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Longitudinal oscillations of a sphere in a micropolar fluid

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Summary. A general expression for the force exerted on a sphere executing longitudinal oscillations, with small amplitude, in an incompressible micropolar fluid is obtained. This is accomplished by using direct integral consequences of the full field and the constitutive equations written in cartesian coordinates. The results which are independent of any boundary conditions are then applied to calculate the hydrodynamic force experienced by a sphere moving with rectilinear oscillating velocity $\boldsymbol{u}(t) = (u_0 e^{i\lambda t}, 0, 0)$ in an unbounded micropolar fluid. As a special case, a general expression for the drag in a similar viscous flow is also derived.

1 Introduction

In 1966, Eringen [1] introduced the theory of micropolar fluids. These fluids are characterized by the existence of a microstructure and a rotation of the microelements. In addition to the traditional Cauchy stress and body forces, these fluids can sustain couple stress and body couples. A complete description of their flow involves, besides the usual velocity variable and the viscosity coefficient, another kinematic variable to account for microrotation and several other material constants. The use of micropolar fluid model has been suggested in the study of flow properties of polymeric fluids, fluids with certain additives, animal blood, etc. Because of the ever-increasing applications of this model, it has been used in the investigation of an enormous number of flow problems.

One of the important problems in fluid dynamics is to find the force exerted on a moving object by the surrounding fluid. A common approach to handle such a problem is to first solve a particular system of differential equations subject to the given boundary conditions of a specific flow situation under consideration and then use this solution to compute the needed force. This is the way Rao et al. [2] derived, inter alia, the drag formula for a sphere oscillating rectilinearly in an incompressible micropolar fluid. In the present investigation, we develop a general expression, independent of any boundary conditions, to calculate the force experienced by a sphere while executing longitudinal oscillations of small amplitude in a homogeneous incompressible micropolar fluid. To achieve this generality, we avoid the common procedure of first trimming the governing equations usually necessitated by a predetermined set of boundary conditions of a paticular flow problem and then seeking their explicit solution. Instead, we use some direct integral consequences of the full field and constitutive equations written in rectangular cartesian coordinates to study a class of flows. Our method is an adaption of a technique developed by Saffman [3] while working on the lift of a small sphere in a slow shear flow. It involves the repeated application of the following basic lemma:

For an arbitrary function ϕ,

$$\int_{r=R} \phi_{,i}\, dS = \frac{d}{dR} \int_{r \leq R} \phi_{,i}\, dV = \frac{d}{dR} \int_{r=R} \frac{\phi x_i}{R}\, dS. \tag{1.1}$$

In (1.1), the first step states that the surface integral is simply the derivative of the volume integral with respect to the radius R of the bounding surface $r = R$, and the second step is a consequence of the divergence theorem and the fact that the unit normal n_i to the sphere equals x_i/R when the origin is at the center of the sphere.

Hills [4] used this technique to obtain an expression for the drag on a sphere immersed in a general slow and steady flow of a dipolar fluid. Our present work extends it to an unsteady flow of a micropolar fluid induced by an oscillating sphere. At the end, we discuss an application of our general results by calculating the force on a sphere performing rectilinear oscillations under given boundary conditions. We also indicate the way to specialize our results to a similar Newtonian flow.

2 The micropolar fluid model

Eringen [1] formulated the constitutive and field equations for a homogeneous incompressible micropolar fluid. For our purposes, we make the following two assumptions about the nature of flow and the general form of flow variables:

a) The amplitude of the oscillations is small so that the convective terms can be dropped from the equations of motion.

b) The velocity and the microrotation vectors and the hydrostatic pressure can be written in the form

$$u_i(x_1, x_2, x_3)\, e^{i\lambda t}, \qquad v_i(x_1, x_2, x_3)\, e^{i\lambda t} \qquad \text{and} \qquad p(x_1, x_2, x_3)\, e^{i\lambda t}$$

respectively. Here λ is the frequency of oscillation.

In light of these assumptions, the constitutive and the field equations take the following form:

constitutive equations:

$$t_{ij} = [-p\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + \varkappa(u_{j,i} - \varepsilon_{ijk} v_k]\, e^{i\lambda t}, \tag{2.1}$$

$$m_{ij} = [\alpha v_{k,k}\delta_{ij} + \beta v_{i,j} + \gamma v_{j,i}]\, e^{i\lambda t}, \tag{2.2}$$

field equations:

$$u_{i,i} = 0, \tag{2.3}$$

$$(\mu + \varkappa)\, u_{i,jj} + \varkappa \varepsilon_{ijk} v_{k,j} - p_{,i} = i\lambda\varrho u_i, \tag{2.4}$$

$$(\alpha + \beta)\, v_{j,ij} + \gamma v_{i,jj} + \varkappa \varepsilon_{ijk} u_{k,j} - 2\varkappa v_i = i\lambda\varrho I v_i. \tag{2.5}$$

Here t_{ij} and m_{ij} are the Cauchy and the couple stress tensors respectively. δ_{ij} is the Kronecker delta, ε_{ijk} the alternating tensor, ϱ the density, I the microinertia, α, β, γ, μ, $\varkappa$ the material constants and comma denotes partial differentiation with respect to a space variable. In writing Eqs. (2.4) and (2.5) we have discarded the body force and the the body couple terms.

The thermodynamical requirements impose the following restrictions on the material constants:

$$2\mu + \varkappa \geqq 0, \qquad \varkappa \geqq 0$$

$$3\alpha + \beta + \gamma \geqq 0, \qquad \gamma \geqq |\beta|. \tag{2.6}$$

3 A general expression for the force

We consider longitudinal oscillations, of small amplitude, of a sphere in a micropolar fluid. Choose a rectangular cartesian coordinate system x_1, x_2, x_3 with origin at the instantaneous position of the center of the sphere. Then the force exerted by the fluid on the sphere is given by

$$f_k = \int_{r=R} t_k \, dS = \int_{r=R} t_{jk} n_j \, dS$$

$$= \left[-\int_{r=R} \frac{p\delta_{jk}x_j}{R} \, dS + \mu \int_{r=R} \frac{u_{j,k}x_j}{R} \, dS + (\mu + \varkappa) \int_{r=R} \frac{u_{k,j}x_j}{R} \, dS - \varkappa \int_{r=R} \frac{\varepsilon_{ijk}v_i x_j}{R} \, dS \right] e^{i\lambda t}, \tag{3.1}$$

where t_k is the traction acting on the surface of a sphere with unit normal n_j and R is the radius of any sphere surrounding and concentric with the particle at the origin. Hereafter, it will be understood, unless otherwise stated, that all the integrals are taken over the surface of the sphere $r = R$. To simplify the integrals appearing in (3.1) we need the following formulas from [4]:

$$\frac{1}{R} \int u_{j,k} x_j \, dS = 0, \tag{3.2}$$

$$\frac{1}{R} \int u_{k,j} x_j \, dS = \left[R \frac{d}{dR} - 1 \right] \int \frac{u_k}{R} \, dS. \tag{3.3}$$

The expression for the force then reduces to

$$f_k = \left[-\int \frac{p x_k}{R} \, dS + (\mu + \varkappa) \left(R \frac{d}{dR} - 1 \right) \int \frac{u_k}{R} \, dS - \varkappa \int \frac{\varepsilon_{ijk} v_i x_j}{R} \, dS \right] e^{i\lambda t}. \tag{3.4}$$

We next proceed to evaluate each of the three integrals in (3.4). First taking the divergence of (2.4) and thereafter employing the continuity equation (2.3), we obtain

$$p_{,ii} = 0. \tag{3.5}$$

Then as demonstrated in [4]

$$\int \frac{p x_k}{R} \, dS = A_k + B_k R^3 \tag{3.6}$$

where A_k and B_k are unknown constant vectors to be determined from the boundary conditions of a particular flow situation.

To evaluate the integral of u_k, we first eliminate the variable $\boldsymbol{v}$ from the field equations (2.4) and (2.5). After performing the curl operation twice on (2.4) and once on (2.5) and taking advantage of the continuity equation (2.3), we can derive the following equation with the variable $\boldsymbol{v}$ absent:

$$u_{k,iijj} - (m^2 + n^2)\, u_{k,ii} + m^2 n^2 u_k = -\frac{m^2 n^2}{i\lambda\varrho}\, p_{,k}, \tag{3.7}$$

where

$$m^2 + n^2 = \frac{\varkappa(2\mu + \varkappa)}{\gamma(\mu + \varkappa)} + i\,\frac{\lambda\varrho(\mu I + \varkappa I + \gamma)}{\gamma(\mu + \varkappa)} \tag{3.8}$$

and

$$m^2 n^2 = -\frac{\lambda^2 \varrho^2 I}{\gamma(\mu + \varkappa)} + i\,\frac{2\lambda\varrho\varkappa}{\gamma(\mu + \varkappa)}. \tag{3.9}$$

We note that the numbers m and n are not pure imaginary. We may assume that their real parts are positive. Integrating each term in (3.7), we arrive at the following differential equation in $\int \frac{u_k}{R}\, dS$:

$$R\,\frac{d^4}{dR^4}\int \frac{u_k}{R}\, dS - (m^2 + n^2)\, R\,\frac{d^2}{dR^2}\int \frac{u_k}{R}\, dS + m^2 n^2 R \int \frac{u_k}{R}\, dS$$
$$= -\frac{m^2 n^2}{i\lambda\varrho}\,\frac{d}{dR}\int \frac{p x_k}{R}\, dS. \tag{3.10}$$

In writing down (3.10), we applied lemma (1.1) and the following formulas from [4]:

$$\int \phi_{,ii}\, dS = R\,\frac{d^2}{dR^2}\int \frac{\phi}{R}\, dS, \tag{3.11}$$

$$\int \phi_{,iijj}\, dS = R\,\frac{d^4}{dR^4}\int \frac{\phi}{R}\, dS. \tag{3.12}$$

The general solution of (3.10) is given by

$$\int \frac{u_k}{R}\, dS = -3\,\frac{B_k}{i\lambda\varrho}\, R + C_k e^{mR} + D_k e^{-mR} + E_k e^{nR} + F_k e^{-nR} \tag{3.13}$$

where C_k, D_k, E_k, and F_k are unknown constant vectors.

Finally an integration of Eq. (2.4) with the help of (1.1) and (3.11) followed by appropriate substitutions for the resulting integrals produces

$$\int \frac{\varepsilon_{ijk} v_i x_j}{R}\, dS = \left(\frac{\mu + \varkappa}{\varkappa}\right)\Big[C_k(mR - 1)\, e^{mR} - D_k(mR + 1)\, e^{-mR}$$
$$+ E_k(nR - 1)\, e^{nR} - F_k(nR + 1)\, e^{-nR}\Big] - \frac{i\lambda\varrho}{\varkappa}\,\psi_k(R) + G_k, \tag{3.14}$$

where

$$\psi_k(R) = C_k\,\frac{1}{m^2}\,(mR - 1)\, e^{mR} - D_k\,\frac{1}{m^2}\,(mR + 1)\, e^{-mR}$$
$$+ E_k\,\frac{1}{n^2}\,(nR - 1)\, e^{nR} - F_k\,\frac{1}{n^2}\,(nR + 1)\, e^{-nR} \tag{3.15}$$

and G_k is another constant vector.

On substituting the values of the integrals from (3.6), (3.13) and (3.14) into the right side of (3.4), the expression for the force f_k reduces to

$$f_k = [-A_k - B_k R^3 + i\lambda\varrho\psi_k(R) - \varkappa G_k]\, e^{i\lambda t}. \tag{3.16}$$

For a further possible reduction of this expression, we now look for some relationships among the various unknown constant vectors. First we form the scalar product of Eq. (3.7) with x_k and then multiply by x_l. A subsequent integration using the following formulas from [4]:

$$\int x_l x_k p_{,k}\, dS = \left(R^2 \frac{d}{dR} - 2R\right) \int \frac{p x_l}{R}\, dS, \tag{3.17}$$

$$\int x_l x_k u_{k,ii}\, dS = \left(R^2 \frac{d}{dR} - R\right) \int \frac{u_l}{R}\, dS, \tag{3.18}$$

$$\int x_l x_k u_{k,iijj}\, dS = \left(R^2 \frac{d^3}{dR^3} - R\frac{d^2}{dR^2}\right) \int \frac{u_l}{R}\, dS, \tag{3.19}$$

in conjunction with the expressions (3.6), (3.13) and another readily verifiable result

$$\int \frac{u_k x_k x_l}{R}\, dS = \psi_l(R) - \frac{B_l}{i\lambda\varrho} R^3 + H_l, \tag{3.20}$$

where H_l is an arbitrary constant vector, yields

$$A_k = \frac{1}{2} i\lambda\varrho H_k. \tag{3.21}$$

Next we show that G_k must vanish. As a first step, we take the curl of (2.4) and substitute for $\nabla \times \nabla \times \boldsymbol{v}$ in (2.5) to obtain

$$v_i = \frac{i\lambda\varrho}{\varkappa m^2 n^2} \varepsilon_{ilk} u_{k,ljj} + \left(\frac{\mu+\varkappa}{\varkappa} - \frac{i\lambda\varrho}{\varkappa m^2} - \frac{i\lambda\varrho}{\varkappa n^2}\right) \varepsilon_{ilk} u_{k,l} + \frac{i\lambda\varrho(\alpha+\beta+\gamma)}{\gamma(\mu+\varkappa)\, m^2 n^2} v_{l,li}. \tag{3.22}$$

Taking its vector product with $\boldsymbol{n}$ and then integrating we find another expression for $\int \frac{\varepsilon_{ijk} v_i x_j}{R}\, dS$:

$$\int \frac{\varepsilon_{ijk} v_i x_j}{R}\, dS = \left(\frac{\mu+\varkappa}{\varkappa}\right) [C_k(mR-1)\, e^{mR} - D_k(mR+1)\, e^{-mR} + E_k(nR-1)\, e^{nR} - F_k(nR+1)\, e^{-nR}] - \frac{i\lambda\varrho}{\varkappa} \psi_k(R). \tag{3.23}$$

In order to derive (3.23), we had to use, in addition to (3.2) and (3.3), the following formulas from [4]:

$$\frac{1}{R}\int x_k u_{i,kjj}\, dS = \left(R\frac{d^3}{dR^3} - \frac{d^2}{dR^2}\right) \int \frac{u_i}{R}\, dS, \tag{3.24}$$

$$\frac{1}{R}\int x_k u_{k,ijj}\, dS = 0. \tag{3.25}$$

We also needed

$$\int \frac{\varepsilon_{ijk} v_{l,li} x_j}{R}\, dS = 0, \tag{3.26}$$

which is an immediate consequence of the basic lemma (1.1).

Now comparison of (3.23) with an earlier result (3.14) leads to the desired conclusion, namely,

$$G_k = 0. \tag{3.27}$$

Utilizing (3.21) and (3.27) in (3.16), our general formula for the force f_k takes the following final form:

$$f_k = \left[-\frac{1}{2}\, i\lambda\varrho H_k - B_k R^3 + i\lambda\varrho\psi_k(R)\right] e^{i\lambda t}. \tag{3.28}$$

We have derived all the foregoing results without reference to any near and/or far boundary conditions. Their generality would thus enable the drag force to be evaluated in a variety of flow situations. In the next section, we shall apply them to calculate the drag force on a sphere oscillating rectilinearly in an unbounded micropolar fluid.

4 An application

We now consider a sphere of radius a executing rectilinear oscillations of small amplitude, with velocity $u_0 e^{i\lambda t}$ along the x_1-axis, in an unbounded incompressible micropolar fluid. We assume that at large distances from the sphere, the fluid is at rest. Therefore, the far conditions are

$$\boldsymbol{u} = \boldsymbol{v} = 0. \tag{4.1}$$

It is seen from (3.13), (3.14) or from (3.20) that the conditions at infinity imply that

$$\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{E} = \boldsymbol{0}. \tag{4.2}$$

With the help of these results, the condition $\boldsymbol{u}(t) = (u_0 e^{i\lambda t}, 0, 0)$ on the surface $r = a$ implies

$$e^{-am}\,\boldsymbol{D} + e^{-an}\,\boldsymbol{F} = (4\pi a u_0, 0, 0) \tag{4.3}$$

and

$$-\frac{1}{m^2}\,(am + 1)\, e^{-am}\boldsymbol{D} - \frac{1}{n^2}\,(an + 1)\, e^{-an}\boldsymbol{F} + \boldsymbol{H} = \left(\frac{4\pi a^3 u_0}{3}, 0, 0\right) \tag{4.4}$$

from (3.13) and (3.20) respectively. Also, the condition of zero microrotation on the surface when applied to (3.14) gives

$$[(\mu + \varkappa)\, m^2 - i\lambda\varrho]\,(am + 1)\,\frac{1}{m^2}\, e^{-am}\boldsymbol{D}$$

$$+ [(\mu + \varkappa)\, n^2 - i\lambda\varrho]\,(an + 1)\,\frac{1}{n^2}\, e^{-an}\boldsymbol{F} = (0, 0, 0). \tag{4.5}$$

Equations (4.3), (4.4) and (4.5) can be solved for $\boldsymbol{D}$, $\boldsymbol{F}$ and $\boldsymbol{H}$ to give

$$D_1 = \frac{4\pi a u_0[(\mu + \varkappa)\, n^2 - i\lambda\varrho]\,(an + 1)\, m^2}{(n - m)\,[a(\mu + \varkappa)\, m^2 n^2 + i\lambda\varrho(amn + m + n)]}\, e^{am}, \tag{4.6}$$

$$F_1 = \frac{-4\pi a u_0[(\mu + \varkappa)\, m^2 - i\lambda\varrho]\,(am + 1)\, n^2}{(n - m)\,[a(\mu + \varkappa)\, m^2 n^2 + i\lambda\varrho(amn + m + n)]}\, e^{an}, \tag{4.7}$$

$$H_1 = \frac{4}{3}\,\pi a^3 u_0 + \frac{4\pi a u_0(\mu + \varkappa)\,(am + 1)\,(an + 1)\,(m + n)}{a(\mu + \varkappa)\, m^2 n^2 + i\lambda\varrho(amn + m + n)} \tag{4.8}$$

All other components of $\boldsymbol{D}$, $\boldsymbol{F}$ and $\boldsymbol{H}$ are zero. Substituting these values of the constants in (3.28) the force on the sphere is given by

$$f_1 = \left[-\frac{2}{3} i\lambda\varrho\pi a^3 u_0 - \frac{6i\lambda\varrho\pi a u_0(\mu + \varkappa)(am + 1)(an + 1)(m + n)}{a(\mu + \varkappa) m^2n^2 + i\lambda\varrho(amn + m + n)} \right] e^{i\lambda t} \tag{4.9}$$

$$f_2 = f_3 = 0.$$

The entire preceding analysis can be specialized so as to apply to a purely viscous flow. In this case we must discard the microrotational aspect of flow. Consequently we work only with Eqs. (2.1), (2.3) and (2.4) with $\varkappa = 0$. A comparison of (2.4) with (3.7) suggests that either $|m|$ or $|n|$ must be chosen arbitrarily large. Then letting, say, $|n|$ approach infinity, m is determined by

$$m^2 = \frac{i\lambda\varrho}{\mu} = 2\delta^2 i. \tag{4.10}$$

In view of these considerations, (3.14) becomes redundant and Eqs. (3.13), (3.20) and (3.28) undergo obvious modifications. Either applying these modified equations or letting $n \to \infty$ in (4.9), we obtain the well-known drag formula for a Newtonian flow, Lamb [5, p. 644]:

$$\text{drag} = -\frac{4}{3} \pi\varrho a^3 \left(\frac{1}{2} + \frac{9}{4a\delta}\right) \frac{du_1}{dt} - 3\pi\lambda\varrho a^3 \left(\frac{1}{a\delta} + \frac{1}{a^2\delta^2}\right) u_1, \tag{4.11}$$

where $u_1 = u_0 e^{i\lambda t}$.

To extract some further useful information from formula (4.9), we first recast it in the following equivalent form:

$$f_1 = -\frac{4}{3} \pi\varrho a^3 \left(\frac{1}{2} + \frac{9}{4}\frac{A}{D}\right) \frac{du_1}{dt} - 3\pi\lambda\varrho a^3 \left(\frac{B}{D}\right) u_1, \tag{4.12}$$

where

$$A = 2\frac{\mu + \varkappa}{a^2} \{\lambda\varrho \operatorname{Im}[(am + 1)(an + 1)(m + n)(a\overline{m}\overline{n} + \overline{m} + \overline{n})] + a(\mu + \varkappa) \operatorname{Re}[(am + 1)(an + 1)(m + n)\overline{m}^2\overline{n}^2]\}, \tag{4.13}$$

$$B = 2\frac{\mu + \varkappa}{a^2} \{\lambda\varrho \operatorname{Re}[(am + 1)(an + 1)(m + n)(a\overline{m}\overline{n} + \overline{m} + \overline{n})] - a(\mu + \varkappa) \operatorname{Im}[(am + 1)(an + 1)(m + n)\overline{m}^2\overline{n}^2]\}, \tag{4.14}$$

$$D = a^2(\mu + \varkappa)^2 |m^2n^2| + \lambda^2\varrho^2 |amn + m + n| + 2a(\mu + \varkappa)\lambda\varrho \operatorname{Im}[(a\overline{m}\overline{n} + \overline{m} + \overline{n}) m^2n^2]. \tag{4.15}$$

We note the analogy of (4.12) with (4.11). This suggests that like the viscous fluid, the first term in (4.12) gives the correction to the inertia of the sphere in the micropolar case. This amounts to the fraction

$$\frac{1}{2} + \frac{9}{4}\frac{A}{D} \tag{4.16}$$

of the mass of fluid displaced, instead of 1/2 as in the case of a frictionless liquid. The second term in each of these formulas gives the respective frictional force varying as the velocity.

Next, we derive from (4.9) a formula for the drag on a sphere moving rectilinearly with uniform velocity in a micropolar fluid. To do this, we first rewrite (3.8) and (3.9) as follows:

$$m^2 + n^2 = i\lambda\varrho \frac{\mu I + \varkappa I + \gamma}{\gamma(\mu + \varkappa)} + L^2, \tag{4.17}$$

$$m^2 n^2 = i\lambda\varrho \frac{2\varkappa + i\lambda\varrho I}{\varkappa(2\mu + \varkappa)} L^2 \tag{4.18}$$

where

$$L^2 = \frac{\varkappa(2\mu + \varkappa)}{\gamma(\mu + \varkappa)}. \tag{4.19}$$

Using (4.18) in (4.9) and then letting the period $2\pi/\lambda$ become infinitely long, we obtain the necessary formula:

$$f_1 = -\frac{6\pi a u_0(\mu + \varkappa)(2\mu + \varkappa)(aL + 1)}{2(\mu + \varkappa) aL + 2\mu + \varkappa}. \tag{4.20}$$

In calculating the above limit, we used the fact that as $\lambda \to 0$,

$$\lim (m^2 + n^2) = L^2 \qquad \text{and} \qquad \lim (m^2 n^2) = 0, \tag{4.21}$$

so that we may take, say, $m = L$ and $n = 0$.

On setting $\varkappa = 0$, we recover the corresponding formula for the the Newtonian fluid, Lamb [5, p. 644]:

$$\text{drag} = -6\pi a \mu u_0. \tag{4.22}$$

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References

[1] Eringen, A. C.: Theory of micropolar fluids. J. Math. Mech. **16**, 1 (1966).
[2] Lakshmana Rao, S. K., Bhujunga Rao, P.: The oscillations of a sphere in a micropolar fluid. Int. J.Engng. Sci. **9**, 651 (1971).
[3] Saffman, P. G.: The lift on a small sphere in slow shear flow. J. Fluid Mech. **28**, 385 (1965).
[4] Hills, R. N.: The slow flow of a dipolar fluid past a sphere. Int. J. Engng. Sci. **5**, 957 (1967).
[5] Lamb, H.: Hydrodynamics, 6th edn., p. 644. New York: Dover 1945.

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