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Long-Time Decay of the Solutions of the Davey–Stewartson II Equations

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Summary. Using the method of inverse scattering, the sup-norms of the solutions of the Davey–Stewartson II equations are shown to decay in the order of $1/|t|$ as $|t|$ goes to infinity. In the focusing case this result is obtained for small initial data, whereas in the defocusing case it is obtained for general initial data.

Key words: Davey–Stewartson II, long-time decay, inverse scattering

1. Introduction

The Davey–Stewartson II (DSII) equations [8, 1, 2, 7, 17] are model equations in the theory of shallow water waves. The Cauchy problem for the DSII equations can be written in the following form:

$$q_t = 2iq_{x_1x_2} - 16i[\mathbf{R}(\pm|q|^2)]q, \tag{1.1a}$$

$$q(x,0) = q_0(x), \tag{1.1b}$$

where $q = q(x_1, x_2, t)$ and the operator $\mathbf{R}$, defined by

$$\widehat{\mathbf{R}f}(\xi) = \frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2}\hat{f}(\xi),$$

is the product of two Riesz transforms in the x variable. The "+" (respectively "−") sign in (1.1a) corresponds to the defocusing (respectively, focusing) case of the Davey–Stewartson II equations. For simplicity we assume that $q_0(x)$ is a rapidly decreasing Schwartz function.

Equation (1.1a) is completely integrable in the sense that it is the compatibility condition of a Lax pair [10, 11]. The Cauchy problem (1.1) was studied in [10], [11], and [3]–[5], and [19]–[21] by the method of inverse scattering. It was also investigated in [14], [15], and [6] by the technique of a priori estimates. Analytic solutions of (1.1) was studied in [9]. The optimal long-time decay estimate of $O(1/|t|)$ was obtained in [6] for a class of nonlinear Schrödinger equations that includes the DSII equations. However, the derivation of the decay estimate in [6] requires the initial data to be small.

In this paper we study the long-time behavior of the solutions of (1.1) by the machinery of inverse scattering. In this approach we have an explicit formula for the solution q:

$$q(x,t) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\exp\big(i[k'\cdot x - 2tk_1'k_2']\big)\overline{v(k',x,t)}\alpha(k'), \tag{1.2}$$

where v satisfies the integral equation

$$v(k,x,t) = 1 \pm \frac{1}{\pi^2}\int_{\mathbb{R}^4} dk_1'\,dk_2'\,dk_1''\,dk_2''\,\exp\big(i[(k'-k'')\cdot x - 2t(k_1'k_2' - k_1''k_2'')]\big) \times \frac{\alpha(k')\overline{\alpha(k'')}}{(k-k')(\overline{k'-k''})}v(k'',x,t). \tag{1.3}$$

The *scattering datum* $\alpha(k)$ is also a Schwartz function.

We will obtain the decay estimate

$$\|q(\cdot,t)\|_{L^\infty} \le \frac{C_{q_0}}{|t|} \qquad \forall t\in\mathbb{R} \tag{1.4}$$

by a careful analysis of (1.2) and (1.3). This estimate is valid for any Schwarz function q_0 in the defocusing case and for small q_0 in the focusing case.

The rest of the paper is organized as follows. In Section 2 we review the method of inverse scattering that gives us (1.2) and (1.3). In Section 3 we establish some preliminary estimates. The proof of (1.4) is given in Section 4. Some lemmas concerning technical details or background material are collected in the Appendix.

For convenience we state here some of the notation used in this paper.

We will identify $x = (x_1, x_2) \in \mathbb{R}^2$ with the complex number $x = x_1 + ix_2 \in \mathbb{C}$. The same holds for $k = (k_1, k_2)$ and $k = k_1 + ik_2$. The complex partial differential operators $\partial\bar{x}$ and ∂x are defined by

$$\frac{\partial}{\partial\bar{x}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

We will also use D_x^l to denote the real partial differential operator $\partial^{l_1+l_2}/\partial x_1^{l_1}\,\partial x_2^{l_2}$.

The Fourier transform $\mathcal{F}$ of a function $f(y_1,\ldots,y_n)$ in n real variables is given by

$$(\mathcal{F}f)(\eta_1,\ldots,\eta_n) = \hat{f}(\eta) = \int_{\mathbb{R}^n} dy_1\cdots dy_n\, e^{-i\eta\cdot y}f(y).$$

The inverse Fourier transform is given by

$$(\mathcal{F}^{-1}g)(y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d\eta_1\cdots d\eta_n\, e^{iy\cdot\eta}g(\eta).$$

The space of 2×2 complex matrices is denoted by $M_{2\times 2}$. $\Pi_o(A)$ is the off-diagonal part of the 2×2 matrix A. The 2×2 identity matrix is denoted by I.

$\mathcal{S}(\mathbb{R}^2)$ is the space of Schwartz functions. If X is a Banach space, then $C_b(\mathbb{C}, X)$ is the space of bounded continuous X-valued functions on $\mathbb{C}$, $C_0(\mathbb{C}, X)$ is the subspace of continuous X-valued functions that vanish at ∞, and $\mathcal{L}(X)$ [respectively, $\mathcal{L}_c(X)$] is the space of bounded-linear (respectively, conjugate-linear) operators on X.

We will use the letter C with or without subscripts to represent a generic positive constant that may take different values at different places. Such constants depend only on the quantities listed in their subscripts.

2. Inverse Scattering

We outline here the method of inverse scattering for the Cauchy problem (1.1). Details can be found in [19]–[21]. An equivalent method was used in [10], [11], and [3]–[5].

The relevant scattering/inverse scattering problem for (1.1) is associated with the following elliptic system on the plane:

$$\frac{\partial \psi}{\partial \bar{x}} = Q\bar{\psi}, \tag{2.1}$$

where

$$Q(x) = \begin{bmatrix} 0 & Q_{12}(x) \\ Q_{21}(x) & 0 \end{bmatrix}. \tag{2.2}$$

For simplicity we assume that Q_{12} and $Q_{21} \in \mathcal{S}(\mathbb{R}^2)$.

When $Q = 0$, $\psi = e^{i\bar{k}x/2}I$ is an exponential solution of (2.1) for each $k \in \mathbb{C}$. For $Q \neq 0$, we consider $M_{2\times 2}$-valued solutions of (2.1) in the form of

$$\psi(x, k) = e^{i\bar{k}x/2}\mu(x, k) \tag{2.3}$$

and we require that

$$\lim_{|x|\to\infty} \mu(x, k) = I. \tag{2.4}$$

The effect of Q on the solutions of (2.1) is measured by the difference between $\mu(x, k)$ and I for $|x|$ large. The differential equation satisfied by μ is

$$\frac{\partial \mu}{\partial \bar{x}} = \exp\left(\frac{-i(\bar{k}x + k\bar{x})}{2}\right) Q(x)\bar{\mu}(x, k). \tag{2.5}$$

Using the fundamental solution $1/(\pi x)$ of the $\partial_{\bar{x}}$ operator we can convert (2.5) and (2.4) into the following integral equation:

$$\mu(x, k) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} dx_1'\, dx_2'\, \frac{\exp(-i(\bar{k}x' + k\overline{x'})/2)}{x - x'} Q(x')\overline{\mu(x', k)}. \tag{2.6}$$

For each $k \in \mathbb{C}$ equation (2.6) is uniquely solvable in the space $C_b(\mathbb{C}, M_{2\times 2})$ if Q is small or if $Q_{12} = Q_{21}$. The following asymptotic expansion of μ for large $|x|$ can be derived from (2.6):

$$\mu(x, k) = I + \frac{\mu_1(k)}{x} + \frac{\mu_2(k)}{x^2} + \cdots, \tag{2.7}$$

where

$$\mu_j(k) = \frac{1}{\pi}\int_{\mathbb{R}^2} dx_1'\,dx_2'\,\exp\!\left(\frac{-i(\bar{k}x' + k\overline{x'})}{2}\right)(x')^{j-1}Q(x')\overline{\mu(x',k)}. \tag{2.8}$$

The *scattering datum* $T(k)$ is defined by

$$T(k) = \frac{i}{2}\Pi_o\mu_1(k) = \frac{i}{2\pi}\Pi_o\int_{\mathbb{R}^2} dx_1'\,dx_2'\,\exp\left(\frac{-i(\bar{k}x' + k\overline{x'})}{2}\right)Q(x')\overline{\mu(x',k)}. \tag{2.9}$$

It can be shown that $T_{12}(k)$ and $T_{21}(k) \in \mathcal{S}(\mathbb{R}^2)$. The construction of $T(k)$ from $Q(x)$ through (2.6) and (2.9) is called the *direct scattering* of (2.1).

If we define $\nu(x,k)$ by

$$\nu(x,k) = \begin{bmatrix} \overline{\mu_{11}(x,k)} & \mu_{12}(x,k)\exp(i(\bar{k}x + k\bar{x})/2) \\ \mu_{21}(x,k)\exp(i(\bar{k}x + k\bar{x})/2) & \overline{\mu_{22}(x,k)} \end{bmatrix}, \tag{2.10}$$

then we can also derive from (2.6) the following asymptotic expansion of ν for $|k|$ large:

$$\nu(k,x) = I + \frac{\nu_1(x)}{k} + \frac{\nu_2(x)}{k^2} + \cdots, \tag{2.11}$$

where

$$\Pi_o\nu_1(x) = 2iQ(x). \tag{2.12}$$

Moreover, $\nu(k,x)$ satisfies

$$\frac{\partial\nu}{\partial\bar{k}}(k,x) = \exp\!\left(\frac{i(\bar{k}x + k\bar{x})}{2}\right)\overline{\nu(k,x)}T(k) \tag{2.13}$$

and

$$\lim_{|k|\to\infty}\nu(k,x) = I. \tag{2.14}$$

Again we can convert (2.13) and (2.14) into the following integral equation:

$$\nu(k,x) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{\exp(i(\overline{k'}x + k'\bar{x})/2)}{k - k'}\overline{\nu(k,x)}T(k'). \tag{2.15}$$

For symmetric or small T, equation (2.15) can be solved uniquely in $C_b(\mathbb{C}, M_{2\times 2})$ for each fixed $x \in \mathbb{C}$. We can then derive from (2.15) the following asymptotic expansion of ν for $|k|$ large:

$$\nu(k,x) = I + \frac{\tilde{\nu}_1(x)}{k} + \frac{\tilde{\nu}_2(x)}{k^2} + \cdots, \tag{2.16}$$

where

$$\tilde{\nu}_j(x) = \frac{1}{\pi}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\exp(i(\overline{k'}x + k'\bar{x})/2)(k')^{j-1}\overline{\nu(k',x)}T(k'). \tag{2.17}$$

By comparing (2.16) and (2.17) with (2.11) and (2.12), we find the following reconstruction formula for Q:

$$\begin{aligned} Q(x) &= \frac{1}{2i}\Pi_o\nu_1(k,x) \\ &= \frac{1}{2\pi i}\Pi_o\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\exp\!\left(\frac{i(\overline{k'}x + k'\bar{x})}{2}\right)\overline{\nu(k',x)}T(k'). \end{aligned} \tag{2.18}$$

The reconstruction of $Q(x)$ from $T(k)$ through (2.15) and (2.18) is the *inverse scattering* of (2.1). It can be shown that $Q^t = \pm Q$ if and only if $T^t = \pm T$.

The Cauchy problem (1.1) can be solved by the following procedure.

(I) Let

$$Q_0(x) = \begin{bmatrix} 0 & q_0(x) \\ \pm q_0(x) & 0 \end{bmatrix}.$$

Solve (2.6) using $Q_0(x)$ and obtain the corresponding scattering datum

$$T_0(k) = \begin{bmatrix} 0 & \alpha(k) \\ \pm\alpha(k) & 0 \end{bmatrix}$$

through (2.9).

(II) Let $T(k,t) = \exp(-2itk_1k_2)T_0(k)$ be the solution of the following linear Cauchy problem:

$$\frac{\partial T}{\partial t} = -2ik_1k_2T, \tag{2.19a}$$

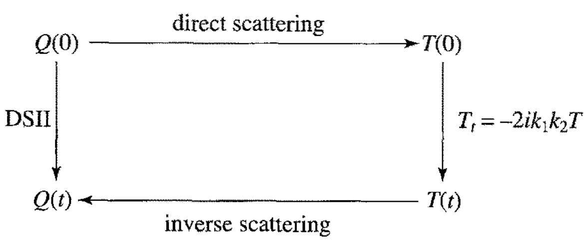
$$T(k,0) = T_0(k). \tag{2.19b}$$

(III) Let $\nu(k,x,t)$ be the solution of (2.15), where the scattering datum $T(k,t)$ is now time-dependent, and let $Q(x,t)$ be defined in terms of $\nu(k,x,t)$ and $T(k,t)$ through (2.18). Then

$$Q(x,t) = \begin{bmatrix} 0 & q(x,t) \\ \pm q(x,t) & 0 \end{bmatrix}$$

and $q(x,t)$ solves the Cauchy problem (1.1).

The solution procedure above can be schematically represented by the following commutative diagram:



In other words, the nonlinear DSII equations are linearized by the direct scattering of the elliptic system (2.1), and then q is recovered from T through inverse scattering.

For Schwartz class initial data, we have the following theorem [4, 21].

Theorem 2.1. *Let $q_0(x) \in \mathcal{S}(\mathbb{R}^2)$. Then (1.1) has a unique global solution q such that $t \longrightarrow q(\cdot, t)$ is a C^∞ map from $\mathbb{R}$ into $\mathcal{S}(\mathbb{R}^2)$, provided the sign in (1.1a) is positive (defocusing case) or*

$$\|\hat{q}_0\|_{L^1(\mathbb{R}^2)}\|\hat{q}_0\|_{L^\infty(\mathbb{R}^2)} < \frac{\pi^3}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2 \tag{2.20}$$

when the sign is negative (focusing case). The solution can be obtained by the procedure (I)–(III).

The equation (2.15) with $T(k,t) = \exp(-2itk_1k_2)T_0(k)$ can be written as

$$\begin{aligned} \nu(k,x,t) = I &+ \frac{1}{\pi}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{\exp\left(i[k'\cdot x - 2tk_1'k_2']\right)}{k-k'}T_0(k') \\ &+ \frac{1}{\pi^2}\int_{\mathbb{R}^4} dk_1'dk_2'dk_1''dk_2''\frac{\exp\left(i[(k'-k'')\cdot x - 2t(k_1'k_2' - k_1''k_2'')]\right)}{(k-k')(\overline{k'-k''})} \\ &\times \nu(k'',x,t)\overline{T_0(k'')}T_0(k'). \end{aligned} \tag{2.21}$$

Let $v(k,x,t) = \nu_{11}(k,x,t)$. Then (1.2) and (1.3) follow from the procedure (I)–(III) and (2.21). Note that $v(k,x,t)$ is a smooth function in $\mathbb{C}\times\mathbb{C}\times\mathbb{R}$.

3. Preliminary Estimates

Let $\gamma(k) \in \mathcal{S}(\mathbb{R}^2)$. We define the operator $\mathbf{H}^\gamma_{x,t}$ by

$$(\mathbf{H}^\gamma_{x,t}f)(k) = \frac{1}{\pi}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{\exp\left(i[x\cdot k' - 2tk_1'k_2']\right)}{k-k'}\gamma(k')\overline{f(k')}. \tag{3.1}$$

It is easy to see ([19], Lemma A.1, and [22]) that $\mathbf{H}^\gamma_{x,t}$: $C_b(\mathbb{C}) \longrightarrow C_0(\mathbb{C})$ and

$$\|\mathbf{H}^\gamma_{x,t}\|_{\mathcal{L}_c(C_b(\mathbb{C}))} \le \left(\frac{8}{\pi}\|\gamma\|_{L^1(\mathbb{R}^2)}\|\gamma\|_{L^\infty(\mathbb{R}^2)}\right)^{1/2}. \tag{3.2}$$

Equation (1.3) can be written as

$$v = 1 \pm (\mathbf{H}^\alpha_{x,t})^2 v. \tag{3.3}$$

We have ([19], Corollaries 2.4 and 2.12)

$$\mathbf{I} - (\mathbf{H}^\alpha_{x,t})^2 \text{ is invertible on } C_b(\mathbb{C}) \qquad \forall x \in \mathbb{C},\ t\in\mathbb{R}. \tag{3.4a}$$

If condition (2.20) is satisfied, then $\|\mathbf{H}^\alpha_{x,t}\|_{\mathcal{L}_c(C_b(\mathbb{C}))} < 1$ ([20], Theorem 4.7), and we have

$$\mathbf{I} + (\mathbf{H}^\alpha_{x,t})^2 \text{ is invertible on } C_b(\mathbb{C}) \qquad \forall x \in \mathbb{C},\ t\in\mathbb{R}. \tag{3.4b}$$

For large $|t|$ or $|x|$, the high frequency oscillation of the exponential function leads to the decay of $\|(\mathbf{H}^\alpha_{x,t})^2\|_{\mathcal{L}(C_b(\mathbb{C}))}$. The proof of the following lemma can be found in [19], Lemma 2.8, and [21], Lemma 5.13.

Lemma 3.1. *Given any $\epsilon > 0$, there exists $t_* > 0$ and $R > 0$ such that*

$$\|(\mathbf{H}^{\alpha}_{x,t})^2\|_{\mathcal{L}(C_b(\mathbb{C}))} \leq \epsilon, \tag{3.5}$$

for $|x| > R$ or $|t| > t_*$.

Combining (3.4), (3.5), and the continuous dependence of $\mathbf{H}^{\alpha}_{x,t}$ on (x, t), we have

$$\big\|\big[\mathbf{I} \mp (\mathbf{H}^{\alpha}_{x,t})^2\big]^{-1}\big\|_{\mathcal{L}(C_b(\mathbb{C}))} \leq C_\alpha \qquad \forall (x,t) \in \mathbb{C} \times \mathbb{R}. \tag{3.6}$$

We can rewrite (1.2) as

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi i}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\exp\big(i[k'\cdot x - 2tk_1'k_2']\big)\alpha(k') \\ &\quad + \frac{1}{2\pi i}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\exp\big(i[k'\cdot x - 2tk_1'k_2']\big)\overline{w(k',x,t)}\alpha(k'), \end{aligned} \tag{3.7}$$

where $w(k,x,t) = v(k,x,t) - 1$.

The first term on the right-hand side of (3.7) can be estimated by the following lemma.

Lemma 3.2. *Let f be a function of two real variables. If $f, \hat{f} \in L^1(\mathbb{R}^2)$, then*

$$\left|\int_{\mathbb{R}^2} dy_1 dy_2 \exp\big(i[y\cdot x - 2ty_1y_2]\big) f(y)\right| \leq C\|\hat{f}\|_{L^1(\mathbb{R}^2)}|t|^{-1}. \tag{3.8}$$

Proof. For $t \neq 0$, the Fourier transform of $\exp(-2ity_1y_2)$ is $\pi|t|^{-1}\exp[i\eta_1\eta_2/(2t)]$ ([16], p. 206]). Hence

$$\int_{\mathbb{R}^2} dy_1\,dy_2\,\exp(i[y\cdot x - 2ty_1y_2])f(y) = \frac{1}{4\pi|t|}\int_{\mathbb{R}^2} d\eta_1 d\eta_2 \exp[i\eta_1\eta_2/(2t)]\hat{f}(-x-\eta)$$

and (3.8) follows. □

The bulk of the proof of the long-time decay therefore falls on estimating the second term on the right-hand side of (3.7). It follows from (3.3) that the equation satisfied by w is

$$w = \pm(\mathbf{H}^{\alpha}_{x,t})^2 1 \pm (\mathbf{H}^{\alpha}_{x,t})w.$$

Hence we have

$$w = \pm[\mathbf{I} \mp (\mathbf{H}^{\alpha}_{x,t})^2]^{-1}(\mathbf{H}^{\alpha}_{x,t})^2 1. \tag{3.9}$$

The rest of this section will be devoted to estimating functions of the form $\mathbf{H}^{\gamma}_{x,t}1$, where $\gamma \in \mathcal{S}(\mathbb{R}^2)$. Note that $\mathbf{H}^{\gamma}_{x,t}1$ is a smooth function of $(k,x,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{R}$.

Let the function $g_{\gamma,k}$ be defined by

$$g_{\gamma,k}(k') = \sum_{j=0}^{2}\frac{(\partial_{\bar{k}}\gamma)(k)}{j!}(\overline{k'-k})^j\exp(-|k'-k|^2/2).$$

Then we have

$$\mathbf{H}^{\gamma}_{x,t}1 = \frac{1}{\pi}\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\frac{\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)}{k-k'}(\gamma(k') - g_{\gamma,k}(k'))$$

$$+\frac{1}{\pi}\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\frac{\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)}{k-k'}g_{\gamma,k}(k'). \tag{3.10}$$

Let $F_{\gamma,k}(k') = [\gamma(k') - g_{\gamma,k}(k')]/(k-k')$. It follows from Lemma A.1 in the Appendix and the Sobolev lemma ([12], p. 243) that $\hat{F}_{\gamma,k} \in L^1(\mathbb{R}^2)$ and $\|\hat{F}_{\gamma,k}\|_{L^1(\mathbb{R}^2)}$ is bounded by a constant C_γ that is independent of $k \in \mathbb{C}$. We deduce from Lemma 3.2 that

$$\left|\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\frac{\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)}{k-k'}(\gamma(k') - g_{\gamma,k}(k'))\right| \le \frac{C_\gamma}{|t|}. \tag{3.11}$$

It remains to estimate

$$\frac{1}{\pi}\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)\frac{g_{\gamma,k}(k')}{k-k'}$$

$$= \frac{-1}{\pi}\sum_{j=0}^{2}\frac{(\partial_{\bar{k}}\gamma)(k)}{j!}$$

$$\times\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)\frac{(\overline{k'-k})^j}{k'-k}\exp\left(\frac{-|k'-k|^2}{2}\right). \tag{3.12}$$

For the $j=2$ term in (3.12) we proceed as follows. Corollary A.3 implies that, for each $k \in \mathbb{C}$, the Fourier transform of $[(\overline{k'-k})^2/(k'-k)]\exp(-|k'-k|^2/2)$ belongs to $L^1(\mathbb{R}^2_{k'})$, with the L^1-norm independent of k. We again deduce from Lemma 3.2 that

$$\left|\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)\frac{(\overline{k'-k})^2}{k'-k}\exp\left(\frac{-|k'-k|^2}{2}\right)\right| \le \frac{C}{|t|}. \tag{3.13}$$

For the $j=0$ term in (3.12), we obtain by the substitution $k' = k + re^{i\theta}$ that

$$\int_{\mathbb{R}^2} dk'_1\,dk'_2\,\exp\bigl(i[x\cdot k' - 2tk'_1k'_2]\bigr)\frac{\exp(-|k'-k|^2/2)}{k'-k}$$

$$= \exp\bigl(i[x\cdot k - 2tk_1k_2]\bigr)\int_0^{2\pi} d\theta\, e^{-i\theta}\int_0^\infty dr\,\exp\left(\frac{-(1+i2t\sin 2\theta)r^2}{2}\right)$$

$$\times\exp\bigl(ir[(x_1 - 2tk_2)\cos\theta + (x_2 - 2tk_1)\sin\theta]\bigr)$$

$$= \exp\bigl(i[x\cdot k - 2tk_1k_2]\bigr)\int_0^{2\pi} d\theta\, e^{-i\theta}\int_0^\infty dr\,\phi(r,\theta,x,t)$$

$$\times\exp(-i2rt(k_2\cos\theta + k_1\sin\theta)), \tag{3.14}$$

where

$$\phi(r,\theta,x,t) = \exp\left(\frac{-(1+i2t\sin 2\theta)r^2}{2}\right)\exp(ir(x_1\cos\theta + x_2\sin\theta)). \tag{3.15}$$

For $l = 0, 1, 2, \ldots$, the following estimates clearly hold:

$$\int_{\mathbb{R}} |dr\, r^l \phi(r,\theta,x,t)| \le C_l. \tag{3.16}$$

The Fourier transform $\mathcal{F}_r\phi$ of ϕ in r is given by ([16], p. 206)

$$(\mathcal{F}_r\phi)(\xi,\theta,x,t) = \left(\frac{2\pi}{1+i2t\sin\theta}\right)^{1/2}\exp\left(-\frac{[\xi-(x_1\cos\theta+x_2\sin\theta)]^2}{2(1+i2t\sin\theta)}\right).$$

Note that

$$\mathrm{Re}\frac{-1}{2(1+i2t\sin\theta)} = \frac{-1}{2(1+4t^2\sin^2\theta)}.$$

Hence there exist positive constants C_l ($l = 0, 1, 2, \ldots$) independent of (ξ, θ, x, t) such that

$$\left|\left(D_\xi^l \mathcal{F}_r\phi\right)(\xi,\theta,x,t)\right| \le \frac{C_l}{|1+i2t\sin\theta|^{1/2}}. \tag{3.17}$$

Also, we have

$$\left\|\left(\mathcal{F}_r\phi\right)(\cdot,\theta,x,t)\right\|_{L^p(\mathbb{R})} \le \frac{C_p}{|1+i2t\sin\theta|^{1/2-1/p}}, \tag{3.18}$$

for any $p \in [1,\infty]$. Therefore, it follows from (3.16), (3.17), (3.18), and Lemma A.4 in the Appendix that for any $p \in [1,\infty)$, we have

$$\left|\int_0^{2\pi} d\theta\, e^{-i\theta}\int_0^\infty dr\,\phi(r,\theta,x,t)\exp(-i2rt(k_2\cos\theta+k_1\sin\theta))\right| \le \frac{C_p}{|t|^{1/2-1/p}}, \tag{3.19}$$

where $C_p > 0$ is independent of (t, k, x).

Similarly, for the $j = 1$ term in (3.12), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} dk_1'\, dk_2'\, \exp\big(i[x\cdot k' - 2tk_1'k_2']\big)\frac{\overline{(k'-k)}}{k'-k}\exp(-|k'-k|^2/2)\\
&\quad = \exp\big(i[x\cdot k - 2tk_1k_2]\big)\int_0^{2\pi} d\theta\, e^{-2i\theta}\int_0^\infty dr\, r\exp\left(\frac{-(1+i2t\sin 2\theta)r^2}{2}\right)\\
&\qquad \times \exp\big(ir[(x_1-2tk_2)\cos\theta + (x_2-2tk_1)\sin\theta]\big)\\
&\quad = \exp\big(i[x\cdot k - 2tk_1k_2]\big)\int_0^{2\pi} d\theta\, e^{-2i\theta}\\
&\qquad \times \int_0^\infty dr\, r\phi(r,\theta,x,t)\exp(-i2rt(k_2\cos\theta+k_1\sin\theta)).
\end{aligned} \tag{3.20}$$

For any $p \in [1, \infty)$, Lemma A.4 again implies that

$$\left| \int_0^{2\pi} d\theta\, e^{-2i\theta} \int_0^\infty dr\, r\phi(r, \theta, x, t) \exp(-i2rt(k_2 \cos\theta + k_1 \sin\theta)) \right|$$
$$\leq \frac{C_p}{|t|^{1/2-1/p}}, \tag{3.21}$$

where $C_p > 0$ is independent of (t, k, x).

In summary, we have proved most of the following proposition.

Proposition 3.3. *Let $\gamma \in \mathcal{S}(\mathbb{R}^2)$. Then*

$$\mathbf{H}^\gamma_{x,t}1 = \Phi_1(k, x, t) + \exp\bigl(i[x \cdot k - 2tk_1k_2]\bigr)[\gamma(k)\Phi_2(k, x, t) + (\partial_{\bar{k}}\gamma)(k)\Phi_3(k, x, t)],$$

where the functions Φ_j are smooth in all variables, and

$$|\Phi_1(k, x, t)| \leq \frac{C_\gamma}{|t|},$$
$$|\Phi_2(x, k, t)|, |\Phi_3(k, x, t)| \leq \frac{C_p}{|t|^{1/2-1/p}},$$

for all $(k, x, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ and $p \in [1, \infty)$. Moreover, we have

$$\Phi_2(k, x, t) = \frac{-1}{\pi} \int_0^{2\pi} d\theta\, e^{-i\theta} \int_0^\infty dr\, \phi(r, \theta, x, t) \exp(-i2rt(k_2 \cos\theta + k_1 \sin\theta)),$$

and

$$\Phi_3(k, x, t) = \frac{-1}{\pi} \int_0^{2\pi} d\theta\, e^{-2i\theta} \int_0^\infty dr\, r\phi(r, \theta, x, t) \exp(-i2rt(k_2 \cos\theta + k_1 \sin\theta)),$$

where ϕ is defined in (3.15).

Proof. It only remains to discuss the smoothness of the functions Φ_j. The functions Φ_2 and Φ_3 are clearly smooth by (3.16). The smoothness of Φ_1 then follows from the smoothness of $\mathbf{H}^\gamma_{x,t}1$. □

Corollary 3.4. *Let $\gamma \in \mathcal{S}(\mathbb{R}^2)$ and $\epsilon > 0$. Then there exists a positive constant $C_{\gamma,\epsilon}$ such that*

$$|\mathbf{H}^\gamma_{x,t}1| \leq \frac{C_{\gamma,\epsilon}}{|t|^{1/2-\epsilon}} \qquad \forall\, (k, x, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}.$$

4. Long-Time Decay

From (3.1), (3.2), and Proposition 3.3 we have

$$(\mathbf{H}^{\alpha}_{x,t})^2 1 = \mathbf{H}^{\alpha}_{x,t}(\mathbf{H}^{\alpha}_{x,t}1) = \sum_{j=1}^{3} \Psi_j(k,x,t), \tag{4.1}$$

where

$$|\Psi_1(k,x,t)| \le \frac{C_\alpha}{|t|}, \tag{4.2}$$

$$\Psi_2(k,x,t) = \frac{1}{\pi^2}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{|\alpha(k')|^2}{k'-k}\int_0^{2\pi} d\theta\, e^{i\theta} \times \int_0^\infty dr\,\overline{\phi(r,\theta,x,t)}\exp(i2rt(k_1'\sin\theta + k_2'\cos\theta)), \tag{4.3}$$

and

$$\Psi_3(k,x,t) = \frac{1}{\pi^2}\int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{\alpha(k')\overline{(\partial_{\bar{k}}\alpha)(k')}}{k'-k}\int_0^{2\pi} d\theta\, e^{2i\theta} \times \int_0^\infty dr\,\overline{r\phi(r,\theta,x,t)}\exp(i2rt(k_1'\sin\theta + k_2'\cos\theta)). \tag{4.4}$$

Note that by (3.15) the functions Ψ_2 and Ψ_3 are smooth in (k,x,t). Since $(\mathbf{H}^{\alpha}_{x,t})^2 1$ is smooth in all variables, the function Ψ_1 is also smooth in (k,x,t).

We first investigate the function $\Psi_2(k,x,t)$. We can write (4.3) as

$$\begin{aligned}
\Psi_2&(k,x,t)\\
&= \frac{1}{\pi^2}\int_0^{2\pi} d\theta\, e^{i\theta}\int_0^\infty dr\,\overline{\phi(r,\theta,x,t)}\\
&\quad\times \int_{\mathbb{R}^2} dk_1'\,dk_2'\, f(k',k)\exp(i2rt(k_1'\sin\theta + k_2'\cos\theta))\\
&\quad+ \frac{|\alpha(k)|^2}{\pi^2}\int_0^{2\pi} d\theta\, e^{i\theta}\int_0^\infty dr\overline{\phi(r,\theta,x,t)}\\
&\quad\times \int_{\mathbb{R}^2} dk_1'\,dk_2'\,\frac{\exp(-|k'-k|^2/2)}{k'-k}\exp(i2rt(k_1'\sin\theta + k_2'\cos\theta))\\
&\quad+ \frac{(\partial_{\bar{k}}|\alpha|^2)(k)}{\pi^2}\int_0^{2\pi} d\theta\, e^{i\theta}\int_0^\infty dr\,\overline{\phi(r,\theta,x,t)}\int_{\mathbb{R}^2} dk_1'\,dk_2'
\end{aligned}$$

$$\times \frac{\overline{(k'-k)}}{k'-k} \exp\left(\frac{-|k'-k|^2}{2}\right) \exp(i2rt(k_1' \sin\theta + k_2' \cos\theta))$$

$$= \mathrm{I}_1 + |\alpha(k)|^2 \mathrm{I}_2 + \big((\partial_{\bar{k}}|\alpha|^2)(k)\big)\mathrm{I}_3, \tag{4.5}$$

where

$$f(k', k) = \frac{|\alpha(k')|^2 - \big[|\alpha(k)|^2 + \big((\partial_{\bar{k}}|\alpha|^2)(k)\big)\overline{(k'-k)}\big] \exp(-|k'-k|^2/2)}{k'-k}.$$

By Lemma A.5 in the Appendix, the function $f(\cdot, k)$ is continuous in $\mathbb{C}$ and C^1 in $\mathbb{C}\backslash\{k\}$, for each $k \in \mathbb{C}$, and there exists positive C_α independent of $k \in \mathbb{C}$ and $\omega \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} dk_2' \left(\int_{\mathbb{R}} dk_1' \sum_{|l| \le 1} \left| (D_{k'}^l f)(k' e^{i\omega}, k) \right| \right)^2 \le C_\alpha \qquad \forall (k, \omega) \in \mathbb{C} \times \mathbb{R}. \tag{4.6}$$

By the substitution $k' = k'' e^{-i\theta}$ we can rewrite I_1 as

$$I_1 = \frac{1}{\pi^2} \int_0^{2\pi} d\theta\, e^{i\theta} \int_0^\infty dr\, \overline{\phi(r, \theta, x, t)} \int_{\mathbb{R}} dk_2'' \exp(i2rtk_2'') \int_{\mathbb{R}} dk_1''\, f(k'' e^{-i\theta}, k)$$

$$= \frac{1}{\pi^2} \int_0^{2\pi} d\theta\, e^{i\theta} \int_0^\infty dr\, \overline{\phi(r, \theta, x, t)}\, \tilde{f}(k, \theta, 2rt), \tag{4.7}$$

where

$$\tilde{f}(k, \theta, s) = \int_{\mathbb{R}} dk_2'' \exp(isk_2'') \int_R dk_1''\, f(k'' e^{-i\theta}, k). \tag{4.8}$$

The Sobolev lemma ([12], p. 243) and (4.6) imply that

$$\int_{\mathbb{R}} ds |\tilde{f}(k, \theta, s)| \le C_\alpha \qquad \forall (k, \theta) \in \mathbb{C} \times \mathbb{R}. \tag{4.9}$$

Clearly, from (3.15) we have for $l \ge 0$,

$$|r^l \phi(r, \theta, x, t)| \le C_l. \tag{4.10}$$

Therefore, we obtain from (4.7), (4.9), and (4.10) (with $l = 0$) that

$$|I_1| \le \frac{C_\alpha}{|t|}, \tag{4.11}$$

where $C_\alpha > 0$ is independent of (k, x, t).

We now turn to the integral I_2. The substitution $k' = k + k'' e^{-i\theta}$ gives

$$\mathrm{I}_2 = \frac{1}{\pi^2} \int_0^{2\pi} d\theta\, e^{2i\theta} \int_0^\infty dr\, \overline{\phi(r, \theta, x, t)} \exp(i2rt(k_1 \sin\theta + k_2 \cos\theta))$$

$$\times \int_{\mathbb{R}^2} dk_1''\, dk_2'' \frac{\exp(-|k''|^2/2)}{k''} \exp(i2rtk_2'')$$

$$= \frac{1}{\pi^2} \int_0^{2\pi} d\theta\, e^{2i\theta} \int_0^\infty dr\, \overline{\phi(r, \theta, x, t)} \exp(i2rt(k_1 \sin\theta + k_2 \cos\theta))\eta(2rt), \tag{4.12}$$

where

$$\eta(s) = \int_{\mathbb{R}^2} dk_1'' \, dk_2'' \, \frac{\exp(-|k''|^2/2)}{k''} \exp(isk_2'').$$

We can evaluate $\eta(s)$ by (A.11) in the Appendix and obtain

$$\eta(s) = 2\pi \frac{1 - \exp(-s^2/2)}{s}. \tag{4.13}$$

Let $p \in (1, \infty]$. It is clear from (4.13) that

$$\int_{\mathbb{R}} ds |\eta(s)|^p < \infty. \tag{4.14}$$

For $t > 0$ we can now rewrite I_2 as

$$\mathrm{I}_2 = \int_{S^1 \times \mathbb{R}^+} dm_{x,t} \exp\big(i[k_1(r \sin\theta) + k_2(r\cos\theta)]\big), \tag{4.15}$$

where the measure $dm_{x,t}$ on $S^1 \times \mathbb{R}^+$ is defined by

$$dm_{x,t} = \frac{1}{2\pi^2 t} e^{2i\theta} \overline{\phi\left(\frac{r}{2t}, \theta, x, t\right)} \eta(r) \, dr \, d\theta. \tag{4.16}$$

It is clear from (3.15) that for any $p \in [1, \infty]$ and $l \geq 0$ there exists a positive $C_{p,l}$ such that

$$\left(\int_0^\infty dr \left| \left(\frac{r}{2t}\right)^l \phi\left(\frac{r}{2t}, \theta, x, t\right) \right|^p \right)^{1/p} \leq C_{p,l} \, t^{1/p}. \tag{4.17}$$

Combining (4.14), (4.16), and (4.17) (with $l = 0$) we find

$$\int_{S^1 \times \mathbb{R}^+} d|m_{x,t}| \leq \frac{C_p}{t^{1-1/p}} \qquad \forall\, (x,t) \in \mathbb{C} \times \mathbb{R}^+ \text{ and } \forall\, p \in [1, \infty). \tag{4.18}$$

Similarly we can rewrite I_3 as

$$\begin{aligned}
\mathrm{I}_3 &= \frac{1}{\pi^2} \int_0^{2\pi} d\theta \, e^{3i\theta} \int_0^\infty dr \, \overline{\phi(r,\theta,x,t)} \exp(i2rt(k_1 \sin\theta + k_2\cos\theta)) \\
&\quad \times \int_{\mathbb{R}^2} dk_1'' \, dk_2'' \, \frac{\overline{k''}}{k''} \exp\left(\frac{-|k''|^2}{2}\right) \exp(i2rtk_2'') \\
&= \frac{1}{\pi^2} \int_0^{2\pi} d\theta \, e^{3i\theta} \int_0^\infty dr \, \overline{\phi(r,\theta,x,t)} \exp(i2rt(k_1 \sin\theta + k_2\cos\theta)) \tilde{\eta}(2rt),
\end{aligned} \tag{4.19}$$

where

$$\tilde{\eta}(s) = \int_{\mathbb{R}^2} dk_1'' \, dk_2'' \, \frac{\overline{k''}}{k''} \exp\left(\frac{-|k''|^2}{2}\right) \exp(isk_2'').$$

Again we can evaluate $\tilde{\eta}(s)$ by (A.11) and obtain

$$\tilde{\eta}(s) = 4\pi \frac{1 - [1 + (s^2/2)] \exp(-s^2/2)}{s^2}. \tag{4.20}$$

Hence we have

$$\int_{\mathbb{R}} ds|\tilde{\eta}(s)| < \infty. \tag{4.21}$$

By (4.10), (4.19), and (4.21) we have

$$|I_3| \leq \frac{C}{|t|} \qquad \forall (x, k, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}. \tag{4.22}$$

Summing up, we have shown that, for $t > 0$, $\Psi_2(k, x, t) = \Psi_{21}(k, x, t) + \Psi_{22}(k, x, t)$ such that

(i) $|\Psi_{21}(k, x, t)| \leq \dfrac{C_\alpha}{t}$,

(ii) $\Psi_{22}(k, x, t) = |\alpha(k)|^2 \displaystyle\int_{S^1 \times \mathbb{R}^+} dm_{x,t} \exp\big(i[k_1(r \sin\theta) + k_2(r \cos\theta)]\big)$,

where $dm_{x,t}$ is defined by (4.16).

The function Ψ_3 can be analyzed similarly. The analog of I_2 for Ψ_3 is the integral

$$\frac{1}{\pi t} \int_0^{2\pi} d\theta\, e^{3i\theta} \int_0^\infty dr\, \overline{\phi(r, \theta, x, t)} \exp(i2rt(k_1 \sin\theta + k_2 \cos\theta))[1 - \exp(-2r^2t^2)],$$

which in view of (3.16) is bounded by C/t. It follows that

$$|\Psi_3(k, x, t)| \leq \frac{C_\alpha}{t}. \tag{4.23}$$

The following proposition has therefore been established.

Proposition 4.1. *Let $\alpha \in \mathcal{S}(\mathbb{R}^2)$ and $p \in [1, \infty)$. Then for $t > 0$ we have*

$$(\mathbf{H}^\alpha_{x,t})^2 1 = f_1(k, x, t) + |\alpha(k)|^2 f_2(k, x, t), \tag{4.24}$$

where the functions f_j are smooth, and

$$|f_1(k, x, t)| \leq \frac{C_\alpha}{t}, \tag{4.25}$$

$$f_2(k, x, t) = \int_{S^1 \times \mathbb{R}^+} dm_{x,t} \exp\big(i[k_1(r \sin\theta) + k_2(r \cos\theta)]\big). \tag{4.26}$$

The measure $m_{x,t}$ on $S^1 \times \mathbb{R}^+$ is defined by (4.16) and it satisfies the estimate (4.18).

We now return to (3.9). From Proposition 4.1 we obtain

$$w = w_1(k, x, t) + w_2(k, x, t), \tag{4.27}$$

where

$$w_1(k, x, t) = \pm[\mathbf{I} \mp (\mathbf{H}^\alpha_{x,t})^2]^{-1} f_1(k, t, x), \tag{4.28}$$

$$w_2(k, x, t) = \pm[\mathbf{I} \mp (\mathbf{H}^\alpha_{x,t})^2]^{-1}(|\alpha(k)|^2 f_2(k, x, t)). \tag{4.29}$$

It is not hard to see from the explicit form of f_2 in (4.26) that w_2 is smooth in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$. The smoothness of w_1 then follows from that of w and w_2.

We deduce from (3.6) and (4.25) that

$$|w_1(k, x, t)| \le \frac{C_\alpha}{t} \qquad \forall\,(k, x, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+. \tag{4.30}$$

The function w_2 can be rewritten as

$$\begin{aligned} w_2(k, x, t) &= \pm|\alpha(k)|^2 f_2(k, x, t) + [\mathbf{I} \mp (\mathbf{H}^\alpha_{x,t})^2]^{-1}(\mathbf{H}^\alpha_{x,t})^2(|\alpha(k)|^2 f_2(k, x, t)) \\ &= w_{21}(k, x, t) + w_{22}(k, x, t). \end{aligned} \tag{4.31}$$

Let ϵ be any positive number. From Corollary 3.4, (4.26), and (4.18) we have

$$\begin{aligned} |\mathbf{H}^\alpha_{x,t}(|\alpha(k)|^2 f_2(k, x, t))| &= \left|\int_{S^1 \times \mathbb{R}^+} d\overline{m_{x,t}}\, \mathbf{H}^\alpha_{x,t}\big(|\alpha(k)|^2 \exp\big(i[k_1(r\sin\theta) + k_2(r\cos\theta)]\big)\big)\right| \\ &= \left|\int_{S^1 \times \mathbb{R}^+} d\overline{m_{x,t}}\, \mathbf{H}^{\alpha|\alpha|^2}_{x-ire^{-i\theta},t} 1\right| \\ &\le \frac{C_{\alpha,\epsilon}}{t^{3/2-\epsilon}}. \end{aligned} \tag{4.32}$$

We then find by (3.2), (3.6), and (4.32) that

$$|w_{22}(k, x, t)| \le \frac{C_{\alpha,\epsilon}}{t^{3/2-\epsilon}} \quad \text{for any } \epsilon > 0 \text{ and } \forall\,(k, x, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+. \tag{4.33}$$

Note that w_{21} is clearly smooth, and hence w_{22} is also smooth.

We are finally ready to prove the long-time decay estimate.

Theorem 4.2. *Under the same conditions in Theorem 2.1, there exists a positive constant C_{q_0} such that the solution $q(x, t)$ of (1.1) satisfies*

$$|q(x, t)| \le \frac{C_{q_0}}{|t|} \qquad \forall\,(x, t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{4.34}$$

Proof. By (4.27) and (4.31) we have, for $t > 0$,

$$\int_{\mathbb{R}^2} dk_1'\, dk_2'\, \exp\big(i[x \cdot k' - 2tk_1'k_2']\big)\overline{w(k', x, t)}\alpha(k') = K_1 + K_{21} + K_{22}, \tag{4.35}$$

where

$$K_1 = \int_{\mathbb{R}^2} dk_1'\, dk_2'\, \exp\big(i[x \cdot k' - 2tk_1'k_2']\big)\overline{w_1(k', x, t)}\alpha(k') \tag{4.36}$$

and

$$K_{2j} = \int_{\mathbb{R}^2} dk_1'\, dk_2'\, \exp\big(i[x \cdot k' - 2tk_1'k_2']\big)\overline{w_{2j}(k', x, t)}\alpha(k'). \tag{4.37}$$

It follows from (4.30) and (4.33) that

$$|K_1| + |K_{22}| \le \frac{C_\alpha}{t} \qquad \forall\,(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+. \tag{4.38}$$

Using (4.26) the integral K_{21} can be written as

$$K_{21} = \pm \int_{S^1 \times \mathbb{R}^+} d\overline{m_{x,t}}$$

$$\times \int_{\mathbb{R}^2} dk_1'\, dk_2' \exp\bigl(i[k_1'(x_1 - r\sin\theta) + k_2'(x_2 - r\cos\theta) - 2tk_1'k_2']\bigr)|\alpha(k')|^2\alpha(k').$$

Lemma 3.2 and (4.18) then imply

$$|K_{21}| \le \frac{C_\alpha}{t} \qquad \forall\,(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+. \tag{4.39}$$

The estimate (4.34) for $t > 0$ now follows from (3.7), Lemma 3.2, (4.35), (4.38), (4.39), and the fact that $\alpha(k)$ comes from the scattering datum of

$$Q_0 = \begin{bmatrix} 0 & q_0(x) \\ \pm q_0(x) & 0 \end{bmatrix}.$$

A similar analysis for $t < 0$ then establishes (4.34). □

Appendix

Lemma A.1. *Let $\gamma \in \mathcal{S}(\mathbb{R}^2)$ and*

$$F_{\gamma,k}(k') = \frac{\gamma(k') - \sum_{j=0}^{2} \bigl[(\partial_{\bar k}\gamma)(k)/j!\bigr]\,(\overline{k'-k})^j \exp(-|k'-k|^2/2)}{k'-k}.$$

Then for each $k \in \mathbb{C}$, $F_{\gamma,k}$ belongs to the Sobolev space $H^2(\mathbb{R}^2)$ and

$$\|F_{\gamma,k}\|_{H^2(\mathbb{R}^2)} \le C\bigl(\|\gamma\|_{C^3(\mathbb{R}^2)} + \|\gamma\|_{H^2(\mathbb{R}^2)}\bigr), \tag{A.1}$$

where $C > 0$ is independent of γ and k.

Proof. Let $\Omega_1 = \{k'\colon |k'-k| > 1/2\}$ and $\Omega_2 = \{k'\colon |k'-k| < 1\}$. Clearly we have

$$\|F_{\gamma,k}\|_{H^2(\Omega_1)} \le C\bigl(\|\gamma\|_{H^2(\mathbb{R}^2)} + \|\gamma\|_{C^2(\mathbb{R}^2)}\bigr). \tag{A.2}$$

A complex form of Taylor's theorem is

$$\gamma(k') = \sum_{j=0}^{N}\sum_{l=0}^{j} \frac{[(\partial_{\bar k}^{l}\gamma)(k)][(\partial_k^{j-l}\gamma)(k)]}{l!(j-l)!}(\overline{k'-k})^l(k'-k)^{j-l}$$

$$+ \sum_{m=0}^{N+1} R_m(k',k)\frac{(\overline{k'-k})^m(k'-k)^{N+1-m}}{m!(N+1-m)!}, \tag{A.3}$$

where the remainders R_m are C^∞ in k' and

$$\|R_m(\cdot,k)\|_{C^r(\mathbb{R}^2)} \le \|\gamma\|_{C^{N+1+r}(\mathbb{R}^2)}, \qquad r = 0, 1, 2 \dots. \tag{A.4}$$

In view of (A.3), (A.4), and the series expansion

$$\exp\left(\frac{-|k'-k|^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} |k'-k|^{2n}, \tag{A.5}$$

the function $F_{\gamma,k}(k')$ is C^1 on Ω_2, C^2 on $\Omega_2\backslash\{k\}$, and for $|l| \le 2$,

$$\left|(D^l_{k'} F_{\gamma,k})(k')\right| \le C\|\gamma\|_{C^3(\mathbb{R}^2)} \qquad \forall k' \in \Omega_2\backslash\{k\}. \tag{A.6}$$

Therefore, $F_{\gamma,k} \in H^2(\Omega_2)$ and

$$\|F_{\gamma,k}\|_{H^2(\Omega_2)} \le C\|\gamma\|_{C^3(\mathbb{R}^2)}. \tag{A.7}$$

The estimate (A.1) follows from (A.2) and (A.7). □

Lemma A.2. *The following convolution formula holds:*

$$\frac{1}{x} * \left(\bar{x}^n \exp\left(-\frac{|x|^2}{2}\right)\right) = \pi 2^{n+1} n! \frac{1 - \left(\sum_{j=0}^n |x|^{2j}/(2^j j!)\right)\exp(-|x|^2/2)}{x^{n+1}}. \tag{A.8}$$

Proof. We will establish (A.8) by mathematical induction. We have for $x \neq 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} dy_1\,dy_2 \frac{\exp(-|y|^2/2)}{x-y} &= \int_{\mathbb{R}^2} dy_1\,dy_2 \frac{1}{(x-y)y}(y\exp(-|y|^2/2)) \\
&= \frac{2}{x}\int_{\mathbb{R}^2} dy_1\,dy_2 \left(\frac{1}{x-y} + \frac{1}{y}\right)\left(-\frac{\partial}{\partial \bar{y}} \exp\left(-\frac{|y|^2}{2}\right)\right) \\
&= \frac{2\pi}{x}\left(1 - \exp\left(-\frac{|x|^2}{2}\right)\right).
\end{aligned} \tag{A.9}$$

The case $n = 0$ of (A.8) follows by continuity.

Let ϕ_n be the left-hand side of (A.8). We have for $x \neq 0$, and $n \ge 1$,

$$\begin{aligned}
\phi_n &= \frac{1}{x}\int_{\mathbb{R}^2} dy_1\,dy_2 \left(\frac{1}{x-y} + \frac{1}{y}\right)\left(y\bar{y}^n \exp\left(\frac{-|y|^2}{2}\right)\right) \\
&= \frac{2}{x}\int_{\mathbb{R}^2} dy_1\,dy_2 \left(\frac{1}{x-y} + \frac{1}{y}\right)\left[-\frac{\partial}{\partial \bar{y}}\left(\bar{y}^n \exp\left(\frac{-|y|^2}{2}\right)\right) + n\bar{y}^{n-1}\exp\left(\frac{-|y|^2}{2}\right)\right] \\
&= \frac{2\pi}{x}\left(-\bar{x}^n \exp\left(\frac{-|x|^2}{2}\right)\right) + \frac{2n}{x}\phi_{n-1}.
\end{aligned} \tag{A.10}$$

In the calculation above we have used the identities

$$\int_{\mathbb{R}^2} dy_1\,dy_2 \frac{\bar{y}^{n-1}}{y} \exp\left(\frac{-|y|^2}{2}\right) = 0, \quad \text{for } n \ge 1,$$

which are established by switching to polar coordinates.

The induction step follows from $(A.10)$ and continuity. □

Since the Fourier transforms of $1/x$ and $\bar{x}^n e^{-|x|^2/2}$ are, respectively ([13], Appendix B1.7), $-2\pi i/\xi$ and $(-i)^n(2\pi)\bar{\xi}^n \exp(-|\xi|^2/2)$ (where $\xi = \xi_1 + i\xi_2$), the following corollary follows immediately.

Corollary A.3. *The following formula holds:*

$$\mathcal{F}\left(\frac{\bar{x}^n \exp(-|x|^2/2)}{x}\right) = (-i)^{n-1}\pi 2^{n+1} n! \frac{\left[\left(\sum_{j=0}^{n} |\xi|^{2j}/(2^j j!)\right)\exp(-|\xi|^2/2)\right] - 1}{\xi^{n+1}}. \tag{A.11}$$

Lemma A.4. *If $f(x) \in L^1(\mathbb{R})$, $\hat{f}(\xi) \in C^1(\mathbb{R}) \cap L^p(\mathbb{R})$, for some $p \in [1,\infty)$, and $\|\hat{f}'\|_{L^\infty(\mathbb{R})} < \infty$, then*

$$\left|\int_0^\infty dx\ f(x)e^{-ix\xi}\right| \le C_p(\|\hat{f}\|_{L^\infty(\mathbb{R})} + \|\hat{f}'\|_{L^\infty(\mathbb{R})} + \|\hat{f}\|_{L^p(\mathbb{R})}). \tag{A.12}$$

Proof. Let $\xi \in \mathbb{R}$. We have the following Sokhotski–Plemelj formula [18]:

$$\lim_{\epsilon\downarrow 0} \frac{1}{2\pi i}\int_{\mathbb{R}} dy\, \frac{g(y)}{y-(\xi+\epsilon i)} = \text{p.v.}\frac{1}{2\pi i}\int_{\mathbb{R}} dy\, \frac{g(y)}{y-\xi} + \frac{g(\xi)}{2}. \tag{A.13}$$

We can rewrite (A.13) as

$$\frac{1}{2\pi}\int_0^\infty dx\, \hat{g}(x)^{ix\xi} = \text{p.v.}\frac{1}{2\pi i}\int_{\mathbb{R}} dy\, \frac{g(y)}{y-\xi} + \frac{g(\xi)}{2}. \tag{A.14}$$

Applying (A.14) to $g = \mathcal{F}^{-1} f$, we find

$$\int_0^\infty dx\ f(x)e^{-ix\xi} = \frac{\hat{f}(\xi)}{2} - \frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{R}} dy\, \frac{\hat{f}(y)}{y-\xi}. \tag{A.15}$$

The estimate (A.12) follows from (A.15). □

Lemma A.5. *Let $g \in \mathcal{S}(\mathbb{R}^2)$ and*

$$f(k',k) = \frac{g(k') - \left[g(k) + \left(\partial_{\bar{k}} g\right)(k)\left(\overline{k'-k}\right)\right]\exp(-|k'-k|^2/2)}{k'-k}.$$

Then $f(\cdot,k)$ is continuous in $\mathbb{C}$ and C^1 in $\mathbb{C}\setminus\{k\}$, for each fixed k, and there exists a positive constant C_g such that

$$\int_{\mathbb{R}} dk_2'\left(\int_{\mathbb{R}} dk_1' \sum_{|l|\le 1}\left|\left(D_{k'}^l f\right)(k'e^{i\omega},k)\right|\right)^2 \le C_g \qquad \forall\,(k,\omega)\in\mathbb{C}\times\mathbb{R}. \tag{A.16}$$

Proof. The continuity and differentiability of $f(k', k)$ follow immediately from Taylor's theorem and (A.5).

If $|k' - ke^{-i\omega}| \geq 1$, then we can control $(D^l_{k'} f)(k'e^{i\omega}, k)$ by the functions $g(k'e^{i\omega})$, $[g(k) + ((\partial_{\bar{k}} g)(k))(k'e^{i\omega} - k)] \exp(-|k'e^{i\omega} - k|^2/2)$, and their derivatives. We have

$$|(D^l_{k'} f)(k'e^{i\omega}, k)| \leq C_g \left\{ \frac{1}{(1 + |k'_2|)(1 + |k'_1|^2)} + \frac{1}{[1 + |k'_2 - \mathrm{Im}(ke^{-i\omega})|][1 + |k'_1 - \mathrm{Re}(ke^{-i\omega})|^2]} \right\}.$$

On the other hand, Taylor's theorem also implies that $|(D^l_{k'} f)(k'e^{i\omega}, k)|$ is bounded by $C \sum_{|m|=0}^{2} \|D^m g\|_{L^\infty(\mathbb{R}^2)}$ if $|k' - ke^{-i\omega}| < 2$.

The estimate (A.16) now follows from the following splitting of the integral:

$$\int_{|k'_2 - \mathrm{Im}(ke^{-i\omega})|<1} dk'_2 \left[\int_{|k'_1 - \mathrm{Re}(ke^{-i\omega})|<1} dk'_1 (\cdot) + \int_{|k'_1 - \mathrm{Re}(ke^{-i\omega})|\geq 1} dk'_1 (\cdot) \right]^2 + \int_{|k'_2 - \mathrm{Im}(ke^{-i\omega})|\geq 1} dk'_2 \left[\int_{\mathbb{R}} dk'_1 (\cdot) \right]^2 . \qquad \square$$

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