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Economic Theory

Exposita notes

Existence of steady–state equilibrium in an overlapping-generations model with production*

Hideo Konishi[1] **and Fernando Perera-Tallo**[2]

[1] Department of Economics, Southern Methodist University, Dallas, TX 75275-0496, USA
[2] Innocenzo Gasparini Institute for Economic Research, Abbazia di Mirasole, I-20090 Opera (MI), ITALY

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Summary. This paper establishes an existence theorem of a non-trivial (positive capital stock) steady–state equilibrium in Diamond's (1965) overlapping-generations model with production by employing the steady–state consumption curve introduced in Ihori (1978). The assumptions on preferences and production technologies that ensure the existence of a nontrivial steady–state equilibrium are separated from each other, unlike in Galor and Ryder (1989). We also provide two simple examples which illustrate the importance of two conditions in the theorem.

1 Introduction

Recently, Galor and Ryder [4] provided sufficient conditions for the existence of a nontrivial (positive capital) steady–state (S–S) equilibrium in the Diamond's [3] overlapping generation economy with production. However, since these conditions are *joint* requirements on the saving function and the production function, it is not easy to check whether or not their conditions are satisfied in an economy. In this paper, we provide an alternative existence theorem of a nontrivial S–S equilibrium by using two *separate* conditions on the utility and the production functions so that we can easily check whether or not our conditions are satisfied in an economy. The idea of the proof is based on the usage of the S–S consumption curve in Ihori [5], which is the collection of the feasible S–S consumption plans. We provide two simple examples which illustrate the importance of our conditions. As a corollary of the theorem, we

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Correspondence to: H. Konishi

obtain a convenient result for applications: There exists a nontrivial S–S equilibrium, if the utility function is homothetic and the production function satisfies *the nonvanishing labor share* (NLS), which requires that when the capital stock goes to zero labor share does not go to zero. All the proofs are collected in the appendix.

2 The model and the result

Diamond's [3] model is summarized as follows. Consider a perfectly competitive economy with infinite discrete time. At any period of time a single good is produced using two factors, capital and labor. The endowment of labor at time t, L_t, is exogenously given by

$$L_t = (1+n)L_{t-1}, \tag{1}$$

where $n > -1$ is the rate of population growth. The endowment of capital at time t, K_t, is equal to the resources not consumed in the preceding period,

$$K_t = Y_{t-1} + K_{t-1} - C_{t-1}, \tag{2}$$

where Y_{t-1} and C_{t-1} are the aggregate production and consumption, respectively, at time $t-1$. Production occurs within a period according to a constant returns to scale production function which is invariant through time. The output produced at time t, Y_t, is governed by a production function with constant returns to scale,

$$Y_t = F(K_t, L_t) = L_t f(k_t); \quad k_t = K_t/L_t. \tag{3}$$

The production function f is continuous, concave, strictly monotonic, and twice continuously differentiable for $k > 0$. These conditions imply the following:

$$f'(k) > 0, \quad f''(k) \leq 0, \quad \text{for } k > 0, \tag{4}$$

where $f'(k) = df(k)/dk$, and $f''(k) = d^2 f(k)/dk^2$. To concentrate on a more interesting case, we assume $f(0) = 0$.

Competition in the markets for capital and labor services ensures that each factor is paid its marginal product. Hence,

$$r_t = f'(k_t), \tag{5}$$

$$w_t = f(k_t) - k_t f'(k_t), \tag{6}$$

where w_t and r_t are the wage and rental rate respectively, at time t; output is the numeraire.

In every period t, L_t individuals are born. Individuals are identical within as well as across time. Individuals live for two periods. In the first period they work and earn the competitive market wage w_t, and in the second they retire from the labor market. Individuals born at time t are characterized by their intertemporal utility function $u(c_t^1, c_{t+1}^2)$ defined over non-negative consumption during the first and second periods of their lives. The intertemporal utility function is twice continuously differentiable, and quasi-concave on the consumption set $\mathbb{R}_+^2$. We assume that $u_i(c) > 0$ for any $c \in \mathbb{R}_{++}^2$, where $u_i(c) = \partial u(c)/\partial c^i$ for $i = 1, 2$. This implies the strict monotonicity of preferences in $\mathbb{R}_{++}^2 (u(c) > u(c')$ if $c, c' \in \mathbb{R}_{++}^2$ and $c > c')$.[1] For simplicity, we assume $\lim_{c^1 \to 0} u_1(c^1, c^2) = \infty$ for $c^2 > 0$, and $\lim_{c^2 \to 0} u_2(c^1, c^2) = \infty$ for $c^1 > 0$. These are boundary conditions. We assume perfect foresight by consumers as in Diamond [3] and in Galor and Ryder [4]. This implies that consumers born at time t have the following budget constraint:

$$c_t^1 + c_{t+1}^2/(1 + r_{t+1}) = w_t. \tag{7}$$

Consumers maximize their utility subject to (7). Finally, capital accumulation is described by the following equation:

$$k_{t+1} = (w_t - c_t^1)/(1 + n). \tag{8}$$

Note that $(w_t - c_t^1)$ denotes the saving by young generation at period t.

Now, we will introduce the S–S consumption curve in Ihori [5]. The idea behind the proof of the existence theorem entirely relies on this curve. The S–S consumption curve is the collection of the feasible S–S consumption plans which are consistent with consumers' budget constraints. Using equations (5)–(8) and the S–S conditions $k_t = k_{t+1} = k$, we can find the candidate of a feasible S–S consumption plan $(\mathbf{c}^1(k), \mathbf{c}^2(k)) \in \mathbb{R}^2$ for each capital stock level $k \in \mathbb{R}_{++}$:

$$\mathbf{c}^1(k) = f(k) - f'(k)k - (1 + n)k, \tag{9}$$

$$\mathbf{c}^2(k) = (1 + n)(k + f'(k)k). \tag{10}$$

Since the consumption set is $\mathbb{R}_+^2$, the set of feasible S–S consumption plans is characterized by the intersection of $\mathbb{R}_+^2$ and the image of the function $\mathbf{c}\colon \mathbb{R}_+ \to \mathbb{R}^2$, where $\mathbf{c}(k) = (\mathbf{c}^1(k), \mathbf{c}^2(k))$, where $\mathbf{c}(0) = \lim_{k \to 0} \mathbf{c}(k) = (0, 0)$. Define the set of feasible S–S consumption plans by $Z = \mathbf{c}(\mathbb{R}_+) \cap \mathbb{R}_+^2$. We can prove the compactness of Z by using the arguments in Boldrin [2] and Jones and Manuelli [6]. Let $Z_1 = \mathbf{c}([0, \bar{k}])$, where $\bar{k} = \inf\{k > 0 \colon \mathbf{c}^1(k) \leq 0\}$. Since Z is compact, Z_1 is also compact ($\mathbf{c}$ is continuous in $\mathbb{R}_{++}$). Roughly speaking, Z_1 denotes the first piece of Z (the piece which contains the origin, see Figure 1).

We will prove the existence of a nontrivial S–S equilibrium, by investigating the properties of Z_1. Galor and Ryder [4] propose a condition which is stronger than the Inada condition ($\lim_{k \to 0} f'(k) = \infty$):

The strengthened Inada condition (SI): $\lim_{k \to 0}[-kf''(k)] > 1 + n$.

Condition SI is necessary for the existence of a nontrivial S–S equilibrium in the sense that we can find a production function which only has a trivial S–S

[1] The definitions of inequalities on vectors $(c, c' \in \mathbb{R}^2)$ are the following: $c \geq c'$ if $c^i \geq c^{i\prime}$ for $i = 1, 2$; $c > c'$ if $c \geq c'$ and $c^i > c^{i\prime}$ for $i = 1$ or 2; $c \gg c'$ if $c^i > c^{i\prime}$ for $i = 1, 2$.

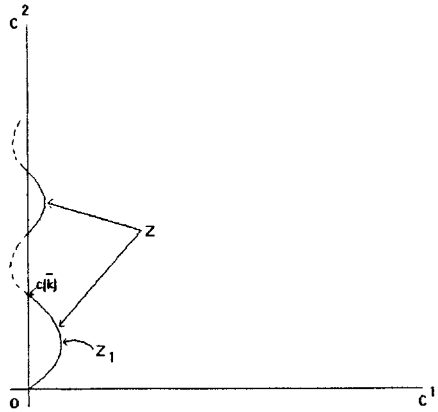

Figure 1.

equilibrium for any preferences without it (Galor and Ryder [4], Proposition 3). Condition SI requires $w(k) \equiv f(k) - f'(k)k > (1+n)k$ for k close to zero. Since the investment level cannot exceed the wage income in our model, SI guarantees the existence of a sustainable S–S capital stock. This observation leads us to the following result.

Proposition 1. If SI is satisfied, then $Z_1 \neq \{0\}$.

Our strategy to prove the existence of a nontrivial S–S equilibrium is to find out an element of Z_1 which satisfies the first order conditions of the consumer's maximization problem by using *the intermediate value theorem* (see Figure 2). Let MRS: $\mathbb{R}^2_{++} \to \mathbb{R}_+$ be such that $\text{MRS}(c) = u_1(c)/u_2(c)$ for $c \in \mathbb{R}^2_{++}$. Under our assumptions on the utility function, MRS(c) is a continuous function in $\mathbb{R}^2_{++}$, and satisfies $\lim_{c^1 \to 0} \text{MRS}(c) = \infty$ for $c^2 > 0$, and $\lim_{c^2 \to 0} \text{MRS}(c) = 0$ for

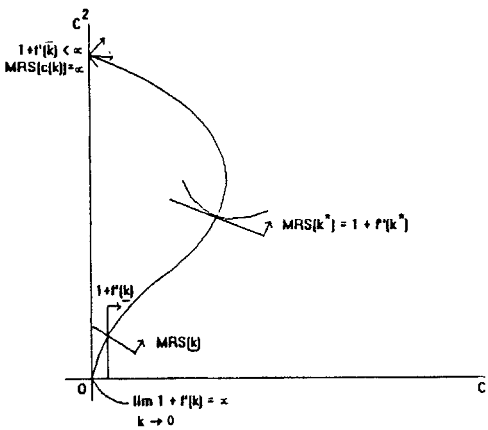

Figure 2.

$c^1 > 0$. The following proposition is general, but the assumptions on production and utility functions are not separated from each other.

Proposition 2. There exists a nontrivial S–S equilibrium under the following conditions: (i) SI and (ii) $\limsup_{k\to 0} \mathrm{MRS}(\mathbf{c}(k)) < \infty$.

Although condition (ii) looks awkward, it can be restated by using the Engel curves. Let $p \in (0, \infty)$ be the relative price of the first period consumption ($p = 1 + r$). The consumer's demand is defined by $x(p, I) \equiv \operatorname{argmax}_c u(c)$ s.t. $pc^1 + c^2 = I$, where $I \geq 0$ is an income level. The Engel curve for p is a collection of $x(p, I)$ for all income levels. Condition (ii) requires that there exists $p < \infty$ such that the Engel curve for p is above the S–S consumption curve locally around the origin (except for the origin).

Now, we will find a set of sufficient conditions which are separately stated on the utility and production functions. We introduce a condition on the production function:

The nonvanishing labor share (NLS): $\lim_{k\to 0}(f'(k)k/f(k)) \in [0, 1)$.

This condition says that when the capital stock goes to zero labor share does not go to zero. The role of NLS is to prevent the S–S consumption curve from converging to the vertical axis when the capital stock goes to zero (see the proof of Theorem 1 below). We can prove that NLS is stronger than SI by L'Hospital's rule. The following is our main result:

Theorem 1. There exists a nontrivial S–S equilibrium, if (i) NLS is satisfied, and (ii) the Engel curves are locally concave around the origin.[2]

Condition (ii) says that the second period consumption is not more income elastic than the first period consumption around the origin. The following two simple (and pretty natural) examples demonstrate the importance of these conditions. If one of them is dropped, there might not exist a nontrivial S–S equilibrium.

Example 1 (Figure 3).[3] Suppose that an economy is described by the following production and utility functions:

$$f(k) = \begin{cases} 0 & k = 0 \\ -\alpha k \ln k & 0 < k \leq \kappa \\ k^\beta & \kappa \leq k \end{cases}$$

where $\alpha = (1-\beta)\left[\exp\left(\frac{1}{1-\beta}\right)\right]^{1-\beta}$ and $\kappa = \left[\exp\left(\frac{1}{1-\beta}\right)\right]^{-1}$,

and $$u(c^1, c^1) = c^1 c^2.$$

If $\beta = 1/2$ and $n = 0$, then there is no nontrivial S–S equilibrium.

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[2] Condition (ii) can be replaced by the following weaker but less friendly condition:

For any $c \in \mathbb{R}^2_{++}$ and $\lambda > 0$, $\limsup_{\lambda\to 0} \mathrm{MRS}(\lambda c) < \infty$.

[3] At κ, f is continuously differentiable but is not twice continuously differentiable. However, it is possible to approximate f by a twice continuously differentiable function.

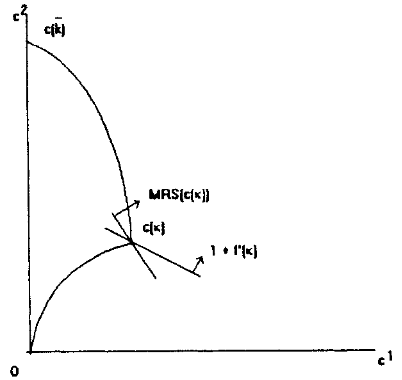

Figure 3.

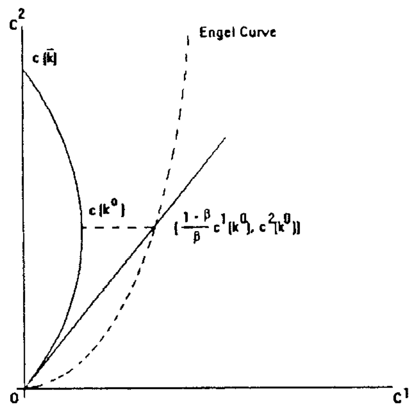

Figure 4.

In this example, f satisfies SI but does not satisfy NLS. It says that without NLS a nontrivial S–S equilibrium might fail to exist even for a Cobb–Douglas utility function.

Example 2 (Figure 4). Suppose that an economy is described by the following production and utility functions:

$$f(k) = k^{\beta},$$

and
$$u(c^1, c^2) = (c^1)^{\mu} + (c^2)^{\nu},$$

where $0 < \beta < 1$, and $0 < \mu < \nu < 1$. If $\beta = 0.8$, $\mu = 1/3$, $\nu = 2/3$, and $n = 0$, then there is no nontrivial S–S equilibrium.

In this example, the utility function u implies that for any $p < \infty$, the Engel curve is convex and converges to the horizontal axis. It says that a nontrivial

S–S equilibrium might fail to exist even for a Cobb–Douglas production function without local concavity of the Engel curves. The following corollary is a trivial consequence of Theorem 1. This result might be important for applications.

Corollary 1. There exists a nontrivial S–S equilibrium under the following conditions: (i) NLS, and (ii) the utility function is homothetic.

3 Concluding remarks

(a) In this paper, we assumed $f(0) = 0$ to deal with interesting cases. If $f(0) > 0$, then, under a mild requirement, we can prove $\mathbf{c}(0) = (f(0), 0)$, which assures the existence of a nontrivial S–S equilibrium.
(b) Variations of examples 1 and 2 can possibly generate interesting dynamics although they are quite simple: a poverty trap, and a multiplicity of S–S equilibria (Azariadis [1], Boldrin [2]).
(c) It is possible to prove the uniqueness of a nontrivial S–S equilibrium under the following conditions: (i) NLS, (ii) the utility function is homothetic, (iii) $f''(k) < 0$ for $k > 0$, (iv) $n \geq 0$, and (v) $(f'(k)k + k)/f(k)$ is non-decreasing. This is proved by assuring that $\mathrm{MRS}(\mathbf{c}(k))$ is monotonically increasing. The details are available upon request.

Appendix

Proof of Proposition 1. Differentiating $\mathbf{c}^1$ by k, we obtain,

$$d\mathbf{c}^1/dk = -f''(k)k - (1+n).$$

By SI, $\lim_{k \to 0} d\mathbf{c}^1/dk > 0$. By the definition of $\mathbf{c}^2 \colon \mathbb{R}_+ \to \mathbb{R}, \mathbf{c}^2(k) > 0$ for any $k > 0$. Since $\mathbf{c}(0) \in \mathbb{R}^2_+$, there exists $\hat{k} > 0$ such that $\mathbf{c}(k) \gg 0$ for all $k \in (0, \hat{k}]$. Therefore, $Z_1 \neq \{0\}$.//

Proof of Proposition 2. There exists $\bar{k} > 0$ such that $Z_1 = \mathbf{c}([0, \bar{k}])$ by (i) (Proposition 1). Note that f' is continuous, non-increasing, and satisfies $\lim_{k \to 0} f'(k) = \infty$. Therefore, in a neighborhood of the origin there exists $\underline{k} > 0$ such that $\mathrm{MRS}(\mathbf{c}(\underline{k})) < 1 + f'(\underline{k})$ by (ii). By the boundary conditions, $\lim_{c^1 \to 0} \mathrm{MRS}(c) = \infty$ for $c^2 > 0$. Since $\mathbf{c}^1(\bar{k}) = 0$ and $\mathbf{c}^2(\bar{k}) > 0$, $\mathrm{MRS}(\mathbf{c}(\bar{k})) = \infty > 1 + f'(\bar{k})$. Since $\mathrm{MRS}(\mathbf{c}(k))$ and f' are continuous on $\mathbb{R}_{++}$, by the intermediate value theorem there exists $k^* \in (\underline{k}, \bar{k})$ such that $\mathrm{MRS}(\mathbf{c}(k^*)) = 1 + f'(k^*)$ (see Figure 2). By construction of $\mathbf{c} \colon \mathbb{R}_+ \to \mathbb{R}^2, \mathbf{c}(k^*)$ satisfies the S–S condition. Obviously, the first order conditions are met. Hence, k^* is a nontrivial S–S capital stock level.//

Proof of Theorem 1. First, we will prove that NLS implies $\lim_{k \to 0} \mathbf{c}^2(k)/\mathbf{c}^1(k) < \infty$; i.e., the S–S consumption curve does not converge to the vertical axis. Let $\beta \equiv \lim_{k\to 0}(f'(k)k/f(k))$. By l'Hospital's rule, we get,

$$\begin{aligned}\gamma \equiv \lim_{k\to 0}\frac{\mathbf{c}^2(k)}{\mathbf{c}^1(k)} &= \lim_{k\to 0}\frac{(1+n)(k+f'(k)k)}{f(k)-f'(k)k-(1+n)k}\\ &= \lim_{k\to 0}\frac{(1+n)(1+f'(k)+f''(k)k)}{-f''(k)k-(1+n)}\\ &= \lim_{k\to 0}\frac{(1+n)(1/f''(k)k+f'(k)/f''(k)k+1)}{-1-(1+n)/f''(k)k}\end{aligned}$$

By l'Hospital's rule, $\lim_{k\to 0}f''(k)k/f'(k)=\beta-1$. Since NLS implies $\lim_{k\to 0}f'(k)=\infty$ (via SI), it also implies $\lim_{k\to 0}f''(k)k=-\infty$. Therefore, $\gamma=\lim_{k\to 0}(\mathbf{c}^2(k)/\mathbf{c}^1(k))= (1+n)\{1/(1-\beta)-1\}=(1+n)\beta/(1-\beta)<\infty$, since $\beta\in[0,1)$.

Let $\mu>0$ be such that the Engel curves are concave in $B_\mu(0)\equiv\{c\in\mathbb{R}^2_+:\|c\|<\mu\}$. By taking $\varepsilon\in(0,\mu)$ small enough, we can find $v\in[\gamma,\infty)$ such that for any $k>0$ such that $\|\mathbf{c}(k)\|<\varepsilon$, $\mathbf{c}^2(k)<v\mathbf{c}^1(k)$ by NLS. Since the preferences are strictly monotonic, $\mathrm{MRS}(\mathbf{c}^1(k),v\mathbf{c}^1(k))<\infty$. Let $p\in(\mathrm{MRS}(\mathbf{c}^1(k),v\mathbf{c}^1(k)),\infty)$. Since the Engel curve for p is concave and the preferences are convex, it is located above the S–S consumption curve locally around the origin (except for the origin itself). Since NLS implies SI, conditions in Proposition 2 are all met.//

Example 1 First, we will check that $f(k)$ satisfies SI, but not NLS. Since for $0<k<\kappa, f'(k)=-\alpha-\alpha\ln k$ and $f''(k)=-\alpha/k$, so $\lim_{k\to 0}(-f''(k)k)=\alpha$. If $\beta=1/2$, then $\alpha=1.35914>1+n=1$, and SI is satisfied. On the other hand, we can easily check that NLS is violated. To see this, we calculate the S–S consumption curve:

$$\mathbf{c}^1(k)=\begin{cases}(\alpha-1)k & k\le\kappa\\ (1-\beta)k^\beta-k & \kappa\le k\le\bar{k}\end{cases}$$

$$\mathbf{c}^2(k)=\begin{cases}-(\alpha-1)k-\alpha k\ln k & k\le\kappa\\ k+\beta k^\beta & \kappa\le k\le\bar{k}\end{cases}$$

where $\bar{k}=(1-\beta)^{1/(1-\beta)}$. By this, it is straightforward to calculate the limit of the ratio of $\mathbf{c}^1(k)$ and $\mathbf{c}^2(k)$:

$$\frac{\mathbf{c}^2(k)}{\mathbf{c}^1(k)}=\frac{-(\alpha-1)k-\alpha k\ln k}{(\alpha-1)k}=-1-\frac{\alpha}{\alpha-1}\ln k.$$

Since $\alpha>1$, $\lim_{k\to 0}(\mathbf{c}^2(k)/\mathbf{c}^1(k))=\infty$, and NLS is violated (see Theorem 1).

Now, we demonstrate that there is no nontrivial S–S equilibrium in this economy. Note that for some $k>0$ to be a S–S capital stock level, $\mathrm{MRS}(\mathbf{c}(k))\equiv u^1(\mathbf{c}^1(k))/u^2(\mathbf{c}^2(k))=1+f'(k)$. Thus, if we can show that for any $0<k<\bar{k}$, $\mathrm{MRS}(\mathbf{c}(k))>1+f'(k)$, then we are done. First, consider the interval $(0,\kappa]$. Since $\mathrm{MRS}(\mathbf{c}(k))=-1-\left(\frac{\alpha}{1-\alpha}\right)\ln k$, and $1+f'(k)=1-\alpha-\alpha\ln k$, the difference $\alpha-2-\alpha(2-\alpha)\ln k/(\alpha-1)$ is positive for all $k\in(0,\kappa)$ (note that

$\alpha = 1.35914$). Second, consider the interval $(\kappa, \bar{k}]$. It is easy to check that $\mathbf{c}^2(k)/\mathbf{c}^1(k)$ is monotonically decreasing in the interval, and it implies that $\mathrm{MRS}(\mathbf{c}(k))$ is monotonically increasing. On the other hand, obviously $1 + f'(k)$ is decreasing. As a result, $\mathrm{MRS}(\mathbf{c}(k)) - 1 - f'(k)$ is monotonically increasing in the interval $(\kappa, \bar{k})$. Since $\mathrm{MRS}(\mathbf{c}(\kappa)) > 1 + f'(\kappa)$, $\mathrm{MRS}(\mathbf{c}(k)) > 1 + f'(k)$ for any $k \in (0, \bar{k})$. Hence, there is no nontrivial S–S equilibrium.//

Example 2 It is easily to check that the Engel curves are convex and converge to the horizontal axis in this example. Let k^0 be such that $\mathbf{c}^1(k^0) \geq \mathbf{c}^1(k)$ for any $k \in (0, \bar{k})$. Then, $d\mathbf{c}^1(k)/dk = 0$ at $k = k^0$. Thus, $-f''(k^0)k^0 - 1 = 0$ implies $k^0 = (\beta(1-\beta))^{1/(1-\beta)}$. Note that $\lim_{k \to 0} \mathbf{c}^2(k)/\mathbf{c}^1(k) = \beta/(1-\beta)$. Convexity of preferences implies $\mathrm{MRS}(\mathbf{c}(k)) > \mathrm{MRS}\left(\frac{1-\beta}{\beta}\mathbf{c}^2(k), \mathbf{c}^2(k)\right)$. Since the Engel curves are upward sloping, $\mathrm{MRS}(\mathbf{c}(k)) > \mathrm{MRS}\left(\frac{1-\beta}{\beta}\mathbf{c}^2(k^0), \mathbf{c}^2(k^0)\right)$ for any $k \in (0, \bar{k})$. Let $\rho(k) \equiv \mathrm{MRS}\left(\frac{1-\beta}{\beta}\mathbf{c}^2(k), \mathbf{c}^2(k)\right)/(1 + f'(k))$. Obviously, if $\rho(k^0) > 1$, then there is no S–S equilibrium for any $k \in [k^0, \bar{k}]$. A messy calculation says that

$$\rho(k) \equiv (\mu/v)\left[\left(\frac{1-\beta}{\beta}\right)^{1-\mu}(1 + \beta k^{\beta-1})^{1+v-\mu}k^{v-\mu}\right]^{-1}.$$

By substituting the parameter values, we can check $\rho(k^0) = 2.45 > 1$.

Finally, what is left for us is to show that there is no S–S equilibrium for any $k \in (0, k^0)$. For this, it is enough to show that $\rho(k)$ is decreasing. Taking the derivative of the contents of the bracket, we obtain,

$$\begin{aligned}(1 + v - \mu)\beta(\beta - 1)k^{\beta-2}(1 + \beta k^{\beta-1})^{v-\mu}k^{v-\mu} + (1 + \beta k^{\beta-1})^{1+v-\mu}(v-\mu)k^{v-\mu-1} \\ = (1 + \beta k^{\beta-1})^{v-\mu}k^{v-\mu-1}\{-(1 + v - \mu)\beta(1-\beta)k^{\beta-1} + (1 + \beta k^{\beta-1})(v-\mu)\}.\end{aligned}$$

If this term is positive, then $\rho'(k) < 0$. Concentrating on the brace term, we obtain,

$$\{\cdot\} = \beta k^{\beta-1}[(v-\mu)\beta - (1-\beta)] + (v-\mu).$$

Our parameter values tell us $\{\cdot\} > 0$.//

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