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Economic Theory

More on money as a medium of exchange

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Summary. We extend the analysis of Kiyotaki and Wright, who study economies where the commodities that serve as media of exchange (or, commodity money) are determined endogenously. Kiyotaki and Wright consider only steady-state, pure-strategy equilibria; here we allow dynamic and mixed-strategy equilibria. We demonstrate that symmetric, steady-state equilibria in mixed-strategies always exist, while sometimes no such equilibria exist in pure-strategies. We prove that the number of symmetric steady-state equilibria is generically finite. We also show, however, that for some parameter values there exists a continuum of dynamic equilibria. Further, some equilibria display cycles.

1. Introduction

Kiyotaki and Wright (1989) study economies where the objects that serve as media of exchange, or money, are determined endogenously. It is shown that different objects can play the role of money, depending on parameters that describe their intrinsic properties, and that there can sometimes exist multiple equilibria with different monies for given parameter values. The analysis in that paper is incomplete, however, in the sense only symmetric, steady-state, pure-strategy equilibria are considered. We extend the model here to allow mixed-strategy equilibria (which can also be interpreted as pure-strategy but nonsymmetric equilibria) and to allow dynamic (that is, not necessarily steady-state) equilibria.

This allows us to do several things. First, we show by construction that symmetric, steady-state equilibria always exist in mixed-strategies, while for some parameter values it was discovered in Kiyotaki and Wright (1989) that no such

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equilibria exist in pure-strategies. Second, we show that new mixed-strategy steady-state equilibria can arise when there is a unique pure-strategy equilibrium, and these equilibria imply different monies. However, we prove that generically in parameter space the number of steady-state equilibria is finite; this means that the model does not allow just anything to happen in terms of determining the equilibrium money, and is also important for the usual reasons concerning comparative statics and related issues (see, for example, the discussion in Kehoe 1985).

When we extend the analysis beyond steady-states, the set of equilibria becomes considerably richer. We demonstrate that in certain regions of parameter space there can exist a robust continuum of equilibrium paths. That is, for certain nonempty open sets of parameters, there is a stationary steady-state equilibrium such that for any initial values of the predetermined variables in some nonempty open set, there exists a continuum of initial strategies such that the economy converges to the steady-state along a path from these initial conditions. Hence, given the predetermined variables, there is a continuum of initial conditions for trading strategies all of which are consistent with equilibrium. We also show that the model can display equilibria that are stable limit cycles. In these equilibria, the trading strategies and, therefore, the commodity monies vary cyclically even though the fundamentals of the model are stationary.

It is worth remarking that several of our results are reminiscent of those derived for overlapping generations economies, and that we exploit the same types of tools that are used in the study of those economies. For example, using similar techniques, Kehoe and Levine (1984) demonstrate the generic finiteness of the number of steady-state equilibria in overlapping generations economies; Kehoe and Levine (1985) analyze the possibility of a continuum of dynamic equilibria converging to a steady-state; and Benhabib and Day (1982), Grandmont (1985), and Azariadis and Guesnerie (1986) analyze the possibility of cycles and even more complex dynamics. It is also worth pointing out that our results are not special cases of general theorems in the repeated game literature. In particular, the model we analyze is an anonymous sequential game (in the sense of Jovanovic and Rosenthal 1988), which means that any results hinging on "reputation" are irrelevant here.

The remainder of the paper is organized as follows. In Section 2 we review the basic model and describe the results in Kiyotaki and Wright (1989) on the existence of symmetric, steady-state, pure-strategy, commodity money equilibria. (That paper also considered economies with fiat money, but we restrict attention to the commodity money case.) In Section 3 we extend the model to include mixed-strategies and dynamics. In Section 4 we construct a steady-state equilibrium in mixed-strategies in the region of parameter space for which no equilibria exist in pure-strategies. In Section 5 we prove that the number of steady-state equilibria is generically finite. In Section 6 we demonstrate that a robust continuum of dynamic equilibria can arise. In Section 7 we construct cyclic equilibria. In Section 8 we conclude.

2. The basic model

Time is discrete and continues forever. There are three indivisible goods, labeled $i = 1, 2, 3$. There is a continuum of agents of unit mass, with equal proportions of three types: type i consumes only good i and produces only good $i+1$ modulo 3 (type 1 produces good 2, type 2 produces good 3, and type 3 produces good 1). For each type i, u is the utility of consuming good i, and c_{ij} is the disutility of storing good j for one period. The cost of production is normalized to zero, and $\beta \in (0, 1)$ is the discount factor. Assume that only one good at a time can be stored, and assume for now that storage costs are ordered $c_{i3} > c_{i2} > c_{i1}$ for all i. This is the consumption-production-storage specification called Model A in Kiyotaki and Wright (1989). That paper also describes a version called Model B, which reorders production so that i produces $i-1$ rather than $i+1$, or, equivalently, reorders storage costs. It will be more convenient here to have i always produce $i+1$, and to differentiate alternative versions of the model by reordering storage costs.

Agents meet randomly in pairs at each date and trade bilaterally if and only if it is mutually agreeable. When type i acquires good i, he immediately consumes it, produces a new unit of good $i+1$, and stores it until the next date. Hence, in equilibrium type i always enters a trading period with an inventory of either good $i+1$ or good $i+2$, and never good i. This means that $p(t) = [p_1(t), p_2(t), p_3(t)]$, where $p_i(t)$ is the proportion of type i agents holding good $i+1$ at date t, completely describes the distribution of inventories at a point in time. A steady-state distribution satisfies $p(t) = p$ for all t. As in Kiyotaki and Wright (1989), we restrict attention in this section to steady-states.

Agents choose strategies for deciding when to trade, given the strategies of others and p. For now, as in Kiyotaki and Wright (1989), we consider only time-invariant pure-strategies. Thus, a strategy for i is a function $\tau_i: \{1, 2, 3\}^2 \to \{0, 1\}$, where $\tau_i(j, k) = 1$ if i wants to trade good j for good k and $\tau_i(j, k) = 0$ otherwise. We assume u is large enough that agents always trade for and consume their own consumption goods, so that $\tau_i(j, i) = 1$ for all j. (A simple sufficient condition for this is $(1-\beta)u > c_{ij} - c_{ik}$ for all i, j, k; much weaker conditions would suffice, but this one is easy to check, as shown in Kiyotaki and Wright 1989.) We also assume that $\tau_i(j, k) = 0$ if and only if $\tau_i(k, j) = 1$, which means that if an agent trades j for k then he will not trade k for j. Therefore, each type i's strategy is completely specified once we decide whether $\sigma_i = \tau_i(i+1, i+2)$ is 1 or 0. If $\sigma_i = 1$, then i is willing to trade his production good $i+1$ for the intermediate good $i+2$, which he later uses to buy his consumption good; if $\sigma_i = 0$, then i keeps his production good until he can trade for his consumption good directly. In other words, choosing σ_i amounts to choosing whether type i uses good $i+2$ as a medium of exchange.

A symmetric, steady-state, pure-strategy equilibrium is defined to be a vector of inventories $p = (p_1, p_2, p_3)$ and a vector of strategies $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ such that: (1) when agents use strategies σ, the resulting steady-state inventory distribution is p; and (2) for all i, σ_i maximizes expected discounted utility from consumption net of storage costs, given σ and p. The following result from Kiyotaki and Wright (1989) describes the set of such equilibria when $c_{i3} > c_{i2} > c_{i1}$. In order to reduce notation in our statement of this result, we normalize utility (with no loss in generality) so that $\beta u/3 = 1$. Given this, it turns out that everything depends on the single parameter $\delta_1 = c_{13} - c_{12}$, which is the difference in storage costs type 1 would incur if he acquired good 3 as a medium of exchange.

Proposition 1. Suppose $c_{i3} > c_{i2} > c_{i1}$. Then if $\delta_1 \geq 1/2$, $\sigma = (0, 1, 0)$ is the unique symmetric, steady-state, pure-strategy equilibrium; if $\delta_1 \leq \sqrt{2} - 1$, $\sigma = (1, 1, 0)$ is the unique such equilibrium; if $\sqrt{2} - 1 < \delta_1 < 1/2$, there exists no such equilibrium.

The intuition behind this result is straightforward. Assume that $\sigma_2 = 1$ and $\sigma_3 = 0$, and consider the best response problem of a type 1 agent. (It is easy to show that $\sigma_2 = 1$ and $\sigma_3 = 0$ are best responses for type 2 and type 3, for all parameter values, when either $\sigma_1 = 0$ or 1 – see below.) The instantaneous cost to type 1 of trading good 2 for good 3 is δ_1, the increase in one-period storage disutility. The instantaneous benefit is the increase in the probability of meeting someone with good 1 next period who is willing to trade, $[p_3 - (1 - p_2)]/3$, times the discounted utility of consumption, βu. Now $\sigma_1 = 1$ if and only if the cost is less than the benefit, which reduces to the condition $\delta_1 \leq p_3 - (1 - p_2)$. But p_2 and p_3 depend on strategies. Simple calculations reveal that $\sigma = (0, 1, 0)$ implies $p_3 - (1 - p_2) = 1/2$, and so $\sigma_1 = 0$ is the best response as long as $\delta_1 \geq 1/2$ (if good 3 is much more costly to store than good 2, type 1 opts for direct barter rather than using a medium of exchange). Also, $\sigma = (1, 1, 0)$ implies $p_3 - (1 - p_2) = \sqrt{2} - 1$, and so $\sigma_1 = 1$ is the best response as long as $\delta_1 \leq \sqrt{2} - 1$ (if good 3 is not too much more costly to store, type 1 opts to use it as a medium of exchange).

If $\sqrt{2} - 1 < \delta_1 < 1/2$, then no symmetric, steady-state, pure-strategy equilibrium exists. When all type 1 agents refuse to accept good 3, type 2 agents end up holding more of good 3 and less of good 1, which means type 1 agents ought to accept good 3 to facilitate trade with type 3, given δ_1. On the other hand, when all type 1 agents accept good 3, type 2 agents end up holding less of good 3 and more of good 1, which means type 1 agents do not need to trade with type 3 and ought to refuse to accept good 3. Apparently, to get an equilibrium we require that some but not all type 1 agents accept good 3, or, equivalently, that type 1 agents accept good 3 with probability strictly between 0 and 1. We analyze this situation in Section 4. Another possibility would be to have type 1 agents accept good 1 at some dates and not others, a situation we consider in Section 7.

3. The generalized model

Let $s_i(t)$ be the probability that type i plays strategy $\sigma_i = 1$ at date t, and let $s(t) = [s_1(t), s_2(t), s_3(t)]$. If the probability of agent i trading good $i+1$ for good $i+2$ at date t is $s_i(t)$, then the probability of him trading good $i+2$ for good $i+1$ is $1 - s_i(t)$. This implies that whether an agent prefers good $i+1$ or good $i+2$ at date t does not depend on the good with which he enters the period. Given $s(t)$, the "trading matrices" in Figure 1 depict the probability of exchange in any particular meeting – excluding cases where two agents of the same type meet, since we can assume with no loss in generality that individuals never trade with their own type.

Given any path for the strategy vector $s(t)$, the transition equation for the inventory distribution is given by $p(t+1) = \gamma[s(t), p(t)]$, where $\gamma: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. By looking at the trading matrices one can compute the explicit functional form of γ, which we write as

$$\gamma(s, p) = p - \frac{1}{3} G(s, p), \tag{3.1}$$

		Type 2: 3	Type 2: 1
Type 1	2	s_1	1
	3	0	$1-s_2$

Type 1 meets type 2

		Type 3: 1	Type 3: 2
Type 2	3	s_2	1
	1	0	$1-s_3$

Type 2 meets type 3

		Type 1: 2	Type 1: 3
Type 3	1	s_3	1
	2	0	$1-s_1$

Type 3 meets type 1

Figure 1. Training matrices.

where $G(s,p) = [G_1(s,p), G_2(s,p), G_3(s,p)]$ and

$$G_i(s,p) = p_i p_{i+1} s_i - (1-p_i)[(1-p_{i+2})(1-s_i) + p_{i+2} + (1-p_{i+1})(1-s_{i+1})]. \tag{3.2}$$

The first term on the right-hand side of (3.2) represents the measure of type i agents who switch from good $i+1$ to good $i+2$ through exchange, while the second term represents the measure of type i agents who switch from good $i+2$ to good $i+1$ through exchange, consumption and production.

We now describe the individual decision problem. Let $V_{ij}(t)$ be the expected discounted utility at the end of period t for type i given an inventory of good j (the payoff, or value, function). If we define $\Delta_i(t) = V_{i,i+1}(t) - V_{i,i+2}(t)$, the maximizing choice of $s_i(t)$ satisfies:

$$s_i(t) \in \begin{cases} \{0\} & \text{if } \Delta_i(t) > 0 \\ [0,1] & \text{if } \Delta_i(t) = 0 \\ \{1\} & \text{if } \Delta_i(t) < 0 \end{cases} \tag{3.3}$$

For example, for type 1, if $\Delta_1(t) = V_{12}(t) - V_{13}(t) > 0$ then he should set $s_1(t) = 0$ and not trade good 2 for good 3; if $\Delta_1(t) < 0$ then he should set $s_1(t) = 1$ and trade good 2 for good 3 whenever he can; and if $\Delta_1(t) = 0$ he is indifferent and may choose the probability $s_1(t)$ to be anything between 0 and 1.

To illustrate the technique we explicitly derive $\Delta_1(t)$. Consider a type 1 agent with good 2 at the end of period t. He first pays his storage cost c_{12} and, next period, he meets an agent of type 1, 2, or 3, each with probability 1/3. If he meets another type 1, he does not trade and leaves with $V_{12}(t+1)$. Now suppose he meets a type 2, who will always want to trade given our type 1 agent holds good 2. With probability $p_2(t+1)$, the type 2 agent has good 3 and our agent chooses the probability of trade $s_1(t+1)$, while with probability $1 - p_2(t+1)$ the type 2 agent has good 1, there is a double coincidence of wants, and our agent definitely trades, consumes and produces a new unit of good 2. Now suppose he meets a type 3. With probability $1 - p_3(t+1)$ both agents have good 2 and they cannot trade, while with probability $p_3(t+1)$ the type 3 agent has good 1 and our agent wants to trade, so type 3 chooses the probability $s_3(t+1)$.

If we write this out explicitly, the payoff for type 1 with good 2 at t is given by

$$\begin{aligned} V_{12}(t) = &-c_{12} + \frac{\beta}{3}[V_{12}(t+1) + p_2(t+1)\{s_1(t+1)V_{13}(t+1) \\ &+ [1 - s_1(t+1)]V_{12}(t+1)\} + [1 - p_2(t+1)][u + V_{12}(t+1)] \\ &+ [1 - p_3(t+1)]V_{12}(t+1) + p_3(t+1)\{s_3(t+1)[u + V_{12}(t+1)] \\ &+ [1 - s_3(t+1)]V_{12}(t+1)\}]. \end{aligned}$$

Simplification yields

$$\begin{aligned} V_{12}(t) = &-c_{12} + 1 - p_2(t+1) + p_3(t+1)s_3(t+1) + \beta V_{12}(t+1) \\ &- \frac{\beta}{3} p_2(t+1)s_1(t+1)\Delta_1(t+1), \end{aligned}$$

using our normalization $\beta u/3 = 1$. A similar analysis for the case where type 1 has good 3 at t yields

$$\begin{aligned} V_{13}(t) = &-c_{13} + [1 - p_2(t+1)][1 - s_2(t+1)] + p_3(t+1) + \beta V_{13}(t+1) \\ &+ \frac{\beta}{3}\{[1 - p_2(t+1)][1 - s_2(t+1)] + 1 - [1 - p_3(t+1)]s_1(t+1)\}\Delta_1(t+1). \end{aligned}$$

Substracting these two equations, we arrive at

$$\Delta_1(t) = \delta_1 + [1 - p_2(t+1)]s_2(t+1) - p_3(t+1)[1 - s_3(t+1)]$$
$$+ \beta[1 - \tfrac{1}{3}\{p_2(t+1)s_1(t+1) + [1 - p_2(t+1)][1 - s_2(t+1)]$$
$$+ p_3(t+1) + [1 - p_3(t+1)][1 - s_1(t+1)]\}]\Delta_1(t+1).$$

By symmetry, for any type i, we can write

$$\Delta_i(t) = F_i[s(t+1), p(t+1), \delta], + \beta Q_i[s(t+1), p(t+1)]\Delta_i(t+1), \tag{3.4}$$

where $\delta_i = c_{i,i+2} - c_{i,i+1}$, $\delta = (\delta_1, \delta_2, \delta_3)$, and we define

$$F_i(s, p, \delta) = \delta_i + (1 - p_{i+1})s_{i+1} - p_{i+2}(1 - s_{i+2}) \tag{3.5}$$

$$Q_i(s, p) = 1 - \tfrac{1}{3}[p_{i+1}s_i + (1 - p_{i+1})(1 - s_{i+1}) + p_{i+2} + (1 - p_{i+2})(1 - s_i)]. \tag{3.6}$$

It will be convenient below to write $\Delta(t) = [\Delta_1(t), \Delta_2(t), \Delta_3(t)]$ and $F(s,p) = [F_1(s,p), F_2(s,p), F_3(s,p)]$. Given any path for $[s(t), p(t)]$, a path for $\Delta_i(t)$ satisfying (3.4) implies the maximizing choice of $s_i(t)$ at every date via the best response condition (3.3). Notice that $[s(t), p(t)]$ does not pin down the sequence $\Delta_i(t)$, however, without some condition on $\Delta_i(0)$, and without $\Delta_i(0)$ there is nothing to pin down the initial choice of $s_i(0)$ in the model; this will be important in Section 6.

In any case, we now have the following definition. A (symmetric) equilibrium, given an initial distribution of inventories $p(0)$, is a path $[s(t), p(t), \Delta(t)]$ such that: (1) given strategies $s(t)$, $p(t)$ satisfies the transition equation (3.1) for all t; and (2) given $[s(t), p(t)]$, $\Delta_i(t)$ and $s_i(t)$ satisfy the best response conditions (3.3) and (3.4) for all t. A steady-state equilibrium can be defined by $[s(t), p(t), \Delta(t)] = (s, p, \Delta)$ for all t, such that: (1) given s, p is a fixed point of the transition equation; and (2) given (s, p), Δ_i and s_i satisfy the best response conditions. Notice that when $\Delta(t) = \Delta$ for all t, (3.4) implies

$$[1 - \beta Q_i(s, p)]\Delta_i = F_i(s, p, \delta). \tag{3.7}$$

Since $\beta Q_i(s, p) < 1$, all that matters for the best response condition in steady-state is the sign of $F_i(s, p, \delta)$.

4. Steady-state equilibria

In this section and the next we restrict attention to steady-state equilibria. Our immediate goal is to fill in the gap in Proposition 1 by constructing a mixed-strategy, symmetric equilibrium (or, equivalently, a pure-strategy but nonsymmetric equilibrium) in the region of parameter space for which no pure-strategy, symmetric equilibrium exists.

We will construct a steady-state equilibrium in which $s_2 = 1$, $s_3 = 0$, and $s_1 \in [0, 1]$ will be determined as a function of the parameters. Because Proposition 1 refers to the case where $\delta_1 > 0$, $\delta_2 < 0$, and $\delta_3 > 0$, in this equilibrium type 2 always trade their production good 3 for a lower storage cost good 1, type 3 never trade their production good 1 for a higher storage cost good 2, and type 1 may or may not trade their production good 2 for a higher storage cost good 3. That this may be an equilibrium is suggested by the observations at the end of Section 2; it is also suggested by Theorem 7 in Aiyagari and Wallace (1991) (see also Theorem 7 in Gintis 1989), which says that there always exists a mixed-strategy, steady-state equilibrium in fairly general versions of this model in which every agent always accepts the lowest storage cost good.

If we substitute $s_2 = 1$ and $s_3 = 0$ into (3.5) then

$$F_1(s, p, \delta) = \delta_1 + (1 - p_2) - p_3$$
$$F_2(s, p, \delta) = \delta_2 - p_1(1 - s_1)$$
$$F_3(s, p, \delta) = \delta_3 + (1 - p_1)s_1.$$

For any $\delta_2 < 0$ we have $F_2 < 0$, which implies $s_2 = 1$ is a best response for type 2; similarly, for any $\delta_3 > 0$ we have $F_3 > 0$, which implies $s_3 = 0$ is a best response for type 3. For type 1, the sign of F_1 depends on p. Solving for the steady-state inventory distribution $p = \gamma(s, p)$ as a function of s_1, we find

$$p = [\sqrt{1 + s_1}/(1 + s_1), (\sqrt{1 + s_1} - 1)/s_1, 1].$$

(This holds for $s_1 > 0$; for $s_1 = 0$, take the limit.) Hence,

$$F_1 = \delta_1 - (\sqrt{1 + s_1} - 1)/s_1.$$

A strategy s_1 is a best response as long as it satisfies condition (3.3). Combinations of δ_1 and s_1 consistent with (3.3) are computed to be

$$\delta_1 \leq \sqrt{2} - 1 \quad \text{and} \quad s_1 = 1$$
$$\sqrt{2} - 1 \leq \delta_1 \leq 1/2 \quad \text{and} \quad s_1 = (1 - 2\delta_1)/\delta_1^2$$
$$1/2 \leq \delta_1 \quad \text{and} \quad s_1 = 0,$$

as shown in Figure 2. Hence, an equilibrium exists for all $\delta_1 > 0$, filling in the gap in Proposition 1. Notice that the two pure-strategy equilibria in Proposition 1 reappear for appropriate values of δ_1, and are connected by mixed-strategy equilibria. We also point out that it is equivalent to reinterpret our symmetric mixed-strategy equilibria as nonsymmetric pure-strategy equilibria, where the fraction s_1 of type 1 agents play strategy $\sigma_1 = 1$ with probability 1 while the fraction $1 - s_1$ play $\sigma_1 = 0$ with probability 1.

Introducing mixed-strategies leads to other new possibilities. For example, suppose $s_1 = 1$, $s_2 = 0$, and $s_3 \in (0, 1)$. It is not difficult to check $F_1 < 0$, $F_2 > 0$, and $F_3 = 0$, so that these are indeed best responses, if and only if the following conditions are satisfied:

$$\sqrt{2} - 1 < \delta_3 < 1/2 \quad \text{and} \quad s_3 = (1 - 2\delta_3)/\delta_3^2$$
$$\delta_1 \leq (\delta_3^2 + 2\delta_3 - 1)/(\delta_3 - \delta_3^2)$$
$$\delta_2 \geq -(1 - 2\delta_3)^2/(\delta_3^2 - \delta_3^3).$$

Hence, in a certain region of parameter space, this is an equilibrium. But, from Proposition 1, there already exists an equilibrium with $s_1 = 1$, $s_2 = 1$, and $s_3 = 0$, as

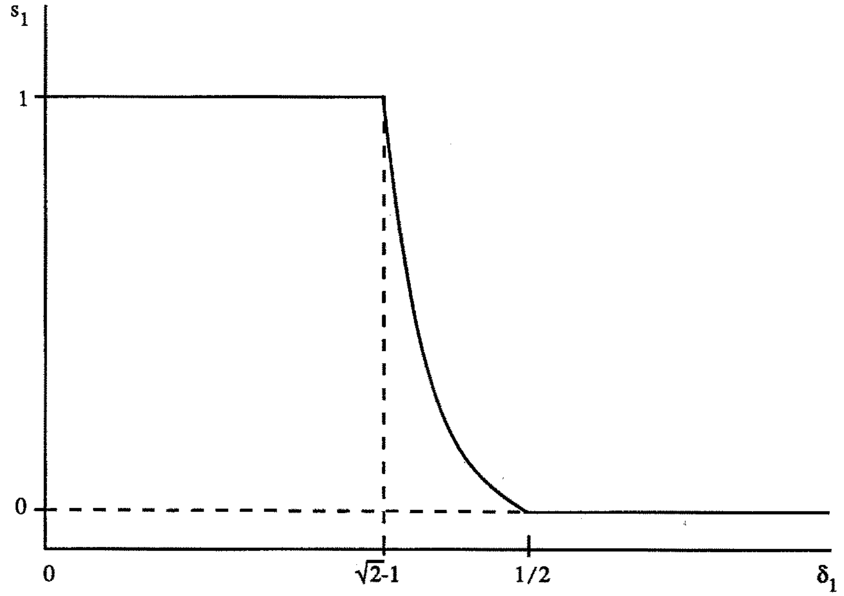

Figure 2. Equilibrium value of s_1 as a function of δ_1.

long as $\delta_1 \leq \sqrt{2} - 1$. For a nonempty open set of parameter values these equilibria exist simultaneously. This raises the question of just how many steady-state equilibria there might be. In the next section, we prove the number is finite.

5. Generic finiteness of steady-state equilibria

To prove generic finiteness of the number of steady-state equilibria we utilize the transversality theorem of differential topology (see Guillemin and Pollack 1974, pp. 68–69, or Hirsch 1976, pp. 74–77). Similar results have been obtained for finite n-person normal form games by Harsanyi (1973) and van Damme (1983). Because of the interaction of p and s in the payoff functions, however, their results do not apply directly to this model.

We use the following notation: If $F(x, \alpha)$ is a function of a vector of variables x and a vector of parameters α, then we write $f_\alpha(x) = F(x, \alpha)$ for fixed α.

Transversality Theorem. Let $F: X \times A \to Y$, where $X \subset \mathbb{R}^l$ is contained in the closure of an open set and $A \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are open sets. Suppose that F is continuously differentiable of order r, where $r > \max(l - n, 0)$. Suppose too that, if $(x, \alpha) \in X \times A$ satisfies $F(x, \alpha) = 0$, then $DF(x, \alpha)$ has rank n. Then $f_\alpha(x) = 0$ implies $Df_\alpha(x)$ has rank n for all α in a subset of A of full Lebesgue measure.

Notice that, if $l < n$, then the $n \times l$ matrix $Df_\alpha(x)$ cannot possibly have rank n. The conclusion of the theorem, in this case, is that for all α in a subset of A of full Lebesgue measure, there is no $x \in X$ such that $f_\alpha(x) = 0$.

The intuition behind this theorem is one of counting equations and unknowns. If $l < n$, then there are more equations than unknowns; we therefore would not expect there to be any solutions if we have sufficient freedom to perturb the equations. If $l = n$, then there are the same numbers of equations and unknowns; we therefore would expect any solutions to the equations to be locally unique if we have sufficient freedom to perturb the equations. Indeed, if the $n \times n$ matrix $Df_\alpha(\bar{x})$ has rank n, then the inverse function theorem tells us that the solution $\bar{x}$ to the equation $f_\alpha(\bar{x}) = 0$ is locally unique. If $l > n$, then there are more unknowns than equations; we would therefore expect there to be an infinite number of solutions. Although we could use this theorem and the implicit function theorem to count degrees of freedom and parameterize the set of solutions, we are concerned here only with situations where $l = n$. In this case, the formal criterion for sufficient freedom to perturb the equations is that the $n \times (l + m)$ matrix $DF(x, \alpha) = [D_1F(x, \alpha), D_2F(x, \alpha)]$ has rank n whenever $F(x, \alpha) = 0$.

Applying this theorem to our commodity money economy, we are forced to consider different cases: $0 < s_i < 1$ for all i; $s_i = 0$ for some i; and $s_i = 1$ for some i. In each case, equilibria are solutions to a different set of equations. In the case where $0 < s_i < 1$ for all i, equilibria are solutions to $E(s, p, \delta) = 0$ where $E: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is given by the rule $E(s, p, \delta) = [F(s, p, \delta), G(s, p)]$ and F and G are given as in Section 3. In cases where $s_i = 0$ or $s_i = 1$ for some i, we replace equation i of $F(s, p, \delta) = 0$ with $s_i = 0$ or $s_i - 1 = 0$. Therefore, here we solve $E(s, p, \delta) = 0$, where $E(s, p, \delta) = [\bar{F}(s, p, \delta), G(s, p)]$ and $\bar{F}$ is formed appropriately. In all of these cases, $(s, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ is the vector of endogenous variables and $\delta \in \mathbb{R}^3$ is the vector of parameters. Let

$$D = \{\delta \in \mathbb{R}^3 \mid \delta_1 > 0, \delta_2 < 0, \delta_3 \neq 0\}$$

be the set of parameters. Since the important distinction between different versions of the model is whether two of the δ_i are negative and one positive or vice-versa (this is the distinction between Model A and Model B in Kiyotaki and Wright 1989), we accommodate all relevant cases with this specification.

Proposition 2. For all δ in a set of full Lebesgue measure in D, there is a finite number of steady-state equilibria.

Proof. The strategy of proof is simple: We first apply the transversality theorem to each of the cases discussed above, where (s, p) is allowed to range over a set contained in the closure of an open set. We then use the inverse function theorem to show that solutions to each system of equations are locally unique for all δ in a set of full Lebesgue measure in D. Finally, we restrict (s, p) to a compact set and argue that the number of equilibria is, in fact, finite.

All equilibria lie in the set

$$S = \{(s, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid 0 \leq s_i \leq 1, 0 \leq p_i \leq 1\},$$

which is obviously compact. Observe that there can be no equilibrium in which $p_i = 0$ for some i, since the steady-state condition $p = \gamma(s, p)$ implies that in this case $p_1 = p_2 = p_3 = 0$ and $s_1 = s_2 = s_3 = 1$, which cannot be best responses for any $\delta \in D$. Indeed, no matter what p is, $s_1 = s_2 = s_3 = 1$ cannot be best responses. In applying the transversality theorem, we therefore restrict our attention to the set

$$S' = \{(s, p) \in S \mid 0 < p_i \text{ for all } i, s_i < 1 \text{ for some } i\},$$

which is contained in the closure of the open set defined by letting all the inequalities be strict.

First, consider the case where $E(s,p,\delta)=[F(s,p,\delta),G(s,p)]$, which corresponds to equilibria where $0<s_i<1$ for all i. In this case,

$$DE(s,p,\delta)=\begin{bmatrix} D_1F(s,p,\delta) & D_2F(s,p,\delta) & D_3F(s,p,\delta) \\ D_1G(s,p) & D_2G(s,p) & 0 \end{bmatrix}.$$

Notice $D_3F(s,p,\delta)$ is the 3×3 identity matrix. If we could show that the 3×3 matrix $D_1G(s,p)$ has rank three, we would then know that $DE(s,p,\delta)$ has rank six. The transversality theorem would imply that, for all δ in a set of full Lebesgue measure, the 6×6 matrix

$$De_\delta(s,p)=\begin{bmatrix} D_1f_\delta(s,p) & D_2f_\delta(s,p) \\ D_1G(s,p) & D_2G(s,p) \end{bmatrix}$$

has rank 6 whenever $e_\delta(s,p)=0$. The inverse function theorem would then imply that any such solution has an open neighborhood in $\mathbb{R}^3\times\mathbb{R}^3$ in which it is the only solution. To see that $D_1G(s,p)$ has rank three, we compute it:

$$D_1G(s,p)$$
$$=\begin{bmatrix} p_1p_2+(1-p_1)(1-p_3) & (1-p_1)(1-p_2) & 0 \\ 0 & p_2p_3+(1-p_2)(1-p_1) & (1-p_2)(1-p_3) \\ (1-p_3)(1-p_1) & 0 & p_3p_1+(1-p_3)(1-p_2) \end{bmatrix}.$$

This has rank three, as required, because it has sign pattern

$$\begin{bmatrix} + & +\text{ or }0 & 0 \\ 0 & + & +\text{ or }0 \\ +\text{ or }0 & 0 & + \end{bmatrix}.$$

The cases where we replace $F_i(s,p,\delta)=0$ with $s_i=0$ or $s_i-1=0$ are more complicated. We can still argue that $D\bar{F}(s,p,\delta)$ has rank three because it contains three linearly independent columns, those columns in $D_1\bar{F}$ for which $\bar{F}_i(s,p,\delta)$ equals s_i or s_i-1 and those columns in $D_3\bar{F}$ for which $\bar{F}_i(s,p,\delta)$ equals $F_i(s,p,\delta)$. We now need to find three linearly independent columns in $[D_1G(s,p),D_2G(s,p)]$ that do not include the columns in $D_1G(s,p)$ for which $\bar{F}_i(s,p,\delta)$ equals s_i or s_i-1.

The case where $s_i=0$ for some i is the easiest. Here, we can argue that the matrix

$$D_2G(s,p)$$
$$=\begin{bmatrix} \begin{matrix} p_2s_1+(1-p_2)(1-s_2) \\ \quad +p_3+(1-p_3)(1-s_1) \end{matrix} & p_1s_1+(1-p_1)(1-s_2) & -(1-p_1)s_1 \\ -(1-p_2)s_2 & \begin{matrix} p_3s_2+(1-p_3)(1-s_3) \\ \quad +p_1+(1-p_1)(1-s_2) \end{matrix} & p_2s_2+(1-p_2)(1-s_3) \\ p_3s_3+(1-p_3)(1-s_1) & -(1-p_3)s_3 & \begin{matrix} p_1s_3+(1-p_1)(1-s_1) \\ \quad +p_2+(1-p_2)(1-s_3) \end{matrix} \end{bmatrix}$$

has rank three because it has sign pattern

$$\begin{bmatrix} + & + \text{ or } 0 & - \text{ or } 0 \\ - \text{ or } 0 & + & + \text{ or } 0 \\ + \text{ or } 0 & - \text{ or } 0 & + \end{bmatrix},$$

and at least one of the potentially negative elements is zero.

In cases where $s_i - 1 = 0$ for some i, we need to consider combinations of columns from D_1G and D_2G. When $s_i - 1 = 0$, for example, we choose the second and third column from D_1G and the first column from D_2G. These columns form a 3×3 matrix with sign pattern

$$\begin{bmatrix} + \text{ or } 0 & 0 & + \\ 0 & + \text{ or } 0 & - \text{ or } 0 \\ 0 & + & + \text{ or } 0 \end{bmatrix},$$

which has rank three. The cases where $s_2 - 1 = 0$ or $s_3 - 1 = 0$ are similar. In the cases where $s_i - 1 = 0$ for two i, we combine one column form D_1G with two from D_2G. We have already ruled out the case where $s_i - 1 = 0$ for all i.

Now consider the set of pairs $(s, p) \in S'$ that satisfy any of the various combinations of equations $E(s, p, \delta) = 0$. This set includes the set of equilibria, but may be larger since there is no guarantee that the appropriate inequality in the best response condition (3.3) is satisfied if $s_i = 0$ or $s_i - 1 = 0$. The set of δ such that all of the solutions to these equations are locally unique has full Lebesgue measure since it is the intersection of a finite number (the number of possible cases) of sets of full Lebesgue measure. Consequently, for almost all δ the set of equilibria consists of locally unique points.

Suppose now that we allow (s, p) to range over all of S. Could there be an infinite number of equilibria? If there were, then there would be a convergent subsequence of equilibria since S is compact. There would then be two possibilities: this convergent subsequence could converge to $(s, p) \in S'$ or it could converge to (s, p) for which $s_1 = s_2 = s_3 = 1$. If $(s, p) \in S'$, then it too is an equilibrium, but it would not be locally unique, which is a contradiction. If $s_1 = s_2 = s_3 = 1$, any sequence converging to (s, p) would eventually violate the best response conditions. Consequently, for all δ in a subset of D with full Lebesgue measure, there is a finite number of steady-state equilibria. □

Several extensions of the result are possible. First, since S is compact, we could argue that the set of δ for which the number of equilibria is finite is also open. Second, it is easy to see that for almost all δ, the inequalities in the best response condition (3.3) must be strict if $s_i = 0$ or $s_i = 1$; otherwise, the equilibrium would be a solution to a system with more equations than unknowns. Third, using an index theorem, we could argue that for almost all δ the number of equilibria is odd (see, for example, Mas-Colell 1985).

6. Dynamic equilibria

We now turn our attention to equilibria in which the strategies $s(t)$ and inventories $p(t)$ vary over time. In this section, we show that there may be a robust continuum of dynamic equilibria, in contrast to the case of steady-states. Although many different types of dynamic equilibria are possible, for simplicity we will restrict attention for now to the case where $0 < s_i(t) < 1$ for all i and t, and look for dynamic equilibria that converge to a steady-state. The construction here was suggested by an example in Aiyagari and Wallace (1992) with fiat money.

The fact that $0 < s_i(t) < 1$ for all t requires $\Delta_i(t) = 0$ for all t, or $F_i[s(t), p(t), \delta] = 0$ for all t, by equation (3.4). Notice that the condition $F[s(t), p(t), \delta] = 0$ is actually linear in $s(t)$, and can be written

$$\begin{bmatrix} 0 & 1 - p_2(t) & p_3(t) \\ p_1(t) & 0 & 1 - p_3(t) \\ 1 - p_1(t) & p_2(t) & 0 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} p_3(t) - \delta_1 \\ p_1(t) - \delta_2 \\ p_2(t) - \delta_3 \end{bmatrix}.$$

As argued in the previous section, $p_i > 0$ for all i in any steady state equilibrium. Thus, if we start with $p_i(0) > 0$ sufficiently close to a stable steady state, then $p_i(t) > 0$ for all t and the above equation can be solved to yield $s(t) = \varphi[p(t)]$ for all t (given a fixed δ). In particular, suppose that $\Delta(t+1) = 0$ and every agent i is indifferent between goods $i+1$ and $i+2$ at date $t+1$. Then we can choose $s(t+1)$ arbitrarily, and if we choose $s(t+1) = \varphi[p(t+1)]$, subject to the condition $s_i(t+1) \in (0,1)$ for all i, this guarantees $\Delta(t) = 0$. In other words, if agents are willing to randomize at $t+1$, then as long as we choose $s(t+1)$ appropriately they will also be willing to randomize at t.

We can use this logic to construct a continuum of dynamic equilibria, given the initial inventory distribution $p(0)$ (which is fixed by nature). First, choose $s(0)$ so that $s_i(0) \in (0,1)$ for all i. Given $[s(0), p(0)]$, the transition equation implies $p(1) = \gamma[s(0), p(0)]$. Now set $s(1) = \varphi[p(1)]$, so that $F[s(1), p(1), \delta] = 0$, which means that $\Delta(0) = 0$ as long as $\Delta(1) = 0$, and our original choices of $s_i(0)$ are indeed best responses as long as $\Delta(1) = 0$. Continuing in this manner, $p(2) = \gamma[s(1), p(1)]$, and we can set $s(2) = \varphi[p(2)]$ to guarantee that $\Delta(1) = 0$ as long as $\Delta(2) = 0$. This implies the transition dynamics for $p(t)$,

$$p(t+1) = \gamma[\varphi(p(t)), p(t)] = T[p(t)].$$

Notice that $s(0)$ is not pinned down in any way here. Consequently, since $p(1) = \gamma[s(0), p(0)]$, even given $p(0)$, $p(1)$ is not pinned down.

Any path satisfying $p(t+1) = T[p(t)]$ and $s(t) = \varphi[p(t)]$ for all t is an equilibrium with all agents mixing, as long as $0 < s_i(t) < 1$ for all t. A steady-state $(\bar{s}, \bar{p})$ solves $\bar{p} = T(\bar{p})$ and $\bar{s} = \varphi(\bar{p})$. The linearization of $T(\cdot)$ around a steady-state $\bar{p}$ is

$$[p(t+1) - \bar{p}] = DT(\bar{p})[p(t) - \bar{p}],$$

where $DT = D_1\gamma D\varphi + D_2\gamma$. We are interested in constructing an example where all of the eigenvalues of the 3×3 matrix $DT(\bar{p})$ are less than one in modulus. Given such an example, we can use the implicit function theorem and the local stable manifold theorem (see Irwin 1980) to argue that, for all $[s(0), p(0)]$ in an open neighborhood of $(\bar{s}, \bar{p})$, there exists an equilibrium path $[s(t), p(t)]$ satisfying $p(t+1) = T[p(t)]$ and $s(t) = \varphi[p(t)]$.

For one such example, consider the economy where $\delta = (0.05, -0.05, 0.05)$. It has a steady-state $(\bar{s}, \bar{p})$ with

$$\bar{s} = (0.7270, 0.5538, 0.6850), \ \bar{p} = (0.6349, 0.7070, 0.6740).$$

At this steady-state, the eigenvalues of $DT(\bar{p})$ are

$$\lambda = 0.2151, 0.5021 + 0.0850i, 0.5021–0.0850i,$$

each of which is less than one in modulus. Hence, the local stable manifold theorem implies that, for all $p(1)$ in some open set containing $\bar{p}$, $p(t+1) = T[p(t)]$ converges to $\bar{p}$. It is easy to verify that the conditions of the implicit function theorem are satisfied. Thus, for all $[s(0), p(0), p(1)]$ in some open set containing $(\bar{s}, \bar{p}, \bar{p})$, the vector $s(0)$ satisfying $p(1) = \gamma[s(0), p(0)]$ varies continuously with $[p(0), p(1)]$. Furthermore, for fixed $p(0)$, the implicit function $s(0) = \psi[p(0), p(1)]$ is an invertible function of $p(1)$. Since $p(1)$ can vary over an open set and still produce a path that converges to $\bar{p}$, $s(0)$ can also vary over an open set and, together with a fixed $p(0)$, produce a dynamic equilibrium that converges to $(\bar{s}, \bar{p})$.

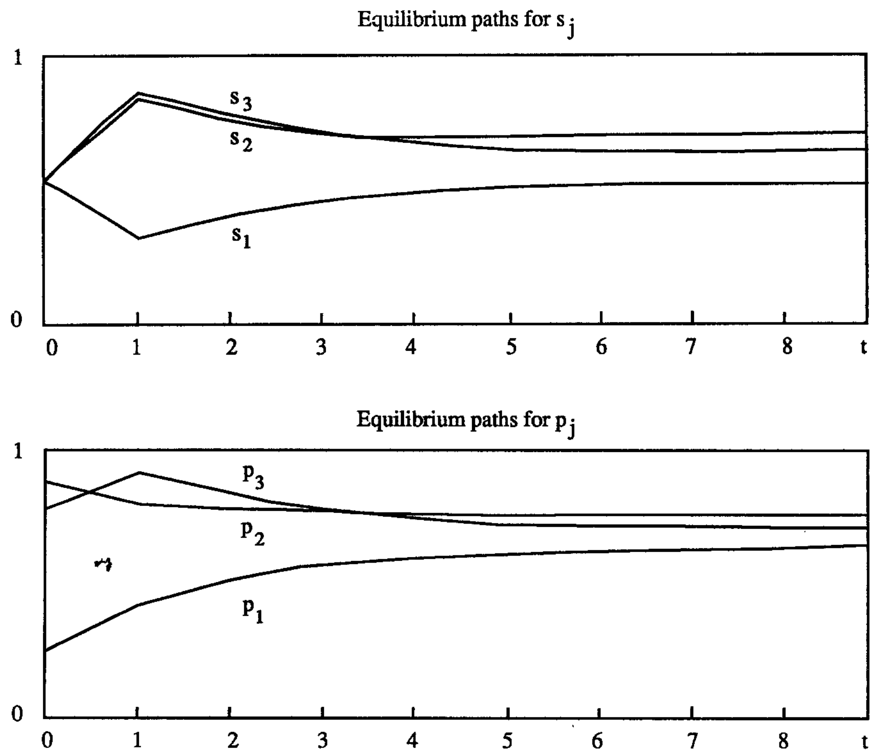

Figure 3. Equilibrium paths for strategies and inventories.

Figure 3 shows the dynamic path of the above example beginning from $p(0) = (0.2, 0.9, 0.8)$ and $s(0) = (0.5, 0.5, 0.5)$. Given $p(0)$, the system converges for a fairly wide range of $s(0)$, although for other values of $s(0)$ it does not, and eventually some or all $s_i(t)$ leave $[0, 1]$. This is true for a fairly wide range of $p(0)$. Finally, this example of a continuum of equilibria is robust. It is easy to verify that the 6×6 matrix $De_\delta(s, p)$ has full rank. Consequently, the parameters $\delta = (0.05, -0.05, 0.05)$ constitute a regular economy, and the implicit function theorem implies that the steady-state equilibrium $(\bar{s}, \bar{p})$ varies continuously with δ. Small perturbations in δ produce small perturbations in the matrix $DT(\bar{p})$, and the continuity of the eigenvalues in the elements of this matrix therefore implies that small perturbations in δ still yield economies in which all three eigenvalues are less than one in modulus. Hence, all economies with δ close to (0.05, -0.05, 0.05) will display qualitatively the same three dimensional indeterminancy.

Although we do not present the details here, it is easy to produce examples with a lower dimension of indeterminancy. Suppose, for example, DT has two eigenvalues less than 1 and the third greater than 1 in modulus. Then the local stable manifold theorem says that there is a two dimensional manifold of inventories $p(1)$ near $\bar{p}$ that lead to convergence to $\bar{p}$. The implicit function theorem implies that, for fixed $p(0)$, this corresponds to a two dimensional manifold of initial strategies $s(0)$. Similarly, we could produce examples with no indeterminacy or with one dimension of indeterminacy; everything depends on the numer of stable eigenvalues of DT. Furthermore, in this section we have only considered dynamic equilibria where $s_i(t) \in (0, 1)$ for all i and for all t. One could also consider dynamic equilibria where some types use pure-strategies while others use mixed-strategies, or where a given type fluctuates between strategies. We take a step in this direction in the next section.

7. Cyclic equilibria

Here we construct a dynamic equilibrium where $s_1(t) = 1$ if t is odd and $s_1(t) = 0$ if t is even, while $s_2(t) = 1$ for all t and $s_3(t) = 0$ for all t. In this equilibrium, $\Delta_1(t)$ will fluctuate between positive and negative, and so type 1 agents will be willing to trade good 2 for good 3 in one period but not the next. (Note that, when $\Delta_1 > 0$, they are not willing to dispose of good 3 and produce a new unit of good 2, as long as there is a sufficiently high cost to doing so.) Given these strategies, it is easy to confirm that the dynamical system $p(t + 1) = \gamma[s(t), p(t)]$ converges to a two-cycle: $p(t) = p^e$ if t is even and $p(t) = p^0$ if t is odd, where

$$p^e = (0.7741, 0.4531, 1.0), \quad p^0 = (0.8494, 0.4432, 1.0).$$

We now verify that for certain parameter values the above strategies are best responses. It is easy to check this for types 2 and 3 whenever $\delta_2 < 0$ and $\delta_3 > 0$. For type 1, we have

$$\Delta_1(t) = \delta_1 - p_2(t + 1) + \frac{\beta}{3}[2 - p_2(t + 1)s_1(t + 1)]\Delta_1(t + 1).$$

In a two-period cycle, $\Delta_1(t) = \Delta^e$ if t is even and $\Delta_1(t) = \Delta^0$ if t is odd; that is,

$$\Delta^e = \delta_1 - p_2^0 + \frac{\beta}{3}(2 - p_2^0 s_1^0)\Delta^0,$$

$$\Delta^0 = \delta_1 - p_2^e + \frac{\beta}{3}(2 - p_2^e s_1^e)\Delta^e.$$

This can be solved to yield

$$\Phi\Delta^e = \delta_1 - p_2^0 + \frac{\beta}{3}(2 - p_2^0 s_1^0)(\delta_1 - p_2^e),$$

$$\Phi\Delta^0 = \delta_1 - p_2^e + \frac{\beta}{3}(2 - p_2^e s_1^e)(\delta_1 - p_2^0),$$

where $\Phi > 0$. The cyclic strategy $s_1(t) = 0$ if t is even and $s_1(t) = 1$ if t is odd is a best response for type 1 as long as $\Delta^e \geq 0 \geq \Delta^0$. Since s_1^e, s_1^0, p_2^e and p_2^0 are known, these inequalities depend only on δ_1 and β, and Figure 4 shows the region of (β, δ_1) space in which $\Delta^e \geq 0$ and $\Delta^0 \leq 0$ both hold. In this region all of the equilibrium conditions are satisfied.

The same procedure can be used to construct equilibrium cycles of other periodicities, and Figure 4 also shows the region of (β, δ_1) space in which there exists a three-cycle equilibrium, with $s_2(t) = 1$ for all t, $s_3(t) = 0$ for all t, $s_1(t) = (0, 0, 1, 0, 0, 1, \ldots)$. Notice both of these cyclical equilibria exist only for values of δ_1 that do not allow a steady-state equilibrium in pure-strategies – that is, $\sqrt{2} - 1 < \delta_1 < 1/2$. Finally, we point out that these cycles are stable. Given cyclic strategies, $p(t)$ locally converges to a cyclic distribution, and as long as $\Delta_1(t)$ alternates in

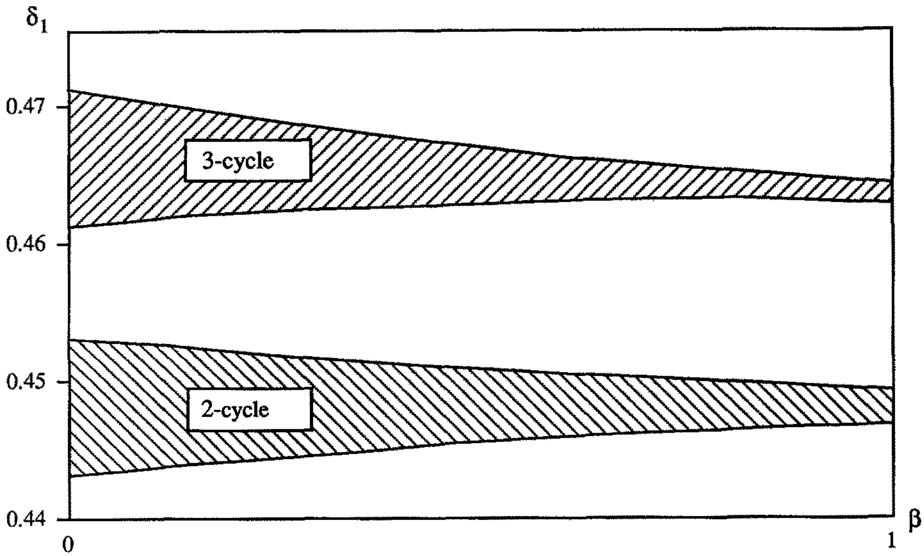

Figure 4. Regions where cycles exist.

sign in the right way in the limit, it will alternate in sign close to the limit. Thus, the cyclic strategies will also be best responses in the neighborhood of the limit cycle. Given $p(0)$, there will be cyclic strategies that imply $p(t)$ converges to a limit cycle, and in certain regions of parameter space these strategies are best responses along the entire path.

8. Concluding remarks

We have generalized the commodity money model in Kiyotaki and Wright (1989) by introducing mixed-strategies and dynamic equilibria. This allows us to construct mixed-strategy, steady-state equilibria in regions of parameter space for which no pure-strategy, stready-state equilibria exist. We have established generic finiteness of the set of steady-state equilibria. We have also described some interesting dynamic equilibria. As pointed out in the introduction, this model displays several properties that have been established for overlapping generations models, and we can analyze the models using very similar techniques.

One issue not addressed here is the existence of an equilibrium for arbitrary initial inventory distributions. It is straightforward to prove the existence of such an equilibrium by adapting an approach used in overlapping generations models, for example, by Balasko, Cass and Shell (1980): truncate the economy at some finite date $\bar{t}$ by arbitrarily choosing the value functions $V_{ij}(\bar{t})$, prove existence for the finite economy using a standard fixed point argument, let $\bar{t}$ increase to infinity, and take the appropriate limits. However, as pointed out in Aiyagari and Wallace (1991), the hard part in this model is establishing that the equilibrium discovered in such a manner is nondegenerate. An extension of the technique they use for steady state equilibria would have to be used to show that nondegenerate dynamic equilibria exist (subject to certain parameter restrictions, of course; if storage costs are sufficiently high then all agents will want to freely dispose of their initial inventories and drop out of the game).

Another issue not addressed here is the existence of sunspot equilibria, where the equilibrium strategies and hence commodity monies fluctuate randomly over time even though the fundamentals of the model are deterministic. Such equilibria can arise in overlapping generations models, of course. Exploring the possible relevance of sunspots in this model is left to future research.

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