Econ. Theory 5, 1–18 (1995)

Economic Theory

Research articles

Dynamic coordination games*

Douglas Gale
Economics Department, Boston University, 270 Bay State Road, Boston, MA 02215, USA

Received: November 5, 1993; revised version January 15, 1994

Summary. Gains from coordination provide incentives for delay. In this paper, the extent of delay is studied in a dynamic, N-person, coordination game. There is no social gain from delay, so an equilibrium with delay is always inefficient. For fixed N, there is no coordination failure when the period length is short: all equilibrium outcomes converge to the Pareto efficient outcome as the period length converges to zero. On the other hand, holding period length fixed, there exist equilibria in which delay is proportional to N, for arbitrarily large values of N. In addition, it can be shown that the possibility of delay depends on the "timing" of strategic complementarities. However, under certain conditions, delay is shown to be a robust phenomenon, in the sense that "well-behaved" equilibria exhibit infinite delay for N sufficiently large.

Keywords: Coordination, delay, strategic complementarities, dynamic games.

0. Introduction

Firms may delay decisions because they want to coordinate their actions. For example, an investment may be more profitable if it is made at a time when the aggregate level of investment is high (Shleifer (1986)). Strategic delay can also occur because of informational externalities (Caplin and Leahy (1992), Chamley and Gale (1994)), but here I am only concerned with the coordination motive. A problem arises if every firm decides to delay, trying to find the optimal place in the decision-making queue, since someone has to go first. A "coordination failure" more

* This paper grew out of discussions with Christophe Chamley. While writing it I benefited from discussions with Ken Binmore, Russell Cooper, Bob Rosenthal and Michael Manove. Joe Farrell, Drew Fudenberg, Martin Hellwig and Sawoong Kang made very useful comments on an earlier version that led to substantial improvements. Helpful comments were also made by seminar participants at the London School of Economics, the SUNY at Stoney Brook, the NBER Summer Institute, Northwestern University, and the University of Chicago. I would like to thank Nick Yannelis and an anonymous referee for their editorial advice. Financial support for this research was provided by the National Science Foundation under Grant No. SES 9196061.

result. For example, in an economic downturn, waiting until other firms start to invest may simply deepen the recession.

Coordination games have become increasingly popular as a way of modeling macroeconomic phenomena (see Cooper and John (1988) for a survey). The salient feature of these games is strategic complementarity, i.e., one agent's optimal activity level is an increasing function of the general level of activity. In such games, there are typically many equilibria and sometimes they can be Pareto ranked. Models of "coordination failure" are found in Diamond (1982), Bryant (1983), Heller (1986), Shleifer (1986), Chatterjee and Cooper (1989, 1990), Durlauf (1993).

This paper explores some of the game-theoretic issues that arise in a model with delay and strategic complementarities. The analysis requires care because there are many equilibria and the results are sensitive to the details of the modeling. Rather than study the most general games, I use a dynamic version of a familiar example of a static coordination game. Even so, the analysis turns out to be subtle. In fact, much of the interest of the paper lies in exploring the conditions that are required for delay.

The basic model has a finite number of players $i = 1, \ldots, N$. Each player makes a binary decision, whether or not to make an investment of fixed size. The player can invest at any date but he can only invest once. Investment has a fixed cost $c > 0$ and generates a stream of future revenues. The revenue flow at each date is an increasing function of the cumulative aggregate investment and a player's payoff is the expected present value of this flow, net of the fixed cost c. A player who never invests receives 0.

Delay is inefficient because players discount the future and there is no social gain from delay. In equilibrium it may be optimal for an individual to delay – he is better off than if he had invested in advance of other players – but they would all be better off if they invested immediately.

The extent of delay is quite sensitive to the specification of the model and to variations in the parameters. The first result, Theorem 1, shows that when the period length, which can be taken as a measure of a player's reaction time, goes to zero, all equilibrium outcomes converge to the unique efficient outcome. This may suggest that making the timing of investment endogenous somehow "solves" the coordination problem, but this conclusion would be premature. Theorem 1 depends crucially on holding fixed the number of players. Theorem 2, on the other hand, shows that there always exist subgame perfect equilibria in which delay is proportional to the number of players. So increasing the number of players may increase the difficulty of coordination, for any given period length. By varying the parameters of the game, we can clearly generate quite different results.

The extent of delay also depends on the timing of complementarities. The basic model assumes that the cumulative investment at date t determines the revenue flow at date t. With *leading* complementarities, where cumulative investment at $t + 1$ determines the revenue flow at t, the unique subgame perfect outcome involves no delay. With *lagging* complementarities, where cumulative investment at $t - 1$ determines the revenue flow at t, analogues of Theorems 1 and 2 continue to hold.

The variety of different results obtained and the evident sensitivity of the model's predictions present a confusing picture. It is important to show in what sense delay is a robust outcome, to bring some order out of this confusion. Suppose we consider only the case where complementarities are contemporaneous or lagging, the number of players is large and the game is perturbed to prevent perfect coordination. Then for a large class of well-behaved equilibria, it can be shown that prolonged delay is the only possible outcome (Theorem 6). This is a fairly strong result, and suggests conditions under which coordination failure will be robust in more general games. However, the complexity of the analysis of this special case warns us that more work is needed and that care must taken in applying ideas from the study of simple coordination games to macroeconomics.

The rest of the paper is organized as follows. Section 1 contains a description of the model. The effect of the period length on the equilibrium set is considered in Section 2. The effect of the number of players is considered in Section 3. Section 4 explores the impact of the timing of complementarities. Section 5 considers a perturbation of the game and shows that, under certain circumstances, only prolonged delay is consistent with equilibrium. Some open questions are discussed in Section 6.

1. The basic model

Consider a game with N players, indexed by $i = 1, \ldots, N$. The play of the game occurs at a countable set of dates, indexed by $t = 1, \ldots, \infty$. Each player has an indivisible investment opportunity, which can be exercised at any date. The uncommitted players at each date simultaneously decide whether to invest.

In making their decisions, players have complete information about the other players and the previous moves of the game. Let $x_{it} = 0$ if player i has not invested by the end of the date t and let $x_{it} = 1$ if he has. The *state* of the game at the end of the date t is a vector $x_t \equiv (x_{1t}, \ldots, x_{Nt}) \in \{0, 1\}^N$. The *history* of the game at date t is a sequence of states $h = (x_1, \ldots, x_{t-1})$, satisfying

$$x_s \geq x_{s-1} \quad \text{for} \quad s = 1, \ldots, t-1.$$

Let H_t denote the set of histories at date t and let $H = \bigcup H_t$ denote the set of all histories. A *behavioral strategy* for player i is a function $f_i : H \to [0, 1]$, where $f_i(h)$ is the probability that player i invests at the information set h. Strategies must satisfy the condition:

$$[x_{i,t-1} = 1] \Rightarrow [f_i(h) = 0],$$

for any history $h = (x_1, \ldots, x_{t-1})$. In other words, player i cannot invest at date t if he has already invested at some date $s < t$.

Since this is a game of complete information and perfect recall, it will be analyzed using the concept of *subgame perfect equilibrium* (SPE).

When a player invests, he pays a fixed cost $c > 0$ and receives a stream of revenues which depend on the number of other players who have already invested. The player's payoff is the net present value of this stream of payments. In the basic game, strategic complementarities are assumed to be *contemporaneous*, that is, the revenue flow at date t depends on the cumulative investment at date t. If x_t is the state of the game at date t, the cumulative investment is

$$\alpha_t \equiv N^{-1} \cdot \sum_{i=1}^{N} x_{it}.$$

Cumulative investment is expressed as a fraction of the total possible investment to avoid scale effects when N is allowed to vary. This entails no essential loss of generality for a fixed value of N. The revenue flow at date t is denoted by $v(\alpha_t)$ and the payoff to a player who invests at date t is

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_s) - \delta^{t-1} c,$$

where $0 < \delta < 1$ is the common discount factor. The parameters δ, v and c are assumed to satisfy the following conditions.

Assumption 1. (a) $\bar{V} \equiv v(1)/(1-\delta) > c$; (b) $v(0)/(1-\delta) < c$; (c) $v(\alpha)$ is continuous, increasing and non-negative.

Assumption 1(a) ensures that if all players invest they will make positive profits. This is clearly necessary for investment to take place. Assumption 1(b) implies that any player who invests before everyone else will make a loss, at least initially, when N is large. The last part of Assumption 1 ensures that there are strategic complementarities.

Under these assumptions, the unique Pareto-efficient equilibrium outcome is the one in which all players invest immediately. Typically, there are also many equilibria involving delay. For example, let α^* denote the proportion of players who must invest before it becomes profitable to do so. Assumption 1(c) implies that α^* is uniquely determined by the condition

$$\frac{v(\alpha^*)}{1-\delta} = c$$

and Assumptions 1(a) and 1(b) imply that $0 < \alpha^* < 1$. For any N satisfying $\alpha^* N \geq 2$, there is a SPE in which every player invests at date 2. The equilibrium strategies require all players to invest at date 2 along the equilibrium path and, in any subgame involving a deviation, the uncommitted players invest immediately. If a player deviates by investing at date 1, the other players react by investing at date 2. By assumption, $v(1/N) < (1-\delta)c$ so the deviating player is worse off.

2. The effect of period length on equilibrium delay

Delay is clearly possible, but the amount of delay may not be significant. We see this when we consider what happens as the period length becomes vanishingly short. Although there are equilibria with delay, they are all approximately efficient. In the limit, there is neither multiplicity nor coordination failure.

The intuition behind the theorem relies on backward induction on the number of players who have already invested. Let n^* be the smallest integer greater than $\alpha^* N$. If $n \geq n^* - 1$ players have invested, it is a dominant strategy for all remaining players to invest. Then if $n^* - 2$ players have invested, a player knows that by investing he precipitates a subgame in which $n^* - 1$ have invested and the rest must invest immediately. This puts a lower bound on a player's payoff, once $n^* - 2$ have invested, and an upper bound on the amount of delay. In the same way, if we assume that all players will invest within a short period after $N - k$ have invested, we can extend the bound to $N - k - 1$ and hence by induction to all subgames. Note that the theorem does not say anything about the *number of periods* that elapse, only about the amount of "real-time" delay.

This result is true for all SPE but the proof is given here for an equilibrium in pure strategies. The straightforward extension to mixed strategies can be found in Gale (1992). Let the period length be $\tau > 0$. Then the revenue flow per period is $v(\alpha)\tau$ and the discount factor is $\delta = e^{-\rho\tau}$, where $\rho > 0$ is the fixed discount rate.

Theorem 1. For any $\varepsilon > 0$ and some $\eta > 0$, if the period length $\tau < \eta$, then with probability $1 - \varepsilon$ all players invest within ε of the start of the game, in any subgame perfect equilibrium.

Proof: If we consider only pure strategy equilibria, the qualification "with probability $1 - \varepsilon$" in the statement of the theorem can be ignored.

Let Γ_n denote the equivalence class of subgames in which exactly n players have already invested. Without loss of generality, the game can be assumed to begin at date 1, with n players already committed. When it is necessary to emphasize the period length, write $\Gamma_n(\tau)$ instead of Γ_n. Let $T_n(\tau)$ denote the superemum, taken over all SPE of $\Gamma_n(\tau)$, of the time taken for all players to invest.

As the induction hypothesis, assume that for $n = k, \ldots, N$, $T_n(\tau) \to 0$ as $\tau \to 0$. The induction hypothesis is clearly true for $k = N - 1$. By Assumption 1(a), as soon as $N - 1$ players have invested, it becomes a dominant strategy for the remaining player to invest immediately. This means that $T_{N-1}(\tau) = \tau$.

Suppose the induction hypothesis is true for $k < N$. We need to show that it holds for $k - 1$. If a positive number of players invest in the game Γ_{k-1}, they precipitate a subgame Γ_n with $n > k - 1$. By the induction hypothesis, delay in this game is bounded by $T_n(\tau)$, so we can prove the induction hypothesis for $k - 1$ by establishing a bound on the time until the first player invests in the subgame Γ_{k-1}.

The proof is by contradiction. For any SPE of Γ_{k-1}, let d denote the time taken for the first player to invest. We want to prove that as τ converges to zero, d converges uniformly to zero for all SPE. Suppose the contrary. Then we can find a sequence of period lengths $\{\tau_r\}$ and corresponding equilibria $\{f_r\}$ with the property that τ_r converges to zero as $r \to \infty$ and $d_r \geq d > 0$ for all r sufficiently large.

Consider some fixed but arbitrary equilibrium f_r and consider the following deviation from the equilibrium strategies. Suppose that some player, call him player 1, deviates by investing at the first date, thus precipitating the subgame Γ_k beginning at the second date. By hypothesis, once Γ_k has started, all the players will invest within $T_k(\tau_r)$ periods. So the payoff to the deviating player must be at least

$$\exp\{-\rho(T_k(\tau_r) + \tau_r)\}\frac{v(1)\tau_r}{1 - e^{-\rho\tau_r}} - c.$$

On the other hand, in the original equilibrium, the best that can be hoped for is that all the players invest after d_r time units. In that case player 1 himself does not invest until d_r time units have passed and his payoff cannot exceed

$$\exp\{-\rho d_r\}\left\{\frac{v(1)\tau_r}{1-e^{-\rho\tau_r}}-c\right\}.$$

To prevent a profitable deviation, then,

$$\exp\{-\rho(T_k(\tau_r)+\tau_r)\}\frac{v(1)\tau_r}{1-e^{-\sigma\tau_r}}-c\leq\exp\{-\rho d_r\}\left\{\frac{v(1)\tau_r}{1-e^{-\rho\tau_r}}-c\right\},$$

for every r. Taking limits as $r\to\infty$ yields

$$\frac{v(1)}{\rho}-c\leq e^{-\rho d}\left(\frac{v(1)}{\rho}-c\right),$$

a contradiction. This establishes the desired result and, by induction, proves that $T_n(\tau)$ converges to 0 as τ converges to 0, for $n=0,1,\ldots,N$. ■

Discussion

A. Theorem 1 has a family resemblence to no-delay results in the bargaining literature. In Rubinstein's (1982) alternating offers model, agreement is instantaneous. In the durable goods monopoly problem (Gul and Sonnenschein (1988)), all trade takes place instantaneously in the limit as the period length becomes vanishingly short, but only if players use weak Markov strategies. The structure of these models is very different, however, and depends crucially on the assumption of two players, a continuous strategy space and so on. They also lack the common interest property which is characteristic of coordination games. A closely related model is found in Admati and Perry (1991). Admati and Perry show that bargaining over contributions to a joint project results in almost efficient provision when the period length is very short.

B. We can interpret the period length as a measure of the players' "reaction time". The shorter the period length, the faster a player can react to the moves of the other players. It is common in the bargaining literature to interpret the period length as the time required to make a decision and to assume that this period should be fairly short. It is not clear that this is always an appropriate assumption in a macroeconomic context. "Time to build" implies a lag between the observation of one player's completed investment and the completion of another investment. A satisfactory treatment of time to build would require an extension of the model, in which the players' reaction time is distinguished from the rate at which capital formation takes place. See the discussion in Section 6.

C. Bryant (1983) has pointed out that coordination problems may be solved by sequencing decisions. Rauch (1993) applies this idea to location of cities and industries. This result follows trivially by backward induction when each player is given a fixed position in the decision-making queue. When the timing of decisions is endogenous, players compete for the most advantageous position in the queue. Then the game cannot be solved by backward induction and this makes Theorem 1 far from obvious.

Farrell and Saloner (1985) analyze several variants of a dynamic game of technology adoption. Although issues of timing arise, structural differences (no discounting, heterogeneous preferences over the technologies to be adopted, etc.) make their results hard to compare with the present setup. In an intriguing aside, they suggest that in a variant of their model with "endogenous timing", it cannot take more than N periods for N players to commit themselves. In the present framework, N players can delay for far more than N periods, as we shall see, so the relationship of Farrel and Saloner's conjecture to Theorem 1 is not clear.

In subsequent work, Farrell (1987), Farrell and Saloner (1988), Farrell and Bolton (1990) and Farrell (1993) focused on the use of different mechanisms involving communication to encourage efficient adoption of a technology or standard.

D. The use of induction in the proof requires a finite number of players. A game with a continuum of players would be quite different. It is natural to assume that a game with a continuum of players is anonymous, since strategies are only specified up to sets of measure zero. This means players observe only the aggregate investment at each date. An equilibrium of the continuum game is a monotonically non-decreasing sequence $\{\alpha_t\}$ with the property that $\alpha_t > \alpha_{t-1}$ implies that t maximizes

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_s) - \delta^{t-1} c.$$

(By convention, $\alpha_0 \equiv 0$). This game has many equilibria. For example, let $\{\alpha_t\}$ be defined by

$$\alpha_t = \begin{cases} 0 & \text{if } t < k \\ \bar{\alpha} & \text{if } t \geq k \end{cases}$$

for any $t = 1, \ldots, \infty$. This sequence defines an equilibrium for any positive integer k and any $\bar{\alpha} \in \{0, \alpha^*, 1\}$. There may be other equilibria, but this set illustrates the possible variety. The important point to note is that this set of equilibria is invariant to the length of the time period, in the sense that holding τk constant as $\tau \to 0$ leaves each equilibrium essentially unchanged.

Krugman (1991) and Matsuyama (1991) study dynamic models of coordination with a continuum of players. An intriguing question is whether their results would be materially altered if the number of players were finite?

3. The effect of the number of players on delay

Theorem 1 suggests that the coordination problem disappears when the players' reaction time is very short. This is only half the story, however. Recall that n^* is the smallest integer greater than α^*N. Clearly, n^* is non-decreasing in N and $n^* \to \infty$ as $N \to \infty$. Theorem 2 shows how to construct a SPE in which delay is proportional to n^*. Again, the proof is by induction. Suppose that once k players invest, the rest will delay for $n^* - k$ periods and then invest. Now suppose $k - 1$ have invested and the uncommitted players intend to invest after $n^* - k + 1$ periods. If anyone deviates by investing immediately, then in the subgame that starts one period after the deviation there exists an equilibrium in which the remaining players will delay for $n^* - k$ periods. Since players cannot react to a deviation until at least one period has elapsed, the total delay is $n^* - k + 1$ periods and the deviant is no better off.

Theorem 2. There exists a SPE in which all players invest at date n^*.

Proof: Let Γ_n denote the equivalence class of subgames in which exactly n players have already invested when the subgame begins. The notional starting date of each subgame is taken to be date 1. Suppose the subgame Γ_n has just begun, that is, the number of players already invested has reached n for the first time. Define equilibrium strategies as follows:

–: if $n = 1, \ldots, n^* - 1$, have all the remaining players invest at date $n^* - n$, as long as there is no further deviation;

–: if $n \geq n^*$, have the remaining players invest at date 1.

Note that whatever the play of the game, this procedure uniquely defines a best response for all remaining players. We can show that this defines a SPE by induction.

For $n \geq n^* - 1$, it is a dominant strategy to invest immediately so the proposed strategies constitute a SPE in Γ_n for this case. Suppose that for every $n = k, \ldots, n^* - 1$, the strategies define a SPE of Γ_n. We want to show that the same is true for $n = k - 1$.

The equilibrium strategy for Γ_{k-1} requires every player to invest at date $n^* - k + 1$ unless someone invests earlier, in which case the equilibrium strategy for the appropriate subgame applies. If a player deviates from this strategy, he will not want to invest later than date $n^* - k + 1$ because this clearly makes him worse off. If he invests before date $n^* - k + 1$, on the other hand, he precipitates the subgame Γ_k and all the remaining players will invest $n^* - k$ periods later. Then the best deviation for him is to invest in the first period. Because the other players do not anticipate his deviation the subgame Γ_k begins in period 2 and the remaining players will therefore invest in period $n^* - k + 1$ as before. His payoff from the deviation is

$$
\begin{aligned}
& -c + v\left(\frac{k}{N}\right) + \delta v\left(\frac{k}{N}\right) + \cdots + \delta^{n^*-k}\left(\frac{v(1)}{1-\delta}\right) \\
&= -c + v\left(\frac{k}{N}\right)\frac{1-\delta^{n^*-k}}{1-\delta} + \delta^{n^*-k}\left(\frac{v(1)}{1-\delta}\right) \\
&= (1-\delta^{n^*-k})\left(\frac{v(k/N)}{1-\delta} - c\right) + \delta^{n^*-k}\left(\frac{v(1)}{1-\delta} - c\right) \\
&\leq \delta^{n^*-k}\left(\frac{v(1)}{1-\delta} - c\right),
\end{aligned}
$$

the equilibrium payoff. This shows that it does not pay to deviate and, by induction, the strategies constitute an equilibrium in every subgame Γ_n. ■

Corollary. For N sufficiently large, there exists an equilibrium in which no player ever invests.

Proof: Choose N^* to be smallest value of N such that $\delta^{n^*-1}v(1) < c$. (Recall that n^* is a non-decreasing function of N.) Suppose the play of the game has just entered the subgame Γ_n. Choose the equilibrium strategies as follows:

–: if $n = 0$, players never invest;
–: if $n = 1, \ldots, n^* - 1$, all remaining players invest at date $n^* - n$;
–: if $n \geq n^*$ all remaining players invest at date 1.

For $n > 0$, these are SPE strategies by Theorem 2. For $n = 0$, if a single player invests, he precipitates a subgame Γ_1 in which all the players will invest at date $n^* - 1$ but since this game begins in the period after he deviates, they are investing in period n^* of the original game. The payoff to the deviating player is $\delta^{n^*-1}v(1) - c$, which is negative for all $N \geq N^*$. ■

Discussion

A. In bargaining models, robust delay occurs when there are more than three players (Shaked (1987)) or when players can strike while continuing to bargain (Fernandez and Glazer (1991)) or when there are externalities (Jéhiel and Moldovanu (1992)) or when there is uncertainty about the timing of offers (Ma and Manove (1993)). The effect of the number of players does not seem to have been investigated beyond games with very small numbers, e.g., two or three players.

Farrell and Saloner (1985) suggest that in a related game with "endogenous timing", it cannot take more than N periods for N players to make their commitments. With this motivation, they go on to study an incomplete information game, which exhibits delay. As the Theorem 2 and its corollary make clear, there is no need for incomplete information to obtain delay in a coordination game.

Delay also occurs in the war of attrition, but in that case increasing numbers reduces delay (Bliss and Nalebuff (1984)).

B. The proof of Theorem 2 is complicated by the fact that the amount of delay in each subgame depends on the number of players left in the game. It is worth noting that the naive strategy of requiring every player to invest at date n^* is not subgame perfect. This is why an inductive argument is required to construct the equilibrium.

C. The argument used to prove Theorem 2 actually proves a stronger result, namely, there exists an equilibrium in which all players invest at date t, for each $t = 1, \ldots, n^*$.

The equilibria mentioned so far have the property that all players invest at the same date. This is not at all necessary. For example, there exists an equilibrium in which player i invests at date i, for all $i = 1, \ldots, n^* - 2$ and the remaining uncommitted players invest at date $n^* - 1$. Because immediate investment becomes a dominant strategy once cumulative investment reaches n^*, the investment process always ends with a "bang"; but the "build up" can be slow.

Obviously, this equilibrium will only exist if $N < N^*$. There is a limit to how "spread out" investment can be. Consider the case of a pure strategy equilibrium, for example, where $v(\alpha) \approx 0$ for all $\alpha < \alpha^*$. The first player to invest makes losses until at least n^* players have invested, at which point it becomes a dominant strategy to invest. It will be optimal for the first player to invest only if $\delta^{T-1}v(1) \geq (1-\delta)c$, where T is the number of periods elapsing between the first player's investment and the last player's.

D. Some care is required in the interpretation of Theorem 2, because it only guarantees the *existence* of equilibria with long delays. There also exist equilibria with little or no delay. With this caveat, Theorem 2 tells us two things. First, it says that Theorem 1 does not necessarily offer a solution to the coordination problem. For any given period length, there may still be significant delays if N is sufficiently large. Second, it shows that the severity of the coordination problem depends on the number of players involved. This accords with intuition and is something that cannot be inferred from the static game.

4. Leading and lagging complementarities

The results in the preceding two sections illustrate the difficulty of obtaining sharp characterizations of the extent of delay. Sharp results are available in some circumstances, but they are sensitive to changes in the parameters.

A further indication of the sensitivity of the predictions of the model appears when we change the *timing* of strategic complementarities. Specifically, consider the case of a game with *leading complementarities*. The extensive form is the same as in the preceding section. Only the payoffs differ. Here the revenue flow at date t is a function of the cumulative investment at date $t+1$. A player investing at date t will receive a payoff equal to

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_{s+1}) - \delta^{t-1} c.$$

All the other assumptions are maintained.

With leading strategic complementarities, players have an incentive to invest "ahead of the pack". In every equilibrium there is no delay.

Theorem 3. In any SPE of the game with leading complementarities, all players invest with probability 1 at the first date.

Proof: The proof is by induction. Take as the induction hypothesis that all players invest with probability 1 at the first date in any subgame of Γ_n, for $n = k, \ldots, N$. This is obviously true for $k \geq n^*$. Now consider the game Γ_{k-1}. Anyone who invests at the first date precipitates a subgame Γ_n at the second date, for some $n \geq k$. By the induction hypothesis, any remaining uncommitted players will invest at the second date. The feasibility condition, Assumption 1(a), together with the form of the payoff function, implies that it is strictly optimal to invest at the first date. This proves that all players invest at the first date in any SPE of Γ_{k-1} and the theorem follows by induction. ■

Discussion

A. The assumption that revenue flow at date t depends on cumulative investment at date $t+1$ may seem odd, but it is formally equivalent to assuming that strategic complementarities are *contemporaneous* and investment must occur one period before the revenue flow begins. To see this put $\tilde{v}(\alpha) \equiv \delta^{-1}v(\alpha)$ and rewrite the payoff function as

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_{s+1}) - \delta^{t-1} c = \sum_{s=t+1}^{\infty} \delta^{s-1} \tilde{v}(\alpha_s) - \delta^{t-1} c.$$

The expression on the right assumes that strategic complementarities are contemporaneous, but requires players to invest in advance.

B. Although the proof of the theorem is simple and conforms to intuition, there is nonetheless something a little surprising about it, if one looks at it in the right way. To see this, start with the game with contemporaneous externalities. Now change the payoffs by zeroing out the revenue to investment in the first period. For example, let the government tax it away. Will this reduction in revenue increase or decrease investment, bring it forward or delay it? It seems surprising that reducing returns encourages investment, but Theorem 3 gives the unambiguous answer as long as Assumption 1 continues to hold. The explanation, of course, is that a player's decision is determined not by the total return to investment, assuming investment is still profitable, but rather by the marginal return to delay.

C. In games with contemporaneous complementarities, many different equilibrium outcomes can be supported by appropriate off-the-equilibrium-path responses. The uniqueness of equilibrium with leading complementarities is in striking contrast. To get this result, one needs both leading complementarities and a finite number of players. The reader can convince himself of this by thinking about the continuum game. Equilibrium with a continuum of players is defined in the same way as for contemporaneous complementarities. There are three types of equilibria with leading complementarities, however. In one, all players invest immediately ($\alpha_t = 1$, $\forall t$); in another, they never invest ($\alpha_t = 0$, $\forall t$); finally, there is a class of equilibria of the form $\alpha_t = 0$, for $t < k$ and $\alpha_t = \alpha^*$ for $t \geq k$, for any fixed but arbitrary k. The first corresponds to the equilibrium in Theorem 3; the others have no counterpart in the finite game.

The difference between the equilibrium sets of the finite game and the continuum game represents a failure of lower hemi-continuity (as opposed to a failure of upper hemi-continuity) and so is not very surprising, perhaps. But, once again, it reminds us of the possible dangers of assuming a continuum of players without sufficient care and attention.

Lagging complementarities

Lagging complementarities give players an incentive to invest after most other players have invested. However, in contrast to the case of leading complementarities, this does not lead to a unique equilibrium outcome. In fact, analogues of Theorems 1 and 2 continue to hold in this case. The extensive form is the same as before, but the payoff to a player who invests at date t is now

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_{s-1}) - \delta^{t-1} c,$$

where α_t is the proportion of players who have invested by the end of date t. Even if all players invest at the first date, the effect of this investment on payoffs is not felt until the second date, so the discounted revenue flow is

$$\bar{V} = v(0) + \delta v(1) + \delta^2 v(1) + \cdots = v(0) + \delta v(1)/(1 - \delta).$$

For this reason, the feasibility condition in Assumption 1(a) must be changed to

$$v(0) + \frac{\delta v(1)}{1 - \delta} > c.$$

All other assumptions are maintained.

The fact that complementarities are "lagging" means the first players to invest are at a disadvantage compared with those who invest later, but the same is true in the case of contemporaneous complementarities when the investment process begins slowly. The same arguments can be used in both cases to show that there is no delay in the limit as the period length becomes vanishingly small. Proofs are omitted, since the arguments are every similar to those found in Sections 3 and 4.

Theorem 4. For any $\varepsilon > 0$ there exists $\eta > 0$ such that, in any SPE, with probability $1 - \varepsilon$, all players invest within ε of the start of the game, whenever the period length is less than η.

Theorem 5. There exists a SPE in which no players invest before date $n^* - 1$.

The construction used to prove Theorem 5 differs slightly from the proof of Theorem 2. In particular, we cannot assume that all players invest at the same date (see Gale (1992)).

Corollary. For sufficiently large N there exists a SPE in which no player ever invests.

5. The robustness of equilibrium delay

The results from the preceding sections emphasize the sensitivity of the equilibrium set to changes in parameter values and the specification of payoffs. These results are interesting for the insight they give into the factors affecting delay, but they may leave the impression that, while delay sometimes occurs, it is not a robust phenomenon. In this section, I argue that delay is, in a precise sense, robust. To see why, consider an equilibrium of the continuum game. For simplicity, I only consider the case of lagging complementarities, but a similar argument is made informally for the case of contemporaneous complemetarities in the discussion at the end of this section.

An equilibrium for the continuum game is a monotonically non-decreasing sequence $\{\alpha_t\}$ with the property that $\alpha_t > \alpha_{t+1}$ implies t maximizes

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_{s-1}) - \delta^{t-1} c.$$

Let t be the first date at which $\alpha_t > 0$. Then $\alpha_{t-1} = 0$ and

$$\sum_{s=t}^{\infty} \delta^{s-1} v(\alpha_{s-1}) - \delta^{t-1} c = \delta^{t-1}(v(0) - (1-\delta)c) + \sum_{s=t+1}^{\infty} \delta^{s-1} v(\alpha_{s-1}) - \delta^{t} c$$

$$< \sum_{s=t+1}^{\infty} \delta^{s-1} v(\alpha_{s-1}) - \delta^{t} c,$$

contradicting the equilibrium condition. In other words, since $v(0) < (1-\delta)c$, players who invest at date t make an initial (flow) loss and would be better off waiting until date $t+1$. Since each player thinks his delay has no effect on the other players, they will all choose to delay and there will be no investment in equilibrium.

In a finite game, players have the same incentive to delay because of the lagging complementarity, but there is a countervailing incentive to invest early. By investing early, a player may encourage others to invest. For example, consider the SPE in which players $i = 1, \ldots, n^*$ invest at the first date and the remaining $N - n^*$ invest at the second date. In any subgame Γ_n, the remaining players invest immediately if $n > n^*$. If $n < n^*$, the $n^* - n$ players with the lowest indexes invest immediately and the remaining $N - n^*$ players invest at the next date. Since players invest in the same fixed order in each subgame, a player cannot change his position in the queue by delaying. He simply delays his payoff. Although there is a small amount of delay in this equilibrium, everyone invests pretty rapidly, no matter how large the number of players.

The critical feature of this equilibrium is that every player in the first group regards himself as pivotal, no matter how large N becomes. In the limit, when there is a continuum of players, this equilibrium disappears. That is, under the assumption of anonymity, there is no equilibrium in which all players invest by the second period. This failure of upper hemi-continuity suggests a lack of robustness in some of the equilibria. To test this hypothesis, I introduce some "noise" and see what effect it has on the equilibrium set. Compared to the previous results, the analysis is rather involved, however, and a number of simplifying assumptions are adopted to make it tractable.

The game is assumed to be *anonymous*, that is, players can only observe the total number of players who invest at each date, and not the identity of the individual investors. In an anonymous game, a history has the form $h = (n_1, \ldots, n_{t-1})$, where n_t is the total number investing at date t. Since actions only need to be specified for those who have not yet invested, a strategy can still be described by a function $f: H \to [0, 1]$.

An equilibrium is *monotonic* if, at each date, every player's equilibrium payoff is monotonically non-decreasing in the number of players who have already invested. Note that all the equilibria constructed so far have been monotonic.

The game is perturbed to capture the fact that perfect coordination is impossible. The simplest way to do this is to restrict players to choosing mixed strategies when they invest. Formally, for some small value of $\varepsilon > 0$, when players choose the strategy f they receive the payoff corresponding to $(1-\varepsilon)f$. For some fixed but arbitrary value of ε consider a sequence $\{f^N\}$, where f^N is a monotonic equilibrium of the perturbed game with N players.

Theorem 6. For any $\varepsilon > 0$ sufficiently small and any information set $h \in H, f^N(h) = 0$ for all N sufficiently large.

The proof is contained in the appendix.

Discussion

A. A *weak Markov strategy* makes an uncommitted player's probability of investment at date t depend only on the date and the cumulative total of players who have invested at the previous date. Thus, a weak Markov strategy for player i can be represented by a sequence of functions $f_i = \{f_{it}\}$ where $f_{it}: \mathbb{N} \to [0, 1]$. Player i invests with probability $f_{it}(n_{t-1})$ at date t if he has not already invested and if n_{t-1} other players have invested by the end of date $t-1$. A strategy is *monotonic* if f_{it} is monotonically non-decreasing. It is easy to see that if the equilibrium strategies are weak Markov and monotonic, then the equilibrium must be monotonic, but these conditions are not necessary.

B. An equilibrium is called *symmetric* if every player chooses the same equilibrium strategy. In Gale (1992), an analogue of Theorem 6 is proved under the assumption that equilibria are symmetric rather than monotonic. In particular, there is no need to assume that strategies are weak Markov and monotonic. Also, there is no need to make use of the trembling hand in the case of symmetric equilibria. Mixed strategies provide all the randomness we need. (I am indebted to Sawoong Kang for this observation).

C. The ideas developed in this section can also be applied to games with contemporaneous complementarities, but in that case an additional assumption is needed to rule out the equilibrium in which everyone invests immediately. This can be done by introducing a small incentive to delay until after the first players have invested. For example, we could strengthen the assumption that perfect coordination is impossible by putting a limit on the total number of players that can invest in any one period. This is meant to capture the idea of capacity constraints on the supply of investment goods. If this limit is less than $\alpha^* N$, there will be a disadvantage in being the first to invest.

Introducing a capacity constraint may seem to be a "large" perturbation of the game, but clearly, if the period length is short, this constraint does not by itself impose a significant amount of delay. The significant delay comes from coordination failure.

6. Open questions

For ease of exposition, I have used the simplest possible model in this paper, but some of the formal results may extend to a richer framework. For example, suppose that each player in the game controls K investments, each of unit size. It appears that the methods used to prove Theorems 1 through 5 would apply in this context, too. In fact, Theorem 1 appears to hold in quite general strategy spaces. Theorem 6 is more problematical, simply because of the difficulty of extending an already complex argument to a richer strategy space.

Extending the model would allow one to analyze some other interesting questions. One limitation of the present framework is that the reaction lag is the same as the "time to build": both are equal to one period. This seems unrealistic. On the one hand, it is not clear why decisions should be time-consuming in this model; on the other, it seems reasonable to assume that physical capital formation takes time. In the extended model, we could distinguish between the two lags. Allowing a player to control K projects, each of unit size, is formally equivalent to assuming the player controls a project of unit size, but can make investments in increments of size $\Delta = 1/K$. In this interpretation, we could add the constraint that only one increment per period is allowed, so a unit of investment takes K periods. We can then ask what happens as the reaction lag becomes very small, holding constant the "time to build". Since τ and Δ vanish at the same rate, we may get results that are quite different from Theorem 1, which assumes Δ is constant as τ converges to zero.

Another interesting question is what would happen if agents had access to institutions that permitted some degree of cooperation. Obviously, if the grand coalition could form and impose an efficient outcome, there would be no coordination problem. This would not be an interesting resolution of the problem. Indeed, any cooperative solution that simply assumes that coalitions behave efficiently is begging the question of how the coalition solves its own, internal, coordination problem. Nonetheless, as a first cut, it might be interesting to see what happens if small coalitions were allowed to form and maximize some aggregate of their members' welfare.

Some insight can be gleaned by considering the model in which each player controls K projects. For then each player is like a coalition of K players who have agreed to share their payoff equally. What this suggests is that as long as the maximum size of the coalitions is bounded above, delay will be robust when the number of players is large, which is exactly what one would expect. Of course, one would ideally like to make the coalition structure endogenous, to explain which coalitions form, to explain what frictions might prevent large coalitions from forming and, ultimately, to explain how coalitions solve their own, internal, coordination problem.

The issue of communication also deserves attention. Even in games of complete and perfect information there may be a role for cheap talk. However, unlike the models studied by Farrell (1987), Farrell and Saloner (1988) and Farrell (1993), there is no danger of players making "incompatible" choices and so no gain from coordinating through cheap talk. To provide a more interesting role for communication one might want to consider games in which a player's investment decision is only imperfectly observed by other players, or is only observed with a lag. Players would have an incentive to make false announcements to encourage the others. Then other players might want to see bricks and mortar before they were convinced that an investment had taken place. This would be like an increase in the reaction lag which, as we have seen, tends to increase the potential for delay.

Appendix

Proof of Theorem 6

Begin by considering two alternative strategies for a distinguished player, say, player 1. Strategy A is to invest at date 1, independently of what the other players do. Strategy B is to wait until date 2 and then invest, independently of what other players do.

Step 1. Consider a fixed but arbitrary equilibrium f^N. The first step is to obtain a lower bound on the payoff to strategy B. Let $V_B(n)$ denote the equilibrium payoff to player 1 from investing at date 2 when exactly n players invest at date 1. Let $p(n)$ be the probability that n players other than player 1 invest at date 1. Then the payoff from investing at date 2 is

$$V_B \equiv \sum_{n=0}^{N-1} p(n) V_B(n).$$

Player 1 cannot be sure of investing at date 2, but he does so with probability $1 - \varepsilon$, so the payoff from strategy B is at least $(1 - \varepsilon)V_B$.

Step 2. The next step is to obtain an upper bound on the payoff to strategy A. If player 1 invests at date 1, he must make a loss of $l = v(0) - (1 - \delta)c$ in the first period. In that case, his payoff cannot be greater than $V_A + l$, where

$$V_A \equiv \sum_{n=0}^{N-1} p(n) V_A(n+1),$$

and $V_A(n + 1)$ is the payoff to player 1 from date 2 onwards if $n + 1$ players (including player 1) invest at date 1. Again, player 1 cannot be sure of investing at date 1, but if he does not manage to invest, his payoff is bounded above by $\bar{V}$, say, so the payoff to strategy A cannot be greater than $(1 - \varepsilon)(V_A + l) + \varepsilon\bar{V}$.

Step 3. The next step is to show that $V_A(n) \leq V_B(n)$ for any n. To do this, consider two situations, A and B. In situation A, player 1 and $n - 1$ players $i_1, \ldots, i_{n-1}$ invest at date 1. In situation B, player 1 adopts strategy B and players $i_1, \ldots, i_n$ invest at date 1. In each case, the number of players who invest at date 1 is the same, but in situation A players $i_n, \ldots, i_{N-1}$ are left to randomize at date 2 and in situation B it is players 1 and $i_{n+1}, \ldots, i_{N-1}$ who are left to randomize at date 2. Players $i_{n+1}, \ldots, i_{N-1}$ use the same probabilities in each situation at date 2, since the game is anonymous and in each case the number of investors at date 1 is n. However, player i_n may choose to invest with probability less than 1 whereas we know that player 1 will choose to invest with probability 1. So the total number of investments made by the end of date 2 in situation B will stochastically dominate the total number of investments made by the end of date 2 in situation A. Since the equilibrium is assumed to be monotonic, the equilibrium payoffs at date 2 are non-decreasing in the total number of investors. This implies that $V_A(n) \leq V_B(n)$. This is the only point in the proof at which we appeal to monotonicity.

Step 4. We can now provide an upper bound for $V_A - V_B$. Direct calculation gives

$$V_A - V_B \leq \sum_{n=0}^{N-1} (V_A(n+1) - V_A(n))p(n)$$

$$= V_A(N)p(N-1) - V_A(0)p(0) + \sum_{n=1}^{N-1} V_A(n)(p(n-1) - p(n)).$$

Let $p(m) = \max_n p(n)$. Since the density function $p(n)$ is single-peaked,

$$\sum_{n=1}^{N-1} V_A(n)(p(n-1) - p(n)) \leq \sum_{n=m+1}^{N-1} V_A(n)(p(n-1) - p(n))$$

$$\leqslant \bar{V}(p(m) - p(N-1)).$$

Then

$$V_A - V_B \leq \bar{V}p(N-1) + \bar{V}(p(m) - p(N-1)) = \bar{V}p(m).$$

Step 5. Using the inequality from Step 4, we can show the sequence $\{\sum_i f_i^N(h)\}$ is bounded for any fixed history h. The proof is by contradiction. Consider the unique history $h = 0$ at date 1. Suppose that along some subsequence, which can be taken to be the original sequence, $\sum_i f_i^N(0) \to \infty$. Then $p(m) \to 0$ as $N \to \infty$. In the limit we have

$$(1-\varepsilon)(V_A + l) + \varepsilon\bar{V} - (1-\varepsilon)V_B \leq \varepsilon\bar{V} + (1-\varepsilon)l < 0,$$

for $\varepsilon > 0$ sufficiently small. Note that ε can be chosen independently of the information set and the number of players.

We have shown that for N sufficiently large, player 1 strictly prefers strategy B to strategy A. What is true for player 1 is true for all of the players, so no one will invest at date 1, contradicting the hypothesis that $\{\sum_i f_i^N(0)\}$ is unbounded.

Essentially the same argument works for any fixed history *h*, *mutatis mutandis*, since the fraction of players who have already invested becomes negligible in the limit.

Step 6. It follows from Step 5 that, for any fixed $h \in H_t$, the number of players investing at the information set h is given by the Poisson approximation to the binomial distribution. For any fixed date t, the cumulative investment α_t^N converges to 0 in probability as N diverges to ∞. Clearly, no one will be willing to invest at the first date or, by the same argument, at any subsequent date. This completes the proof of the theorem. ■

References

Admati, A., Perry, M.: Joint projects without commitment. Rev. Econ. Stud. **58**, 159–276 (1991)

Bliss, B., Nalebuff, B.: Dragon-slaying and ballrom dancing: the private supply of a public good. J. Publ. Econ. **25**, 1–12 (1984)

Bolton, P., Farrell, J.: Decentralization, duplication and delay, J. Polit. Econ. **98**, 801–826 (1990)
Bryant, J.: A simple rational expectations Keynes-type model. Quart. J. Econ. **97**, 525–529 (1983)
Caplin, A., Leahy, J.: Business as usual, market crashes and wisdom after the fact. Columbia University, unpublished (1992)
Chamley, C., Gale, D.: Information revelation and strategic delay in a model of investment. Econometrica, forthcoming (1994)
Chatterjee, S., Cooper, R.: Multiplicity of equilibria and fluctuations in dynamic perfectly competitive economies. AEA Papers Proc. **79**, 353–357 (1989)
Chatterjee, S., Cooper, R., Ravikumar, B.: Participation dynamics: sunspots and cycles. NBER Working Paper Series No. 3438 (1990)
Cooper, R., John, A.: Coordinating coordination failures in a Keynesian model. Quart. J. Econ. **103**, 441–464 (1988)
Diamond, P.: Aggregate demand management in search equilibrium. J. Polit. Econ. **90**, 881–894 (1982)
Durlauf, S.: Non-ergodic growth theory. Rev. Econ. Stud. **60**, 349–366 (1993)
Farrell, J., Saloner, G.: Standardization, compatibility and innovation. Rand J. Econ. **16**, 70–83 (1985)
Farrell, J.: Cheap talk, coordination and delay. Rand J. Econ. **18**, 34–39 (1987)
Farrell, J., Saloner, G.: Coordination through committees and markets. Rand J. Econ. **19**, 235–252 (1988)
Farrell, J.: Choosing the rules for formal standardization. UC Berkeley, unpublished (1993)
Fernandez, R., Glazer, J.: Striking for a bargain between two completely informed agents. Amer. Econ. Rev. **81**, 240–252 (1991)
Gale, D.: Dynamic coordination games. Boston University, unpublished (1992)
Gul, F., Sonnenschein, H.: On delay in bargaining with one-sided uncertainty. Econometrica **56**, 81–95 (1988)
Heller, W.: Coordination failure under complete markets with application to effective demand. In: Heller, W., Starr, R., Starrett, D. (eds.) Equilibrium analysis: essays in honor of Kenneth J. Arrow, vol. II. Cambridge: Cambridge University Press 1986
Krugman, P.: History versus expectations. Quart. J. Econ. **106**, 651–667 (1991)
Jéhiel, P., Moldovanu, B.: Cyclical delay in bargaining with externalities. University of Bonn, unpublished (1992)
Ma, A., Manov, M.: Bargaining with deadlines and imperfect player control. Econometrica **61**, 1313–1340 (1993)
Matsuyma, K.: Increasing returns, industrialization and indeterminacy of equilibrium. Quart. J. Econ. **106**, 617–650 (1991)
Rauch, J.: Does history matter only when it matters little? the case of city industry location. Quart. J. Econ. **108**, 843–867 (1993)
Rubinstein, A.: Perfect equilibrium in a bargaining model. Econometrica **50**, 97–110 (1982)
Shaked, A.: Opting out: bazaars vs "high tech" markets. London School of Economics, STICERD Discussion Paper (1987)
Shleifer, A.: Implementation cycles. J. Polit. Econ. **94**, 1163–1190 (1986)