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Sample function behavior of increasing processes of class L

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Summary. We consider increasing processes $\{X(t) : t \geqq 0\}$ of class L, that is, increasing self-similar processes with independent increments. Let $h(t)$ be an increasing positive function on $(0, \infty)$ with $h(0+) = 0$ and $h(\infty) = \infty$. By virtue of the zero-one laws, there exists c (resp. C) $\in [0, \infty]$ such that $\liminf$ (resp. $\limsup$) $X(t)/h(t) = c$ (resp. C) a.s. both as t tends to 0 and as t tends to ∞. We decide a necessary and sufficient condition for the existence of $h(t)$ with c or $C = 1$ and explicitly construct $h(t)$ in case $h(t)$ exists with c or $C = 1$. Moreover, we give a criterion to classify functions $h(t)$ with c (or C) $= 0$ and $h(t)$ with c (or C) $= \infty$ in case $h(t)$ does not exist with c (or C) $= 1$.

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1 Introduction and results

Distributions of class L on $\mathbb{R}^d$ are defined in Gnedenko and Kolmogorov [8] for $d = 1$ and in Sato [11] for general d. A necessary and sufficient condition for a distribution on $\mathbb{R}^d$ to be of class L is that it is self-decomposable. Sato [12] introduces self-similar processes with independent increments and proves that their distributions are of class L and that conversely, for each distribution η of class L, there exists a unique (up to equivalence in law) self-similar process with independent increments such that its distribution at time 1 is η. So he calls a self-similar process with independent increments a *process of class L*. Moreover he investigates in [12] the sample function behavior of increasing processes $\{X(t)\}$ of class L, comparing it with increasing self-decomposable processes $\{Y(t)\}$ under the assumption that $X(1)$ and $Y(1)$ have the same distribution. In this paper we shall extend his results on the sample function behavior of increasing processes $\{X(t)\}$ of class L not only in the case of limsup of $X(t)/h(t)$ but also in the case of liminf of $X(t)/h(t)$ for positive increasing functions $h(t)$ both as t tends to 0 and as t tends to ∞. In case $\{X(t)\}$ is an increasing stable process, $\{X(t)\}$ and $\{Y(t)\}$ are equivalent in law and the problems which are treated in this paper were already solved by Fristedt [4, 6]. Our key lemmas (Lemmas 3.1, 3.2, 4.3, and 4.5), which give estimates of the values of liminf and limsup of $X(t)/h(t)$, are originally due to Sato [12] but some of them are improved technically. The unimodality and some analytical properties of distributions of class L, which are proved by Sato and Yamazato [13], Wolfe [16] and Yamazato [17], play important roles in our discussion. Also an integral equation of the density function of one-sided infinitely divisible distribution, which is introduced by Steutel [15], is employed as a basic tool.

A stochastic process $\{X(t) : t \geqq 0\}$ with values in $\mathbb{R}^d$, which is defined on a probability space $(\Omega, \mathscr{F}, P)$, is said to be a *process of class L* with exponent H if it satisfies the following three conditions (i), (ii), and (iii):

(i) $\{X(t)\}$ is self-similar with exponent H, that is, for every $c > 0, \{X(ct)\}$ and $\{c^H X(t)\}$ have the identical finite-dimensional distributions.

(ii) $\{X(t)\}$ has independent increments, that is, $X(t_1) - X(t_0)$, $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent for $0 \leqq t_0 < t_1 < t_2 < \cdots < t_n$.

(iii) Almost surely $X(t)$ is right-continuous in $t \geqq 0$ and has left limits in $t > 0$.

Here H is a positive constant. Note that a process of class L is not assumed to have stationary increments. A probability measure μ on $\mathbb{R}^d$ is said to be *self-decomposable* if, for every $a \in (0, 1)$, there exists a probability measure μ_a such that the characteristic functions $\hat{\mu}(z)$ and $\hat{\mu}_a(z)$ satisfy

$$\hat{\mu}(z) = \hat{\mu}(az)\hat{\mu}_a(z) \quad \text{for } z \in \mathbb{R}^d . \tag{1.1}$$

A stochastic process $\{Y(t) : t \geqq 0\}$ with values in $\mathbb{R}^d$ is said to be a self-decomposable process if it is a Lévy process and the distribution of $Y(t)$ is self-decomposable for each t. In this paper we use the words "increase" and "decrease" in the wide sense. From now on, let $d = 1$ and let $\{X(t)\}$ be an increasing process of class L with exponent 1, which is not a deterministic motion. Note that the exponent of a process of class L can be changed by time change. Let μ be the distribution of $X(1)$. Then μ is self-decomposable by Sato [12] and the characteristic function $\hat{\mu}(z)$ is represented as

$$\hat{\mu}(z) = \exp(\psi(z)) \qquad \psi(z) = i\gamma_0 z + \int_0^\infty (e^{izx} - 1)x^{-1}k(x)\, dx , \tag{1.2}$$

where $\gamma_0 \geqq 0$ and $k(x)$ is a nonnegative decreasing function on $(0, \infty)$ with $\int_0^\infty (1+x)^{-1}k(x)dx < \infty$. Denote $\lambda = k(0+)$. If $\lambda < \infty$, we define the function $K_\lambda(x)$ on $(0, \infty)$ as

$$K_\lambda(x) = (x \wedge 1)^\lambda \exp\left(\int_{x \wedge 1}^1 (\lambda - k(u))u^{-1}\, du \right) . \tag{1.3}$$

Define the functions $F(x), G(x)$, and $G_a(x)$ for $a \in (0, 1)$ as

$$F(x) = P(X(1) \leqq x), \qquad G(x) = P(X(1) \geqq x) , \tag{1.4}$$

and

$$G_a(x) = P(X(1) - X(a) \geqq x)\,. \tag{1.5}$$

Let $f(x)$ and $g(x)$ be measurable functions on $(0, \infty)$. A relation $f(x) \sim g(x)$ is defined as $\lim_{x\to\infty} f(x)/g(x) = 1$. A relation $f(x) \asymp g(x)$ is defined as $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$ and $\liminf_{x\to\infty} |f(x)/g(x)| > 0$. Let $f_0(x)$ be a measurable function on $(0, \infty)$, which is positive on (A, ∞) for some $A \geqq 0$. A function $f_0(x)$ is said to be slowly varying if, for every $\rho > 0$, $\lim_{x\to\infty} f_0(\rho x)/f_0(x) = 1$. A function $f_0(x)$ is said to be rapidly varying if, for every $\rho > 1, \lim_{x\to\infty} f_0(\rho x)/f_0(x) = 0$ or ∞. A function $f_0(x)$ is said to belong to the class OR if, for each $\rho > 1, \limsup_{x\to\infty} f_0(\rho x)/f_0(x) < \infty$ and $\liminf_{x\to\infty} f_0(\rho x)/f_0(x) > 0$. Denote by $\mathscr{H}_0$ the totality of positive increasing functions $h(t)$ on $(0, \infty)$ with $h(0+) = 0$ and $\lim_{t\to\infty} h(t) = \infty$. By virtue of the zero-one laws, there are c (resp. C) $\in [0, \infty]$ for $h(t) \in \mathscr{H}_0$ such that

$$\liminf (\text{resp. } \limsup) X(t)/h(t) = c \ (\text{resp. } C) \quad \text{a.s.} \tag{1.6}$$

both as time t tends to 0 and as t tends to ∞. Main problems with which we shall be concerned are as follows:

(i) What is a necessary and sufficient condition for the existence of $h(t) \in \mathscr{H}_0$ satisfying (1.6) with c or $C = 1$?

(ii) In case $h(t)$ satisfying (1.6) with c or $C = 1$ exists, how is $h(t)$ given?

(iii) In case $h(t)$ satisfying (1.6) with c or $C = 1$ does not exist, what is a criterion to classify functions $h(t)$ with c (or $C) = 0$ and $h(t)$ with c (or $C) = \infty$?

In the case of liminf we shall answer the problems above completely. Denote by $\mathscr{H}_1$ the totality of functions $h(t) \in \mathscr{H}_0$ such that $h(t)/t$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. In the case of lim sup we shall answer the problems above for functions $h(t)$ in $\mathscr{H}_1$. Namely our results are as follows. The functions $H_0(t)$ and $H_1(t)$ below are explicitly constructed in Sect. 2. The function $H_0(t)$ belongs to $\mathscr{H}_0$ and the function $H_1(t)$ to $\mathscr{H}_1$.

Theorem 1.1 (i) *If* $\gamma_0 > 0$, *then*

$$\liminf X(t)/t = \gamma_0 \quad \text{a.s.} \tag{1.7}$$

both as $t \downarrow 0$ *and* $t \to \infty$.

(ii) *If* $\gamma_0 = 0$ *and* $\lambda = \infty$, *then*

$$\liminf X(t)/H_0(t) = 1 \quad \text{a.s.} \tag{1.8}$$

both as $t \downarrow 0$ *and* $t \to \infty$.

Theorem 1.2 *Let* $h(t) \in \mathscr{H}_0$. *Suppose that* $\gamma_0 = 0$ *and* $\lambda < \infty$.

(i) *If*

$$\int_0^1 t^{-1} K_\lambda(h(t)/t)\, dt < \infty \,(\text{resp.} = \infty)\,, \tag{1.9}$$

then

$$\liminf_{t\downarrow 0} X(t)/h(t) = \infty \ (\text{resp.} = 0) \quad \text{a.s.} \tag{1.10}$$

(ii) *If*

(1.11) $$\int_1^{\infty} t^{-1} K_\lambda(h(t)/t)\,dt < \infty \text{ (resp.} = \infty) ,$$

then

(1.12) $$\liminf_{t\to\infty} X(t)/h(t) = \infty \text{ (resp.} = 0) \quad \text{a.s.}$$

Remark. 1.1 Let $h(t) \in \mathscr{H}_1$. Define $\tilde{X}(t) = X(t) - \gamma_0 t$. Note from Proposition 4.5 of Sato [12] or Lemma 4.3 that

$$\limsup X(t)/t = \infty \quad \text{and} \quad \limsup X(t)/h(t) = \limsup \tilde{X}(t)/h(t) \quad \text{a.s.}$$

both as $t \downarrow 0$ and as $t \to \infty$. Thus, in the case of lim sup of $X(t)/h(t)$ for $h(t) \in \mathscr{H}_1$, we may assume without loss of generality that $\gamma_0 = 0$.

Theorem 1.3 *Suppose that* $k(x) \notin$ OR *and* $\gamma_0 = 0$. *Then we have*

(1.13) $$\limsup X(t)/H_1(t) = 1 \quad \text{a.s.}$$

both as $t \downarrow 0$ *and* $t \to \infty$.

Remark. 1.2 Suppose that $\gamma_0 = 0$. If $k(x)$ is either rapidly varying or there is $b > 0$ such that $k(x) = 0$ on (b, ∞), then $k(x) \notin$ OR and (1.13) holds both as $t \downarrow 0$ and $t \to \infty$.

Theorem 1.4 *Let* $h(t) \in \mathscr{H}_1$. *Suppose that* $k(x) \in$ OR.
(i) *If*

(1.14) $$\int_0^1 t^{-1} k(h(t)/t)\,dt < \infty \text{ (resp.} = \infty) ,$$

then

(1.15) $$\limsup_{t\downarrow 0} X(t)/h(t) = 0 \text{ (resp.} = \infty) \quad \text{a.s.}$$

(ii) *If*

(1.16) $$\int_1^{\infty} t^{-1} k(h(t)/t)\,dt < \infty \text{ (resp.} = \infty) ,$$

then

(1.17) $$\limsup_{t\to\infty} X(t)/h(t) = 0 \text{ (resp.} = \infty) \quad \text{a.s.}$$

Remark. 1.3 If there are a slowly varying function $l(x)$ on $(0, \infty)$ and a nonnegative number α such that

(1.18) $$k(x) \asymp x^{-\alpha} l(x) ,$$

then $k(x) \in$ OR and Theorem 1.4 holds.

Organization of this paper is as follows. In Sect. 2 we define the functions $H_0(t)$, $H_1(t)$, and $h_\alpha(t)$, and state known facts which are necessary for the proof of the theorems above. In Sect. 3 we prove Theorems 1.1, 1.2, and one more theorem, which is a law of iterated logarithm type. In Sect. 4 we prove Theorems 1.3 and 1.4. In Sect. 5 we give an example of the theorems above. Section 6 is an appendix; we prove there Proposition 4.1, which is stated but not proved in Sect. 4, together with a Tauberian theorem.

We add that sample function behavior of increasing Lévy processes $\{Z(t)\}$ is investigated in the case of limsup of $Z(t)/h(t)$ by Fristedt [5, 6] and in the case of liminf of $Z(t)/h(t)$ by Fristedt and Pruitt [7]. But the latter case is not solved completely even for increasing self-decomposable processes. Comparison with the sample function behavior of increasing self-decomposable processes will be discussed in the future.

2 Preliminaries

We continue to assume that $\{X(t)\}$ is an increasing process of class L with exponent 1, which is not a deterministic motion, and μ is the distribution of $X(1)$. A probability measure η on $\mathbb{R}$ is said to be unimodal with mode a if

$$\eta(dx) = f(x)\,dx + c\delta_a(dx)\,, \tag{2.1}$$

where $c \geqq 0, \delta_a(dx)$ is the delta measure at a, and $f(x)$ is increasing on $(-\infty, a)$ and decreasing on (a, ∞). If η is unimodal, we denote the mode by a_η; we choose the least mode as a_η when the set of modes of η is not a one point but a closed interval. At first we state unimodality of μ. A remarkable fact that all self-decomposable distributions are unimodal is proved by Yamazato [17]. But we do not use the two-sided case.

Lemma 2.1 (Sato and Yamazato [13] and Wolfe [16]) *The distribution μ is absolutely continuous and unimodal. Denote a density function of μ by $f(x)$. Then the following holds*:

(i) $f(x) = 0$ *on* $(-\infty, \gamma_0)$ *and* $f(x) > 0$ *on* (γ_0, ∞).
(ii) *If* $\gamma_0 = 0$ *and* $0 < \lambda \leqq 1$, *then* $a_\mu = 0$.
(iii) *If* $\gamma_0 = 0$ *and* $1 < \lambda \leqq \infty$, *then* $a_\mu > 0$ *and* $f(x)$ *is continuous on* $\mathbb{R}$.

Hereafter, let $f(x)$ be the density function of μ.

Lemma 2.2 (Steutel [15] or Sato and Yamazato [13]) *Suppose that* $\gamma_0 = 0$. *Then we have*

$$xF(x) = \int_0^x F(u)\,du + \int_0^x F(x-u)k(u)\,du \tag{2.2}$$

and

$$xf(x) = \int_0^x f(x-u)k(u)\,du\,. \tag{2.3}$$

We define a constant A_λ as

(2.4)

$$A_\lambda = \Gamma(\lambda+1)^{-1}\exp\left\{\lambda\int_0^1 (e^{-u}-1)u^{-1}\,du + \lambda\int_1^\infty e^{-u}u^{-1}\,du - \int_1^\infty k(u)u^{-1}\,du\right\}.$$

If $\lambda < \infty$, then the behavior of $F(x)$ as $x \downarrow 0$ is determined by $K_\lambda(x)$ as follows.

Lemma 2.3 (Sato and Yamazato [13]) *Suppose that* $\gamma_0 = 0$ *and* $\lambda < \infty$. *Then we have*

$$F(1/x) \sim A_\lambda K_\lambda(1/x)\,. \tag{2.5}$$

Now let us define the function $H_0(t)$ under the assumption that $\gamma_0 = 0$ and $\lambda = \infty$. Noting (iii) of Lemma 2.1, we can choose a real number b such that

$$0 < b < (2^{-1}a_\mu) \wedge 1, \qquad k(b) \geqq 2 \quad \text{and} \quad 4f(2b) < 1\,. \tag{2.6}$$

Further we can find a continuously differentiable function $k_0(x)$ on $(0,b)$ such that $k_0(0+) = \infty$, $k(x) \geqq k_0(x) > 0$, and $k_0'(x) < -1$ on $(0,b)$. Define a positive function $g_0(x)$ on $(0,b)$ as

$$g_0(x) = -\int_x^b \frac{k_0'(u)}{F(u)k_0(u)^2}\,du\,. \tag{2.7}$$

Since $g_0(x)$ is strictly decreasing on $(0,b)$ and $g_0(0+) = \infty$ (see Lemma 2.4), there exists the inverse function $g_0^{-1}(x)$ of $g_0(x)$ such that $g_0^{-1}(x)$ is positive and strictly decreasing on $(0,\infty)$. We define $H_0(t)$ as

$$H_0(t) = t g_0^{-1}(|\log t|)\,. \tag{2.8}$$

Lemma 2.4 *Suppose that* $\gamma_0 = 0$ *and* $\lambda = \infty$. *Then* $g_0(0+) = \infty$ *and* $H_0(t) \in \mathscr{H}_0$.

Proof. We find from (2.3) and (iii) of Lemma 2.1 that

$$a_\mu f(a_\mu) \geqq x f(x) = \int_0^x f(x-u)k(u)\,du \geqq k_0(x)F(x) \tag{2.9}$$

for $0 < x \leqq b$. Hence we see that

$$g_0(0+) \geqq -\int_0^b \frac{k_0'(u)}{a_\mu f(a_\mu)k_0(u)}\,du = \infty\,.$$

Obviously the function $H_0(t)$ is positive on $(0,\infty)$ and increasing on $(0,1)$, and $H_0(0+) = 0$. Let $u(t) = t^{-1}H_0(t)$. Then we find from (2.8) that

$$0 < u(t) \leqq b \quad \text{on } (0,\infty) \quad \text{and} \quad g_0(u(t)) = \log t \quad \text{on } [1,\infty)\,.$$

Differentiating the equation above, we get

$$H_0'(t) = u(t) + \frac{F(u(t))k_0(u(t))^2}{k_0'(u(t))} \quad \text{on } [1,\infty)\,. \tag{2.10}$$

Noting that $2u(t) \leqq 2b \leqq a_\mu$, we see as in (2.9) that

$$2u(t)f(2u(t)) \geqq \int_0^{u(t)} f(2u(t)-y)k(y)\,dy \geqq u(t)f(u(t))k_0(u(t)) \quad \text{on } [1,\infty)\,.$$

Hence, using (2.9) and $4f(2b) < 1$,

$$\frac{F(u(t))k_0(u(t))^2}{k_0'(u(t))} \geqq -u(t)f(u(t))k_0(u(t)) > -2^{-1}u(t) \quad \text{on } [1,\infty).$$

Therefore, we obtain from (2.10) that $H_0'(t) > 2^{-1}u(t) > 0$ on $[1,\infty)$ and $\lim_{t\to\infty} H_0(t) = \infty$. It follows that $H_0(t) \in \mathscr{H}_0$. The proof of Lemma 2.4 is complete.

Next we consider the following condition of regular variation:

(R_α) $k(x) = x^{-\alpha}l(1/x)$ on $(0,\infty)$, where $0 < \alpha < 1$, and $l(x)$ is slowly varying as $x \to \infty$ satisfying that, for some $\rho > 1$,

$$(l(\rho x)/l(x) - 1)\log l(x) \to 0 \quad \text{as } x \to \infty. \tag{2.11}$$

We define a slowly varying function $\tilde{l}(x)$ as

$$x^{-\alpha(1-\alpha)}\tilde{l}(x) = \sup_{x \leqq u < \infty} u^{-\alpha(1-\alpha)}l(u). \tag{2.12}$$

Let $v = ((2-\alpha)/\alpha) \vee \log(|\log t| \vee 1)$. Under the condition (R_α) on $k(x)$ we define a function $h_\alpha(t)$ on $(0,\infty)$ as

$$h_\alpha(t) = t((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}\Gamma(1-\alpha)^{1/\alpha}v^{-(1-\alpha)/\alpha}\tilde{l}(v^{1/\alpha})^{1/\alpha} \quad \text{on } (0,1] \tag{2.13}$$

and

$$h_\alpha(t) = h_\alpha(1) \vee [t((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}\Gamma(1-\alpha)^{1/\alpha}v^{-(1-\alpha)/\alpha}l(v^{1/\alpha})^{1/\alpha}] \quad \text{on } (1,\infty).$$

Then obviously $h_\alpha(t) \in \mathscr{H}_0$. The following lemma is a direct consequence of Theorems 1.5.13, 2.3.3, and 8.2.2 of Bingham et al. [1].

Lemma 2.5 *Suppose that the condition (R_α) holds and that $\gamma_0 = 0$. Let $\beta = 1/(1-\alpha)$. Then we have*

$$-\log F(1/x) \sim (1-\alpha)\alpha^{-1}\Gamma(1-\alpha)^{\beta}x^{\alpha\beta}l(x^{\beta})^{\beta} \tag{2.14}$$

and

$$l(xl(x)^a) \sim l(x) \qquad \textit{for each } a \in \mathbb{R}. \tag{2.15}$$

We state Corollary 2.0.6 of Bingham et al. [1] as Lemma 2.6.

Lemma 2.6 *Let $g(x)$ be a positive decreasing function on $(0,\infty)$. Then $g(x) \in$ OR if and only if, for some $\rho > 1$, $\liminf_{x\to\infty} g(\rho x)/g(x) > 0$.*

Finally let us define the function $H_1(t)$ under the assumption that, for every $a \in (0,1)$, $G_a(x) \notin$ OR and $\gamma_0 = 0$. We shall see in Sect. 4 that if $k(x) \notin$ OR and $\gamma_0 = 0$, then this assumption holds. Denote $a_k = (k+2)^{-1}$ for integers $k \geqq 0$. Then, by Lemma 2.6, there are two sequences $\{u_k\}_{k=0}^{\infty}$ and $\{\rho_k\}_{k=0}^{\infty}$ satisfying that $u_0 = 1$, u_k is increasing and $\lim_{k\to\infty} u_k = \infty$, ρ_k is decreasing and $\lim_{k\to\infty} \rho_k = 1$, and

$$\sum_{k=0}^{\infty} G_{a_k}(u_k)[G_{a_k}(\rho_k^{-1}u_k)]^{-1} < \infty. \tag{2.16}$$

Denote $r_k = [G_{a_k}(\rho_k^{-1}u_k)]^{-1}$. Define a decreasing sequence $\{c_k\}_{k=0}^{\infty}$ such that $c_0 = 1$,

$$\log(c_{2k}/c_{2k+1}) = r_k, \quad \text{and} \quad c_{2k+1}u_k = c_{2k+2}u_{k+1} \quad \text{for } k \geqq 0 . \tag{2.17}$$

Also define an increasing sequence $\{d_k\}_{k=0}^{\infty}$ such that $d_0 = 1$ and

$$\log(d_{k+1}/d_k) = r_k \quad \text{for } k \geqq 0 . \tag{2.18}$$

We define $H_1(t)$ as follows:

$$\begin{aligned} H_1(t) &= t u_k \quad \text{on } [c_{2k+1}, c_{2k}] \quad \text{and} \quad [d_k, d_{k+1}) , \\ H_1(t) &= H_1(c_{2k+1}) = c_{2k+1}u_k \quad \text{on } [c_{2k+2}, c_{2k+1}] \end{aligned} \tag{2.19}$$

for all integers $k \geqq 0$. Then obviously $H_1(t) \in \mathscr{H}_1$.

3 The case of lim inf $X(t)/h(t)$

We shall prove all lemmas and all theorems in this section only for $t \downarrow 0$; the proof for $t \to \infty$ is similar and omitted. At first we shall prove two basic lemmas which play essential roles for the proof of Theorems 1.1 and 1.2.

Lemma 3.1 *Let* $h(t) \in \mathscr{H}_0$.

(i) *If*

$$\int_0^1 t^{-1}F(h(t)/t)\,dt < \infty , \tag{3.1}$$

then

$$\liminf_{t\downarrow 0} X(t)/h(t) \geqq 1 \quad \text{a.s.} \tag{3.2}$$

(ii) *If*

$$\int_1^\infty t^{-1}F(h(t)/t)\,dt < \infty , \tag{3.3}$$

then

$$\liminf_{t\to\infty} X(t)/h(t) \geqq 1 \quad \text{a.s.} \tag{3.4}$$

Proof. Let a be an arbitrary real number in $(0,1)$. We have

$$P(X(t) \leqq h(t)) \geqq P(X(a^{n+2}) \leqq a^2 h(a^{n+1}))$$

for $a^{n+1} \leqq t < a^n$. Hence

$$\sum_{n=0}^{\infty} P(X(a^{n+2}) \leqq a^2 h(a^{n+1}))$$
$$\leqq \sum_{n=0}^{\infty} a^{-n}(1-a)^{-1} \int_{a^{n+1}}^{a^n} P(X(t) \leqq h(t))\, dt$$
$$\leqq (1-a)^{-1} \int_0^1 t^{-1} F(h(t)/t)\, dt < \infty .$$

So, by the first Borel–Cantelli lemma, we see that

$$X(a^{n+1}) > a^2 h(a^n) \quad \text{a.s.} \tag{3.5}$$

for all large n. Note that $X(a^{n+1}) > a^2 h(a^n)$ implies

$$X(t) > a^2 h(t) \quad \text{for } a^{n+1} \leqq t < a^n .$$

Hence we obtain (3.2) from (3.5) and from the arbitrariness of a in $(0, 1)$.

Lemma 3.2 *Let* $h(t) \in \mathscr{H}_0$.

(i) *If*

$$\int_0^1 t^{-1} F(h(t)/t)\, dt = \infty , \tag{3.6}$$

then

$$\liminf_{t \downarrow 0} X(t)/h(t) \leqq 1 \quad \text{a.s.} \tag{3.7}$$

(ii) *If*

$$\int_1^{\infty} t^{-1} F(h(t)/t)\, dt = \infty , \tag{3.8}$$

then

$$\liminf_{t \to \infty} X(t)/h(t) \leqq 1 \quad \text{a.s.} \tag{3.9}$$

Proof. Without loss of generality, we can assume that $\sup_{t>0} h(t)/t \leqq \gamma_0 + 1$ since, if necessary, we can change $h(t)$ by $h(t) \wedge ((\gamma_0 + 1)t)$ in (3.6) and (3.8). Let a be an arbitrary real number in $(0, 1)$. We have

$$P(X(t) \leqq h(t)) \leqq P(X(a^{n-1}) \leqq a^{-2} h(a^n)) \quad \text{for } a^{n+1} \leqq t < a^n .$$

Hence

$$\sum_{n=0}^{\infty} P(X(a^{n-1}) \leqq a^{-2} h(a^n)) \tag{3.10}$$
$$\geqq \sum_{n=0}^{\infty} a^{-n}(1-a)^{-1} \int_{a^{n+1}}^{a^n} P(X(t) \leqq h(t))\, dt$$
$$\geqq a(1-a)^{-1} \int_0^1 t^{-1} F(h(t)/t)\, dt = \infty .$$

Denote $A_n = \{\omega : X(a^{n-1}) \leqq a^{-2}h(a^n)\}$. Let m and n be integers satisfying $0 \leqq m \leqq n-1$ and define

$$S(m,n) = P(X(a^{j-1}) - X(a^{n-1}) > a^{-2}h(a^j) \text{ for all } j \text{ satisfying } m \leqq j \leqq n-1)\,.$$

We shall prove the following assertion.

(a) There exist increasing sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ such that $0 \leqq m_k \leqq n_k - 1$, and $m_k, n_k \to \infty$ and $S(m_k, n_k) \to 0$ as $k \to \infty$.

Suppose, on the contrary, that there exists $\delta > 0$ such that $S(m,n) \geqq \delta$ for all sufficiently large integers m and n. Then we see from the independence of increments that

$$1 \geqq P\left(\bigcup_{n=m+1}^{\infty} A_n\right) \geqq \sum_{n=m+1}^{\infty} P\left(\left(\bigcup_{j=m}^{n-1} A_j\right)^c \cap A_n\right)$$
$$\geqq \sum_{n=m+1}^{\infty} P(A_n)S(m,n) \geqq \delta \sum_{n=m+1}^{\infty} P(A_n)\,.$$

This contradicts (3.10) and hence the assertion (a) is true. Denote $S(k) = S(m_k, n_k)$ and

$$T(k) = P(X(a^{j-1}) > a^{-2}h(a^j) \text{ for all } j \text{ satisfying } m_k \leqq j \leqq n_k - 1)\,.$$

Define

$$p_k(x) = P(X(a^{j-1}) - X(a^{n_k-2}) > a^{-2}h(a^j) - x \text{ for all } j \text{ satisfying } m_k \leqq j \leqq n_k - 2)\,.$$

Note that $0 \leqq p_k(x) \leqq 1$ and $p_k(x)$ is increasing in x. We shall prove that

$$\lim_{k\to\infty} T(k) = 0\,. \tag{3.11}$$

Denote the distribution of $X(1) - X(a)$ by η and the distribution of $X(a^{n_k-2}) - X(a^{n_k-1})$ by η_k for $k \geqq 0$. Let $v_k = a^{-2}h(a^{n_k-1})$ and let $w_k = a^{-n_k}h(a^{n_k-1})$. Then $S(k)$ and $T(k)$ are expressed as

$$S(k) = \int_{v_k}^{\infty} p_k(x)\eta_k(dx) = \int_{w_k}^{\infty} p_k(a^{n_k-2}x)\eta(dx) \tag{3.12}$$

and

$$T(k) = \int_{w_k}^{\infty} p_k(a^{n_k-2}x)\mu(dx)\,. \tag{3.13}$$

Note that

$$S(k) \geqq p_k(a^{n_k-2}N)\int_{N}^{\infty} \eta(dx) > 0 \quad \text{for } N \geqq (\gamma_0 + 1)a\,.$$

Hence the assertion (a) implies that

$$\lim_{k\to\infty} p_k(a^{n_k-2}N) = 0 \quad \text{for every } N \geqq (\gamma_0 + 1)a\,. \tag{3.14}$$

On the other hand, note that

$$T(k) \leqq \int_0^N p_k(a^{n_k-2}N)\mu(dx) + \int_N^\infty \mu(dx)\,.$$

Hence, letting $k \to \infty$ and then $N \to \infty$, we get (3.11) by (3.14). Denote

$$B_k = \{\omega : X(t) \leqq a^{-2}h(t) \text{ for some } t \in (0, a^{m_k})\}\,.$$

Note that the set B_k is decreasing as k increases and satisfies $P(B_k) \geqq 1 - T(k)$. Hence we see from (3.11) that $P(\bigcap_{k=1}^\infty B_k) = 1$, which yields that

$$\liminf_{t\downarrow 0} X(t)/h(t) \leqq a^{-2} \quad \text{a.s.}$$

Therefore, we obtain (3.7) from the arbitrariness of a in $(0, 1)$. The proof of Lemma 3.2 is complete.

Proof of Theorem 1.1 We first show (i). Let $h(t) = \gamma t$. Since

$$\int_0^1 t^{-1}F(h(t)/t)\,dt = \int_0^1 t^{-1}F(\gamma)\,dt = 0 \quad \text{for } 0 < \gamma < \gamma_0\,,$$

we see from Lemma 3.1 that

$$\liminf_{t\downarrow 0} X(t)/t \geqq \gamma_0 \quad \text{a.s.} \tag{3.15}$$

On the other hand, since

$$\int_0^1 t^{-1}F(h(t)/t)\,dt = \int_0^1 t^{-1}F(\gamma)\,dt = \infty \quad \text{for } \gamma_0 < \gamma\,,$$

we get by Lemma 3.2 that

$$\liminf_{t\downarrow 0} X(t)/t \leqq \gamma_0 \quad \text{a.s.} \tag{3.16}$$

Hence we obtain (1.7) from (3.15) and (3.16). Next we prove (ii). Suppose that $\gamma_0 = 0$ and $\lambda = \infty$. Note that

$$\int_0^1 t^{-1}F(H_0(t)/t)\,dt = -\int_0^{H_0(1)} F(u)g_0'(u)\,du = -\int_0^{H_0(1)} k_0'(u)/k_0(u)^2\,du < \infty\,.$$

Hence we see from Lemma 3.1 that

$$\liminf_{t\downarrow 0} X(t)/H_0(t) \geqq 1 \quad \text{a.s.} \tag{3.17}$$

Let θ be an arbitrary real number in $(1, 2)$. We obtain from (2.2) that

$$\theta x F(\theta x) > \int_0^{(\theta-1)x} F(\theta x - u)k(u)\,du \geqq k((\theta-1)x)(\theta-1)xF(x)\,.$$

Since $k((\theta-1)x) \geqq k_0(x)$ on $(0,b)$,

$$\int_0^1 t^{-1}F(\theta H_0(t)/t)\,dt = -\int_0^{H_0(1)} F(\theta u)g_0'(u)\,du$$
$$= -\frac{1}{\theta}\int_0^{H_0(1)} \frac{\theta u F(\theta u)k_0'(u)}{uF(u)k_0(u)^2}\,du$$
$$\geqq -\frac{\theta-1}{\theta}\int_0^{H_0(1)} \frac{k_0'(u)}{k_0(u)}\,du = \infty\,.$$

Hence we see from Lemma 3.2 and from the arbitrariness of θ in $(1,2)$ that

(3.18) $$\liminf_{t\downarrow 0} X(t)/H_0(t) \leqq 1 \quad \text{a.s.}$$

Combining (3.17) with (3.18), we establish (1.8). The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2 Suppose that $\gamma_0 = 0$ and $\lambda < \infty$. Define

(3.19) $$I(h) = \int_0^1 t^{-1}K_\lambda(h(t)/t)\,dt \quad \text{for } h(t) \in \mathscr{H}_0\,.$$

Then we see from Lemma 2.3 that

(3.20) $$I(h) < \infty \quad \text{if and only if} \quad \int_0^1 t^{-1}F(h(t)/t)\,dt < \infty\,.$$

Note from the regular variation of $K_\lambda(x)$ that

(3.21) $$I(h) < \infty \quad \text{if and only if} \quad I(\delta h) < \infty \quad \text{for every } \delta > 0\,.$$

Therefore Theorem 1.2 is proved by the use of Lemmas 3.1 and 3.2.

Next we show a theorem of a law of iterated logarithm type under the assumption (R_α) defined in Sect. 2.

Theorem 3.1 *Suppose that the condition (R_α) holds and that $\gamma_0 = 0$. Then we have*

(3.22) $$\liminf X(t)/h_\alpha(t) = 1 \quad \text{a.s.}$$

both as $t \downarrow 0$ and $t \to \infty$.

Proof. By virtue of Lemma 2.5 we find that

(3.23) $$\int_0^1 t^{-1}F(\delta h_\alpha(t)/t)\,dt = \infty \quad \text{for every } \delta > 1$$

and

(3.24) $$\int_0^1 t^{-1}F(\delta h_\alpha(t)/t)\,dt < \infty \quad \text{for every } \delta \in (0,1)\,.$$

Hence we obtain (3.22) from Lemmas 3.1 and 3.2. The proof of Theorem 3.1 is complete.

4 The case of lim sup $X(t)/h(t)$

At first we prove some lemmas which are necessary for the proof of Theorems 1.3 and 1.4.

Lemma 4.1 *Let $g(x)$ be a nonnegative and decreasing function on $[a,\infty)$. If $a \leqq A < B \leqq D$ and $A \leqq C < D$, then*

$$\frac{1}{B-A}\int_A^B g(x)\,dx \geqq \frac{1}{D-C}\int_C^D g(x)\,dx\,. \tag{4.1}$$

Proof. Define

$$x_1(u) = A + (B-A)u \quad \text{and} \quad x_2(u) = C + (D-C)u \quad \text{for } 0 \leqq u \leqq 1\,. \tag{4.2}$$

Then $x_2(u) \geqq x_1(u) \geqq a$ and

$$\frac{1}{B-A}\int_A^B g(x)dx = \int_0^1 g(x_1(u))\,du \geqq \int_0^1 g(x_2(u))\,du = \frac{1}{D-C}\int_C^D g(x)\,dx\,.$$

Lemma 4.2 *Let $0 < a \leqq b < 1$.*

(i) *Suppose that $\gamma_0 = 0$. Then we have*

$$G(x) - G(x/a) \leqq M_a G_a((1-a)x) \quad on\ [0,\infty)\,, \tag{4.3}$$

and

$$aG_a(x) \leqq N_b G_b((1-b)x) \quad on\ [0,\infty)\,. \tag{4.4}$$

(ii) *We have*

$$\int_1^\infty x^{-1}G_a(x)\,dx < \infty\,. \tag{4.5}$$

Here M_a and N_b are positive constants depending only on a and b, respectively.

Proof. Suppose that $\gamma_0 = 0$. Denote the distribution of $X(1) - X(a)$ by μ_a. Let

$$\mu_a(dx) = f_a(x)\,dx + a^\lambda\delta_0(dx)\,.$$

Since

$$f(x) = a^{-1}\int_0^\infty f_a(x-y)f(y/a)\,dy + a^{\lambda-1}f(x/a)\,,$$

we get

$$\begin{aligned} G(u) &= a^{-1}\int_0^\infty f(y/a)\,dy \int_u^\infty f_a(x-y)\,dx + a^{\lambda-1}\int_u^\infty f(x/a)\,dx \\ &= \int_0^{u/a} f(v)G_a(u-av)\,dv + G(u/a)\,. \end{aligned} \tag{4.6}$$

Hence

$$(4.7)\qquad G(u)-G(u/a)=\int_0^{u/a} f(v)G_a(u-av)\,dv \geqq KG_a(u) \quad \text{on } [1,\infty)\,,$$

where $K=\int_0^1 f(v)\,dv$. We can choose a positive number A such that $A > a_\mu$ and $G(A) < 1/4$. Then we get, for $u \geqq 2A/(1-a)$, that

$$(4.8)\qquad \int_0^{u/a} f(v)G_a(u-av)\,dv = J_1+J_2+J_3\,,$$

where

$$J_1=\int_0^u f(v)G_a(u-av)\,dv,\qquad J_2=\int_u^{(u-A)/a} f(v)G_a(u-av)\,dv\,,$$

and

$$J_3=\int_{(u-A)/a}^{u/a} f(v)G_a(u-av)\,dv\,.$$

Note that

$$J_1 \leqq G_a(u(1-a)),\qquad J_2 \leqq G_a(A)(G(u)-G(u/a)) \leqq (1/4)(G(u)-G(u/a))\,,$$

and, from Lemma 4.1, that

$$J_3 \leqq \frac{u/a-(u-A)/a}{u/a-u}\int_u^{u/a} f(v)\,dv \leqq (1/2)(G(u)-G(u/a))\,.$$

Hence we obtain from (4.6) that

$$(4.9)\qquad G(u)-G(u/a) \leqq 4G_a(u(1-a)) \quad \text{for } u \geqq 2A/(1-a)\,,$$

which implies (4.3). We see from Lemma 4.1 that

$$\frac{1}{x/a-x}\int_x^{x/a} f(u)\,du \leqq \frac{1}{x/b-x}\int_x^{x/b} f(u)\,du \quad \text{for } x \geqq a_\mu\,.$$

Hence by (4.3) and (4.7)

$$(4.10)\qquad aG_a(x) \leqq K^{-1}a(G(x)-G(x/a)) \leqq K^{-1}b(1-b)^{-1}M_bG_b((1-b)x)$$

for $x \geqq a_\mu \vee 1$, which means (4.4). We get by (4.7) that

$$(4.11)\qquad \int_1^\infty x^{-1}G_a(x)\,dx \leqq K^{-1}\int_1^\infty u^{-1}(G(u)-G(u/a))\,du$$

$$= K^{-1}\int_1^{1/a} u^{-1}G(u)\,du < \infty\,.$$

Obviously (4.5) is true for $\gamma_0 > 0$. Thus we have proved Lemma 4.2.

Hereafter, as in Sect. 3, we shall prove all lemmas and all theorems only for $t \downarrow 0$. The following lemma is essentially due to Sato [12].

Lemma 4.3 *Let $h(t) \in \mathscr{H}_0$.*

(i) *If*

$$\int_0^1 t^{-1} G_a(h(t)/t)\,dt = \infty \quad \textit{for some } a \in (0,1)\,, \tag{4.12}$$

then

$$\limsup_{t \downarrow 0} X(t)/h(t) \geqq 1 \quad \text{a.s.} \tag{4.13}$$

(ii) *If*

$$\int_1^\infty t^{-1} G_a(h(t)/t)\,dt = \infty \quad \textit{for some } a \in (0,1)\,, \tag{4.14}$$

then

$$\limsup_{t \to \infty} X(t)/h(t) \geqq 1 \quad \text{a.s.} \tag{4.15}$$

Proof. There exists a sequence $\{t_n\}_{n=1}^\infty$ such that

$$a^{n+1} \leqq t_n < a^n \quad \text{and} \quad \sup_{a^{n+1} \leqq t < a^n} G_a(h(t)/t) \leqq 2G_a(h(t_n)/t_n)\,.$$

Hence

$$\begin{aligned}\infty = \int_0^1 t^{-1} G_a(h(t)/t)\,dt &\leqq \sum_{n=0}^\infty a^{-(n+1)} \int_{a^{n+1}}^{a^n} G_a(h(t)/t)\,dt \\ &\leqq 2(a^{-1}-1) \sum_{n=0}^\infty G_a(h(t_n)/t_n)\,,\end{aligned} \tag{4.16}$$

which implies that

$$\sum_{n=0}^\infty G_a(h(t_{2n})/t_{2n}) = \infty \tag{4.17}$$

or

$$\sum_{n=0}^\infty G_a(h(t_{2n+1})/t_{2n+1}) = \infty\,. \tag{4.18}$$

Since both cases of (4.17) and (4.18) are treated in the same way, we assume (4.17), which is equivalent to

$$\sum_{n=0}^\infty P(X(t_{2n}) - X(at_{2n}) \geqq h(t_{2n})) = \infty\,.$$

Note that $\{X(t_{2n}) - X(at_{2n})\}_{n=0}^{\infty}$ are independent. Hence, by virtue of the second Borel–Cantelli lemma, we have

$$X(t_{2n}) - X(at_{2n}) \geqq h(t_{2n}) \quad \text{i.o.} \tag{4.19}$$

which means (4.13). The proof of Lemma 4.3 is complete.

Lemma 4.4 *Let $h(t) \in \mathscr{H}_1$. Suppose that*

$$\int_0^1 t^{-1} G_a(h(t)/t)\, dt < \infty \quad \textit{for every } a \in (0,1)\,. \tag{4.20}$$

Then there exists a function $h_*(t) \in \mathscr{H}_1$ such that $h_*(t) \leqq h(t)$ on $(0,1)$ and, for some $\delta \in (0,1)$, $h_*(e^{-n-1})/h_*(e^{-n}) \leqq \delta$ holds for every integer $n \geqq 0$, and

$$\int_0^1 t^{-1} G_a(h_*(t)/t)\, dt < \infty \quad \textit{for any } a \in (0,1)\,. \tag{4.21}$$

Proof. Put $0 < \beta < 1$ and $e^{-\beta} = \delta$. Denote $M_k = e^{k\beta} h(e^{-k})$ for integers $k \geqq 0$. We define $\phi_k(t)$ on $(0, e^{-k}]$ as

$$\phi_k(t) = M_k t^{\beta}\,. \tag{4.22}$$

Note that $\phi_k(e^{-k}) = h(e^{-k})$. In the following, we shall define an increasing sequence $\{k_n\}_{n=0}^{\infty}$ and a function $h_*(t)$ by induction. On $[1,\infty)$, we may define $h_*(t) \in \mathscr{H}_1$ arbitrarily.

(I) We define as

$$k_0 = 0 \quad \text{and} \quad h_*(t) = \phi_0(t) \wedge h(t) \quad \text{on } [e^{-1}, 1]\,. \tag{4.23}$$

(II) Let $n \geqq 1$. Assume that k_{n-1} and $h_*(t)$ on $[e^{-n}, 1]$ are already defined. We define k_n and $h_*(t)$ on $[e^{-n-1}, e^{-n}]$ considering two possible cases.

Case (i). Suppose that

$$h_*(e^{-n}) = \phi_{k_{n-1}}(e^{-n}) < h(e^{-n})\,. \tag{4.24}$$

Then we set

$$k_n = k_{n-1} \quad \text{and} \quad h_*(t) = \phi_{k_{n-1}}(t) \wedge h(t) \quad \text{on } [e^{-n-1}, e^{-n}]\,. \tag{4.25}$$

Case (ii). Suppose that

$$h_*(e^{-n}) = h(e^{-n})\,. \tag{4.26}$$

Then we set

$$k_n = n \quad \text{and} \quad h_*(t) = \phi_n(t) \wedge h(t) \quad \text{on } [e^{-n-1}, e^{-n}]\,. \tag{4.27}$$

Thus the definition of $\{k_n\}$ and $h_*(t)$ is complete. It is easy to see that $h_*(t) \in \mathscr{H}_1$ and $h_*(t) \leqq h(t)$ on $(0,1]$. Since

$$\phi_k(e^{-n-1})/\phi_k(e^{-n}) = e^{-\beta} = \delta \quad \text{for } n \geqq k\,,$$

we have

$$h_*(e^{-n-1})/h_*(e^{-n}) \leqq \delta \quad \text{for every } n \geqq 0 .$$

Thus only nontrivial fact to be proved is (4.21). For the proof of (4.21) we first assume that, for some m and n with $n \leqq m$,

$$(4.28) \qquad k_n = n = k_{n+1} = \cdots = k_m < k_{m+1} = m+1 .$$

In general $\{k_n\}$ can be divided into finite or infinite parts such as (4.28). Note that

$$(4.29) \qquad \phi_n(e^{-m}) \leqq h(e^{-m}) \quad \text{and} \quad \phi_n(e^{-m-1}) \geqq h(e^{-m-1}) .$$

Since $\phi_n(t)$ and $h(t)$ are continuous on $[e^{-m-1}, e^{-m}]$, there exists the least number θ on $[e^{-m-1}, e^{-m}]$ satisfying the equation $\phi_n(\theta) = h(\theta)$ so that $h_*(t) = h(t)$ on $[e^{-m-1}, \theta]$. Noting that

$$h_*(t) = \phi_n(t) \wedge h(t) \quad \text{on } [e^{-m-1}, e^{-n}] ,$$

we have

$$(4.30) \qquad \int_{e^{-m-1}}^{e^{-n}} t^{-1} G_a(h_*(t)/t)\, dt \leqq \int_{e^{-m-1}}^{e^{-n}} t^{-1} G_a(h(t)/t)\, dt + \int_{\theta}^{e^{-n}} t^{-1} G_a(\phi_n(t)/t)\, dt .$$

Since $h(t) \in \mathscr{H}_1$, it follows that

$$(4.31) \qquad e^{m+1} h(e^{-m-1}) \geqq h(\theta)/\theta ,$$

and hence

$$(4.32) \qquad \int_{\theta}^{e^{-n}} t^{-1} G_a(\phi_n(t)/t)\, dt = (1-\beta)^{-1} \int_{e^n h(e^{-n})}^{h(\theta)/\theta} s^{-1} G_a(s)\, ds \leqq (1-\beta)^{-1} \int_{e^n h(e^{-n})}^{e^{m+1} h(e^{-m-1})} s^{-1} G_a(s)\, ds ,$$

where we set $s = M_n t^{\beta-1}$. Recalling (4.5) of Lemma 4.2 we see from (4.30) and (4.32) that

$$(4.33) \qquad \int_0^1 t^{-1} G_a(h_*(t)/t)\, dt \leqq \int_0^1 t^{-1} G_a(h(t)/t)\, dt + (1-\beta)^{-1} \int_{h(1)}^{\infty} s^{-1} G_a(s)\, ds < \infty$$

for every $a \in (0,1)$. Thus we have established Lemma 4.4.

Lemma 4.5 *Let* $h(t) \in \mathscr{H}_1$.

(i) *If*

$$\int_0^1 t^{-1} G_a(h(t)/t)\,dt < \infty \quad \textit{for every } a \in (0,1)\,, \tag{4.34}$$

then

$$\limsup_{t \downarrow 0} X(t)/h(t) \leqq 1 \quad \text{a.s.} \tag{4.35}$$

(ii) *If*

$$\int_1^\infty t^{-1} G_a(h(t)/t)\,dt < \infty \quad \textit{for every } a \in (0,1)\,, \tag{4.36}$$

then

$$\limsup_{t \to \infty} X(t)/h(t) \leqq 1 \quad \text{a.s.} \tag{4.37}$$

Remark. 4.1 For the proof of (ii), we do not have to prove the analogue of Lemma 4.4, since $h(at) \geqq ah(t)$ for every $a > 1$ and every $t \geqq 1$ whenever $h(t) \in \mathscr{H}_1$.

Proof of Lemma 4.5 We can assume from Lemma 4.4 that there exists $\delta \in (0,1)$ such that $h(e^{-n-1})/h(e^{-n}) \leq \delta$ for every $n \geqq 0$. Hence, for each $\varepsilon \in (0,1)$, we can choose sufficiently small $a \in (0,1)$ such that

$$\sum_{n=0}^\infty h(xa^n) \leqq (1+\varepsilon)h(x) \quad \text{for } 0 < x \leqq 1\,. \tag{4.38}$$

Let $0 < b < 1$ and $a = b^N$ with a positive integer N. Note that, for $b^{j+1} \leqq t < b^j$,

$$P(X(1) - X(a) \geqq h(t)/t) \geqq P(X(1) - X(a) \geqq h(b^{j+1})/b^{j+1})\,. \tag{4.39}$$

Hence

$$|\log b| \sum_{j=0}^\infty P(X(b^j) - X(ab^j) \geqq h(b^{j+1})/b) \leqq \int_0^1 t^{-1} G_a(h(t)/t)\,dt < \infty\,. \tag{4.40}$$

Therefore, by the first Borel–Cantelli lemma,

$$X(b^j) - X(ab^j) < h(b^{j+1})/b \quad \text{a.s.} \tag{4.41}$$

for all large j. Since $a = b^N$, we see that, for every $n \geqq 0$,

$$X(a^n b^j) - X(a^{n+1} b^j) < h(a^n b^{j+1})/b \quad \text{a.s.} \tag{4.42}$$

for all large j. Suming up (4.42) in n from 0 to ∞, we obtain from (4.38) that

$$X(b^j) < b^{-1} \sum_{n=0}^\infty h(a^n b^{j+1}) \leqq b^{-1}(1+\varepsilon)h(b^{j+1}) \quad \text{a.s.} \tag{4.43}$$

for all large j. This implies that

$$X(t) \leqq b^{-1}(1+\varepsilon)h(t) \quad \text{a.s.}$$

for all small t. Hence from the arbitrariness of b and ε we conclude that (4.35) holds.

For the proof of Theorems 1.3 and 1.4 we state a proposition, whose proof is given in Appendix. Let η be an infinitely divisible distribution on $[0,\infty)$. Then the Laplace transform $L\eta(z)$ is expressed as

$$L\eta(z) = \exp(\Phi(z)), \qquad \Phi(z) = -\gamma z + \int_0^\infty (e^{-zx} - 1)\nu(dx), \tag{4.44}$$

where $\gamma \geqq 0$ and $\int_0^\infty x(1+x)^{-1}\nu(dx) < \infty$. The measure ν is called Lévy measure of η. Denote $\phi(u) = \int_u^\infty \nu(dx)$ and $g(u) = \int_u^\infty \eta(dx)$ for $u > 0$.

Proposition 4.1 (i) $\phi(x) \in \mathrm{OR}$ *if and only if* $g(x) \in \mathrm{OR}$.
(ii) *If* $\phi(x) \in \mathrm{OR}$, *then* $g(x) \asymp \phi(x)$.

Remark. 4.2 Proposition 4.1 is an analogue of Theorem 1 of Embrechts et al. [3], which states that the following assertions (i), (ii) and (iii) are equivalent:
(i) η is subexponential.
(ii) $1 - \phi(x)/\phi(1)$ is subexponential on $[1,\infty)$.
(iii) $g(x) \sim \phi(x)$.
Obviously there is an infinitely divisible distribution η on $[0,\infty)$ such that it is not subexponential but $g(x) \in \mathrm{OR}$. On the other hand, we see from their final example of [3] that the lognormal distribution η is an infinitely divisible distribution on $[0,\infty)$ such that it is subexponential but $g(x) \notin \mathrm{OR}$. It follows that the converse of the assertion (ii) of Proposition 4.1 is not necessarily true.

Remark. 4.3 The distribution of $X(1) - X(a)$ for $a \in (0,1)$ is infinitely divisible with Lévy measure ν_a. Denote $\phi_a(u) = \int_u^\infty \nu_a(dx)$. We have, by Proposition 4.1 of Sato [12],

$$\phi_a(u) = \int_u^{u/a} x^{-1}k(x)\,dx \quad \text{for every } a \in (0,1).$$

Proof of Theorem 1.3 Suppose that $k(x) \notin \mathrm{OR}$ and $\gamma_0 = 0$. Then we see from Proposition 4.1 and Remark 4.3 that $G_a(x) \notin \mathrm{OR}$ for every $a \in (0,1)$. We continue to use the notations in Sect. 2. We find from (4.5) of Lemma 4.2 that

$$\sum_{k=0}^{\infty} \int_{c_{2k+2}}^{c_{2k+1}} t^{-1}G_a(H_1(t)/t)\,dt = \sum_{k=0}^{\infty} \int_{u_{k+1}^{-1}}^{u_k^{-1}} s^{-1}G_a(1/s)\,ds \tag{4.45}$$

$$= \int_1^\infty x^{-1}G_a(x)\,dx < \infty$$

for every $a \in (0,1)$. If $0 < a_m \leqq a$, then, by (2.16) and (2.17),

$$\sum_{k=m}^{\infty} \int_{c_{2k+1}}^{c_{2k}} t^{-1}G_a(H_1(t)/t)\,dt \leqq \sum_{k=m}^{\infty} \int_{c_{2k+1}}^{c_{2k}} t^{-1}G_{a_k}(H_1(t)/t)\,dt < \infty. \tag{4.46}$$

Hence

$$\int_0^1 t^{-1} G_a(H_1(t)/t)\,dt < \infty .$$

Thus, by Lemma 4.5,

$$\limsup_{t\downarrow 0} X(t)/H_1(t) \leqq 1 \quad \text{a.s.} \tag{4.47}$$

On the other hand, for every $\rho > 1$, there exists an integer n such that $\rho^{-1} \leqq (1 - a_n)\rho_n^{-1}$. We obtain from (4.4) of Lemma 4.2 that

$$\begin{aligned} N_{a_n} \sum_{k=n}^{\infty} \int_{c_{2k+1}}^{c_{2k}} t^{-1} G_{a_n}(\rho^{-1} H_1(t)/t)\,dt \\ \geqq \sum_{k=n}^{\infty} \int_{c_{2k+1}}^{c_{2k}} t^{-1} a_k G_{a_k}(\rho_k^{-1} H_1(t)/t)\,dt = \sum_{k=n}^{\infty} a_k = \infty . \end{aligned} \tag{4.48}$$

Hence we see from Lemma 4.3 and from the arbitrariness of ρ that

$$\limsup_{t\downarrow 0} X(t)/H_1(t) \geqq 1 \quad \text{a.s.}$$

Combined with (4.47), this establishes (1.13). The proof of Theorem 1.3 is complete.

Proof of Theorem 1.4 Suppose that $k(x) \in \mathrm{OR}$. Then we find from Proposition 4.1 and Remark 4.3 that

$$G_a(x) \in \mathrm{OR} \quad \text{and} \quad G_a(x) \asymp \phi_a(x) \asymp k(x) \quad \text{for every } a \in (0,1) . \tag{4.49}$$

Define

$$J(h) = \int_0^1 t^{-1} k(h(t)/t)\,dt \quad \text{for } h(t) \in \mathscr{H}_1 . \tag{4.50}$$

Then we see from (4.49) that

(4.51)

$$J(h) < \infty \quad \text{if and only if} \quad \int_0^1 t^{-1} G_a(h(t)/t)\,dt < \infty \quad \text{for every } a \in (0,1) .$$

Note from $k(x) \in \mathrm{OR}$ that

$$J(h) < \infty \quad \text{if and only if} \quad J(\delta h) < \infty \quad \text{for every } \delta > 0 . \tag{4.52}$$

Therefore Theorem 1.4 is proved by the use of Lemmas 4.3 and 4.5.

5 Example

Let us give an example of the theorems above. Sato [12] handled the following example with $\lambda = 1$ and proved (iii) for $t \downarrow 0$. It is interesting that the limsup in the equation (5.5) does not depend on λ.

Example 5.1 Let $h(t) \in \mathscr{H}_0$. Suppose that $\gamma_0 = 0$ and $k(x) = \lambda e^{-x}$ on $(0, \infty)$ with $\lambda > 0$.

(i) If

$$(5.1) \qquad \int_0^1 t^{-\lambda-1} h(t)^\lambda \, dt < \infty \text{ (resp. } = \infty),$$

then

$$(5.2) \qquad \liminf_{t \downarrow 0} X(t)/h(t) = \infty \text{ (resp. } = 0) \quad \text{a.s.}$$

(ii) If

$$(5.3) \qquad \int_1^\infty t^{-\lambda-1} h(t)^\lambda \, dt < \infty \text{ (resp. } = \infty),$$

then

$$(5.4) \qquad \liminf_{t \to \infty} X(t)/h(t) = \infty \text{ (resp. } = 0) \quad \text{a.s.}$$

(iii) We have

$$(5.5) \qquad \limsup X(t)/(t \log(|\log t| \vee e)) = 1 \quad \text{a.s.}$$

both as $t \downarrow 0$ and $t \to \infty$.

Proof. The distribution μ is a Γ distribution, that is, $\mu(dx) = (\Gamma(\lambda))^{-1} x^{\lambda-1} e^{-x} dx$ on $[0, \infty)$. Hence we have

$$(5.6) \qquad F(1/x) \sim (\lambda \Gamma(\lambda))^{-1} x^{-\lambda} \quad \text{and} \quad G(x) \sim (\Gamma(\lambda))^{-1} x^{\lambda-1} e^{-x}.$$

Hence the assertions (i) and (ii) are proved by using directly Lemmas 3.1 and 3.2. Define $H(t) = t \log(|\log t| \vee e)$ on $(0, \infty)$. Then $H(t) \in \mathscr{H}_1$. Though the assertion (iii) holds for $H_1(t)$ instead of $H(t)$, the function $H(t)$ is much simpler than $H_1(t)$. We see from (5.6) that, for every $a \in (0, 1)$ and every $c > 1$,

$$(5.7) \qquad \int_0^1 t^{-1} G_a(cH(t)/t) \, dt \leqq \int_0^1 t^{-1} G(cH(t)/t) \, dt < \infty.$$

Hence, by Lemma 4.5,

$$(5.8) \qquad \limsup_{t \downarrow 0} X(t)/H(t) \leqq 1 \quad \text{a.s.}$$

We find from (4.3) and (5.6) that, for every $a \in (0, 1)$,

$$(5.9) \qquad K_a x^{\lambda-1} e^{-x} \leqq G_a((1-a)x) \quad \text{on } [1, \infty),$$

where K_a is a positive constant depending only on a. Hence we obtain that

$$\int_0^1 t^{-1} G_a((1-a)H(t)/t)\,dt = \infty \quad \text{for every } a \in (0,1). \tag{5.10}$$

We get by Lemma 4.3

$$\limsup_{t \downarrow 0} X(t)/H(t) \geqq 1 \quad \text{a.s.} \tag{5.11}$$

Therefore, we have established (iii) for $t \downarrow 0$. The proof of (iii) for $t \to \infty$ is similar and omitted.

6 Appendix

We prove a Tauberian theorem and, using it, show Proposition 4.1. Let n be a nonnegative integer and let $s > 0$. For a measurable function $g(x)$ on $(0,\infty)$ with $\int_0^\infty e^{-sx}|g(x)|\,dx < \infty$ for every $s > 0$, we define

$$L_n g(s) = \int_0^\infty e^{-sx} x^n g(x)\,dx. \tag{6.1}$$

Theorem 6.1 *Let n be a nonnegative integer and let $g(x)$ be a positive decreasing function on $(0,\infty)$ with $\int_0^1 g(x)\,dx < \infty$.*

(i) *The following conditions* (a), (b), *and* (c) *are equivalent*:

(a)
$$\sup_{t>1} \frac{\int_0^t u^n g(u)\,du}{t^{n+1} g(t)} < \infty.$$

(b)
$$\sup_{t>1} \frac{\int_1^t u^{-1} L_n g(1/u)\,du}{L_n g(1/t)} < \infty.$$

(c)
$$t^{n+1} g(t) \asymp L_n g(1/t).$$

(ii) $g(x) \in \mathrm{OR}$ *if and only if* (a) *in* (i) *holds for some nonnegative integer n.*

Remark. 6.1 We find from Theorem 1 of de Haan and Stadtmüller [9] that, under the assumption that $g(x)$ is nonnegative and increasing on $(0,\infty)$, the three conditions $g(x) \in \mathrm{OR}$, $L_n g(1/t) \in \mathrm{OR}$, and $t^{n+1} g(t) \asymp L_n g(1/t)$ are equivalent for every $n \geqq 0$. Actually it is proved by them for $n = 0$, and consequently true for general n. Cline [2] obtains an analogous Tauberian theorem for subclasses ER and IR of class OR under the same assumption on $g(x)$. On the other hand, we prove Theorem 6.1, which has similarity to the results above, under the assumption that $g(x)$ is positive and decreasing on $(0,\infty)$. In our case, the condition $L_n g(1/t) \in \mathrm{OR}$ is always true for each $n \geqq 0$, and hence the condition $g(x) \in \mathrm{OR}$ is equivalent to the condition $L_n g(1/t) \in \mathrm{OR}$ for no $n \geqq 0$. Further the condition $g(x) \in \mathrm{OR}$ is not necessarily equivalent to the condition $t^{n+1} g(t) \asymp L_n g(1/t)$ for all $n \geqq 0$.

Proof of Theorem 6.1 At first we prove the equivalence of (a) and (c) in (i). We have

$$s^{n+1}L_n g(s) = \int_0^\infty e^{-u}u^n g(u/s)\,du \geqq g(1/s)\int_0^1 e^{-u}u^n\,du\,.$$

Denote $c_n = \int_0^1 e^{-u}u^n\,du$. Then

$$t^{n+1}g(t) \leqq c_n^{-1}L_n g(1/t) \quad \text{on } (0,\infty)\,. \tag{6.2}$$

Note that

(6.3)

$$\frac{L_n g(1/t)}{t^{n+1}g(t)} = \int_0^\infty e^{-x}x^n g(tx)/g(t)\,dx$$
$$\leqq \int_0^1 x^n g(tx)/g(t)\,dx + \int_1^\infty e^{-x}x^n\,dx \leqq \frac{\int_0^t u^n g(u)\,du}{t^{n+1}g(t)} + n!\,.$$

Hence (a) implies (c). Conversely, since

$$\frac{L_n g(1/t)}{t^{n+1}g(t)} \geqq e^{-1}\frac{\int_0^t u^n g(u)\,du}{t^{n+1}g(t)}\,,$$

(c) implies (a). Hence (a) and (c) are equivalent. Obviously (a) and (c) implies (b). Secondly we prove that (b) implies (c). Let

$$M = \sup_{t>1}\frac{\int_1^t u^{-1}L_n g(1/u)\,du}{L_n g(1/t)}\,.$$

Since $L_n g(s)$ is decreasing,

$$M \geqq \frac{\int_t^{Nt} u^{-1}L_n g(1/u)\,du}{L_n g(1/(Nt))} \geqq \frac{(\log N)L_n g(1/t)}{L_n g(1/(Nt))} \tag{6.4}$$

for $N > 1$ and $t \geqq 1$. Hence, choosing N such that $M \leqq (2e)^{-1}\log N$, we get

$$L_n g(1/t) \leqq (2e)^{-1}L_n g(1/(Nt)) \quad \text{for } t \geqq 1\,. \tag{6.5}$$

Suppose that (c) does not hold. Then we see from (6.2) that, for sufficiently large $T > 1$,

$$T^{n+1}g(T) \leqq (3N^{n+1}(n!+1))^{-1}L_n g(1/T)\,. \tag{6.6}$$

Since

$$\int_0^{NT} u^n g(u)\,du = \int_0^T u^n g(u)\,du + \int_T^{NT} u^n g(u)\,du \leqq eL_n g(1/T) + g(T)(NT)^{n+1}\,,$$

it follows that

$$(6.7)\qquad (NT)^{-(n+1)}L_n g(1/(NT))$$
$$= (NT)^{-(n+1)} \int_0^{NT} e^{-u/(NT)} u^n g(u)\, du + \int_1^{\infty} e^{-u} u^n g(NTu)\, du$$
$$\leqq e(NT)^{-(n+1)} L_n g(1/T) + g(T)(n! + 1)\,.$$

Hence we obtain from (6.5), (6.6), and (6.7) that

$$(6.8)\qquad L_n g(1/(NT)) \leqq eL_n g(1/T) + (n! + 1)(NT)^{n+1} g(T) < L_n g(1/(NT))\,.$$

This is a contradiction. Thus we have shown that (b) implies (c). The proof of (i) is complete. The assertion (ii) is evident from Karamata's theorem for OR. (see Theorem A2(b) of Seneta [14] or Theorem 2.6.5 of Bingham et al. [1].) The proof of theorem 6.1 is complete.

Proof of Proposition 4.1 Without harming generality we can assume that $\gamma = 0$ and ν does not vanish identically. The equality (4.44) implies that

$$(6.9)\qquad L_0 g(s) = s^{-1}(1 - \exp(\Phi(s))) \quad \text{and} \quad \Phi(s) = -sL_0\phi(s)\,.$$

Hence

$$(6.10)\qquad L_0 g(s) = \sum_{j=1}^{\infty} (j!)^{-1}(-s)^{j-1}(L_0\phi(s))^j \quad \text{on } (0, \infty)\,.$$

Denote $u_j(s) = s^{j-1}(L_0\phi(s))^j$. Differentiating n times term by term and multiplying $(-1)^n$, we get

$$(6.11)\qquad L_n g(s) = L_n\phi(s) + R_n(s) \quad \text{for } n \geqq 0\,,$$

where

$$(6.12)\qquad R_n(s) = \sum_{j=2}^{\infty} (j!)^{-1}(-1)^{n+j-1}(d/ds)^n u_j(s)\,.$$

We can easily see that

$$(6.13)\qquad L_m\phi(1/t) \asymp \int_0^t x^m \phi(x)\, dx \quad \text{for every } m \geqq 0\,.$$

Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Let

$$\Lambda_j = \{\mathbf{p} : \mathbf{p} = (p_0, p_1, \ldots, p_j) \in \mathbb{Z}_+^{j+1}, p_0 \leqq j - 1, \text{ and } \sum_{k=0}^{j} p_k = n\}$$

for $j \geqq 2$ and $n \geqq 0$. Then we have

$$(6.14)\qquad |(d/ds)^n u_j(s)| \leqq \sum_{\mathbf{p}\in\Lambda_j} n! \binom{j-1}{p_0} \frac{s^{j-p_0-1}}{p_1! p_2! \cdots p_j!} \prod_{k=1}^{j} L_{p_k}\phi(s)\,.$$

If $2 \leqq j \leqq n$, then we have by (6.13)

$$(6.15) \qquad \frac{(1/t)^{j-p_0-1}\prod_{k=1}^{j} L_{p_k}\phi(1/t)}{L_n\phi(1/t)} \asymp \frac{\prod_{k=1}^{j}\int_0^t x^{p_k}\phi(x)\,dx}{t^{j-p_0-1}\int_0^t x^n\phi(x)\,dx}.$$

On the other hand, if $j \geqq n+1$, then

(6.16)

$$\frac{(1/t)^{j-p_0-1}\prod_{k=1}^{j} L_{p_k}\phi(1/t)}{L_n\phi(1/t)} = ((1/t)L_0\phi(1/t))^{j-n}\frac{\prod_{k=1}^{n} L_{q_k}\phi(1/t)}{t^{n-p_0-1}L_n\phi(1/t)}$$

$$\asymp ((1/t)L_0\phi(1/t))^{j-n}\frac{\prod_{k=1}^{n}\int_0^t x^{q_k}\phi(x)\,dx}{t^{n-p_0-1}\int_0^t x^n\phi(x)\,dx}$$

with some nonnegative integers q_k satisfying $p_0 + \sum_{j=1}^{n} q_k = n$. Note from $\lim_{x\to\infty}\phi(x) = 0$ that $\lim_{t\to\infty}(1/t)L_0\phi(1/t) = 0$ and, by using de l'Hôpital's rule, that

$$(6.17) \quad \lim_{t\to\infty}\frac{\prod_{k=1}^{j}\int_0^t x^{p_k}\phi(x)\,dx}{t^{j-p_0-1}\int_0^t x^n\phi(x)\,dx} \leqq \lim_{t\to\infty}\sum_{l=1}^{j}\prod_{k\neq l,0}(1/t)\int_0^t (x/t)^{p_k}\phi(x)\,dx = 0$$

for $j \geqq 2$. It follows that $\lim_{s\downarrow 0} R_n(s)/(L_n\phi(s)) = 0$ for $n \geqq 0$, and hence

$$(6.18) \qquad L_n g(1/t) \sim L_n\phi(1/t) \quad \text{for } n \geqq 0\,.$$

To prove Proposition 4.1 we consider two possible cases. First, suppose that $\phi(x_0) = 0$ for some $x_0 > 0$. Then, by Sato [10], $g(x)$ is rapidly varying, and hence $g(x) \notin \mathrm{OR}$. Secondly, suppose that $\phi(x)$ is positive on $(0,\infty)$. In this case Proposition 4.1 can be proved by using (6.18) and Theorem 6.1. The proof of Proposition 4.1 is complete.

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