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Exponential Decay in the Stark Effect

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Abstract. Let $H = -\Delta + V + Fx_1$ with $V(x_1, x_\perp)$ analytic in the first variable and $V(x_1 + ia, x_\perp)$ bounded and decreasing to zero as $x \to \infty$ for each $a \in \mathbb{R}$. Let ψ be an eigenvector of $-\Delta + V$ with negative eigenvalue. Among our results we show that for $F \neq 0$, $(\psi, e^{-itH}\psi)$ decays exponentially at a rate governed by the positions of the resonances of H. This exponential decay is in marked contrast to "conventional" atomic resonances for which power law decay is the rule.

I. Introduction

The phenomenon of exponential decay associated with resonances is well known in quantum mechanics. Arguments which predict this phenomenon can be found in almost any elementary quantum mechanics text (see, for example, [1]). One imagines (for example) a Hamiltonian of the form $H_0 = -\Delta + V$ to be weakly perturbed by an operator W which causes an eigenvalue E_0 of H_0 to disappear into the continuum of $H = H_0 + W$. If we prepare our system at $t = 0$ in a state ψ_0 with $H_0\psi_0 = E_0\psi_0$, non-rigorous arguments indicate [1, 2] that under rather general conditions, after a very short time one has

$$(\psi_0, e^{-itH}\psi_0) \cong e^{-itE_r} \tag{1.1}$$

where $E_r = E_0 + \Delta E - i\Gamma/2$. (Here we have assumed $(\psi_0, \psi_0) = 1$.) ΔE is the energy shift due to W which can be computed approximately using Rayleigh–Schrödinger perturbation theory and Γ is the transition rate given by Fermi's Golden Rule [1].

The validity of an equation such as (1.1) has been discussed briefly by Simon [3] in the dilation-analytic framework. Simon considers Hamiltonians H which are bounded below. In this case he concludes that the best one could hope for is an approximate validity when t is not too large (nor too small). The reason for the restriction to times which are not too large is easy to understand from the following well known argument: Suppose that a bound of the form

$$|(\psi_0, e^{-itH}\psi_0)| \leqq Ce^{-\alpha|t|} \tag{1.2}$$

were true for some $\alpha > 0$ and all $t \geqq 0$ (and thus by the self-adjointness of H for all

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$t \in \mathbb{R}$). It would then follow by Fourier transformation that the spectral measure $d(\psi_0, E_H(\xi)\psi_0)$ was absolutely continuous with Radon–Nikodym derivative $f(\xi)$ analytic in the strip $\{z : |\mathrm{Im}\, z| < \alpha\}$. Since such an f cannot vanish on a set of positive measure we must conclude $\sigma(H) = \mathbb{R}$.

We consider the Stark effect Hamiltonian in $L^2(\mathbb{R}^3)$:

$$H = H_0 + V, \quad H_0 = -\Delta + Fx_1$$

The assumptions we make about V are stated precisely at the beginning of Sect. II. They are satisfied, for example, if V is translation analytic [4], and for all $a \in \mathbb{R}$, $V(x_1 + ia, x_\perp)$ is bounded and $\lim_{x \to \infty} |V(x_1 + ia, x_\perp)| = 0$. Unfortunately they are not satisfied for the Coulomb potential. However as noted in [4] they are satisfied if the Coulomb potential $\frac{1}{r}$ is replaced by $\rho * \frac{1}{r}$, where ρ is a Gaussian charge distribution.

Under our assumptions for $F > 0$, H has purely absolutely continuous spectrum filling all of $\mathbb{R}$ and thus the objections to a bound of the form (1.2) are no longer valid. In Section II we show that the resonances of H in the lower half-plane can be numbered so that their widths $\Gamma_j = -2\mathrm{Im}\, E_j$ satisfy $0 < \Gamma_1 \leqq \Gamma_2 \leqq \cdots \Gamma_j \leqq \cdots$ and that for translation entire vectors ψ and ϕ satisfying certain domain conditions we have the expansion (as $t \to \infty$)

$$(\psi, e^{-itH}\phi) = \sum_{\Gamma_j \leqq \alpha} C_j(\psi, \phi) e^{-itE_j} + O(e^{-t(\alpha+\varepsilon)/2})$$

The constants C_j are computed in terms of the projections onto resonance eigenfunctions.

In Section III we discuss the connection between the translation and dilation analytic frameworks.

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Before we begin it seems fitting to say something about the connection of our results with the Lax–Phillips theory of scattering [5] where local exponential decay results have been known for some time (see [6] and [7] and references given there). On a fundamental level the Stark operator and the operators considered by Lax and Phillips are very similar. As shown in [8], for a large class of V, $-\Delta + V + Fx_1$ is unitarily equivalent to $-i\frac{d}{dx} \otimes I$ if $F \neq 0$ and this is also true of the generators of the Lax–Phillips unitary propagators. On the other hand we do not see how to fit the Stark operator into the Lax–Phillips theory although this may be more a function of our ignorance than the unsuitability of their framework.

II. Asymptotics of $(\psi, e^{-itH}\psi)$

We begin by stating our conditions on V. Let $H_0 = -\Delta + Fx_1$ in $L^2(\mathbb{R}^3)$, $V_a(x) = V(x_1 + a, x_\perp)$.

a). $V(x)$ is a real measurable function such that for almost all $x_\perp$, $V(z, x_\perp)$ is an entire function of z and the operator $V_z(H_0 + i)^{-1}$ is compact and analytic for all $z \in \mathbb{C}$.

b). For each $a \in \mathbb{R}$ and $\varepsilon > 0$ there is a $p \in (3/2, \infty)$ and two functions V^1 and V^2 with

$$V_{ia} = V^1 + V^2, \quad V^1 \in L^p(\mathbb{R}^3), \quad \|V^2\|_\infty < \varepsilon$$

c). $H = H_0 + V$ has purely continuous spectrum.

We remark that a) is just the statement that V is "translation analytic" in $\mathbb{C}$ as defined in [4]. Also for purposes of orientation it is useful to note that if $V(z, x_\perp)$ is entire for each $x_\perp$ with $V(x_1 + ia, x_\perp)$ bounded and decreasing to zero as $x = (x_1, x_\perp) \to \infty$ for all $a \in \mathbb{R}$, then a), b), and c) are all satisfied. (The last condition is satisfied because the Cauchy formula implies $\frac{\partial}{\partial x_1} V(x_1, x_\perp)$ is bounded and decreases to zero as $|x_1| \to \infty$ uniformly in $x_\perp$, and this implies absence of bound states [4].)

We summarize the relevant results from [4]. Define $H(\lambda) = H_0 + V_\lambda + F_\lambda$. Then the family of operators $\{H(\lambda) : \lambda \in \mathbb{C}\}$ is type A analytic in the sense of Kato [9]. The spectrum of $H(\lambda)$ is as follows. We have $\sigma_{\text{ess}}.\ (H(\lambda)) \subseteq \mathbb{R} + iF(\operatorname{Im} \lambda)$ and for $F > 0$, $\operatorname{Im} \lambda < 0$ we have $\sigma(H(\lambda)) \subseteq \{z : 0 > \operatorname{Im} z \geqq F(\operatorname{Im} \lambda)\}$. The spectrum of $H(\lambda)$ in $\{z : 0 > \operatorname{Im} z > F(\operatorname{Im} \lambda)\}$ consists of discrete eigenvalues of finite algebraic multiplicity. These eigenvalues do not depend on λ as long as the line $\mathbb{R} + iF(\operatorname{Im} \lambda)$ does not intersect them. We call all eigenvalues in $\bigcup_{\operatorname{Im} \lambda < 0} \sigma_{\text{disc.}}(H(\lambda))$ resonances of H. It is shown in [10] that if V is both translation analytic and dilation analytic the resonances defined in [4] and [10] coincide.

The following estimate is useful in controlling any local singularities which V may have.

Lemma 2.1 *Suppose f and g are in $L^p(\mathbb{R}^3)$ with $p > 3$. Then there is a constant C independent of F so that*

$$\sup_{E \in \mathbb{R}} \| f(H_0 - E + i\gamma)^{-1} \| \leqq C \| f \|_p |\gamma|^{-1 + 3/2p} \tag{2.1}$$

$$\sup_{E \in \mathbb{R}} \| f(H_0 - E + i\gamma)^{-1} g \| \leqq C \| f \|_p \| g \|_p |\gamma|^{-1 + 3/p} \tag{2.2}$$

Proof. The basic technique is that used by Kato [11] to prove that certain multiplication operators are smooth with respect to $-\Delta$. We use the formula [4]

$$e^{itH_0} = e^{itFx_1/2} e^{-it\Delta} e^{itFx_1/2} e^{it^3F^2/12} \tag{2.3}$$

to reduce expressions involving H_0 to those involving only $-\Delta$. Thus for example

$$\| f e^{itH_0} g \| = \| f e^{-it\Delta} g \|. \tag{2.4}$$

We then use (following Kato [11]) the fact that

$$\| f e^{it\Delta} g \| \leqq C(p) \| f \|_p \| g \|_p t^{-3/p} \tag{2.5}$$

This gives (for $\gamma > 0$)

$$\|f(H_0 - E + i\gamma)^{-1}g\| \leqq C(p)\int_0^\infty \|fe^{-it\Delta}g\|e^{-\gamma t}dt$$
$$\leqq C(p)\gamma^{-1+3/p}\|f\|_p\|g\|_p \tag{2.6}$$

A similar result clearly holds if $\gamma < 0$ so that (2.2) is proved. To establish (2.1) we write

$$\|f(H_0 - E + i\gamma)^{-1}\|^2 = \|f(H_0 - E + i\gamma)^{-1}(H_0 - E - i\gamma)^{-1}\bar{f}\|$$
$$= \frac{1}{2|\gamma|}\|f[(H_0 - E + i\gamma)^{-1} - (H_0 - E - i\gamma)^{-1}]\bar{f}\|$$
$$\leqq |\gamma|^{-1}\|f(H_0 - E - i\gamma)^{-1}\bar{f}\| \tag{2.7}$$

The proof is therefore complete.

The crucial estimate for our result is the following.

Proposition 2.2. *Suppose f and g are bounded and have compact support in $\mathbb{R}^3$. Then for $F \geqq 0$*

$$\lim_{E \to \pm\infty} \|f(H_0 - E - i\gamma)^{-1}g\| = 0$$

uniformly for γ in compacts of $\mathbb{R}\backslash\{0\}$.

Proof. We write (for $\gamma > 0$)

$$f(H_0 - E + i\gamma)^{-1}g = \int_0^\infty (fe^{itH_0}g)e^{-\gamma t}e^{-iEt}dt = \int_0^\infty F(t)e^{-iEt}dt$$
$$= \int_0^\varepsilon F(t)e^{-iEt}dt + \int_\varepsilon^\infty F(t)e^{-iEt}dt$$

The first term has norm $\leqq \|f\|_\infty \|g\|_\infty \varepsilon$ while the second can be integrated by parts to give

$$\int_\varepsilon^\infty F(t)e^{-iEt}dt = -iE^{-1}F(\varepsilon)e^{-iE\varepsilon} + \frac{1}{iE}\int_\varepsilon^\infty F'(t)e^{-iEt}dt$$

We write $F(t)$ using Eqn. (2.3) as

$$F(t) = fe^{itFx_1/2}e^{-it\Delta}e^{itFx_1/2}ge^{-\gamma t}e^{it^3F^2/12}$$

and note that $F'(t)$ is an integral operator with kernel

$$Ce^{-\gamma t}f(x)g(y)e^{itF(x_1+y_1)/2}e^{it^3F^2/12}t^{-3/2}e^{-i|x-y|^2/4t}$$
$$\cdot\left\{\frac{i}{2}F(x_1 + y_1) + \frac{it^2}{4}F^2 - \frac{3}{2}t^{-1} + \frac{i}{4}t^{-2}|x - y|^2 - \gamma\right\}$$

If $\gamma \geqq \delta > 0$, then the Hilbert–Schmidt norm of $\frac{d}{dt}F(t)$ is bounded by

$$C(\delta)t^{-7/2}e^{-\delta t/2}$$

Thus for each $\varepsilon > 0$,

$$\| f(H_0 - E + i\gamma)^{-1} g \| \leqq (\varepsilon + |E|^{-1}) \| f \|_\infty \| g \|_\infty + |E|^{-1} C(\delta) \int_\varepsilon^\infty t^{-7/2} e^{-\delta t/2} dt$$

This is easily seen to give the result.

We remark that a similar technique can be used to prove the same result uniformly for γ in $(0, \infty)$ or $(-\infty, 0)$. Thus for $\gamma > 0$ one uses Eqn. (2.5) to show that $\| \int_0^\varepsilon F(t) e^{-iEt} dt + \int_{1/\varepsilon}^\infty F(t) e^{-iEt} dt \| \to 0$ as $\varepsilon \downarrow 0$ uniformly in γ for $\gamma > 0$ and then integrates $\int_\varepsilon^{1/\varepsilon} F(t) e^{-iEt} dt$ by parts as above. We will have no need for this result however.

Proposition 2.3. *Suppose W is a measurable function such that for each $\varepsilon > 0$ there exist W_1 and W_2 with $W = W_1 + W_2$ and $W_1 \in L^p(\mathbb{R}^3)$ with $\infty > p > 3/2$ while $\| W_2 \|_\infty < \varepsilon$. Then*

$$\lim_{E \to \pm\infty} \| |W|^{1/2} (H_0 - E - i\gamma)^{-1} |W|^{1/2} \| = \lim_{E \to \pm\infty} \| |W|^{1/2} (H_0 - E - i\gamma)^{-1} \| = 0$$

uniformly for γ in compacts of $\mathbb{R} \backslash \{0\}$.

Proof. If f and g are in L^q with $q > 3$, a simple approximation argument shows that $\lim_{E \to \pm\infty} \| f(H_0 - E - i\gamma)^{-1} g \| = 0$ with the stated uniformity. In addition Eqn. (2.3) shows that $\lim_{E \to \pm\infty} \| f(H_0 - E - i\gamma)^{-1} \| = 0$ with the same uniformity. Using the fact that $|W|^{1/2} \leqq |W_1|^{1/2} + |W_2|^{1/2}$ and that $|f_1| \geqq |f_2|$, $|g_1| \geqq |g_2| \Rightarrow \| f_1 (H_0 - E - i\gamma)^{-1} g_1 \| \geqq \| f_2 (H_0 - E - i\gamma)^{-1} g_2 \|$ we easily derive the stated result.

Theorem 2.4. *Suppose $F > 0$ and that V satisfies the conditions a), b), and c) stated at the beginning of this section. Then if $a > 0$ is given, there exists an $N(a) > 0$ so that $H(-2ia)$ has no eigenvalues in $B_a = \{ z : 0 \geqq \operatorname{Im} z \geqq -aF,\ |\operatorname{Re} z| \geqq N(a) \}$ and in addition for any $b > a$ we have*

$$\sup_{z \in \tilde{B}_a} \| (z - H(-ib))^{-1} \| < \infty$$

where $\tilde{B}_a = B_a \cup \{ z : 0 \leqq \operatorname{Im} z \leqq a \}$.

Proof. Write $V_{-ib} = AB$ where $A = |V_{-ib}|^{1/2}$ and $B = |V_{-ib}|^{1/2} \operatorname{sgn} V_{-ib}$. Let $H_0(\gamma) = -\Delta + Fx_1 + F\gamma$, $R_0 = (z - H_0(-ib))^{-1}$, $R = (z - H(-ib))^{-1}$. Choose $N(a)$ so that $\| B R_0 A \| \leqq 1/2$ for $z \in \tilde{B}_a \cap \{ z : |\operatorname{Re} z| \geqq N(a) \} \equiv C_a$. This is possible by proposition (2.3). Then in C_a, the Neumann series

$$\sum_{n=0}^\infty R_0 (V R_0)^n$$

converges since we have

$$\sum_{n=0}^{\infty} R_0(VR_0)^n = R_0 + R_0A\left(\sum_{n=0}^{\infty} (BR_0A)^n\right)BR_0$$

Thus in C_a we have

$$\|R\| \leqq \|R_0\| + \|R_0A\| \, \|BR_0\|(1-1/2)^{-1}$$

Since $\|R_0\| \leqq ((b-a)F)^{-1}$ and $\|R_0A\| \quad \|BR_0\|$ is bounded in C_a by proposition 2.3, we have

$$\sup_{z\in C_a} \|(z-H(-ib))^{-1}\| < \infty$$

The bound $\sup_{z\in \tilde{B}_a\backslash C_a} \|(z-H(-ib))^{-1}\| < \infty$ follows from the analyticity of $(z-H(-ib))^{-1}$ in the upper half plane and on the real axis.

Theorem 2.4 implies that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of different resonances in the lower half plane then $\operatorname{Im} E_j \to -\infty$. We number the points of $\bigcup_{a>0} \sigma_{\text{disc.}}(H(-ia))$ so that

$$0 > \operatorname{Im} E_1 \geqq \operatorname{Im} E_2 \cdots \geqq \operatorname{Im} E_j \geqq \cdots$$

For $E_j \in \sigma_{\text{disc.}}(H(-ia))$ we write

$$P_j(-ia) = \frac{1}{2\pi i}\oint_{|z-E_j|=\varepsilon} (z-H(-ia))^{-1}\,dz$$

where ε is small enough so E_j is the only point of $\sigma(H(-ia))$ in $|z-E_j| \leqq \varepsilon$.

Our main result is

Theorem 2.5. *Suppose ψ, ϕ, $H_0\psi$, and $H_0\phi$ are entire vectors for the translation group $U(b)$ $(U(b)f(x) = f(x_1+b, x_\perp) \equiv f_b(x))$. Then given $\alpha > 0$ we have for $t \geqq 0$*

$$(\psi, e^{-itH}\phi) = \sum_{-\operatorname{Im} E_j \leqq \alpha/2} (\psi_{\bar{ia}}, P_j(-ia)\phi_{-ia})e^{-itE_j} + r(t)$$

where

$$|r(t)| \leqq \text{const}\exp\left(-\frac{t}{2}(\alpha+\varepsilon)\right).$$

for some $\varepsilon > 0$. Here $a > \alpha/2F$. The quantities $C_j = (\psi_{\bar{z}}, P_j(z)\phi_z)$ are independent of z as long as $F \operatorname{Im} z < \operatorname{Im} E_j$.

Remark. If ψ_0 is a negative eigenvalue of $-\Delta + V$ where V satisfies the assumptions of Theorem 2.5, then ψ is a translation entire vector. In addition, by the Combes–Thomas argument [12] $(\psi_0)_z$ is in the domain of $e^{\alpha x_1}$ for small $|\alpha|$ so $H_0\psi_0$ is also translation entire. Thus Theorem 2.5 holds for $\psi = \phi = \psi_0$.

Proof. Let $F(z) = (\psi, (z - H)^{-1}\phi)$ for $\mathrm{Im}\, z > 0$. F has a meromorphic continuation to $\mathbb{C}$ which we again denote by $F(z)$. For $\mathrm{Im}\, z > -aF$ this continuation is given explicitly by $F(z) = (\psi_{ia}, (z - H(-ia))^{-1}\phi_{-ia})$. Similarly let $G(z) = (\psi, (z - H)^{-1}\phi)$ for $\mathrm{Im}\, z < 0$. G has a meromorphic continuation to $\mathbb{C}$ which we again denote by $G(z)$. For $\mathrm{Im}\, z < aF$ this continuation is given explicitly by $G(z) = (\psi_{-ia}, (z - H(ia))^{-1}\phi_{ia})$.

Let

$$\begin{aligned} Q(\lambda) &= \lim_{\epsilon \downarrow 0} \left(-\frac{1}{2\pi i}\right)(\psi, [(\lambda + i\varepsilon - H)^{-1} - (\lambda - i\varepsilon - H)^{-1}]\phi),\ \lambda \in \mathbb{R} \\ &= -(2\pi i)^{-1}(F(\lambda) - G(\lambda)) \end{aligned} \tag{2.8}$$

We have by the spectral theorem,

$$(\psi, e^{-itH}\phi) = \int_{-\infty}^{\infty} Q(\lambda) e^{-it\lambda} d\lambda \tag{2.9}$$

The function Q has a meromorphic continuation to $\mathbb{C}$ given by

$$Q(z) = -(2\pi i)^{-1}(F(z) - G(z))$$

which by assumption *c*) of the beginning of this section and proposition 2.4 is analytic in a (possibly narrow) strip $|\mathrm{Im}\, z| < \delta$.

We use the identity (for large $|E|$ and $0 \leqq \gamma < a$)

$$\begin{aligned} &(E - i\gamma - H(-ia))^{-1} = (E - i\gamma)^{-1} + (E - i\gamma)^{-2} H(-ia) \\ &\quad + (E - i\gamma)^{-2} H(-ia)(E - i\gamma - H(-ia)^{-1} H(-ia) \end{aligned}$$

in the expression $F(z) = (\psi_{ia}, (z - H(-ia))^{-1}\phi_{-ia})$ and find

$$\begin{aligned} F(E - i\gamma) &= (E - i\gamma)^{-1}(\psi_{ia}, \phi_{-ia}) + O(|E|^{-2}) \\ &= (E - i\gamma)^{-1}(\psi, \phi) + O(|E|^{-2}) \end{aligned}$$

for $E \in \mathbb{R}$ and $|E|$ large, uniformly for γ in compacts of $(-\delta, a)$. Similarly

$$G(E - i\gamma) = (E - i\gamma)^{-1}(\psi, \phi) + O(|E|^{-2})$$

uniformly for γ in compacts of $(-\delta, a)$. Thus for large $|E|$ and γ in closed intervals of $(-\delta, a)$

$$Q(E - i\gamma) = O(|E|^{-2}) \tag{2.10}$$

We can therefore shift the contour of integration from the real axis in the integral of Eqn. (2.9) downward to a line parallel to the real axis picking up contributions from poles in the standard way.

We find

$$\begin{aligned} (\psi, e^{-itH}\phi) = &\sum_{-\mathrm{Im}\, E_j \leqq \alpha/2} \mathrm{Res}\, F(z)|_{z = E_j} e^{-itE_j} \\ &+ e^{-t(\alpha + \varepsilon)/2} \int_{-\infty}^{\infty} Q(\lambda - i(\alpha + \varepsilon)/2) e^{-it\lambda} d\lambda \end{aligned}$$

where ε is chosen so that all E_k with $-\operatorname{Im} E_k > \alpha/2$ also satisfy $-\operatorname{Im} E_k > \frac{\alpha+\varepsilon}{2}$. Since $Q(\lambda - i(\alpha+\varepsilon)/2)$ is in L^1 by the estimate (2.10), the result follows.

III. Dilation Analytic Potentials

If V is only assumed to be dilation analytic with $V(\Theta)(-\Delta+1)^{-1}$ compact and analytic in a strip $|\operatorname{Im}\Theta| < \Theta_0$, it can be shown that [10,13] the functions F and G defined in the proof of Theorem 2.5, have meromorphic continuations to $\mathbb{C}$. (Here, of course, ϕ and ψ must be dilation analytic vectors.) However because

$$\lim_{E\to+\infty} \|(-\Delta e^{-2i\Phi} + Fx_1 e^{i\Phi} - E + i\gamma)^{-1}\| = \infty$$

for any $\Theta \in (0, \pi/3)$ and $\gamma \in \mathbb{R}$ (see [10]), a proof following that given for Theorem 2.4 that $Q(z)$ is L^1 along lines parallel to the real axis must surely fail. We still believe, however, that a result analogous to Theorem 2.5 holds for the Coulomb potential.

Suppose that V satisfies the assumption of Theorem 2.5 and in addition is dilation analytic with $V(\Theta)(-\Delta+1)^{-1}$ compact and analytic in a strip $|\operatorname{Im}\Theta| < \Theta_0$. Suppose ψ is a negative eigenvalue of $-\Delta + V$ with non-degenerate eigenvalue $E(0)$. Then the functions F and G can also be written for any Θ with $0 < \Theta < \operatorname{Max}\{\Theta_0, \pi/3\}$

$$F(z) = (\psi(-i\Theta), (z - H(i\Theta))^{-1}\psi(i\Theta))$$
$$G(z) = (\psi(i\Theta), (z - H(-i\Theta))^{-1}\psi(i\Theta))$$

for all $z \in \mathbb{C}$. Here $\psi(\Theta)(x) = e^{3\Theta/2}\psi(e^{\Theta}x)$ for $\Theta \in \mathbb{R}$. Clearly then

$$(\psi_{ia}, P_j(-ia)\psi_{-ia}) = (\psi(-i\Theta), \tilde{P}_j(i\Theta)\psi(i\Theta))$$

where $\tilde{P}_j(i\Theta)$ is the spectral projection of $H(i\Theta)$ corresponding to the eigenvalue E_j. For F small, there is one and only one eigenvalue E_{j_0} of $H(i\Theta)$ close to $E(0)$ and

$$\lim_{F\to 0} (\psi(-i\Theta), P_{j_0}(i\Theta)\psi(i\Theta)) = (\psi, \psi)$$

(see [10] for a proof). We presume (but do not know how to prove that if F is small we can write

$$(\psi, e^{-itH}\psi) = (\psi, \psi)e^{-itE_{j_0}} + r(t, F)$$

where $|r(t, F)|$ can be made arbitrarily small uniformly in $t > 0$ if F is chosen small enough. This would be an interesting result.

References

1. Merzbacher, E.: Quantum mechanics. New York Wiley, 1961
2. Weisskopf, V. F., Wigner, E. P.: Z. Physik **63**, 54 (1930)
3. Simon, B.: Ann. Math. **97**, 247–274 (1973)
4. Avron, J., Herbst, I.: Commun. Math. Phys. **52**, 239–254 (1977)
5. Lax, P. D., Phillips, R. S.: Scattering theory. New York: Academic Press 1967

6. Lax, P. D., Morawetz, C. S., Phillips, R. S.: Commun. Pure Appl. Math. **16**, 477–486 (1963)
7. Lax, P. D., Phillips, R. S.,: Commun. Pure Appl. Math. **22**, 737–787 (1977)
8. Herbst, I.: Math Z. **155**, 55–70 (1977)
9. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1976
10. Herbst, I.: Commun. Math. Phys. **64**, 279–298 (1979)
11. Kato, T.: Math. Ann. **162**, 258–279 (1966)
12. Combes, J. M., Thomas, L.: Commun. Math. Phys. **34**, 251–270 (1973)
13. Herbst, I., Simon, B.: Dilation analyticity in constant electric field. II. The N-body problem, Borel summability. To be submitted to Commun. Math. Phys.

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