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Probability Theory and Related Fields

A class of quasilinear stochastic partial differential equations of McKean–Vlasov type with mass conservation*

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Summary. A system of N particles in $\mathbf{R}^d$ with mean field interaction and diffusion is considered. Assuming adiabatic elimination of the momenta the positions satisfy a stochastic ordinary differential equation driven by Brownian sheets (microscopic equation), where all coefficients depend on the position of the particles and on the empirical mass distribution process. This empirical mass distribution process satisfies a quasilinear stochastic partial differential equation (SPDE). This SPDE (mezoscopic equation) is solved for general measure valued initial conditions by "extending" the empirical mass distribution process from point measure valued initial conditions with total mass conservation. Starting with measures with densities in $L_2(\mathbf{R}^d, dr)$, where dr is the Lebesgue measure, the solution will have densities in $L_2(\mathbf{R}^d, dr)$ and strong uniqueness (in the Itô sense) is obtained. Finally, it is indicated how to obtain (macroscopic) partial differential equations as limits of the so constructed SPDE's.

Mathematics Subject Classification 60H15, 81G99, 35K55

1 Introduction

In this section we give the basic definitions of the microscopic model of N interacting particles (1.1), *driven by noise which is white in time and correlated in space* (*SODE*(1.12)). *Moreover we derive from the microscopic model the mezoscopic one as the empirical process,* (1.2), *which by Itô's formula* (*Lemma* 1.3) *satisfies the quasilinear SPDE* (1.25).

Let N point particles be distributed over $\mathbf{R}^d$, where d is arbitrary and fixed. The position of the ith particle at time t will be denoted by $r^i(t)$, which is by assumption in $\mathbf{R}^d$ for all i. We will assume that by some kind of adiabatic

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elimination we can neglect the momenta (cf. e.g. [17]). Thus we consider the position of the N-particle system at time t as a point in $\mathbf{R}^{dN}$, i.e.,

$$(r^1(t),\ldots,r^N(t)) \in \mathbf{R}^{dN} , \tag{1.1}$$

and a description of its time evolution will be a microscopic model for the particle distribution. Typically one derives from mechanical principles ordinary differential equations (ODE's) or stochastic ordinary differential equations (SODE's) for (1.1) (cf. e.g. [19, 12, 22, 17] etc.). We will call such equations (describing the position of each particle, usually in dependence on the position of the other particles) *microscopic equations*. Now let the mass of the ith particle be a_i. Then the empirical mass distribution at time t of the N-particle system is given by

$$\mathscr{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t))} , \tag{1.2}$$

where δ_r is the point measure, concentrated in $r \in \mathbf{R}^d$. In other words, $\mathscr{X}_N(t)$ is a measure process and for $A \subset \mathbf{R}^d$, $\mathscr{X}_N(t, A) = \Sigma_{i=1}^N a_i 1_{\{r^i(t)\in A\}}$ describes the mass in A at t irrespective of which particular particles are in A at t, i.e., $\mathscr{X}_N(t)$ reduces information given by (1.1) to the relevant one as far as the mass distribution is concerned. On the other hand, if we agree to call a mass distribution *macroscopic* if the particle structure cannot be seen and stochastic fluctuations are absent then we may call a description of the time evolution of (1.2) a *mezoscopic model*, in particular, if stochastic fluctuations are present in (1.1), which we will always assume. Our goal is to derive a stochastic partial differential equation (SPDE) for (1.2), which will be called a *mezoscopic equation*. Under physically and mathematically reasonable assumption on (1.1) this SPDE will be extendible from sums of weighted point measures to measures having densities with respect to the Lebesque measure dr on $\mathbf{R}^d$. Then as a certain parameter, the correlation length of the fluctuation forces in (1.1), tends to 0 the solution of the SPDE will tend to the solution of an (integro-) partial differential equation (PDE), which we will call a *macroscopic equation* for the mass distribution.

Our approach is motivated by the author's derivation of a stochastic Navier–Stokes equation for the vorticity of a two-dimensional fluid [16]. The corresponding mezoscopic description $\mathscr{X}_N(t)$ in that case is a sum of point measures multiplied by intensities $a_i \in \mathbf{R}$, i.e., by point measures in the position of point vortices with positive and negative intensities. To include this case as well as other possible signed measure valued cases (cf. [10]) we will allow the a_i in (1.2) to be positive and negative and just call them positive and negative "masses". So let $a^+ > 0$ be the total "positive" mass of (1.2), $a^- \geqq 0$ be the total negative mass and $a := a^+ + a^-$. These quantities will be fixed throughout the paper. Set

$$\mathbf{M} := \{\mu\colon\ \mu \text{ is a finite signed Borel measure on } \mathbf{R}^d,\ \mu^{\pm}(\mathbf{R}^d) = a^{\pm}\} ,$$

where μ^+ and μ^- is the Jordan decomposition of μ and for any Borel set $A \subset \mathbf{R}^d$ and nonnegative numbers b^+, b^- $\mu^{\pm}(A) = b^{\pm}$ if and only if $\mu^+(A) = b^+$ and $\mu^-(A) = b^-$. First we define a metric on $\mathbf{R}^d$ by

$$\rho(r,q) := (\bar{c}|r-q| \wedge 1)\,, \tag{1.3}$$

where $r, q \in \mathbf{R}^d$, $|r-q|$ is the Euclidean distance on $\mathbf{R}^d$, $\bar{c}$ some positive constant and "$\wedge$" denotes "minimum". If $\mu, \tilde{\mu} \in \mathbf{M}$ we will call positive Borel measures $Q^{\pm}$ on $\mathbf{R}^{2d}$ joint representations of $(\mu^+, \tilde{\mu}^+)$, resp $(\mu^-, \tilde{\mu}^-)$ if $Q^{\pm}$ $(A \times \mathbf{R}^d) = \mu^{\pm}(A)a^{\pm}$ and $Q^{\pm}(\mathbf{R}^d \times B) = \tilde{\mu}^{\pm}(B)a^{\pm}$ for arbitrary Borel sets $A, B \subset \mathbf{R}^d$. The set of all joint representations of $(\mu^+, \tilde{\mu}^+)$, resp. $(\mu^-, \tilde{\mu}^-)$ will be denoted by $C(\mu^+, \tilde{\mu}^+)$, resp. $C(\mu^-, \tilde{\mu}^-)$. For $\mu, \tilde{\mu} \in \mathbf{M}$ and $m = 1, 2$ set

$$\gamma_m(\mu,\tilde{\mu}) := \left[\inf_{Q^+ \in C(\mu^+,\tilde{\mu}^+)} \int\int Q^+(dr,dq)\varrho^m(r,q) + \inf_{Q^- \in C(\mu^-,\tilde{\mu}^-)} \int\int Q^-(dr,dq)\varrho^m(r,q) \right]^{1/m}\,, \tag{1.4}$$

where the integration is taken over $\mathbf{R}^d \times \mathbf{R}^d$. (We will not indicate the integration domain when integrating over $\mathbf{R}^d$.) By the boundedness of ϱ and the Cauchy–Schwarz inequality

$$\gamma_1(\mu,\tilde{\mu}) \geqq \gamma_2^2(\mu,\tilde{\mu}) \geqq \frac{1}{a^+ \vee a^-}\gamma_1^2(\mu,\tilde{\mu})\,, \tag{1.5}$$

where "$\vee$" denotes the maximum of two numbers.

After normalizing the measures by $\mu^{\pm} \to \mu^{\pm}/a^{\pm}$ (setting $\mu^-/a^- = 0$ if $a^- = 0$) the Kantorovich-Rubinstein theorem implies $\gamma_2(\mu,\tilde{\mu}) = 0$ if and only if $\mu^+ = \tilde{\mu}^+$ and $\mu^- = \tilde{\mu}^-$ [8, Chap. 11]. The triangle inequality for $\gamma_2(\mu,\tilde{\mu})$ follows as for the Wasserstein metric (which is $\gamma_1(\mu^+/a^+, \tilde{\mu}^+/a^+)$ for μ^+/a^+ and $\tilde{\mu}^+/a^+$). Hence γ_2 is a metric on $\mathbf{M}$, and $\mathbf{M}$ endowed with γ_2 will be denoted by $(\mathbf{M}, \gamma_2)$. By (1.5), the Prohorov and the Kantorovich–Rubinstein theorems $(\mathbf{M}, \gamma_2)$ is complete (cf. [8, 11.5.5 and 11.8.2]). Moreover, as in the Wasserstein case (cf. [7, Appendix, Lemma 4]) we obtain that the set of linear combinations of signed point measures from $\mathbf{M}$ is dense in $(\mathbf{M}, \gamma_2)$. For $f \in C(\mathbf{R}^d, \mathbf{R})$ the space of real valued continuous functions on $\mathbf{R}^d$, we set

$$|||f|||_L := \sup_{r,q \in \mathbf{R}^d, r \neq q} \left\{ \frac{|f(r) - f(q)|}{\varrho(r,q)} \right\}\,.$$

(1.5) and the Kantorovich–Rubinstein theorem imply

$$\gamma_2^2(\mu,\tilde{\mu}) \geqq \frac{1}{2(a^+ \vee a^-)} \sup_{|||f|||_L \leqq 1} |< \mu - \tilde{\mu}, f >^2|\,. \tag{1.6}$$

Next we introduce the stochastic set-up. $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω and to be $\mathscr{F}_t$-adapted (including all initial conditions in stochastic ordinary differential equations (SODE's) and stochastic partial differential equations (SPDE's). Moreover, the processes are assumed to be $dP \otimes dt$-measurable, where dt is the Lebesgue measure on $[0,\infty)$. Let $w_\ell(r,t)$ be i.i.d. real valued Brownian sheets on $\mathbf{R}^d \times \mathbf{R}_+$, $\ell = 1, \ldots, d$ (cf. [26, 14]) with mean zero and variance $t|A|$, where A is a Borel set in $\mathbf{R}^d$ with finite Lebesgue measure $|A|$. Adaptedness for $w_\ell(r,t)$ means that $\int_A w_\ell(dp,t)$ is adapted for any Borel set $A \subset \mathbf{R}^d$ with $|A| < \infty$. Set $w(p,t) := (w_1(p,t),\ldots,w_d(p,t))^{\mathrm{T}}$, where T denotes the transpose.

For $m \in \mathbf{N}$ let $C_b^m(\mathbf{R}^d,\mathbf{R})$ be the space of m times continuously differentiable bounded real valued functions on $\mathbf{R}^d$, where all derivatives up to order m are bounded, $C_0^m(\mathbf{R}^d,\mathbf{R})$ is the subspace of $C_b^m(\mathbf{R}^d,\mathbf{R})$ whose elements vanish at infinity. If $f \in C_b^m(\mathbf{R}^d,\mathbf{R})$ we set

$$|||f|||_m := \max_{\substack{\ell_1+\cdots+\ell_d=|\ell| \\ |\ell|\leqq m}} \sup_{r\in\mathbf{R}^d} |\partial^{|\ell|}_{\ell_1,\ldots,\ell_d} f(r)| ,$$

where

$$\partial^{|\ell|}_{\ell_1,\ldots,\ell_d} f(r) = \frac{\partial^{\ell_1+\cdots+\ell_d}}{(\partial r_{\ell_1})^{\ell_1}\cdots(\partial r_{\ell_q})^{\ell_d}} f(r) ,$$

and r_{ℓ_i} is the ℓ_ith coordinate of r. If we take only one partial derivative, say with respect to r_ℓ, we will just write ∂_ℓ.
For $p \geqq 1$ let $L_p(\mathbf{R}^d, dr)$ be the space of Borel measurable real valued functions on $\mathbf{R}^d$ such that $\int |f|^p(r)\,dr < \infty$. We set

$$(\mathbf{H}_0, \langle\cdot,\cdot\rangle_0) := (L_2(\mathbf{R}^d,dr), \langle\cdot,\cdot\rangle_0) ,$$

where $\langle\cdot,\cdot\rangle_0$ is the standard L_2-scalar product on $\mathbf{H}_0$, and $|\cdot|_0$ will denote its associated norm. We will describe the interaction of a specific particle with the other particles and with the surrounding random medium through smooth kernels. The contribution of the interaction of this particle with n particles to its motion will be weighted by positive numbers $p_n \geqq 0$, $q_n \geqq 0$, $n \in \mathbf{N}\cup\{0\}$, such that

$$\begin{aligned} c_a &:= \sum_{n=1}^{\infty} p_n n a^n \left(1 + \tfrac{1}{a^+} + \tfrac{1}{a^-}\cdot 1_{\{a^->0\}}\right) < \infty , \\ \tilde{c}_a^2 &:= \left(\sum_{n=0}^{\infty} q_n(n+1)(a^n + a^{n-1} + 1)\left(1 + \tfrac{1}{a^+} + \tfrac{1}{a^-}\cdot 1_{\{a^->0\}}\right)\right)^2 < \infty , \quad (1.7) \end{aligned}$$

where $1_{\{a^->0\}} = 1$ if $a^- > 0$ and $(1/a^-)\ 1_{\{a^->0\}} := 0$ if $a^- = 0$. If $f \in L_1(\mathbf{R}^d, dr)$ and $\mu \in \mathbf{M}$, we set

$$f * \mu^{*n}(r) := \begin{cases} \int f(r-p)\mu^{*n}(dp) \\ := \int\ldots\int f(r-(p_1+\ldots+p_n))\mu(dp_1)\ldots\mu(dp_n) & \text{if } n \geqq 1, \\ f(r), & \text{if } n = 0. \end{cases}$$

The particle–particle interaction is governed by a sequence of kernels $K_n = (K_{1n},\ldots,K_{dn})\colon\ \mathbf{R}^d \to \mathbf{R}^d$ such that

(i) $K_{\ell n} \in C_b^1(\mathbf{R}^d,\mathbf{R}) \cap L_1(\mathbf{R}^d, dr)$ for $n \in \mathbf{N}\cup\{0\}$, $\ell = 1,\ldots,d$,

(ii) $\sup_{n\in\mathbf{N}\cup\{0\}} \max_{\ell=1,\ldots,d} \{|||K_{\ell n}|||_1 + |||K_{\ell n}|||_L + \int |K_{\ell n}|(r)\,dr\} =: c_K < \infty$. (1.8)

Similarly, the particle–medium–particle interaction is governed by a sequence of kernels $\Gamma_n = (\Gamma_{k\ell n})$: $\mathbf{R}^d \to \mathscr{M}_{d\times d}$, the $d \times d$-matrices over $\mathbf{R}$, such that for some $\bar{m} \geqq d/2+1$

(i) $\Gamma_{k\ell n} = \Gamma_{\ell kn} \in C_b^{\bar{m}}(\mathbf{R}^d, \mathbf{R}) \cap \mathbf{H}_0$ for $n \in \mathbf{N} \cup \{0\}$, $k, \ell = 1, \ldots, d$,

(ii) $\sup_{n\in\mathbf{N}\cup\{0\}} \max_{\ell=1,\ldots,d} \{|||\Gamma_{k\ell n}|||_{\bar{m}}^2 + |||\Gamma_{k\ell n}|||_{L,0}^2 + ||\Gamma_{k\ell n}||_2^2\} =: c_\Gamma^2 < \infty$, (1.9)

where for suitable f: $\mathbf{R}^d \to \mathbf{R}$

$$|||f|||_{L,0}^2 := \sup_{r,q} \int \frac{|f(r-p) - f(q-p)|^2}{\rho^2(r,q)} dp ,$$

$$||f||_2^2 := ||f||_0^2 + \sum_{\ell=1}^{d} ||\partial_\ell f||_0^2 + \sum_{k,\ell=1}^{d} ||\partial_{k\ell}^2 f||_0^2 .$$

Example 1.1 Let $\varepsilon > 0$ and

$$\tilde{\Gamma}_\varepsilon(r-q) := \left\{ \frac{1}{(2\pi\varepsilon)^{d/2}} \exp(-\frac{|r-q|^2}{2\varepsilon}) \right\}^{\frac{1}{2}} .$$

Further, let $\rho(r,q) := ((|r-q|/\sqrt{8\varepsilon}) \wedge 1)$. (1.9) can be easily verified for

$$\Gamma_{k\ell n} := \tilde{\Gamma}_\varepsilon \delta_{k\ell} \tag{1.10}$$

for all n where $\delta_{k\ell} = 1$, if $k = \ell$, and $= 0$, if $k \neq \ell$.
Let us now introduce the following abbreviations:

$$F(\mu, r) := \sum_{n=0}^{\infty} p_n K_n * \mu^{*n}(r) ,$$

$$\mathscr{J}(\mu, r) := \sum_{n=0}^{\infty} q_n \Gamma_n * \mu^{*n}(r) , \tag{1.11}$$

where $\mu \in \mathbf{M}$. In what follows N is fixed, $a_i \in \mathbf{R}$, $i = 1, \ldots, N$, $\sum_{a_i \geqq 0} a_i = a^+$, $-\sum_{a_i<0} a_i = a^-$. We will assume that the positions of our N-particle system satisfy the following SODE:

$$dr^i(t) = F(\mathscr{X}_N(t), r^i(t))dt + \int \mathscr{J}(\mathscr{X}_N(t), r^i(t) - p) w(dp, dt)$$

$$r^i(0) = r_0^i, \quad i = 1, \ldots, N, \quad \mathscr{X}_N(t) = \sum_{i=1}^{N} a_i \delta_{r^i(t)} . \tag{1.12}$$

Remark 1.2 (i) Let $\{\tilde{\phi}_n\}_{n\in\mathbf{N}}$ be a complete orthonormal system (CONS) in $\mathbf{H}_0$ and define an $\mathscr{M}_{d\times d}$-valued function ϕ_n whose entries on the main diagonal are all $\tilde{\phi}_n$ and whose other entries are all 0. Then for any adapted processes $\mu(t)$ and $r(t)$ with values in $\mathbf{M}$ and $\mathbf{R}^d$, respectively,

$$\int \mathscr{J}(\mu(t), r(t) - p) w(dp, dt) = \sum_{n=1}^{\infty} \int \mathscr{J}(\mu(t), r(t) - p) \phi_n(p) dp d\beta^n(t) , \tag{1.13}$$

where $\beta^n(t)$ are $\mathbf{R}^d$-valued i.i.d. standard Wiener processes. The right hand side of (1.13) defines the increment $dM(\mu(t), r(t), t)$ of an $\mathbf{R}^d$-valued square integrable continuous martingale. For the verification of this statement it is enough to show that the quadratic variation of the right hand side of (1.13) is finite. Let [M] denote the quadratic variation of $\mathbf{R}$-, $\mathbf{R}^d$- or $\mathbf{H}_0$-valued square integrable martingales M. It will be clear from the context which state space is underlying in the definition of [ ]. The mutual quadratic variation of the one-dimensional components of $dM(\mu(t), r(t), t)$ are (formally) given by

$$\begin{aligned} & d[M_k(\mu(t), r(t), t), M_\ell(\mu(t), r(t), t)] \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{d} \int \mathscr{J}_{kj}(\mu(t), r(t)-p)\tilde{\phi}_n(p)dp \int \mathscr{J}_{\ell j}(\mu(t), r(t)-p)\tilde{\phi}_n(p)dpdt \\ &= \sum_{j=1}^{d} \int \mathscr{J}_{kj}(\mu(t), r(t)-p)\mathscr{J}_{\ell j}(\mu(t), r(t)-p)dpdt \\ &= \sum_{j=1}^{d}\sum_{m,\tilde{m}} q_m q_{\tilde{m}} \int \Gamma_{kjm} * \mu^{*m}(t)(r(t)-p)\Gamma_{\ell j\tilde{m}} * \mu^{*\tilde{m}}(t)(r(t)-p)dpdt\,. \end{aligned} \tag{1.14}$$

Hence, by (1.7) and (1.9)

$$[M_k(\mu(t), r(t), t), M_k(\mu(t), r(t), t)] \leqq dc_\Gamma^2 \bar{c}_a^2 t\,. \tag{1.15}$$

This shows that (1.14) is rigorous and altogether it follows that we may view (1.12) as an (Itô) SODE driven by infinitely many i.i.d. Wiener processes.

(ii) In view of the aforementioned adiabatic elimination we see that the right hand side of (1.12) is the sum of the slowly varying (F) and rapidly varying ($\int \mathscr{J}(\cdot, \cdot - p)w(dp, \cdot)$) components of the forces acting on the ith particle. The dependence of $\mathscr{J}$ on $\mathscr{X}_N(t)$ (the "mean field") reflects the fact that the action of the medium on the ith particle itself depends on the position of the other particles. □

We now check that (1.8) and (1.9) imply suitable Lipschitz conditions on the coefficients in (1.11). Suppose $\mu, \tilde{\mu} \in \mathbf{M}$ and $r, \tilde{r} \in \mathbf{R}^d$. We easily verify

$$|F(\mu, r) - F(\mu, \tilde{r})| \leqq c_K c_a \rho(r, \tilde{r})\,. \tag{1.16}$$

Next let f be a Borel measurable real valued function on $\mathbf{R}^d$ with $|||f|||_L \leqq c < \infty$. Then for any $n \in \mathbf{N}$

$$\begin{aligned} & \left|\int \ldots \int f(r - (\xi_1 + \cdots + \xi_n))(\mu(d\xi_1)\ldots\mu(d\xi_n) - \tilde{\mu}(d\xi_1)\ldots\tilde{\mu}(d\xi_n))\right| \\ & \leqq ca^{n-1} n\sqrt{2}\gamma_2(\mu, \tilde{\mu})\,. \end{aligned} \tag{1.17}$$

Indeed, set

$$P_{n-1}(\mu, \tilde{\mu})(d\xi_2, \ldots, d\xi_n) := \sum_{i=1}^{n} \mu(d\xi_2)\ldots\mu(d\xi_i)\tilde{\mu}(d\xi_{i+1})\ldots\tilde{\mu}(d\xi_n)\,, \tag{*}$$

where by definition the product measure on the right hand side of (*) equals $\tilde{\mu}(d\xi_2)\dots\tilde{\mu}(d\xi_n)$ for $i=1$ and $\mu(d\xi_2)\dots\mu(d\xi_n)$ for $i=n$. Then we easily see that the left hand side of (1.17) equals

$$\begin{aligned}
&|\int\dots\int f(r-(\xi_1+\dots+\xi_n))(\mu(d\xi_1)-\tilde{\mu}(d\xi_1))P_{n-1}(\mu,\tilde{\mu})(d\xi_2,\dots,d\xi_n)\\
&\leqq a^{n-1}n\sup_{\zeta}|\int f(r-\xi+\zeta)(\mu(d\xi)-\tilde{\mu}(d\xi))|\\
&\leqq a^{n-1}n\sup_{\zeta}\sum_{+,-}\tfrac{1}{a^{\pm}}|\int\int f(r-\xi+\zeta)\\
&\quad -f(r-\eta+\zeta)Q^{\pm}(d\xi,d\eta)|
\end{aligned}\tag{1.18}$$

with arbitrary $Q^{\pm}\in C(\mu^{\pm},\tilde{\mu}^{\pm})$ and $\sum_{+,-}b^{\pm}:=b^{+}+b^{-}$. Since $\rho(r,q)=\rho(r-q,0)$, and $Q^{\pm}$ arbitrary in (1.18) we obtain from (1.18) and the Cauchy–Schwarz inequality (1.17). (1.17) in addition to (1.3), (1.7) (implying independence of n), (1.8), and (1.11) implies for any $\tilde{r}$

$$|F(\mu,\tilde{r})-F(\tilde{\mu},\tilde{r})|\leqq c_K c_a\sqrt{2}d\gamma_2(\mu,\tilde{\mu})\,. \tag{1.19}$$

(1.19) in addition to (1.16) implies

$$|F(\mu,r)-F(\tilde{\mu},\tilde{r})|\leqq c_K c_a\{\rho(r,\tilde{r})+\sqrt{2}d\gamma_2(\mu,\tilde{\mu})\}\,. \tag{1.20}$$

Similarly we obtain first

$$\begin{aligned}
&\left[\int(\mathcal{J}(\tilde{\mu},r-p)-\mathcal{J}(\tilde{\mu},\tilde{r}-p))w(dp,dt)\right]\\
&\leqq \sum_{k,j=1}^{d}\sum_{m,\tilde{m}}q_m a^m m q_{\tilde{m}}a^{\tilde{m}}\tilde{m}\sup_{\zeta,\tilde{\zeta}}\Bigg\{\Bigg(\int(\Gamma_{kjm}(r-p+\zeta)\\
&\qquad -\Gamma_{kjm}(\tilde{r}-p+\zeta))^2\,dp\Bigg)^{1/2}\\
&\qquad \left(\int(\Gamma_{kj\tilde{m}}(r-p+\tilde{\zeta})-\Gamma_{kj\tilde{m}}(\tilde{r}-p+\tilde{\zeta}))^2dp\right)^{1/2}\Bigg\}\,dt\\
&\leqq d^2\tilde{c}_a^2c_\Gamma^2\rho^2(r,\tilde{r})\,dt\,.
\end{aligned}\tag{1.21}$$

Then, as in the derivation of (1.18)

$$\begin{aligned}
&\left[\int(\mathcal{J}(\mu,r-p)-\mathcal{J}(\tilde{\mu},r-p))w(dp,dt)\right]\\
&\leqq \sum_{j,k=1}^{d}a\sum_{m,\tilde{m}}q_m a^{m-1}mq_{\tilde{m}}a^{\tilde{m}-1}\tilde{m}\sup_{\zeta,\tilde{\zeta}}\\
&\quad\Bigg\{\sum_{+,-}\tfrac{1}{a^{\pm}}\int\int\int(\Gamma_{kjm}(r-p-\xi+\zeta)-\Gamma_{kjm}(r-p-\eta+\zeta))^2\\
&\quad\times dp\,Q^{\pm}(d\xi,d\eta)\Bigg\}^{1/2}\\
&\quad\Bigg\{\sum_{+,-}\tfrac{1}{a^{\pm}}\int\int\int(\Gamma_{kj\tilde{m}}(r-p-\xi+\tilde{\zeta})-\Gamma_{kj\tilde{m}}(r-p-\eta+\tilde{\zeta}))^2
\end{aligned}$$

$$\left.\times dp\, Q^{\pm}(d\xi, d\eta)\right\}^{1/2} dt$$

$$\leqq d^2 \tilde{c}_a^2 c_\Gamma^2 \sum_{+,-} \int\int \rho^2(\xi,\eta) Q^{\pm}(d\xi, d\eta)\, dt \tag{1.22}$$

with arbitrary $Q^{\pm} \in C(\mu^{\pm}, \tilde{\mu}^{\pm})$. Since $Q^{\pm}$ are arbitrary we obtain from (1.21) and (1.22)

$$\left[\int(\mathscr{J}(\mu, r-p) - \mathscr{J}(\tilde{\mu}, \tilde{r}-p))w(dp, dt)\right] \leqq 2d^2 \tilde{c}_a^2 c_\Gamma^2 \{\rho^2(r, \tilde{r}) + a\gamma_2^2(\mu, \tilde{\mu})\}\, dt\,. \tag{1.23}$$

Next we will derive the (mezoscopic) SPDE for the mass distribution $\mathscr{X}_N(t)$ associated with (1.12), assuming that (1.12) is solvable.
Let $\mu \in \mathbf{M}$ and $r \in \mathbf{R}^d$. Set

$$D_{k\ell}(\mu, r) := \frac{d}{dt}[M_k(\mu, r, t), M_\ell(\mu, r, t)] \tag{1.24}$$

(cf. (1.14)). By assumptions (1.7) and (1.9) $D_{k\ell}(\mu) := D_{k\ell}(\mu, \cdot) \in C_b^2(\mathbf{R}^d, \mathbf{R})$ and $\mathscr{J}_{k\ell}(\mu) := \mathscr{J}_{k\ell}(\mu, \cdot) \in C_b^2(\mathbf{R}^d, \mathbf{R}) \cap \mathbf{H}_0$. Similarly $F(\mu) := F(\mu, \cdot) \in C_b^1(\mathbf{R}^d, \mathbf{R}) \cap L_1(\mathbf{R}^d, \mathbf{R})$ (cf. (1.11)). Let ∇ be the gradient on $\mathbf{R}^d$ and $\cdot$ denote the scalar product on $\mathbf{R}^d$. Consider the following quasilinear SPDE on $\mathbf{M}$

$$d\mathscr{X} =: \left\{ \tfrac{1}{2} \sum_{k,\ell=1}^{d} \partial_{k\ell}^2 (D_{k\ell}(\mathscr{X})\mathscr{X}) - \nabla\cdot(\mathscr{X}F(\mathscr{X})) \right\} dt$$
$$- \nabla\cdot\left(\mathscr{X} \int \mathscr{J}(\mathscr{X}, \cdot - p) w(dp, dt)\right), \tag{1.25}$$

$$\mathscr{X}^{\pm}(\mathbf{R}^d, t) \overset{\text{a.s.}}{=} \mathscr{X}^{\pm}(\mathbf{R}^d, 0) \in \mathbf{M} \text{ (conservation of mass).} \tag{1.26}$$

$\int \mathscr{J}(\mathscr{X}, \cdot - p) w(dp, dt)$ is treated as a density with respect to $\mathscr{X}$, i.e. $\mathscr{X} \int \mathscr{J}(\mathscr{X}, \cdot - p) w(dp, dt)$ is the $\mathbf{R}^d$-valued (signed) measure $\int \mathscr{J}(\mathscr{X}, \cdot - p) w(dp, dt) \mathscr{X}(d\cdot)$. Similarly for $D_{k\ell}(\mathscr{X})\mathscr{X}$ and $\mathscr{X}F(\mathscr{X})$. A weak solution of (1.25) is by definition a continuous $\mathbf{M}$-valued adapted process $\mathscr{X}(t)$ which satisfies

$$d\langle \mathscr{X}, \varphi\rangle =: \left\{ \tfrac{1}{2} \sum_{k,\ell=1}^{d} \langle \mathscr{X}, D_{k\ell}(\mathscr{X}) \partial_{k\ell}^2 \varphi\rangle + \langle \mathscr{X}, F(\mathscr{X})\cdot\nabla\varphi\rangle \right\} dt$$
$$+ \left\langle \mathscr{X}, \int \mathscr{J}(\mathscr{X}, \cdot - p) w(dp, dt)\cdot\nabla\varphi \right\rangle, \tag{1.27}$$

where $\varphi \in C_0^3(\mathbf{R}^d, \mathbf{R})$ and $\langle\cdot,\cdot\rangle$ is the duality between measures and elements from $C_b(\mathbf{R}^d, \mathbf{R})$, which is an extension of $\langle\cdot,\cdot\rangle_0$. The restriction to $C_0^3(\mathbf{R}^2, \mathbf{R})$ is motivated by the fact that for those φ $|||\partial_\ell\varphi|||_L$ and $|||\partial_{k\ell}^2\varphi|||_L < \infty$ for $k, \ell = 1, \dots, d$ and, therefore, the right hand side of (1.27) is defined for $\mathscr{X} \in (\mathbf{M}, \gamma_2)$. The fact that φ vanishes at infinity allows to come from (1.27) to (1.25) through integration by parts, where in (1.25) the derivatives have to be interpreted in the distributional sense. That (1.25)/(1.26) is also the "correct"

SPDE for the mass distribution associated with (1.12) follows from the following lemma.

Lemma 1.3 *Suppose* (1.12) *has a solution* $(r^1(t,r_0),\dots,r^N(t,r_0))$. *Then* $\mathscr{X}_N(t) := \sum_{i=1}^N a_i\delta_{r^i(t)}$ *is a weak solution of* (1.25)/(1.26).

Proof.

(i) $\mathscr{X}_N(t)$ satisfies (1.27) by Itô's formula.
(ii) The mass conservation (1.26) follows from the construction. □

Let us now briefly describe the content of the following sections. In Sect. 2 we show existence and uniqueness for the (microscopic) SODE (1.12). In Sect. 3 we show existence for the (mezoscopic) SPDE (1.25)/(1.26) for any (adapted) initial condition $\mathscr{X}_0$. The solution is constructed via extension by continuity of $\mathscr{X}(t,\mathscr{X}_N(0)) := \mathscr{X}_N(t,\mathscr{X}_N(0)) := \mathscr{X}_N(t)$ to $\mathscr{X}(t,\mathscr{X}_0)$ and by showing that this "extension" is a weak solution of (1.25). In Sect. 4 we show that if $\mathscr{X}_0 \in \mathbf{H}_0$ then $\mathscr{X}(t,\mathscr{X}_0) \in C([0,\infty);\mathbf{H}_0)$ a.s. ($\mathbf{H}_0$-valued continuous functions), and we obtain an equation for $\|\mathscr{X}(t,\mathscr{X}_0)\|_0^{2n}$. This equation shows in particular that (1.25) does not satisfy the coercivity conditions in Pardoux's variational approach [23] and its generalization by Krylov and Rozovskii [18]. In addition to (1.12) we consider in Sect. 2 a microscopic equation where the measure valued input is not the empirical process but some arbitrary process $\mathscr{Y}(t)$. The empirical process associated with this SODE satisfies a "bilinear" SPDE. Results on this "bilinear" SPDE and the quasilinear SPDE are used to prove uniqueness for (1.25)/(1.26) in Sect. 5 provided $\mathscr{X}_0 \in \mathbf{H}_0$. In Sect. 6 we look at a special semilinear case of (1.25)/(1.26), where $\Gamma_{k\ell n}(r) = \tilde{\Gamma}_\varepsilon(r)\delta_{k\ell}$. Referring to [16] we conclude that under these assumptions the solution of (1.25)/(1.26) tends to a macroscopic PDE if $\varepsilon \to 0$ (the correlation length of the fluctuation forces tends to 0) for appropriately chosen initial conditions.

Next we comment on the basic idea of our approach and some related work and on other approaches to SPDE's. The solution of the (mezoscopic) SPDE (1.25) is obtained by solving the (microscopic) SODE (1.12) for an arbitrary number of particles but keeping the total mass constant, and by extending the empirical process associated with (1.12) to other measure-valued initial conditions. The idea to start with microscopic equations (for the positions of branching particles) and then to obtain the mezoscopic mass distribution by some sort of extension appeared already in [5], which laid the basis for a new area, namely "superprocesses". By choosing finitely many uncorrelated Brownian motions the empirical process in Dawson's case does not directly satisfy an SPDE. Indirectly, however, in the one-dimensional case the martingale problem is well-posed for a diffusion limit of Dawson's model. Correlations into the fluctuation forces for a system of N interacting and diffusing particles were to our knowledge first introduced by Vaillancourt [25] but again restricting the noise to N Brownian motions for N particles, which again excludes the empirical process $\mathscr{X}_N(t)$ as a candidate for the solution of an SPDE. Looking at (1.13) we see that an essential difference to Vaillancourt's model (and other particle models) is the perturbation of each of the N particle positions by the same infinitely many Brownian motions. It is this feature which leads to Lemma 1.3 and the extension procedure in Sect. 3 with the result that our solution is strong in the probabilistic sense (on the same $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, same Brownian sheet). In Vaillancourt's model one can get weak solutions (in the probabilistic sense) for some infinite particle limit by solving a martingale problem [6]. Since solving (1.25)/(1.26) is achieved by solving (1.12) and extension we will call this the particle approach to SPDE's. It should be mentioned that in the absence of fluctuation forces, i.e., where (1.12) becomes an ODE, this approach is used to obtain numerical solutions of certain first order PDE's, e.g., of the Euler equation for the vorticity of an incompressible ideal two-dimensional fluid (cf. [20]). Finally, we want to mention Borkar's paper [1], where the SODE for the position of one particle in a random medium (represented by a Brownian sheet and random measures) is considered. Also the time evolution of an associated mass distribution (not the empirical distribution of an N-particle system) has been analyzed in [1].

Apart from the aforementioned relation of our approach to Dawson's and Vaillancourt's work it is essentially different from the techniques (and the spirit) used so far, say the semigroup approach ([3, 26, 13–15], etc.), the variational approach (Pardoux, loc. cit., Krylov and Rozovskii, loc. cit., etc.) as well as from the general functional analytic methods used to derive strong solutions of certain quasilinear SPDE's by Dalecky and Goncharuk [2]. In particular, the assumptions on the fluctuation part in Daletcky's and Goncharuk's paper are quite restrictive and not satisfied by (1.25). Gärtner [11] uses a system of diffusing particles with mean field interaction to obtain quasilinear macroscopic PDE's. An adaptation of his techniques to our framework should yield macroscopic limits for (1.25) also in the general quasilinear case provided that the fluctuation forces for different particles become uncorrelated in the limit. This problem will be investigated in a forthcoming paper.

Let Δ be the Laplacian, considered as a self-adjoint operator on $\mathbf{H}_0$. I is the identity operator on $\mathbf{H}_0$. Set $A := (I - 1/2\Delta)$. Then for any $\alpha \in \mathbf{R}$, A^α is defined through the spectral resolution of A and is self-adjoint. Let $C_0^\infty(\mathbf{R}^d, \mathbf{R})$ the subspace of $C_0^m(\mathbf{R}^d, \mathbf{R})$ whose elements are infinitely often differentiable. For $\alpha \geqq 0$ set

$$\langle \varphi, \psi \rangle_\alpha := \langle A^{\frac{\alpha}{2}} \varphi, A^{\frac{\alpha}{2}} \psi \rangle_0$$

and $\|\varphi\|_\alpha := \langle \varphi, \varphi \rangle_\alpha^{1/2}$, where $\varphi, \psi \in C_0^\infty(\mathbf{R}^d, \mathbf{R})$.

Let $\mathbf{H}_\alpha$ be the completion of $C_0^\infty(\mathbf{R}^d, \mathbf{R})$ in $\mathbf{H}_0$, identify $\mathbf{H}_0$ with its strong dual $\mathbf{H}_0^*$ and denote by $\mathbf{H}_{-\alpha}$ the strong dual of $\mathbf{H}_\alpha$. The norms $\|\cdot\|_{-\alpha}$ on $\mathbf{H}_{-\alpha}$ are Hilbert norms, and we easily see that if $\varphi, \psi \in \mathbf{H}_0$

$$\langle \varphi, \psi \rangle_{-\alpha} = \langle A^{-\frac{\alpha}{2}} \varphi, A^{-\frac{\alpha}{2}} \psi \rangle_0$$

(cf. [13]). Hence we have the scale of Hilbert spaces

$$\mathbf{H}_\alpha \subset \mathbf{H}_\beta \subset \mathbf{H}_0 = \mathbf{H}_0^* \subset \mathbf{H}_{-\beta} \subset \mathbf{H}_{-\alpha} \tag{1.28}$$

for $0 \leqq \beta \leqq \alpha$ with dense continuous inclusions. By [24]

$$\mathbf{H}_\alpha \subset C_b^m(\mathbf{R}^d, \mathbf{R})$$

if $m + d/2 < \alpha$ with continuous inclusions. Since $|||\cdot|||_1$ is stronger than $|||\cdot|||_L$ we obtain

$$(\mathbf{M}, \gamma_2) \subset \mathbf{H}_{-\alpha} \quad \text{for all } \alpha > 1 + d/2\,. \tag{1.29}$$

Finally, some remarks about notation. If $\mathbf{M}_1$ and $\mathbf{M}_2$ are metric spaces $C(\mathbf{M}_1, \mathbf{M}_2)$ is the set of continuous functions from $\mathbf{M}_1$ into $\mathbf{M}_2$. If $\mathbf{B}$ and $\tilde{\mathbf{B}}$ are normed spaces, $\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})$ will be the space of linear bounded operators from $\mathbf{B}$ into $\tilde{\mathbf{B}}$ and $\|\cdot\|_{\mathscr{L}(\mathbf{B},\tilde{\mathbf{B}})}$ the usual operator norm on $\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})$. For $\mathbf{B} = \tilde{\mathbf{B}}$ we will just write $\mathscr{L}(\mathbf{B})$ instead of $\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})$. If we want to specify on which variable a given differential operator acts we will indicate this by writing the variable as a subscript, e.g., $(\Delta_r f)(r, q)$ means the Laplace operator acts on $f(\cdot, q)$ as a function of r, and q is considered a parameter. Finally, we will endow $\mathbf{R}^{dN}$ with the metric $\rho_N(r_N, q_N) := \max_{1 \leqq i \leqq N} \rho(r^i, q^i)$, where $r_N := (r^1, \ldots, r^N)$, $q_N := (q^1, \ldots, q^N) \in \mathbf{R}^{dN}$.

2 The microscopic equations

We derive existence and uniqueness for the (microscopic) SODE (Theorem 2.2). In this derivation we also obtain bounds for some other (microscopic) SODE's which depend on some given measure processes (which are not necessarily the empirical processes associated with those equations ((2.2)).

Set

$$\mathbf{M}_d := \{\mu \in \mathbf{M} : \mu \text{ is a finite linear combination of point measures on } \mathbf{R}^d\}\,.$$

Further, we introduce the following metric spaces of $\mathbf{M}_d$, resp. $\mathbf{M}$ valued $\mathscr{F}_0$-measurable random variables and of continuous adapted $\mathbf{M}$-valued processes:

$$\tilde{\mathscr{M}}_0 := L_2(\Omega; \mathbf{M}_d)\,, \quad \mathscr{M}_0 := L_2(\Omega; \mathbf{M})\,, \quad \mathscr{M}_{[0,T]} := L_2(\Omega; C([0, T]; \mathbf{M}))\,,$$

where the metric on the first two spaces is given by $(E\gamma_2^2(\mu, \tilde{\mu}))^{1/2}$ for $\mu, \tilde{\mu} \in \mathscr{M}_0$ and on the last one by $(E \sup_{0 \leqq t \leqq T} \gamma_2^2(\mu_t, \tilde{\mu}_t))^{1/2}$ for $\mu_t, \tilde{\mu}_t \in \mathscr{M}_{[0,T]}$. Note that $\mathscr{M}_0$ and $\mathscr{M}_{[0,T]}$ are complete, since $\mathbf{M}$ is complete. In what follows we will always assume that our random variables are adapted if this does not automatically follow from their definition. Consider the following $\mathbf{R}^d$-valued SODE:

$$\begin{aligned} dz(t) = {} & F(\mathscr{Y}(t), z(t))\, dt + \int \mathscr{J}(\mathscr{Y}(t), z(t) - p) w(dp, dt)\,, \\ & z(0) = z_0, \quad \mathscr{Y} \in \mathscr{M}_{[0,T]}\,. \end{aligned} \tag{2.1}$$

Denote a solution of (2.1), if it exists, by $z(t, \mathscr{Y}, z_0)$, and let $\mathscr{B}^d$ be the Borel σ-algebra on $\mathbf{R}^d$.

Lemma 2.1 (I) *There is a unique $\mathscr{F}_t$-adapted solution $z(\cdot, \mathscr{Y}, z_0) \in C([0, \infty); \mathbf{R}^d)$ a.s.*

(II) *Let $\mathscr{Y}_1, \mathscr{Y}_2 \in \mathscr{M}_{[0,T]}$ and $z_{0,1}, z_{0,2}$ two initial conditions. Then for any $T > 0$*

$$E \sup_{0\leq t\leq T} \rho^2(z(t,\mathcal{Y}_1,z_{0,1}),z(t,\mathcal{Y}_2,z_{0,2}))$$

$$\leqq 3\exp(c_{a,\Gamma,K}(T^2+T))\left\{E\rho^2(z_{0,1},z_{0,2})+c_{a,\Gamma,K}(T+1)\right.$$

$$\left.\times\int_0^T E\gamma_2^2(\mathcal{Y}_1(s),\mathcal{Y}_2(s))\,ds\right\}, \tag{2.2}$$

where

$$c_{\alpha,\Gamma,K} := \bar{c}^2 d^2\{2c_a^2 c_K^2 + d^2\tilde{c}_a^2 c_\Gamma^2\}\,. \tag{2.3}$$

(III) *For* $\forall t \geqq 0$ *the map* $(\omega,\xi)\mapsto z(t,\mathcal{Y}(\omega),\omega,\xi)$ *from* $\Omega\times\mathbf{R}^d$ *into* $\mathbf{R}^d$ *is* $\mathcal{F}_t\otimes\mathcal{B}^d-\mathcal{B}^d$*-measurable.*

Proof. (i) Let $q_\ell(\cdot)$ be $\mathbf{R}^d$-valued, adapted and $dt\otimes dP$-measurable stochastic processes, $\ell=1,2$. Set for $\ell=1,2$

$$\tilde{q}_\ell(t) := q_\ell(0)+\int_0^t F(\mathcal{Y}_\ell(s),q_\ell(s))\,ds+\int_0^t\int \mathcal{J}(\mathcal{Y}_\ell(s),q_\ell(s)-p)w(dp,ds)\,.$$

(ii) By (1.3) and (1.16)

$$\rho\left(\int_0^t F(\mathcal{Y}_1(s),q_1(s))\,ds,\int_0^t F(\mathcal{Y}_1(s),q_2(s))\,ds\right)\leqq \bar{c}c_K c_a d\int_0^t\rho(q_1(s),q_2(s))\,ds\,.$$

Further, from (1.3), (1.7), and (1.18) we obtain

$$\rho\left(\int_0^t F(\mathcal{Y}_1(s),q_2(s))\,ds,\int_0^t F(\mathcal{Y}_2(s),q_2(s))\,ds\right)$$

$$\leqq \bar{c}c_a\sum_{+,-}\int_0^t\sup_p\left|\int\int\tfrac{1}{a^\pm}Q^\pm(s,d\xi,d\eta)(K(q_2(s)-\xi-p)\right.$$

$$\left.-K(q_2(s)-\eta-p))\right|\,ds \tag{2.4}$$

(with $Q^\pm(s)\in C(\mathcal{Y}_1^\pm(s),\mathcal{Y}_2^\pm(s))$ arbitrary). Moreover, by (1.3) and (1.19)

$$\rho\left(\int_0^t F(\mathcal{Y}_1(s),q_2(s))\,ds,\int_0^t F(\mathcal{Y}_2(s),q_2(s))\,ds\right)$$

$$\leqq \bar{c}c_a c_K\sqrt{2}d\int_0^t\gamma_2(\mathcal{Y}_1(s),\mathcal{Y}_2(s))\,ds\,. \tag{2.5}$$

(iii) By (1.21)

$$\left[\int_0^t\int\{\mathcal{J}(\mathcal{Y}_1(s),q_1(s)-p)-\mathcal{J}(\mathcal{Y}_1(s),q_2(s)-p)\}w(dp,ds)\right]$$

$$\leqq d^2\tilde{c}_a^2c_\Gamma^2\int_0^t\rho^2(q_1(s),q_2(s))\,ds\,,$$

and by (1.22)

$$\left[\int_0^t \int \{\mathscr{J}(\mathscr{Y}_1(s), q_2(s) - p) - \mathscr{J}(\mathscr{Y}_2(s), q_2(s) - p)\} w(dp, ds)\right]$$

$$\leqq \sum_{j,k=1}^{d} a \sum_{m,\tilde{m}} q_m a^{m-1} m q_{\tilde{m}} a^{\tilde{m}-1} \tilde{m} \sup_{\zeta,\tilde{\zeta}}$$

$$\int_0^t \left\{ \sum_{+,-} \frac{1}{a^{\pm}} \int \int \int (\Gamma_{kjm}(q_2(s) - p - \xi + \zeta) - \Gamma_{kjm}(q_2(s) - p - \eta + \zeta))^2 \right.$$

$$\left. \times dp\, Q^{\pm}(s, d\xi, d\eta) \right\}^{1/2}$$

$$\left\{ \sum_{+,-} \frac{1}{a^{\pm}} \int \int \int (\Gamma_{kj\tilde{m}}(q_2(s) - p - \xi + \tilde{\zeta}) - \Gamma_{kj\tilde{m}}(q_2(s) - p - \eta + \tilde{\zeta}))^2 \right.$$

$$\left. \times dp\, Q^{\pm}(s, d\xi, d\eta) \right\}^{1/2} ds \tag{2.6}$$

(with $Q^{\pm}(s) \in C(\mathscr{Y}_1^{\pm}(s), \mathscr{Y}_2^{\pm}(s))$ arbitrary).
Hence, as in step (ii)

$$\left[\int_0^t \int \{\mathscr{J}(\mathscr{Y}_1(s), q_2(s) - p) - \mathscr{J}(\mathscr{Y}_2(s), q_2(s) - p)\} w(dp, ds)\right]$$

$$\leqq d^2 \tilde{c}_a^2 c_\Gamma^2 \int_0^t \gamma_2^2(\mathscr{Y}_1(s), \mathscr{Y}_2(s))\, ds\,. \tag{2.7}$$

(iv) Steps (i)–(iii) in addition to Doob's inequality and (1.3) imply

$$E \sup_{0 \leqq t \leqq T} \rho^2(\tilde{q}_1(t), \tilde{q}_2(t)) \leqq 3E\rho^2(q_1(0), q_2(0))$$
$$+ 3c_{a,\Gamma,K}(T+1) \int_0^T E\rho^2(q_1(s), q_2(s))\, ds$$
$$+ 3c_{a,\Gamma,K}(T+1) \int_0^T E\gamma_2^2(\mathscr{Y}_1(s), \mathscr{Y}_2(s))\, ds\,. \tag{2.8}$$

(v) Choosing first $\mathscr{Y}_1 \equiv \mathscr{Y}_2$ the existence of a unique continuous solution follows from (2.8) and the contraction mapping principle. Having thus established the existence of unique solutions $z(\cdot, \mathscr{Y}_\ell, z_{0,\ell}), \ell = 1, 2$, (2.2) follows from (2.8) and Gronwall's lemma.

(vi) (III) follows from (2.2) (for $\mathscr{Y}_1 \equiv \mathscr{Y}_2$) exactly as in the classical proof of the Markov property for SODE's (cf. [9, Chap. XI, Sect. 2]). □

Theorem 2.2 *To each $\mathscr{F}_0$-adapted initial condition $r_N(0) \in \mathbf{R}^{dN}$* (1.12) *has a unique $\mathscr{F}_t$-adapted solution $r_N(\cdot, r_N(0)) \in C([0,\infty); \mathbf{R}^{dN})$ a.s., which is an $\mathbf{R}^{dN}$-valued Markov process.*

Proof. (i) Let $q_{N,\ell}(\cdot) := (q_\ell^1(\cdot), \ldots, q_l^N(\cdot))$ be $\mathbf{R}^{dN}$-valued adapted and $dt \otimes dP$-measurable stochastic processes, $\ell = 1, 2$. Set for $\ell = 1, 2$,

$$\mathscr{Y}_\ell(t) := \sum_{i=1}^N a_i \delta_{q^i_\ell(t)}$$

and

$$\tilde{q}^i_\ell(t) := q^i_\ell(0) + \int_0^t F(\mathscr{Y}_\ell(s), q^i_\ell(s))\, ds + \int_0^t \int \mathscr{J}(\mathscr{Y}_\ell(s), q^i_\ell(s) - p) w(dp, ds) .$$

(ii) By steps (ii) and (iii) in the proof of Lemma 2.1 (cf. (2.4), (2.6)) we obtain for any $i \in \{1, \ldots, N\}$

$$\begin{aligned} E \sup_{0 \leqq t \leqq T} \rho^2(\tilde{q}^i_1(t), \tilde{q}^i_2(t)) \leqq{} & 3E\rho^2(q^i_1(0), q^i_2(0)) \\ & + 3c_{a,\Gamma,K}(T+1) \int_0^T E\rho^2(q^i_1(s), q^i_2(s))\, ds \\ & + 3c_{a,\Gamma,K}(T+1) \int_0^T E\rho^2_N(q_{N,1}(s), q_{N,2}(s))\, ds , \end{aligned} \tag{2.9}$$

whence (with $\rho_N(r_N, q_N)$ defined at the end of Sect. 1)

$$\begin{aligned} E \sup\nolimits_{0 \leqq t \leqq T} \rho^2_N(\tilde{q}_{N,1}(t), \tilde{q}_{N,2}(t)) \leqq{} & 3NE\rho^2_N(q_{N,1}(0), q_{N,2}(0)) \\ & + 6Nc_{a,\Gamma,K}(T+1) \int_0^T E\rho^2_N(q_{N,1}(s), q_{N,2}(s))\, ds . \end{aligned} \tag{2.10}$$

(iii) The contraction mapping principle implies now the existence of a unique continuous solution. The Markov property follows as for Itô equations with perturbations by finitely many Wiener processes (cf. (III) of Lemma 2.1 and Dynkin, loc. cit.). □

3 The Mezoscopic equation–existence

The empirical process $\mathscr{X}_N(t)$ is considered as a map $\mathscr{X}_N(0) \mapsto \mathscr{X}(t, \mathscr{X}_N(0)) := \mathscr{X}_N(t)$ from the space of $\mathscr{F}_0$-adapted **M**-*valued random variables into the space of adapted continuous* **M**-*valued processes. The first main result is that this map has a unique extension from discrete* **M**-*valued initial conditions $\mathscr{X}_N(0)$ to general ($\mathscr{F}_0$-measurable)* **M**-*valued initial conditions $\mathscr{X}_0$, denoted by $\mathscr{X}(t, \mathscr{X}_0)$ (Theorem 3.4). Then it is shown that this extension is a weak solution of the mezoscopic SPDE (1.25) (Theorem 3.5).*

Let $\mathscr{Y}_l \in \mathscr{M}_{0,T}$ for any $T > 0$ and $z^i_{0,l}$, $i = 1, \ldots, N$ $\mathscr{F}_0$-measurable random variables with values in $\mathbf{R}^d$, $l = 1, 2$. Consider the two systems of SODE's:

$$\begin{aligned} dz^i_l(t) ={}& F(\mathscr{Y}_l(t), z^i_l(t)\, dt + \int \mathscr{J}(\mathscr{Y}_l(t), z^i_l(t) - p) w(dp, dt) \\ z^i_l(0) ={}& z^i_{0,l},\ i = 1, \ldots, N,\ l = 1, 2 . \end{aligned} \tag{3.1}$$

By Lemma 2.1 (3.1) has unique continuous solutions $z_l^i(t)$ for $i = 1, \ldots, N$ and $l = 1, 2$. Set

$$\mathscr{X}_{N,l}(t) := \sum_{i=1}^{N} a_i \delta_{z_l^i(t)} \,. \tag{3.2}$$

Lemma 3.1 *For any $T > 0$*

$$\begin{aligned} &E \sup\nolimits_{0 \leqq t \leqq T} \gamma_2^2(\mathscr{X}_{N,1}(t), \mathscr{X}_{N,2}(t)) \\ &\quad \leqq 3 \exp(c_{a,\Gamma,K}(T^2 + T)) E \gamma_2^2(\mathscr{X}_{N,1}(0), \mathscr{X}_{N,2}(0)) \\ &\quad + 3 \exp(c_{a,\Gamma,K}(T^2 + T)) a^2 c_{a,\Gamma,K}(T + 1) \int_0^T E \gamma_2^2(\mathscr{Y}_1(t), \mathscr{Y}_2(t)) \, dt \,. \end{aligned} \tag{3.3}$$

Proof. (i) Let us first assume that $z_{0,l}^i =: \xi_l^i$ is deterministic, $i = 1, \ldots, N$, $l = 1, 2$. $z_l(t, \mathscr{Y}_l, r)$ is the solution of (2.1) with $\mathscr{Y} = \mathscr{Y}_l$ and $z(0) := r \in \mathbf{R}^d$. If $r = \xi_l^i$, then $z_l(t, \mathscr{Y}_l, \xi_l^i) = z_l^i(t)$, the ith component of the solution of (3.1). Let $Q^{\pm}(0) \in C(\mathscr{X}_{N,1}^{\pm}(0), \mathscr{X}_{N,2}^{\pm}(0))$ and $f \in C_b(\mathbf{R}^{2d}; \mathbf{R})$. Then

$$\int\int Q^{\pm}(t, dr, dq) f(r, q) := \int\int Q^{\pm}(0, dr, dq) f(z_1(t, \mathscr{Y}_1(t), r), z_2(t, \mathscr{Y}_2(t), q)) \tag{3.4}$$

defines $Q^{\pm}(t) \in C(\mathscr{X}_{n,1}^{\pm}(t), \mathscr{X}_{N,2}^{\pm}(t))$. Thus, for arbitrary $Q^{\pm}(0) \in C(\mathscr{X}_{N,1}^{\pm}(0), \mathscr{X}_{N,2}^{\pm}(0))$ using (3.4):

$$\begin{aligned} &E \sup\nolimits_{0 \leqq t \leqq T} \sum_{+,-} \int\int Q^{\pm}(t, dr, dq) \rho^2(r, q) \\ &\quad E \sup\nolimits_{0 \leqq t \leqq T} \sum_{+,-} \int\int Q^{\pm}(0, dr, dq) \rho^2(z_1(t, \mathscr{Y}_1(t), r), z_2(t, \mathscr{Y}_2(t), q)) \\ &\quad \leqq \sum_{+,-} \int\int Q^{\pm}(0, dr, dq) E \sup\nolimits_{0 \leqq t \leqq T} \rho^2(z_1(t, \mathscr{Y}_1(t), r), z_2(t, \mathscr{Y}_2(t), q)) \\ &\quad \leqq \sum_{+,-} \int\int Q^{\pm}(0, dr, dq) 3 \exp(c_{a,\Gamma,K}(T^2 + T)) \\ &\qquad \times \left\{ \rho^2(r, q) + c_{a,\Gamma,K}(T + 1) \int_0^T E \gamma_2^2(\mathscr{Y}_1(s), \mathscr{Y}_2(s)) \, dr \right\} \end{aligned}$$

by (2.2).

(ii) Since $Q^{\pm}(0) \in C(\mathscr{X}_{N,1}^{\pm}(0), \mathscr{X}_{N,2}^{\pm}(0))$ arbitrary we obtain (3.3) for deterministic initial conditions.

(iii) By Lemma 2.1, part 3), $z_l^i(t, \mathscr{Y}_l(t), z_{0,l}^i) = z_l(t, \mathscr{Y}_l(t), r)_{|r = z_{0,l}^i}$, and averaging over the distribution of $(z_{0,1}^1, \ldots, z_{0,1}^N, z_{0,2}^1, \ldots, z_{0,2}^N)$ implies (3.3) for general initial conditions. □

Set

$$\mathscr{X}(t, \mathscr{Y}(t), \mathscr{X}_N(0)) := \mathscr{X}_N(t, \mathscr{Y}, \mathscr{X}_N(0)) \,,$$

where the right hand side is given by (3.2) and $\mathscr{Y} \in \{\mathscr{Y}_1, \mathscr{Y}_2\}, \mathscr{X}_N(0) \in \{\mathscr{X}_{N,1}(0), \mathscr{X}_{N,2}(0)\}$.

Corollary 3.2 *The map $\mathscr{X}_N(0) \mapsto \mathscr{X}(\cdot, \mathscr{Y}, \mathscr{X}_N(0))$ from $\tilde{\mathscr{M}}_0$ into $\mathscr{M}_{[0,T]}$ extends uniquely to a map $\mathscr{X}(0) \mapsto \mathscr{X}(\cdot, \mathscr{Y}, \mathscr{X}_0)$ from $\mathscr{M}_0$ into $\mathscr{M}_{[0,T]}$. Moreover, let $\mathscr{X}_{0,1}, \mathscr{X}_{0,2} \in \mathscr{M}_0$ and $\mathscr{Y}_1, \mathscr{Y}_2 \in \mathscr{M}_{[0,T]}$ for all $T > 0$. Then for any $T > 0$*

$$\begin{aligned} &E \sup_{0\leqq t\leqq T} \gamma_2^2(\mathscr{X}(t, \mathscr{Y}_1(t), \mathscr{X}_{0,1}), \mathscr{X}(t, \mathscr{Y}_2(t), \mathscr{X}_{0,2})) \\ &\quad \leqq 3 \exp(c_{a,\Gamma,K}(T^2 + T))\{E\gamma_2^2(\mathscr{X}_{0,1}, \mathscr{X}_{0,2}) \\ &\quad + a^2 c_{a,\Gamma,K}(T+1)\int_0^T E\gamma_2^2(\mathscr{Y}_1(t), \mathscr{Y}_2(t))\, dt\} \end{aligned} \tag{3.5}$$

with $c_{a,\Gamma,K}$ given by (2.3).

Proof. Since by (3.3) $\mathscr{X}_N(0) \mapsto \mathscr{X}(\cdot, \mathscr{Y}, \mathscr{X}_N(0))$ is uniformly continuous we can extend it by continuity to all $\mathscr{X}_0 \in \mathscr{M}_0$ by the density of $\tilde{\mathscr{M}}_0$ in $\mathscr{M}_0$. (3.5) follows immediately from (3.3). □

Corollary 3.3 *Let $r_N(\cdot, r_N(0))$ and $q_N(\cdot, q_N(0))$ be two solutions of the SODE (1.12) with initial conditions $r_N(0)$ and $q_N(0)$, respectively, and let $\mathscr{X}_N(t)$ and $\mathscr{Y}_N(t)$ be the empirical processes associated with $r_N(t)$ and $q_N(t)$, respectively. Then for any $T > 0$ and all $N \in \mathbf{N}$.*

$$E \sup_{0\leqq t\leqq T} \gamma_2^2(\mathscr{X}_N(t), \mathscr{Y}_N(t)) \leqq \bar{c}_{T,a,\Gamma,K} E\gamma_2^2(\mathscr{X}_N(0), \mathscr{Y}_N(0)), \tag{3.6}$$

with

$$\bar{c}_{T,a,\Gamma,K} := \exp(6a^2 c_{a,\Gamma,K}(T^2+T)\exp(c_{a,\Gamma,K}(T^2+T))). \tag{3.7}$$

Proof. Set $\mathscr{Y}_1(t) := \mathscr{X}_N(t)$ and $\mathscr{Y}_2(t) := \mathscr{Y}_N(t)$ and apply (3.3) and Gronwall's inequality. □

Let $\mathscr{X}_N(t)$ be the empirical process for (1.12) starting at $\mathscr{X}_N(0)$. Set

$$\mathscr{X}(t, \mathscr{X}_N(0)) := \mathscr{X}_N(t).$$

Theorem 3.4 *The map $\mathscr{X}_N(0) \mapsto \mathscr{X}(\cdot, \mathscr{X}_N(0))$ from $\tilde{\mathscr{M}}_0$ into $\mathscr{M}_{[0,T]}$ extends uniquely to a map $\mathscr{X}_0 \mapsto \mathscr{X}(\cdot, \mathscr{X}_0)$ from $\mathscr{M}_0$ into $\mathscr{M}_{[0,T]}$. Moreover, for any $\mathscr{X}_0, \mathscr{Y}_0 \in \mathscr{M}_0$*

$$E \sup_{0\leqq t\leqq T} \gamma_2^2(\mathscr{X}(t, \mathscr{X}_0), \mathscr{X}(t, \mathscr{Y}_0)) \leqq \bar{c}_{T,a,\Gamma,K} E\gamma_2^2(\mathscr{X}_0, \mathscr{Y}_0), \tag{3.8}$$

where $\bar{c}_{T,a,\Gamma,K}$ is given by (3.7).

Proof. The same argument as for Corollary 3.2 using (3.6). □

Next we will show that our extensions from Corollary 3.2 and Theorem 3.4 are weak solutions of the associated SPDE's. First we start with $\mathscr{X}(t, \mathscr{X}_0)$ from Theorem 3.4 and note that (1.26) follows from the definition of $\mathscr{X}(t, \mathscr{X}_0)$.

Theorem 3.5 *For any $\varphi \in C_b^3(\mathbf{R}^d, \mathbf{R})$ and $\mathscr{X}_0 \in \mathscr{M}_0 \langle \mathscr{X}(t, \mathscr{X}_0), \varphi \rangle$ satisfies (1.27), $0 < t < \infty$.*

Proof. (i) By the choice of $\varphi |||\partial_l \varphi|||_L$ and $|||\partial_{lk}^2 \varphi|||_L < \infty$ for $k, l = 1, \ldots, d$. So the right hand side of (1.27) is defined for $\mathscr{X}(t, \mathscr{X}_0)$.

(ii) Set $h_N(t) := \mathscr{X}(t,\mathscr{X}_0) - \mathscr{X}(t,\mathscr{X}_N(0)), f(t) := \mathscr{X}(t,\mathscr{X}_0), f_N(t) := \mathscr{X}(t,\mathscr{X}_N(0))$. Then

$$E\left\{\int_0^t \langle h_N(s), \int \mathscr{J}(f(s), \cdot - p)w(dp,ds)\cdot\nabla\varphi\rangle\right\}^2$$
$$= \sum_{k,l,m=1}^{d} \int_0^t E\int\int h_N(s,dr)h_N(s,dq)\partial_k\varphi(r)\partial_l\varphi(q)$$
$$\times \int \mathscr{J}_{km}(f(s),r-p)\cdot\mathscr{J}_{lm}(f(s),q-p)\,dp\,ds$$

$\to 0$, as $N \to \infty$, by (3.8) and the boundedness of all terms different from h_N.
Next

$$E\left\{\int_0^t \langle f_N(s), \int (\mathscr{J}(f(s),\cdot - p) - \mathscr{J}(f_N(s),\cdot - p)w(dp,ds)\cdot\nabla\varphi\rangle\right\}^2$$
$$\leqq |||\varphi|||_1^2 \int_0^t ds\, E\int\int |f_N|(s,dr)|f_N|(s,dq) \sum_{l,k,m=1}^{d} a\cdot\sum_{n,\tilde{n}} q_n q_{\tilde{n}}\, a^{n-1} n\, a^{\tilde{n}-1}\tilde{n}$$
$$\times\sup_x\left(\sum_{+,-}\frac{1}{a^{\pm}}\int\int Q_N^{\pm}(s,d\xi,d\eta)\int(\Gamma_{kmn}(r-\xi-x-p)\right.$$
$$\left.-\Gamma_{kmn}(r-\eta-x-p))^2\,dp\right)^{\frac{1}{2}}$$
$$\times\sup_y\left(\sum_{+,-}\frac{1}{a^{\pm}}\int\int Q_N^{\pm}(s,d\xi,d\eta)\int(\Gamma_{lm\tilde{n}}(q-\xi-y-p)\right.$$
$$\left.-\Gamma_{lm\tilde{n}}(q-\eta-y-p))^2\,dp\right)^{\frac{1}{2}} \qquad (*)$$

(with $Q_N^{\pm}(s) \in C(f^{\pm}(s), f_N^{\pm}(s))$ arbitrary (cf. (2.6)))

$$\leqq \sum_{+,-}(\tilde{c}_a)^2 d^3 c_\Gamma^2 |||\varphi|||_1^2 E\int_0^t\int\int |f_N|(s,dr)|f_N|(s,dq)$$
$$\times\int\int Q_N^{\pm}(s,d\xi,d\eta)\rho^2(\xi,\eta)\,ds$$
$$\leqq \sum_{+,-}(\tilde{c}_a)^2 d^3 c_\Gamma^2 a^2 |||\varphi|||_1^2 \int_0^t E\int\int Q_N^{\pm}(s,d\xi,d\eta)\rho^2(\xi,\eta)\,ds\,.$$

Hence, by the arbitrariness of $Q_N(s)$ the left hand side of (*) is bounded above by

$$\text{const}\,|||\varphi|||_1^2\cdot\int_0^t E\gamma_2^2(f(s),f_N(s))\,ds \to 0 \quad \text{by (3.8)}\,.$$

(iii) The convergence to 0 of the corresponding deterministic integrals can be proved in the same way. □

Again let $\varphi \in C_b^3(\mathbf{R}^d,\mathbf{R})$ and $\mathscr{Y} \in \mathscr{M}_{[0,T]}$ for any $T > 0$. Consider the following bilinear equation on $\mathbf{M}$ in weak form:

$$
\begin{aligned}
d\langle \mathscr{X}(t),\varphi\rangle &= \frac{1}{2}\sum_{k,l=1}^{d} \langle D_{kl}(\mathscr{Y}(t))\mathscr{X}(t),\partial_{kl}^2\varphi\rangle\, dt \\
&\quad + \langle \mathscr{X}(t),F(\mathscr{Y}(t))\cdot\nabla\varphi\rangle\, dt \\
&\quad + \langle \mathscr{X}(t),\int \mathscr{J}(\mathscr{Y}(t),\cdot - p)w(dp,dt)\cdot\nabla\varphi\rangle \\
&\quad \mathscr{X}(0) = \mathscr{X}_0 \in \mathscr{M}_0 \,. \qquad (3.9)
\end{aligned}
$$

Theorem 3.6 *Let* $\mathscr{X}(\cdot,\mathscr{Y},\mathscr{X}_0)$ *be the extension obtained in Corollary* 3.2. *Then* $\mathscr{X}(\cdot,\mathscr{Y},\mathscr{X}_0)$ *is a weak solution of* (3.9).

The proof is a simplified version of the proof of Theorem 3.5. □

4 The Mezoscopic equation-L_2-results

The main result of this section is Theorem 4.14, *which implies that for initial conditions* $\mathscr{X}_0(dr) = X_0(r)dr$ *(i.e. having a density with respect to the Lebesgue measure* dr*) such that* $X_0 \in L_2(\mathbf{R}^d,dr)$ $\mathscr{X}(t,\mathscr{X}_0)$ *also has a density with respect to the Lebesgue measure in* $L_2(\mathbf{R}^d,dr)$, $X(t,X_0)$, *which satisfies the mezoscopic SPDE* (1.25). *Moreover the explicit representation of* $\|X(t,X_0)\|_0^{2n}$ *in* (4.29), *where* $\|\cdot\|_0$ *is the* L_2*-norm, shows that the variational methods of Pardoux and Krylov and Rozovski cannot be applied to the SPDE* (1.25).

Let $T(t)$ be the heat semigroup on $\mathbf{H}_0$. For $\lambda > 0$ the resolvent of $\frac{1}{2}\Delta$ is defined by $(\lambda - \frac{1}{2}\Delta)^{-1}f = \int_0^\infty e^{-\lambda t}T(t)f\,dt$, where $f \in \mathbf{H}_0$. We will denote by $G(t,r) := (2\pi t)^{-d/2}\exp\left(\frac{-|r|^2}{2t}\right)$ the kernel of $T(t)$. Setting for $f \in \mathbf{H}_0$ $\|f\|_{-2n,\lambda} := \|\lambda^n(\lambda - \frac{1}{2}\Delta)^{-n}f\|_0$, it follows that $\|\cdot\|_{-2n,\lambda}$ and $\|\cdot\|_{-2n}$ are equivalent norms on $\mathbf{H}_0$ and by extension also on $\mathbf{H}_{-2n}$. Let $\overline{m}$ be the smoothness index from (1.9). Since $\overline{m} > d/2 + 1$ $\mathbf{M} \subset \mathbf{H}_{-\overline{m}}$. Moreover, by (1.7)/(1.9) $D_{kl}(f)$ for $f \in \mathbf{M}$ defines a bounded multiplication operator on $\mathbf{H}_{\overline{m}}$ whence both f and $D_{kl}(f)\cdot f$ are in $\mathbf{H}_{-\overline{m}}$, i.e., in the domain of ∂_{kl}^2 considered as an unbounded operator on $\mathbf{H}_{-(\overline{m}+2)}$. Fix $m \geqq \overline{m}/2 + 1$ and set $R_\lambda := \lambda^m(\lambda - \frac{1}{2}\Delta)^{-m}$. We will consider R_λ both as a bounded operator on $\mathbf{H}_0$ and (by extension) as a bounded operator from $\mathbf{H}_{-2m}$ into $\mathbf{H}_0$. Let $\mathscr{Y} \in \mathscr{M}_{[0,T]}$ for all $T > 0$ and $\mathscr{X}(\cdot) := \mathscr{X}(\cdot,\mathscr{Y},\mathscr{X}_0)$ the solution of the bilinear equation (3.9) which was derived in Corollary 3.2 and Theorem 3.6. Set

$$Z_\lambda := R_\lambda \mathscr{X} \,.$$

Abbreviate (cf. (1.24))

$$
\begin{aligned}
\tilde{D}_{kl}(s) &:= D_{kl}(\mathscr{Y}(s))\,, \\
L(s) &:= F(\mathscr{Y}(s))\,, \\
dM(s) &:= \int \mathscr{J}(\mathscr{Y}(s),\cdot - p)w(dp,ds)\,.
\end{aligned}
$$

Then we have the following equation on $\mathbf{H}_0$:

$$
\begin{aligned}
Z_\lambda(t) &= Z_\lambda(0) + \int_0^t \frac{1}{2}\sum_{k,l=1}^{d} \partial_{kl}^2 R_\lambda(\mathscr{X}(s)\tilde{D}_{kl}(s))\,ds \\
&\quad - \int_0^t \nabla\cdot R_\lambda(\mathscr{X}(s)L(s))\,ds - \int_0^t \nabla\cdot R_\lambda(\mathscr{X}(s)dM(s))\,. \qquad (4.1)
\end{aligned}
$$

Itô's formula yields for $n \geqq 1$,

$$
\begin{aligned}
\|Z_\lambda(t)\|_0^{2n} &= \|Z_\lambda(0)\|_0^{2n} \\
&+ n \sum_{k,l=1}^{d} \int_0^t \|Z_\lambda(s)\|_0^{2(n-1)} \langle Z_\lambda(s), \partial_{kl}^2 R_\lambda(\mathscr{X}(s)\tilde{D}_{kl}(s))\rangle_0 \, ds \\
&- 2n \int_0^t \|Z_\lambda(s)\|_0^{2(n-1)} \langle Z_\lambda(s), \nabla \cdot R_\lambda(\mathscr{X}(s)L(s))\rangle_0 \, ds \\
&- 2n \int_0^t \|Z_\lambda(s)\|_0^{2(n-1)} \langle Z_\lambda(s), \nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s))\rangle_0 \\
&+ n \int_0^t \|Z_\lambda(s)\|^{2(n-1)} [\nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s))] \\
&+ n(n-1) \int_0^t \|Z_\lambda(s)\|^{2(n-2)} [\langle Z_\lambda(s), \nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s))\rangle_0] \,.
\end{aligned} \tag{4.2}
$$

The mutual quadratic variation of $dM(s,r)$ and $dM(s,q)$ is given by

$$
\begin{aligned}
d[M_k(s,r), M_l(s,q)] &= \sum_{j=1}^{d} \int \mathscr{J}_{kj}(\mathscr{Y}(s), r-p)\mathscr{J}_{lj}(\mathscr{Y}(s), q-p)\, dp\, ds \\
&=: \tilde{D}_{kl}(s,r,q)\, ds
\end{aligned}
$$

(cf. (1.14)).
Set

$$
\begin{aligned}
\hat{D}_{kl}(s,r,q) := \sum_{j=1}^{d} \frac{1}{2} \int &\{\mathscr{J}_{kj}(\mathscr{Y}(s), r-p) - \mathscr{J}_{kj}(\mathscr{Y}(s), q-p)\} \\
&\times \{\mathscr{J}_{lj}(\mathscr{Y}(s), r-p) - \mathscr{J}_{lj}(\mathscr{Y}(s), q-p)\}\, dp \,.
\end{aligned}
$$

By the same techniques as in (1.21) for all $s \geqq 0, r, q \in \mathbf{R}^d$

$$
|\hat{D}_{kl}(s,r,q)| \leqq d\tilde{c}_a^2 c_\Gamma^2 \frac{\rho^2(r,q)}{2} \,. \tag{4.3}
$$

Clearly

$$
\tfrac{1}{2}\tilde{D}_{kl}(s,q,r) + \tfrac{1}{2}\tilde{D}_{kl}(s,r,q) = -\hat{D}_{kl}(s,r,q) + \tfrac{1}{2}\{\tilde{D}_{kl}(s,r) + \tilde{D}_{kl}(s,q)\} \,. \tag{4.4}
$$

In what follows we will assume that for some $n \geqq 1$ and some $\lambda > 0$,

$$
E\|Z_\lambda(0)\|_0^{2n} < \infty \,. \tag{4.5}
$$

By the equivalence of the norms $\|\cdot\|_{-2m,\lambda}, \lambda > 0$, (4.5) will be satisfied for all $\lambda > 0$ if it holds for some $\lambda > 0$.

Lemma 4.2 *A.s. for all* $t \geqq 0$

$$
\begin{aligned}
&\left[\int_0^t \nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s))\right] + \sum_{k,l=1}^{d} \int_0^t \langle Z_\lambda(s), \partial_{kl}^2 R_\lambda(\mathscr{X}(s)\tilde{D}_{kl}(s))\rangle_0 \, ds \\
&\leqq 2^{\frac{d}{2}} \cdot 16 d^2 \bar{c} c_\Gamma^2 \tilde{c}_a^2 \int_0^t \|R_\lambda|\mathscr{X}|(s)\|_0^2 \, ds \,.
\end{aligned} \tag{4.6}
$$

Proof.

(i)

$$\left[\int_0^t R_\lambda \nabla \cdot (\mathscr{X}(s)dM(s))\right]$$

$$= \sum_{k,l=1}^{d} \lambda^{2m} \int_0^t \int_0^\infty \dots \int_0^\infty e^{-\lambda(u_1 + \cdot + u_{2m})}\, du_1 \dots du_{2m} \tag{4.7}$$

$$\int\int (\partial_{kl,r}^2 G(u_1 + \cdot + u_{2m}, r - q))$$

$$\times\, \mathscr{X}(s,dr)\mathscr{X}(s,dq)\{\hat{D}_{kl}(s,r,q) - \tilde{D}_{kl}(s,r)\}\, ds$$

by (4.4) and the fact that in the above integration the roles of r and q can be interchanged. Hence the left hand side of (4.6) equals

$$\sum_{k,l=1}^{d} \lambda^{2m} \int_0^t \int_0^\infty \dots \int_0^\infty e^{-\lambda(u_1 + \dots + u_{2m})}\, du_1 \dots du_{2m}$$

$$\int\int (\partial_{kl,r}^2 G(u_1 + \dots + u_{2m}, r - q))$$

$$\times\, \mathscr{X}(s,dr)\mathscr{X}(s,dq)\hat{D}_{kl}(s,r,q)\, ds\,. \tag{$*$}$$

(ii) Next,

$$\left|\sum_{k,l=1}^{d} \partial_{kl,r}^2 G(u, r - q)\right| \leqq 2^{\frac{d}{2}} 16d\left(\frac{|r-q|^2}{16u^2} + \frac{1}{4u}\right)\exp\left(\frac{-|r-q|^2}{4u}\right) G(2u, r - q)\,, \tag{4.8}$$

whence by (4.3)

$$\left|\sum_{k,l=1}^{d} (\partial_{kl,r}^2 G(u, r - q))\hat{D}_{kl}(s,r,q)\right| \leqq 2^{\frac{d}{2}} 16 d^2 \bar{c}^2 c_\Gamma^2 \tilde{c}_a^2 G(2u, r - q)\,.$$

(iii) First we change variables $2u_k =: v_k$, $k = 1, \dots, d$, and note that $\|R_{\frac{\lambda}{2}} R_\lambda^{-1}\|_{\mathscr{L}(\mathbf{H}_0)} \leqq 1$ (cf. [4, p. 48]) which implies by unique extendibility to $\mathbf{H}_{-2m}$ for $f \in \mathbf{H}_{-2m}$

$$\|f\|_{-2m,\frac{\lambda}{2}} \leqq \|f\|_{-2m,\lambda}\,. \tag{4.9}$$

This in addition to steps (i)–(ii) implies (4.6). □

In the right hand side of (4.6) the derivative operators from the left hand side have disappeared. If $\mathscr{X} \geqq 0$ (4.6) would suggest a Gronwall estimate of the left hand side. The following observation shows how to get rid of the gradients in the two other integrals in (4.2). For sufficiently smooth functions f and F we obtain for $l = 1, \dots, d$

$$2\langle f, \partial_l(f \cdot F)\rangle_0 = \langle f^2, \partial_l F\rangle_0\,, \tag{4.10}$$

where $f \cdot F$ is pointwise multiplication. (4.10) is equivalent to

$$2\langle f, (\partial_l f) \cdot F\rangle_0 = -\langle f^2, \partial_l F\rangle_0\,. \tag{4.11}$$

Moreover, (4.10) implies (for the stochastic differential by (1.13))

$$2\langle Z_\lambda(s), \nabla\cdot(Z_\lambda(s)L(s))\rangle_0 = \langle Z_\lambda^2(s), \nabla\cdot L(s)\rangle_0\,,$$
$$2\langle Z_\lambda(s), \nabla\cdot(Z_\lambda(s)dM(s))\rangle_0 = \langle Z_\lambda^2(s), \nabla\cdot dM(s)\rangle_0\,. \tag{4.12}$$

Set

$$B_1(s,\lambda) := \langle Z_\lambda(s), \nabla\cdot(Z_\lambda(s)L(s)) - \nabla\cdot R_\lambda(\mathscr{X}(s)L(s))\rangle_0\,.$$

Lemma 4.3 *A.s. for all* $s \geqq 0$

$$|B_1(s,\lambda)| \leqq 2^{\frac{d}{2}}3c_Kc_a d\|R_\lambda|\mathscr{X}|(s)\|_0^2\,. \tag{4.13}$$

Proof.
Let $f \in \mathbf{M}$, $f_\lambda := R_\lambda f$ and F be a sufficiently smooth function from $\mathbf{R}^d$ into $\mathbf{R}^d$. Then

$$\left|\int f_\lambda(r)\int(\nabla_r G(u,r-q))f(dq)\cdot(F(q)-F(r))\,dr\right|$$
$$\leqq 2\cdot 2^{\frac{d}{2}}\max_l |||F_l|||_1 d\int |f_\lambda|(r)G(2u,r-q)|f|(dq)\,dr\,.$$

(Similarly to (4.8)). Moreover,

$$\langle f_\lambda, \nabla\cdot(f_\lambda\cdot F)\rangle_0 = \langle f_\lambda, (\nabla f_\lambda)\cdot F\rangle_0 + \langle f_\lambda^2, \nabla\cdot F\rangle_0\,.$$

This in addition to (4.9) implies (4.13).

Lemma 4.4 *A.s. for all* $0 \leqq s < t < \infty$

$$\int_s^t[\langle Z_\lambda^2(u), \nabla\cdot dM(u)\rangle_0] \leqq d^3\bar{c}_a^2c_\Gamma^2\int_s^t\|Z_\lambda(u)\|_0^4\,du\,. \tag{4.14}$$

Proof.
Set $f := Z_\lambda^2(u)$. Then

$$[\langle f, \nabla\cdot dM(u)\rangle_0]$$
$$= \sum_{l,k,m=1}^d \int\int f(q)f(r)\sum_{n,\tilde{n}=1}^\infty q_n q_{\tilde{n}}\int\ldots\int\mathscr{Y}(u,d\xi_1)\ldots\mathscr{Y}(u,d\xi_n)$$
$$\times\mathscr{Y}(u,d\eta_1)\ldots\mathscr{Y}(u,d\eta_{\tilde{n}})$$
$$\times\int\partial_{k,r}\Gamma_{kmn}(r-p-(\xi_1+\ldots+\xi_n))\cdot$$
$$\partial_{l,q}\Gamma_{lm\tilde{n}}(q-p-(\eta_1+\ldots+\eta_{\tilde{n}}))\,dp\,du\,dr\,dq$$
$$\leqq d^3\bar{c}_a^2c_\Gamma^2\left(\int f(r)dr\right)^2du\,. \quad\square$$

Lemma 4.5 *A.s. for all* $0 \leqq s < t < \infty$,

$$\int_s^t[\langle Z_\lambda(u), \nabla R_\lambda(\mathscr{X}(u)dM(u))\rangle_0] \leqq 2^{\frac{d}{2}}33\bar{c}\bar{c}_a^2c_\Gamma^2d^3\int_s^t\|R_\lambda|\mathscr{X}|(u)\|_0^4\,du\,. \tag{4.15}$$

Proof.
(i)

$$\{\nabla\cdot R_\lambda(\mathscr{X}(s)\,dM(s)) - (\nabla R_\lambda\mathscr{X}(s))\cdot dM(s)\}(r)$$

$$= \sum_{l=1}^{d} \lambda^m \int_0^\infty \dots \int_0^\infty e^{-\lambda(u_1+\dots+u_m)} du_1 \dots du_m \int \frac{2^{\frac{d}{2}} 4(q_l - r_l)}{4(u_1 + \dots + u_m)}$$
$$\times \exp\left(-\frac{|r-q|^2}{4(u_1+\dots+u_m)}\right)$$
$$\times \mathscr{X}(s,dq) \cdot G(2(u_1+\dots+u_m), r-q)(dM_l(s,q) - dM_l(s,r)) .$$

Hence by (4.3) and (4.9)

$$[\nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s)) - (\nabla Z_\lambda(s)) \cdot dM(s)]$$
$$\leqq 2^{\frac{d}{2}} 32 d^2 \bar{c} \bar{c}_a^2 c_\Gamma^2 \|R_\lambda |\mathscr{X}|(s)\|_0^2\, ds .$$

(ii) Now (4.15) follows using (4.12) and (4.14) in addition to

$$[\langle Z_\lambda(s), \nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s)) - (\nabla Z_\lambda(s)) \cdot dM(s)\rangle_0]$$
$$\leqq \|Z_\lambda(s)\|_0^2 \cdot [\nabla \cdot R_\lambda(\mathscr{X}(s)\, dM(s)) - (\nabla Z_\lambda(s)) \cdot dM(s)]$$

(cf. [21, Chap. 2.42]). □

The previous lemmas lead to the following important estimates.

Lemma 4.6 (i) *For any* $t \geqq 0$

$$E\|Z_\lambda(t)\|_0^{2n} \leqq E\|Z_\lambda(0)\|_0^{2n} + c_{n,a,\Gamma,K} \int_0^t E\|R_\lambda|\mathscr{X}|(s)\|_0^{2n}\, ds; \tag{4.16}$$

(ii) *for any* $T \geqq 0$

$$E \sup_{0 \leqq t \leqq T} \|Z_\lambda(t)\|_0^{2n} \leqq E\|Z_\lambda(0)\|_0^{2n} + c_{n,a,\Gamma,K} \int_0^T E\|R_\lambda|\mathscr{X}|(s)\|_0^{2n}\, ds$$
$$+ c_{n,a,\Gamma,K} \left(E \int_0^t \|R_\lambda|\mathscr{X}|(s)\|_0^{4n}\, ds \right)^{\frac{1}{2}} . \tag{4.17}$$

Here

$$c_{n,a,\Gamma,K} := (nd^2 \bar{c} c_\Gamma^2 \bar{c}_a^2((n-1)33d + 64) + 4dnc_a c_K) 2^{\frac{d}{2}} \tag{4.18}$$

both in (I) *and* (II).

Proof. (I) follows from (4.2), (4.12) and Lemmas 4.2, 4.3, and 4.5.
(II) follows from the previous step, (4.2), and the Burkholder–Davis–Gundy inequality. □

Remark 4.7. If we knew that $\int_0^T E\|\mathscr{X}(s)\|_0^{2n}\, ds < \infty$, resp. $\int_0^T E\|\mathscr{X}(s)\|_0^{4n}\, ds < \infty$ Fatou's and Gronwall's lemmas would imply bounds for $E\|\mathscr{X}(t)\|_0^{2n}$, resp. $E \sup_{0 \leqq t \leqq T} \|\mathscr{X}(t)\|_0^{2n}$ by multiples of $E\|\mathscr{X}(0)\|_0^{2n}$, resp. $(E\|\mathscr{X}(0)\|_0^{4n})^{\frac{1}{2}}$. It is actually not even necessary that $\mathscr{X}$ be derived from Corollary 3.2. More precisely, let $\mathscr{X}(\cdot, \mathscr{Y}, Z_0)$ be a solution of (3.9) such that $\mathscr{X}(\cdot, \mathscr{Y}, Z_0)$ is a linear combination of processes from $\mathscr{M}_{[0,T]} \cap C([0,T]; \mathbf{H}_0)$ with $\mathscr{Y} \in \mathscr{M}_{[0,T]}$ for all $T > 0$. If $E\|Z_0\|^{2n} < \infty$, $\mathscr{X}(\cdot, \mathscr{Y}, Z_0)$ will satisfy (4.1) and hence for any $t \leqq T$ there is a $c_T < \infty$ such that

$$E\|\mathscr{X}(t,\mathscr{Y},Z_0)\|_0^{2n} \leqq c_T E\|Z_0\|_0^{2n} < \infty\,. \tag{$*$}$$

On the other hand, if we assume $\mathscr{X}(s) \geqq 0$ we can apply the Gronwall lemma to (4.16) and obtain:

Corollary 4.8. Suppose $\mathscr{X}(s) \geqq 0$ a.s. for all $s \geqq 0$. Then for any $t \geqq 0$,

$$E\|Z_\lambda(t)\|_0^{2n} \leqq \exp(c_{n,a,\Gamma,K}t)E\|Z_\lambda(0)\|_0^{2n}\,. \tag{4.19}$$

Proof. The proof follows from (4.16) and from Gronwall's lemma. □

Corollary 4.9 Suppose $\mathscr{X}(s) \geqq 0$ a.s. for all $s \geqq 0$ in addition to $\mathscr{X}(0) =: Z(0) \in \mathbf{H}_0$ and $E\|Z(0)\|_0^{2n} < \infty$ for some $n \geqq 1$. Then for any $t > 0$, $\mathscr{X}(t) =: Z(t) \in \mathbf{H}_0$ and

$$E\|Z(t)\|_0^{2n} \leqq \exp(c_{n,a,\Gamma,K}t)E\|Z(0)\|_0^{2n}\,. \tag{4.20}$$

The proof follows from (4.19) and Fatou's lemma. □

Remark 4.10 Set $\tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_N^{\pm}(0)) := \mathscr{X}_N^{\pm}(t)$, where $\mathscr{X}_N^{\pm}(t)$ is the Jordan decomposition if the empirical process $\mathscr{X}_N(t)$ defined by (3.2). It follows directly from the definition that both $\tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_N^{+}(0))$ and $\tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_N^{-}(0))$ are solutions of (3.9) with initial conditions $\mathscr{X}_N^{+}(0)$ and $\mathscr{X}_N^{-}(0)$, respectively. Since the extended process from Corollary 3.2 $\mathscr{X}(t,\mathscr{Y}(t),\mathscr{X}_0)$ is obtained by extending both the positive and negative components we obtain "extensions" $\tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_0^{\pm})) = \mathscr{X}^{\pm}(t,\mathscr{Y}(t),\mathscr{X}_0)$ which satisfy (3.9) with initial conditions $\mathscr{X}_0^{\pm}$. Therefore we will assume in what follows

$$\mathscr{X}(t,\mathscr{Y}(t),\mathscr{X}_0) = \sum_{l=1}^{M} \gamma_l \tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_{0,l}^{\pm})\,, \tag{4.21}$$

where $M \in \mathbf{N}$, $\gamma_l \in \mathbf{R}$ and $\tilde{\mathscr{X}}(t,\mathscr{Y}(t),\mathscr{X}_{0,l}^{\pm})$ are positive, resp. negative extensions of empirical processes $\mathscr{X}_N^{\pm}(t)$ with $\mathscr{X}_{0,l} \in \mathscr{M}_0$. Clearly, $\mathscr{X}(t,\mathscr{Y}(t),\mathscr{X}_0)$ satisfies (3.9) with $\mathscr{X}(0) =: \mathscr{X}_0$.

Theorem 4.11 (I) *Suppose* $E\|Z_0\|_0^{2n} < \infty$ *for some* $n \geqq 1$ *in addition to (4.21). Then for any* $t \geqq 0\, \mathscr{X}(t) =: \mathscr{X}(t) \in \mathbf{H}_0$, $Z(t)$ *is adapted and* $Z(t,\omega)$ *is* $2n$*-integrable over* $[0, T] \times \Omega$ *with values in* $\mathbf{H}_0$ *for arbitrary* $T > 0$. *Moreover, for any* $t > 0$

$$E\|Z(t)\|_0^{2n} < \exp(c_{n,a,\Gamma,K}t)E\|Z_0\|_0^{2n}\,. \tag{4.22}$$

(II) *Suppose* $E\|Z_0\|_0^{4n} < \infty$ *for some* $n \geqq 1$ *in addition to* (4.21). *Then for any* $T \geqq 0$,

$$E \sup_{0 \leqq t \leqq T} \|Z(t)\|_0^{2n} \leqq 2\exp(c_{2n,a,\Gamma,K}T)(E\|Z_0\|_0^{4n})^{\frac{1}{2}} \tag{4.23}$$

Here $c_{m,a,\Gamma,K}$, *is given by* (4.18), *where* $m \in \{n, 2n\}$.

Proof. The $2n$-integrability of $Z(t)$ as an $\mathbf{H}_0$-valued process follows from the $2n$-integrability of its positive and negative components $Z(t,\mathscr{Y}(t),\mathscr{X}_{0,l}^{\pm})$, whose integrability properties follow from those corresponding properties of $Z_\lambda(t,\mathscr{Y}(t),Z_{0,l}^{\pm})$ and from (4.20). (4.22) and (4.23) follow from (4.16) and (4.17), respectively. □

In what follows we will derive an expression for $\|Z(t)\|_0^{2n}$ using Itô's formula, where $Z(t)$ is the measurable $\mathbf{H}_0$-valued version of $\mathscr{Z}(t)$ from (4.21), which was obtained in Theorem 4.11. The following lemma will be used at various steps in that derivation.

Lemma 4.12 *Let $f, g \in \mathbf{H}_0 \cap L_1(\mathbf{R}^d, dr)$ and set $g_\lambda := |(\tilde{R}_\lambda - I)g| + |g|$, where $\tilde{R}_\lambda := R_\lambda^n$ for some $n \geqq 1$. Then for any $k \in \mathbf{N}$*

$$\lim_{\lambda\to\infty} \int_0^\infty \int_0^\infty \dots \int e^{-\lambda(u_1+\dots+u_k)} \lambda^k \, du_1 \dots du_k \int G(u_1 + \dots + u_k, r - q)$$
$$|f(q) - f(r)| dq g_\lambda(r)\, dr = 0\,. \tag{4.24}$$

Proof. (i) Let $\delta > 0$ be given and denote the multiple integral on the left hand side of (4.2) by $\wedge(f,g)$. By change of variables $p := \frac{(q-r)}{\sqrt{u}}$

$$\begin{aligned}
&\int\int G(u, r-q)|f(q) - f(r)| g_\lambda(r)\, dq\, dr \\
&= \int\int G(1,p)|f(r + p\sqrt{u}) - f(r)|\, dp\, g_\lambda(r)\, dr \\
&= \int\int_{B_L} G(1,p)|f(r + p\sqrt{u}) - f(r)|\, dp\, g_\lambda(r)\, dr \\
&\quad + \int\int_{B_L^c} G(1,p)|f(r + p\sqrt{u}) - f(r)|\, dp\, g_\lambda(r)\, dr \\
&=: A_L(u) + A_L^c(u)\,,
\end{aligned}$$

where $B_L = \{p \in \mathbf{R}^d : |p| \leqq L\}, B_L^c = \mathbf{R}^d \backslash B_L, L > 0$.
(ii) On B_L^c, $G(1,p) \leqq 2\exp(\frac{-L^2}{4})G(2,p)$ which implies

$$A_L^c(u) \leqq 2\exp\left(\frac{-L^2}{4}\right)\{\|f\|_0^2 + \|g\|_0^2\} \leqq \frac{\delta}{2}$$

for L sufficiently large.
(iii) Let m denote the d-dimensional Lebesgue measure and set

$$F(r,u) := \frac{1}{m(B_{\sqrt{u}L})} \int_{B_{\sqrt{u}L}} |f(r+q) - f(r)|\, dq\,.$$

We have

$$0 \leqq \int_{B_L} G(1,p)|f(r + p\sqrt{u}) - f(r)|\, dp \leqq \text{const. } L^d F(r,u) \to 0 \text{ as } u \to 0\,,$$

m-a.e. (m-almost everywhere) by the Lebesgue differentiation theorem.
(iv) To conclude from (iii) $A_L(\frac{v}{\lambda}) \to 0$, $\lambda \to \infty$ for any $v > 0$ we first set

$$Hf(r) := \sup_{u>0} \frac{1}{m(B_u)} \int_{B_u} |f(r+q)|\, dq\,,$$

which is the Hardy–Littlewood maximal function for f, and

$$\tilde{H}f(r) := Hf(r) + |f|(r).$$

Let $N \in \mathbf{N}$. Then

$$\begin{aligned}\int F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\,dr &= \int_{\{\tilde{H}f \geqq N\}} F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\,dr + \int_{\{\tilde{H}f < N\}} F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\,dr \\ &=: I_\lambda(v,N) + II_\lambda(v,N).\end{aligned}$$

(v) Our assumptions on g imply $g_\lambda \in \mathbf{H}_0 \cap L_1(\mathbf{R}^d, dr)$, and we easily check that both $\{g_\lambda\}$ and $\{g_\lambda^2\}$ are uniformly integrable. Hence, for any $v > 0, N \in \mathbf{N}$,

$$II_\lambda(v,N) \to 0, \quad \text{as } \lambda \to \infty.$$

(vi) $I_\lambda(v,N) \leqq 2(\int 1_{\{\tilde{H}f \geqq N\}}(r) g_\lambda^2(r)\,dr)^{\frac{1}{2}} \|f\|_0 \to 0$, uniformly in v and λ as $N \to \infty$, since $\{g_\lambda^2\}$ is uniformly integrable and $m\{\tilde{H}f \geqq N\} \to 0$, as $N \to \infty$.
(vii) By steps (iv)–(vi) we first choose $N = N(\varepsilon)$ for given $\varepsilon > 0$ such that $I_\lambda(v,N) \leqq \frac{\varepsilon}{2}$ for all v, λ and then choose $\lambda = \lambda(v,N,\varepsilon)$ such that for $\lambda \geqq \lambda(v,N,\varepsilon)$, $II_\lambda(v,N) \leqq \frac{\varepsilon}{2}$. This implies for any $v > 0$, $L > 0$.

$$A_L\left(\frac{v}{\lambda}\right) \to 0, \quad \text{as } \lambda \to \infty.$$

(viii) By change of variables

$$\begin{aligned}\wedge(f,g) &= \int_0^\infty \dots \int_0^\infty e^{-(v_1+\dots+v_m)}\,dv_1 \dots dv_m \Big\{ A_L\Big(\frac{v_1 + \dots + v_m}{\lambda}\Big) \\ &\quad + A_L^c\Big(\frac{v_1 + \dots + v_m}{\lambda}\Big)\Big\} \\ &\leqq \int_0^\infty \dots \int_0^\infty e^{-(v_1+\dots+v_m)}\,dv_1 \dots dv_m \Big(A_L\Big(\frac{v_1 + \dots + v_m}{\lambda}\Big)\Big) + \frac{\delta}{2} \qquad (*)\end{aligned}$$

by step (ii) for sufficiently large L. Since

$$A_L\Big(\frac{v_1 + \dots + v_m}{\lambda}\Big) \leqq \text{const.}\, L^2(\|f\|_0^2 + \|g\|_0^2)$$

(vii) and Lebesgue's dominated convergence theorem imply that for any L the multiple integral in the right hand side of (*) will be less than $\delta/2$ for $\lambda \geqq \lambda(L,\delta) = \lambda(L(\delta),\delta)$. □

Set for $\mathscr{Y} \in \mathscr{M}_{[0,T]}, s \in [0,T]$,

$$\hat{\mathscr{L}}(\hat{D})(\mathscr{Y}(s),r) : \lim_{q \to r} \sum_{k,l=1}^{d} \partial_{k,l=1}^2 \hat{D}_{kl}(s,r,q), \tag{4.25}$$

which exists by assumptions (1.7), (1.9) and the definition of $\hat{D}_{kl}(s,r,q)$ (cf. (4.4)). We obtain from (1.9)

$$\left|\sup_q \sum_{k,l=1}^{d} \partial_{kl,r}^2 \hat{D}_{kl}(s,r,q)\right| \leqq 6d\tilde{c}_a^2 c_\Gamma^2. \tag{4.26}$$

Let us now abbreviate $D_{kl}(\mathscr{Y}(s),r,p) := \frac{d[M_k(s,r),M_l(s,q)]}{ds}$ (instead of $\tilde{D}_{kl}(s,r,q)$).

Theorem 4.13. *Suppose* $Z(t)$ *is given by* (4.21) *and* $E\|Z_0\|_0^{4n} < \infty$ *for some* $n \geqq 1$. *Then*

(I) *for any* $t \geqq 0$

$$\|Z(t)\|_0^{2n} = \|Z_0\|_0^{2n} + n\int_0^t \|Z(s)\|_0^{2(n-1)} \langle Z^2(s), \hat{\mathscr{L}}(\hat{D})(\mathscr{Y}(s))\rangle_0 \, ds$$
$$- n\int_0^t \|Z(s)\|_0^{2(n-1)} \langle Z^2(s), \nabla \cdot F(\mathscr{Y}(s))\rangle_0 \, ds$$
$$- n\int_0^t \|Z(s)\|_0^{2(n-1)} \langle Z^2(s), \nabla \cdot \int \mathscr{J}(\mathscr{Y}(s), \cdot - p) w(dp, ds)\rangle_0$$
$$+ n(n-1)\int_0^t \|Z(s)\|_0^{2(n-2)} \int\int Z^2(s,r) Z^2(s,q)$$
$$\times \left(\sum_{k,l=1}^d \partial^2_{k,r,l,q} D_{kl}(\mathscr{Y}(s),r,q) \right) dr\, dq\, ds\,; \tag{4.27}$$

(II)

$$Z(\cdot) \in C([0,\infty); \mathbf{H}_0) \text{ a.s.} \tag{4.28}$$

Proof. (i) The assumption implies by (4.14) that the stochastic integral in (4.27) defines a real valued square integrable continuous martingale. Since by assumption (1.8) $|||\nabla \cdot F(\mathscr{Y}(s))||| \leqq dc_a c_K$ we obtain from (4.26) that the right hand side of (4.27) defines a continuous real valued process.

(ii) We will first replace the martingale and the last quadratic variation integral on the right hand side of (4.2) by their respective limits (cf. Lemma 4.5). (ii.1) Recall that $Z_\lambda(s) = R_\lambda Z(s)$. Then

$$\langle Z_\lambda(s), \nabla \cdot R_\lambda(Z(s)dM(s)) - (\nabla Z_\lambda(s))) \cdot dM(s)\rangle_0$$
$$= \sum_{l=1}^d \int_0^\infty \ldots \int_0^\infty e^{-\lambda(u_1+\ldots+u_m)} \lambda^m \, du_1 \ldots du_m$$
$$\times \Big\{ \int\int \partial_{l,r} G(u_1+\ldots+u_m, r-q) Z(s,r)$$
$$\times Z_\lambda(s,r)(dM_l(s,q) - dM_l(s,r))\, dq\, dr$$
$$- \int\int \frac{(r_l - q_l)}{(u_1+\ldots+u_m)} |q-r| G(u_1+\ldots+u_m, r-q)(Z(s,q)$$
$$- Z(s,r)) Z_\lambda(s,r) \left\{ \frac{dM_l(s,q) - dM_l(s,r)}{|q-r|} \right\} dq\, dr \Big\}$$
$$=: I_\lambda(ds) + II_\lambda(ds)\,.$$

(ii.2) $I_\lambda(ds) = \sum_{i=1}^4 F_{i,\lambda}(ds)$, where

$$F_{i,\lambda}(ds) = \sum_{l=1}^d \int_0^\infty \ldots \int_0^\infty e^{-\lambda(u_1+\ldots+u_m)} \lambda^m \, du_1 \ldots du_m$$
$$\times \int\int G(u_1+\ldots+u_m, r-q) f_{i,\lambda}(s,q,r) \partial_{l,q} dM_l(s,q)\, dq\, dr$$

and

$$\begin{aligned} f_{1,\lambda}(s,q,r) &:= Z^2(s,q),\\ f_{2,\lambda}(s,q,r) &:= (Z(s,r)-Z(s,q))Z_\lambda(s,r),\\ f_{3,\lambda}(s,q,r) &:= (Z_\lambda(s,r)-Z(s,r))Z(s,q),\\ f_{4,\lambda}(s,q,r) &:= (Z(s,r)-Z(s,q))Z(s,q)\,. \end{aligned}$$

(ii.3) Clearly,

$$F_{1,\lambda}(ds) = \langle Z^2(s), \nabla\cdot dM(s)\rangle_0\,.$$

(ii.4) $\int_0^t [II_\lambda(du)] \to 0$, as $\lambda\to\infty$ by (4.3) and (4.24).
(ii.5) Similarly,

$$\int_0^t [F_{i,\lambda}(du)] \to 0, \text{ as } \lambda\to\infty \text{ for } i=2,4\,.$$

(ii.6)

$$\int_0^t [F_{3,\lambda}(du)] \leqq \bar{c}_a^2\, c_T^2 \int_0^t \|(R_\lambda - I)Z(u)\|_0^2 \|R_\lambda |Z|(u)\|_0^2\, du \quad \to 0, \text{ as } \lambda\to\infty\,.$$

(ii.7) Since $Z_\lambda^2(s,r)\to Z^2(s,r)\, dP\otimes dt\otimes dr$ a.e. and since $Z_\lambda^2(s)$ is uniformly integrable with respect to

$$\int_s^t [\langle Z_\lambda^2(u), \nabla\cdot dM(u)\rangle_0] \to \int_s^t [\langle Z^2(u), \nabla\cdot dM(u)\rangle_0] \text{ as } \lambda\to\infty$$

a.e. uniformly on bounded intervals $[s,t]\subset[0,\infty)$.

Altogether we obtain from (ii.1)–(ii.7) and the definition of $D_{kl}(\mathscr{Y}(s),r,q)$ that the last quadratic variation integral in (4.2) tends a.s. to the last integral in the right hand side of (4.27) uniformly on bounded intervals, where we also use the identity $\langle Z_\lambda(s), (\nabla Z_\lambda(s))\cdot dM(s)\rangle_0 + \frac{1}{2}\langle Z_\lambda^2(s), \nabla\cdot dM(s)\rangle_0 = 0$. By choosing a subsequence $\lambda\to\infty$, we obtain that also the martingale in (4.2) tends to the martingale in (4.27) a.s. uniformly on bounded intervals.

(iii) In view of Lemma 4.2 (cf. (*)) and Lemma 4.12 we see that the first integral plus the quadratic variation integral in (4.2) tends a.s. to the first integral in the right hand side of (4.27) uniformly on bounded intervals as $\lambda\to\infty$.

(iv) Similarly for the deterministic integrals with the gradient operator.

(v.1) By the previous steps the convergence of the right hand side of (4.2) to the right hand side of (4.27) is uniform a.s. So we may assume that on the same measurable set Ω_0 with $P(\Omega_0)=1$ we have: (1) $Z(t)\in \mathbf{H}_0$ uniformly in t; (2) $R_\lambda\mathscr{Z} = R_\lambda Z$ is continuous with values in $\mathbf{H}_0$; (3) $\|Z(\cdot)\|_0$ is continuous.

(v.2) Now we easily see that for $\omega\in\Omega_0\, Z(\cdot)$ is weakly continuous, whence by the continuity of $\|Z(\cdot)\|_0$ we obtain (4.28). □

Next we consider the quasilinear SPDE (1.25)/(1.26) with its weak solution from Theorems 3.4 and 3.5.

Theorem 4.14 *Suppose* $\mathscr{X}(0) =: X_0 \in \mathbf{H}_0$ *and* $E\|X_0\|_0^{4n} < \infty$ *for some* $n\geqq 1$. *Then*

(I) $\mathscr{X}(t) =: X(t)\in\mathbf{H}_0$ *a.s. for all* $t\geqq 0$;

(II)

$$\|X(t)\|_0^{2n} = \|X_0\|_0^{2n} + n\int_0^t \|X(s)\|_0^{2(n-1)} \langle X^2(s), \hat{\mathscr{L}}(\hat{\mathscr{D}})(X(s))\rangle_0\, ds$$

$$-n\int_0^t \|X(s)\|_0^{2(n-1)} \langle X^2(s), \nabla\cdot F(X(s))\rangle_0\, ds$$

$$-n\int_0^t \|X(s)\|_0^{2(n-1)} \langle X^2(s), \nabla\cdot \int \mathscr{J}(X(s),\cdot - p) w(dp,ds)\rangle_0$$

$$+n(n-1)\int_0^t \|X(s)\|_0^{2(n-2)} \int\int X^2(s,r) X^2(s,q)$$

$$\times \left(\sum_{k,l=1}^d \partial^2_{k,r,l,q} D_{kl}(X(s),r,q) \right) dr\, dq\, ds\,, \tag{4.29}$$

with

$$D_{kl}(X(s),r,q) := \sum_{j=1}^d \int \mathscr{J}_{kj}(X(s), r-p) \mathscr{J}_{lj}(X(s), q-p)\, dp\, ds\,;$$

(III)

$$X(\cdot) \in C([0,\infty); \mathbf{H}_0) \text{ a.s.}; \tag{4.30}$$

(IV) *for any* $T > 0$,

$$E \sup_{0\leqq t\leqq T} \|X(t)\|_0^{2n} \leqq 2\exp(c_{2n,a,\Gamma,K} T)(E\|X_0\|^{4n})^{\frac{1}{2}}\,, \tag{4.31}$$

with $c_{2n,a,\Gamma,K}$ *given by* (4.18).

Proof. Set $\mathscr{Y}(\cdot) \equiv X(\cdot)$ and apply Theorem 4.13 and (4.23). □

Remark 4.15 If $n = 1$ in (4.29), it follows that our quasilinear SPDE (1.25)/(1.26) cannot be treated by the usual variational methods on $\mathbf{H}_0$ (cf. [23] and the generalization of Pardoux's variational approach by Krylov and Rozovskii [18]).

5 The Mezoscopic equation–strong uniqueness

We obtain strong (Itô) uniqueness for the (mezoscopic) SPDE if the initial condition X_0 *is in* $L_2(\mathbf{R}^d, dr)$.

Theorem 5.1 *Suppose* $X_0 \in \mathbf{H}_0$ *and* $E\|X_0\|_0^4 < \infty$. *Let* $X(\cdot, X_0)$ *be the weak solution of* (1.25)/(1.26) *from Theorems* 3.4 *and* 3.5 *starting at* X_0. *Let* $Y(\cdot, X_0)$ *be an arbitrary solution of* (1.25)/(1.26) *with* $Y(0) = X_0$ *such that* $Y(\cdot, X_0) \in C([0,\infty); \mathbf{H}_0)$ *a.s. Then a.s.*

$$X(\cdot, X_0) \equiv Y(\cdot, X_0)\,. \tag{5.1}$$

Proof. (i) Let $Z(\cdot, Y, X_0)$ be the solution of (3.9) from Corollary 3.2, where $Y(t) := Y(t, X_0)$. Set

$$\tilde{Z}(\cdot, Y, 0) := Z(\cdot, Y, X_0) - Y(\cdot, X_0).$$

Obviously, $\tilde{Z}$ solves (3.9) with initial condition $\tilde{Z}(0) = 0$. By Lemma 4.6 and Remark 4.7 $\tilde{Z}(\cdot, Y, 0) \equiv 0$ a.s., whence

$$Z(\cdot, Y, X_0) \equiv Y(\cdot, X_0).$$

(ii) Applying now (3.5) we obtain

$$E \sup_{0 \leqq t \leqq T} \gamma_2^2(X(t, X_0), Y(t, X_0)) = E \sup_{0 \leqq t \leqq T} \gamma_2^2(X(t, X_0), Z(t, Y(t), X_0))$$

$$\leqq \text{ const } \int_0^T E\gamma_2^2(X(s, X_0), Y(s, X_0))\, ds.$$

This implies (5.1) by Gronwall's lemma. □

6 The Macroscopic equation

We now assume for simplicity

(i) $q_n = 0, n \geqq 1$;
(ii) Γ is diagonal with $\Gamma_{kk} = \tilde{\Gamma}_\varepsilon, k = 1, \ldots d$, where $\tilde{\Gamma}_\varepsilon$ is the kernel from Example 1.1; (6.1)
(iii) $d \geqq 2$.

Consider the following (macroscopic) PDE on $\mathbf{H}_0$

$$\frac{\partial}{\partial t} X(t) = \frac{1}{2} \Delta X(t) - \nabla(X(t) F(X(t))); \; X_0 \in \mathbf{H}_0 \cap \mathbf{M}, |||X_0||| < \infty. \tag{6.2}$$

Suppose that (6.2) has a unique weak solution $\in C([0, \infty); \mathbf{H}_0)$ such that $|||X(t)||| < \infty$ and $X(t) \in \mathbf{H}_0 \cap \mathbf{M}$ for all $t \geqq 0$. Set

$$\Lambda_N := \{ r_N \in \mathbf{R}^{dN} : \exists (i,j) : 1 \leqq i < j \leqq N \text{ such that } r^i = r^j \}$$

Let $\mathscr{X}_N(0) = \sum_{i=1}^N a_i \delta_{r_0^i} \in \mathbf{M}$ such that $P\{r_N(0) \notin \Lambda_N\} = 1$ for all $N \in \mathbf{N}$, where $r_N(0) = (r_0^1, \ldots, r_0^N)$. Let $\mathscr{X}_\varepsilon(t, \mathscr{X}_N(0))$ be the empirical process for (1.12) (which is a solution of (1.25)/(1.26). Let $\varphi \in C_b(\mathbf{R}^d, \mathbf{R})$. By analogy with Theorem 4.4 of Kotelenez [16] we expect the following:
If $E\langle \mathscr{X}_N(0), \varphi \rangle \to \langle X_0, \varphi \rangle$ then there is a sequence $\varepsilon(N) \to 0$, as $N \to \infty$ such that for any $t > 0$ $N \to \infty$ implies

$$E\langle \mathscr{X}_{\varepsilon(N)}(t, \mathscr{X}_N(0)), \varphi \rangle \to \langle X(t, X_0), \varphi \rangle_0. \tag{6.3}$$

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