\xi+r)^2-(\tau-t)^2]^{\frac{n-1}{2}}}F\left(\frac{n-1}{2},\frac{n-1}{2};1;\frac{(\xi-r)^2-(\tau-t)^2}{(\xi+r)^2-(\tau-t)^2}\right)$$

with F denoting the hypergeometric function (cf. [1], Chapter 15).

Proof. That R is given as above follows from [7], p. 135 and 150. Let

$$v(x,t)=u(x,t)-u(x,0)-t\,u_t(x,0).$$

Then $v(x,0)=0$, $v_t(x,0)=0$, $v_{tt}(x,t)=u_{tt}(x,t)$, and

$$\Delta v=\Delta u(x,t)-\Delta u(x,0)-t\,\Delta u_t(x,0);$$

thus

$$v_{tt}-\Delta v=\Delta u(x,0)+t\,\Delta u_t(x,0)+f(u(x,t)).$$

With

$$\bar{v}(r,t)=\frac{1}{\omega_n}\int_{|\omega|=1} v(\omega r,t)\,d\omega$$

we find as in Lemma (4.1) that

$$\frac{\partial^2\bar{v}}{\partial t^2}-\frac{(n-1)}{r}\frac{\partial\bar{v}}{\partial r}-\frac{\partial^2\bar{v}}{\partial r^2}$$
$$=\frac{1}{\omega_n}\int_{|\omega|=1}\{\Delta u(\omega r,0)+t\,\Delta u_t(\omega r,0)+f(u(\omega r,t))\}\,d\omega$$

with $\bar{v}(r,0)=\bar{v}_t(r,0)=0$. Using Riemann's representation formula, and rewriting the result in terms of $\bar{u}(r,t)$, we obtain the lemma.

We show next that sufficiently far away from its singularity $r=0$ of (4.4), R is positive:

Lemma 4.3. *There exists a positive constant r_0, depending only on the dimension n, such that the Riemann function is nonnegative in the backward characteristic cone $0 \leqq \tau \leqq t$, $|\xi - r| \leqq t - \tau$, if $r \geqq t(r_0 + 1)$.*

Proof. For r as chosen above, the coefficient of F in R is clearly positive, so it suffices to show that $F\left(\frac{n-1}{2}, \frac{n-1}{2}; 1; z\right) \geqq 0$, where we have put

$$z = \frac{(\xi - r)^2 - (\tau - t)^2}{(\xi + r)^2 - (\tau - t)^2}.$$

From [1], p. 556, Formula 15.1.1, we see that $F\left(\frac{n-1}{2}, \frac{n-1}{2}; 1; 0\right) = 1$. Moreover, note that $-1 < z \leqq 0$. We will show that for r as chosen above, z may be made as close to zero as desired; then by the continuity of F, we will be done. Now

$$-z = \frac{(\tau - t)^2 - (\xi - r)^2}{(\xi + r)^2 - (\tau - t)^2} \equiv \frac{z_1}{z_2} \geqq 0$$

with $z_1 \geqq 0$, $z_2 \geqq 0$. We have

$$z_1 \leqq (t - \tau)^2 \leqq t^2;$$

since $\xi \geqq r - (t - \tau) \geqq r - t \geqq r_0 t$, it follows that

$$\xi + r \geqq 2 r_0 t + t; \qquad \text{thus}$$

$$z_2 = (\xi + r)^2 - (\tau - t)^2 \geqq (2 r_0 t + t)^2 - t^2 \geqq 4 r_0^2 t^2.$$

Hence $0 \leqq (-z) = \frac{z_1}{z_2} \leqq \frac{t^2}{4 r_0^2 t^2} = \frac{1}{4 r_0^2}$, completing the proof.

Remark. There are special values of n for which R is positive in the full region $r > t$. For instance, when $n = 5$, we have

$$F(2, 2; 1; z) = \sum_{k=0}^{\infty} \frac{(\Gamma(k+2))^2}{\Gamma(k+1)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} (k+1)^2 z^k$$
$$= \frac{1+z}{(1-z)^3}$$

which is positive because $-1 < z \leqq 0$ whenever $r > t$. However, when $n = 7$, the corresponding function is

$$F(3, 3; 1; z) = \frac{1}{4} \sum_{k=0}^{\infty} (k+1)^2 (k+2)^2 z^k$$
$$= (4(1-z))^{-3} \left[4 + \frac{24z}{1-z} + \frac{24z^2}{(1-z)^2}\right]$$

which changes sign as z traverses the interval $(-1, 0)$. Thus, in general R will not remain positive, and Lemma (4.3) will be necessary.

Corollary. *Under assumptions* i) *and* ii),

$$\bar{u}(r,t) \geqq \alpha > 0 \quad \textit{for any } a, b \textit{ satisfying}$$
$$b \geqq r \geqq a \geqq (r_0+2)\,T_0, \qquad 0 \leqq t \leqq T_0.$$

Proof. Using Lemmas (4.2), (4.3), hypotheses i) and ii), and the convexity of g, we obtain

$$\bar{u}(r,t) \geqq \bar{u}(r,0) + t\,\bar{u}_t(r,0) + \int_0^t \int_{r-(t-\tau)}^{r+(t-\tau)} R(\xi,\tau;r,t)\, g(\bar{u}(\xi,\tau))\, d\xi\, d\tau$$

for such values of (r,t). We have in the above a positive kernel and a locally Lipschitzian function $g(s)$ which is positive for $s \geqq \alpha$. Thus we may proceed exactly as in Theorem 3.1 to complete the proof.

Now let $\psi(r)$ be as defined by (4.3). Note that the operator acting on ψ is just the adjoint of the "meanvalue" operator (in r) appearing on the left side of (4.4). Under the change of variables

$$\psi(r) = r^{\frac{n-1}{2}}\,\Phi(r)$$

the equation for Φ assumes the self-adjoint form $\Phi'' + [\mu - (\nu^2 - \frac{1}{4})\, r^{-2}]\,\Phi = 0$ of the Bessel equation (see [1], p. 362, Formula 9.1.49), where $\nu^2 = \frac{1}{4}(n-2)^2$. Thus from [4], Vol. I, p. 454, $\Phi(r)$ is of one sign on $[a, b]$, so clearly the same is true for $\psi(r)$. We will assume that $\psi(r) \geqq 0$ on $[a, b]$ and, moreover, that

$$\int_a^b \psi(r)\, dr = 1.$$

We may now establish the theorem:

Proof of Theorem 4.1. Given n, let r_0 be the number determined in Lemma 4.3. Let a, b satisfy $b \geqq r \geqq a \geqq T_0(r_0+2)$, and let $\psi(r)$ be as above. From the corollary to Lemma 4.1, we have

$$\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{(n-1)}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\partial^2 \bar{u}}{\partial r^2} \geqq g(\bar{u}).$$

Multiply this by $\psi(r)$, and integrate over $[a, b]$. With

$$\phi(t) = \int_a^b \psi(r)\, \bar{u}(r,t)\, dr$$

we obtain

$$\ddot{\phi} \geqq \int_a^b \psi\, g(\bar{u})\, dr + \int_a^b \left(\frac{n-1}{r}\,\psi\, \bar{u}_r + \psi\, \bar{u}_{rr}\right) dr.$$

We again use the convexity of g in conjunction with Jensen's inequality to obtain

$$\int_a^b g(\bar{u})\,\psi\,dr \geqq g(\phi).$$

The remaining terms are integrated by parts:

$$\int_a^b \psi\,\bar{u}_{rr}\,dr = \psi\,\bar{u}_r\big|_a^b - \int_a^b \psi'\,\bar{u}_r\,dr = -\int_a^b \psi'\,\bar{u}_r\,dr$$

$$= -\psi'\,\bar{u}\big|_a^b + \int_a^b \psi''\,\bar{u}\,dr \geqq \int_a^b \psi''\,\bar{u}\,dr$$

since $\bar{u}>0$ by the corollary to Lemma 4.3, and since, by choice of ψ, $\psi'(a)>0$, $\psi'(b)<0$. The second term is treated similarly:

$$\int_a^b \frac{n-1}{r}\,\psi\,\bar{u}_r\,dr = \frac{n-1}{r}\,\psi\,\bar{u}\big|_a^b - (n-1)\int_a^b \bar{u}\left(\frac{1}{r}\,\psi' - \frac{1}{r^2}\,\psi\right)dr$$

$$= -(n-1)\int_a^b \bar{u}\left(\frac{1}{r}\,\psi' - \frac{1}{r^2}\,\psi\right)dr.$$

Thus we have

$$\ddot{\phi} \geqq g(\phi) + \int_a^b \bar{u}\left(\psi'' - \frac{n-1}{r}\,\psi' + \frac{n-1}{r^2}\,\psi\right)dr,$$

or

$$\ddot{\phi} + \mu\,\phi \geqq g(\phi).$$

Note that

$$\phi(0) = \int_a^b \bar{u}(r,0)\,\psi(r)\,dr \equiv \alpha_1 \geqq \alpha;$$

$$\dot{\phi}(0) = \int_a^b \bar{u}_t(r,0)\,\psi(r)\,dr \equiv \beta_1 \geqq \beta.$$

Lemma 1.1 applies with $h(s)=g(s)-\mu s$, and shows that $\phi(t)$ becomes infinite in a finite time $t_0 \leqq T_1$, where

$$T_1 = \int_{\alpha_1}^{\infty} \left[\mu\,\alpha_1 + \beta_1^2 - \mu\,s^2 + 2\int_{\alpha_1}^{s} g(\xi)\,d\xi\right]^{-\frac{1}{2}} ds$$

$$= \int_{\alpha_1}^{\infty} \left[\beta_1^2 + 2\int_{\alpha_1}^{s} (g(\xi) - \mu\,\xi)\,d\xi\right]^{-\frac{1}{2}} ds.$$

Since $\alpha_1 \geqq \alpha$, $\beta_1 \geqq \beta$, we have by the same reasoning as in Theorem 3.1 that

$$T_1 \leqq \int_{\alpha}^{\infty} \left[\beta^2 + 2\int_{\alpha}^{s} (g(\xi) - \mu\,\xi)\right]^{-\frac{1}{2}} ds.$$

Using ii)a), we find that $2g(\xi)-2\mu\xi \geqq g(\xi)$ for $\xi \geqq \alpha$. Thus

$$T_1 \leqq \int_\alpha^\infty \left[\beta^2+\int_\alpha^s g(\xi)\,d\xi\right]^{-\frac{1}{2}} ds \leqq \int_\alpha^\infty \left[\beta^2+\int_\alpha^s f(\xi)\,d\xi\right]^{-\frac{1}{2}} ds = T_0 .$$

The proof is then completed by noting that

$$\begin{aligned} 0<\phi(t) &= \int_a^b \bar{u}(r,t)\,\psi(r)\,dr \leqq \sup_{a\leqq r\leqq b} |\bar{u}(r,t)| \\ &\leqq \sup_{a\leqq r\leqq b}\ \sup_{|\omega|=1} |u(\omega r,t)| \quad \sup_{a\leqq |x|\leqq b} |u(x,t)| . \end{aligned}$$

To extend this result, we now show that the sign of the nonlinear term $f(u)$ is irrelevant, provided f is an even function. Consider the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+f(u)=0 \qquad (x\in \mathbb{R}^n,\ t>0). \tag{4.5}$$

Let i) and ii) be replaced by

i′) a) for arbitrary $\alpha>0$, $\beta>0$,

$$u(x,0)\leqq -\alpha, \qquad u_t(x,0)\leqq -\beta$$

for $x\in S$, where S and T_0 are as given in i);

b) the data are superharmonic on S.

ii′) $f(s)$ is an *even* function which is bounded below by a locally Lipschitzian convex function $g(s)$ satisfying ii)a) and ii)b).

We then have

Theorem 4.2. *Let $u(x,t)$ be a C^2 solution of* (4.5). *There exists a positive constant r_0, depending only on n, with the property that: if* i′) *and* ii′) *are satisfied for any constants a, b with $b>a\geqq T_0(r_0+2)$, then*

$$\lim_{t\to t_0^-}\ \sup_{a\leqq |x|\leqq b} |u(x,t)| = +\infty$$

for some finite time $t_0\leqq T_0$.

Proof. Let $v(x,t)=-u(x,t)$; then the equation for v becomes

$$\frac{\partial^2 v}{\partial t^2}-\Delta v=f(v)$$

because f is even. Under this simple "sign inversion", i′) and ii′) become hypotheses i) and ii) of Theorem 4.1. It follows that $\sup_x |v|$ blows up in a finite time; hence, so does $\sup_x |u|$.

Note that this result applies equally well to the case $n\leqq 3$.

We now give a brief description of Levine's result [13]. Here the abstract equation

$$P\frac{d^2u}{dt^2} = -A(t)\,u + F(u)$$

$$u(0)=u_0, \qquad u_t(0)=v_0$$

is analyzed, where u is a Hilbert space-valued function of t and $A(t)$ is a symmetric, nonnegative linear operator, and where P is a positive symmetric operator. If $(\cdot,\cdot)$ denotes the inner product in the Hilbert space, then Levine concludes, under certain assumptions on $A(t)$, $F(u)$, and the data, that

$$\lim_{t\to T^-}\big(u(t), P\,u(t)\big) = +\infty$$

for some $T<\infty$ which can be estimated from above. His method is completely different from that above, and has definite advantages; not only does it apply to a much more general class of problems than the nonlinear wave equation, but also the necessity of knowing the (pointwise) positivity of a solution is avoided. Levine's hypotheses on the nonlinear function are perfectly suitable for superlinear power functions $F(u)=f(u)$. If, however, $F(u)=e^u$ or, say, $F(u)=u\,e^{u^2}$, they may be more difficult to verify. In contrast, such nonlinearities are subsumed in the hypotheses of Theorem 3.1. Note also that in the special case of the nonlinear wave equation, the underlying Hilbert space is $L_2(\mathbb{R}^n)$, so that Levine concludes that

$$\lim_{t\to T^-}\int_{\mathbb{R}^n} u^2(x,t)\,dx = +\infty, \qquad T<\infty.$$

On the other hand, our results are "local" in the sense that we show for each p, $1\leqq p\leqq\infty$, the expressions

$$\|u(t)\|_{p,R} \equiv \Big(\int_{|x|\leqq R} |u(x,t)|^p\,dx\Big)^{1/p}$$

all develop singularities in finite time.

5. An "Accretive" Equation

We now turn our attention to the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f\left(\frac{\partial u}{\partial t}\right) \qquad (x\in\mathbb{R}^n,\ t>0). \tag{5.1}$$

If f is decreasing and $s\,f(s)\leqq 0$, the theory of monotone operators provides the existence of global weak solutions for (5.1); see, for example, [17],

p. 101. In this section we shall show, using previous methods, that if f is positive and convex, (5.1) has solutions which blow-up in finite time.

First we consider the case $n=3$. For any given $R>0$, let $\psi(x)$ be defined by Eq. (3.2). Our assumptions are similar to those of Theorem 3.1; namely

i) for arbitrary $\beta>\alpha>0$,

$$u_t(x,0)\geqq\beta>u(x,0)\geqq\alpha>0$$

for all $x\in S=\{x:|x|\leqq R+T_0\}$, where T_0 is given by (1.1) with $\lambda=\mu=\pi^2/R^2$;

ii) $\Delta u(x,0)\geqq 0$ for all $x\in S$;

iii) (H_2) holds with the following additional conditions: $g(s)\geqq 0$ for all s, and the function

$$g(s)-(\mu+1)\,s$$

is nonnegative and nondecreasing for $s\geqq\alpha$.

Under these conditions we can prove

Theorem 5.1. *Let* i)–iii) *hold for a* C^2 *solution* $u(x,t)$ *of* (5.1). *Then*

$$\lim_{t\to t_0^-}\sup_{|x|\leqq R}|u(x,t)|=+\infty$$

for some finite time $t_0\leqq T_0$.

Proof. By hypothesis and the integral equation satisfied by the solution u, we have immediately that $u(x,t)\geqq 0$ for $|x|\leqq R$, $0\leqq t\leqq T_0$. With

$$\phi(t)=\int_{|x|\leqq R}u(x,t)\,\psi(x)\,dx$$

we see that $\phi(t)\geqq 0$ for $t\leqq T_0$ and that

$$\begin{aligned}\ddot\phi+\mu\phi&\geqq\int_{|x|\leqq R}f(u_t)\,\psi\,dx\geqq\int_{|x|\leqq R}g(u_t)\,\psi\,dx\\&\geqq g\Big(\int_{|x|\leqq R}u_t\,\psi\,dx\Big)=g(\dot\phi).\end{aligned}$$

Note that

$$\phi(0)=\int_{|x|\leqq R}\psi(x)\,u(x,0)\,dx\equiv\alpha_1\geqq\alpha;$$

$$\dot\phi(0)=\int_{|x|\leqq R}\psi(x)\,u_t(x,0)\,dx\equiv\beta_1\geqq\beta$$

and that $\beta_1>\alpha_1$. We now claim that $\dot\phi(t)>\phi(t)$ wherever ϕ exists. If this assertion is false, let $t=t_1$ be the first point where $\dot\phi(t_1)=\phi(t_1)$, and set

$$\sigma(t)=\dot\phi(t)-\phi(t).$$

Then by the definition of t_1, we have $\sigma(t)\geqq 0$ on $[0,t_1]$. For $0\leqq t\leqq t_1$, we use this to get

$$\dot\sigma=\ddot\phi-\dot\phi\geqq g(\dot\phi)-\mu\phi-\dot\phi\geqq g(\dot\phi)-(\mu+1)\,\dot\phi.$$

Moreover, the nonnegativity of $\sigma(t)$ on $[0, t_1]$ implies that

$$\dot{\phi}(t) \geqq \phi(t) \geqq 0$$

for such t; thus $\phi(t)$ is increasing and therefore

$$\dot{\phi}(t) \geqq \phi(t) \geqq \alpha_1 \geqq \alpha \qquad \text{for } 0 \leqq t \leqq t_1.$$

It then follows from iii) and the above that $\dot{\sigma}(t) \geqq 0$ on $[0, t_1]$, from which we obtain

$$0 = \sigma(t_1) \geqq \sigma(0) = \beta_1 - \alpha_1 > 0.$$

This contradiction shows that the point t_1 does not exist, and proves the claim. Thus, since $g(s)$ is nondecreasing for $s \geqq \alpha$, the differential inequality

$$\ddot{\phi} + \mu \phi \geqq g(\dot{\phi}) \geqq g(\phi)$$

obtains. We are now reduced to the situation of Theorem 3.1, and the proof proceeds as before.

Corollary. *Under the same conditions,*

$$\sup_{|x| \leqq R} \left| \frac{\partial u}{\partial t}(x, t) \right|$$

also blows up in a finite time $t_2 \leqq t_0 \leqq T_0$.

Proof. Since $0 < \phi(t) < \dot{\phi}(t)$ wherever ϕ exists, we have

$$\dot{\phi}(t) = |\dot{\phi}(t)| \leqq \sup_{|x| \leqq R} \left| \frac{\partial u}{\partial t}(x, t) \right|,$$

which proves the corollary.

The method is similar when $n > 3$. We again consider the mean values

$$\bar{u}(r, t) = \frac{1}{\omega_n} \int_{|\omega|=1} u(\omega r, t)\, d\omega$$

and obtain the partial differential inequality

$$\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{n-1}{r} \frac{\partial \bar{u}}{\partial r} - \frac{\partial^2 \bar{u}}{\partial r^2} \geqq g\left(\frac{\partial \bar{u}}{\partial t}\right)$$

by the convexity of g. The Riemann function for the operator on the left above is given by Lemma 4.2. Thus we may again determine the number r_0 given by Lemma 4.3, which implies the positivity of the kernel in the integral representation. It then follows immediately from the assumed subharmonicity of the data and the positivity of f that $\bar{u}(r, t) \geqq 0$ for the same (r, t) as in the corollary to Lemma 4.3. The proof then goes through as above, with the additional assumption that $u_t(x, 0) > u(x, 0)$ for all $x \in S$.

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