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Influence of large deflections on the dynamic stability of nonlinear viscoelastic plates

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Summary. The dynamic stability analysis of nonlinear viscoelastic plates is presented. The problem is formulated within the large deflections theory for isotropic plates, and the Leaderman representation of nonlinear viscoelasticity for the material behavior. The influence of the various parameters on the stability/instability possible situation is investigated within the concept of the Lyapunov exponents. In addition, it is shown that in some cases the system has a chaotic behavior.

1 Introduction

The subject of the dynamic stability of elastic linear structures was extensively investigated in [1], where the motion is governed by the Mathieu equation and the stability characterizations are given by the Strutt diagram. Further results were given e.g., in [2], [3], in a review paper and a monograph. The same happens, for example, in bridge dynamics or in wing flutter (instability of aircraft in air flow).

When plates are considered, and the deflections are not small compared to the thickness of the plate but still small with respect to the other dimensions, the analysis of the problem should be extended in order to include the plate middle plane strains [4], [5], which leads to the theory of large deflections (see e.g. [6]). The dynamic stability of elastic plates with large deflections was also investigated in [1] for particular cases.

When the structure is made of a viscoelastic material, the problem becomes much more complicated since the equation of motion turns out to be an integro-differential one, rather than an ordinary differential equation as in the elastic case. The solution of this problem in the linear case was given in [7] within the averaging method, and in [8]–[10] by using the spring-dashpot representation. The dynamic stability analysis of viscoelastic homogeneous plates, investigated within the concept of the Lyapunov exponents, was performed in [11]. This procedure was used also in [12] to investigate the dynamic stability of shear deformable viscoelastic laminated plates. In these two studies the Boltzmann superposition principle was incorporated, enabling the modeling of any linear viscoelastic material.

However, it is well known that many materials (polymers, for example) are not linear and should be modeled non-linearily in order to give an adequate description of their behavior. Smart and Wiliams [13] made a comparison investigation about the response of polypropylene and polyvinylchloride, by using three different single integral representations of *nonlinear* viscoelasticity: the Leaderman model [14], the Schapery model [15] and the Bernstein-Kearsley-Zapas model [16], [17]. Their main conclusion was that the Leaderman model is the most useful representation, where prediction and simplicity are concerned.

In a previous work by the authors [18], the dynamic stability of nonlinear viscoelastic homogeneous plates was investigated within the small deflections theory and the concept of the Lyapunov exponents.

In the present investigation we use the large deflections theory to derive the integro-differential equations of motion within the Leaderman model, which are nonlinear from the material and the geometrical points of view and with time-dependent coefficients.

2 Problem formulation

Within the theory of large deflections, the equations of motion of an isotropic plate subjected to in-plane loads are (see e.g., [6])

$$N_{x,x} + N_{xy,y} = \varrho h\ddot{u} \tag{1}$$

$$N_{y,y} + N_{yx,x} = \varrho h\ddot{v} \tag{2}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + (N_x w_{,x})_{,x} + (N_y w_{,y})_{,y} + (N_{xy} w_{,y})_{,x} + (N_{xy} w_{,x})_{,y} = \varrho h\ddot{w} \tag{3}$$

where u, v and w are the displacements of a point on the middle plane in the x, y and z directions (see Fig. 1), ϱ is the material density, and h is the plate thickness.

The in-plane forces N_x, N_y and N_{xy} are given by

$$\begin{aligned} N_x &= N_x{}^0 + \bar{N}_x \\ N_y &= N_y{}^0 + \bar{N}_y \\ N_{xy} &= N_{xy}^0 + \bar{N}_{xy}. \end{aligned} \tag{4}$$

Here $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ are the in-plane applied edge loads while $N_x{}^0$, $N_y{}^0$ and N_{xy}^0 are the in-plane resultants given by

$$[N_x{}^0, N_y{}^0, N_{xy}^0] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \sigma_{xy}]\, dz \tag{5}$$

where σ_x, σ_y and σ_{xy} are the stress components.

The various moments, M_x, M_y, and M_{xy}, are given by

$$[M_x, M_y, M_{xy}] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \sigma_{xy}]\, z\, dz. \tag{6}$$

For nonliner viscoelastic materials, the Leaderman stress-strain constitutive relation is given by [14]

$$\sigma(t) = Q(0)\, g[\varepsilon(t)] + \int_{0^+}^{t} \dot{Q}(t-\tau)\, g[\varepsilon(\tau)]\, d\tau \tag{7}$$

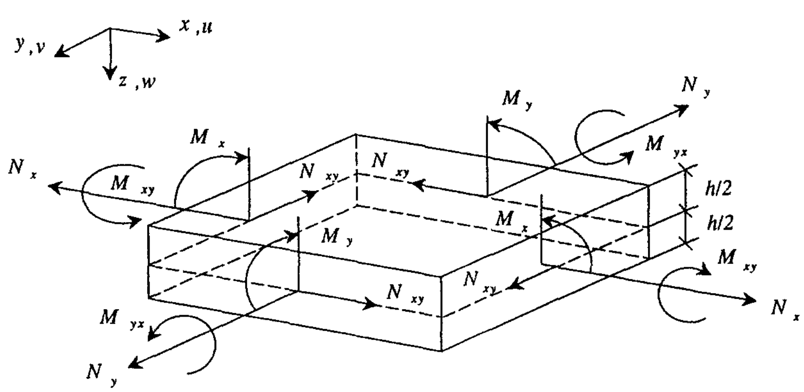

Fig. 1. Resultants and couples in rectangular Cartesian coordinates

where

$$g[\varepsilon(t)] = \varepsilon(t) + \beta\varepsilon(t)^2 + \gamma\varepsilon(t)^3 + \dots$$

and β and γ are constants, such that for small strain $g(\varepsilon) \to \varepsilon$.
In the state of plane stress for isotropic plates

$$Q_{11}(t) = Q_{22}(t) = \frac{E(t)}{1 - \nu(t)^2}$$
$$Q_{12}(t) = \nu(t)\, Q_{11}(t) \tag{8}$$
$$Q_{66}(t) = \frac{1 - \nu(t)}{2}\, Q_{11}(t)$$

where $E(t)$ is a time-dependent relaxation function which at $t = 0$ denotes the initial Young's modulus of the material, while $\nu(t)$ is the time-dependent Poisson's ratio.

The strain-displacement relations for an homogeneous thin plate are given by

$$\varepsilon_x = e_x - zw_{,xx}$$
$$\varepsilon_y = e_y - zw_{,yy} \tag{9}$$
$$\varepsilon_{xy} = e_{xy} - 2zw_{,xy}$$

and the von Karman strains of the middle plane e_x, e_y and e_{xy} by

$$e_x = u_{,x} + \frac{1}{2}\, w_{,x}^2$$
$$e_y = v_{,y} + \frac{1}{2}\, w_{,y}^2 \tag{10}$$
$$e_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y}.$$

By separating the variables, the assumed solution functions for the various displacements are given in terms ot time and spatial functions,

$$w(x, y, t) = f_1(t)\, \varphi(x, y)$$
$$u(x, y, t) = f_2(t)\, \psi(x, y) \tag{11}$$
$$v(x, y, t) = f_3(t)\, \phi(x, y).$$

Substituting Eqs. (5), (7)–(11) into Eq. (4) yields the following expressions for the in-plane forces:

$$N_x = hQ_{11}(0) \left\{ \left[f_2(t)\, \psi_{,x} + \frac{1}{2} (f_1(t)\, \varphi_{,x})^2 \right] + \beta_x \left[f_2(t)\, \psi_{,x} + \frac{1}{2} (f_1(t)\, \varphi_{,x})^2 \right]^2 \right.$$
$$+ \gamma_x \left[f_2(t)\, \psi_{,x} + \frac{1}{2} (f_1(t)\, \varphi_{,x})^2 \right]^3 + \nu \left[f_3(t)\, \phi_{,y} + \frac{1}{2} (f(t)_1\, \varphi_{,y})^2 \right]$$
$$\left. + \nu\beta_y \left[f_3(t)\, \phi_{,y} + \frac{1}{2} (f(t)_1\, \varphi_{,y})^2 \right]^2 + \nu\gamma_y \left[f_3(t)\, \phi_{,y} + \frac{1}{2} (f(t)_1\, \varphi_{,y})^2 \right]^3 \right\}$$

$$+ h \int_{0^+}^{t} \dot{Q}_{11}(t-\tau) \left\{ \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right] + \beta_x \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right]^2 \right.$$

$$+ \gamma_x \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right]^3 + v \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right]$$

$$\left. + v\beta_y \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right]^2 + v\gamma_y \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right]^3 \right\} d\tau + \bar{N}_x \tag{12}$$

$$N_y = hQ_{11}(0) \left\{ \left[f_3(t)\, \phi_{,y} + \frac{1}{2}(f(t)_1\, \varphi_{,y})^2 \right] + \beta_y \left[f_3(t)\, \phi_{,y} + \frac{1}{2}(f(t)_1\, \varphi_{,y})^2 \right]^2 \right.$$

$$+ \gamma_y \left[f_3(t)\, \phi_{,y} + \frac{1}{2}(f_1(t)\, \varphi_{,y})^2 \right]^3 + v \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right]$$

$$\left. + v\beta_x \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right]^2 + v\gamma_x \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right]^3 \right\}$$

$$+ h \int_{0^+}^{t} \dot{Q}_{11}(t-\tau) \left\{ \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right] + \beta_y \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right]^2 \right.$$

$$+ \gamma_y \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}(f_1(\tau)\, \varphi_{,y})^2 \right]^3 + v \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right]$$

$$\left. + v\beta_x \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right]^2 + v\gamma_x \left[f_2(\tau)\, \psi_{,x} + \frac{1}{2}(f_1(\tau)\, \varphi_{,x})^2 \right]^3 \right\} d\tau + \bar{N}_y \tag{13}$$

$$N_{xy} = h\frac{1-v}{2} Q_{11}(0)\{[f_3(t)\, \phi_{,x} + f_2(t)\, \psi_{,y} + f_1(t)^2\, \varphi_{,x}\varphi_{,y}]$$

$$+ \beta_{xy}[f_3(t)\, \phi_{,x} + f_2(t)\, \psi_{,y} + f_1(t)^2\, \varphi_{,x}\varphi_{,y}]^2 + \gamma_{xy}[f_3(t)\, \phi_{,x} + f_2(t)\, \psi_{,y} + f_1(t)^2\, \varphi_{,x}\varphi_{,y}]^3\}$$

$$+ h\frac{1-v}{2} \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\{[f_3(\tau)\, \phi_{,x} + f_2(\tau)\, \psi_{,y} + f_1(\tau)^2\, \varphi_{,x}\varphi_{,y}] + \beta_{xy}[f_3(\tau)\, \phi_{,x} + f_2(\tau)\, \psi_{,y}$$

$$+ f_1(\tau)^2\, \varphi_{,x}\varphi_{,y}]^2 + \gamma_{xy}[f_3(\tau)\, \phi_{,x} + f_2(\tau)\, \psi_{,y} + f_1(\tau)^2\, \varphi_{,x}\varphi_{,y}]^3\}\, d\tau + \bar{N}_{xy} \tag{14}$$

and substituting Eqs. (7)–(11) into Eq. (6) yields

$$M_x = -I_1(\varphi_{,xx} + v\varphi_{,yy}) \left(Q_{11}(0)\, f_1(t) + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)\, d\tau \right)$$

$$- 2\beta_x I_1 \varphi_{,xx} \left\{ Q_{11}(0)\, f_1(t) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right] \right.$$

$$\left. + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right] d\tau \right\}$$

$$- 3\gamma_x I_1 \varphi_{,xx} \left\{ Q_{11}(0)\, f_1(t) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}(f_1(t)\, \varphi_{,x})^2 \right]^2 \right.$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}\left(f_1(t)\, \varphi_{,x}\right)^2 \right]^2 d\tau \Bigg\}$$

$$- I_2(\gamma_x \varphi_{,xx}^3 + \nu\gamma_y \varphi_{,yy}^3) \left(Q_{11}(0)\, f_1(t)^3 + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)^3\, d\tau \right)$$

$$- 2\nu\beta_y I_1 \varphi_{,yy} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_3(t)\, \phi_{,y} + \frac{1}{2}\left(f_1(t)\, \phi_{,y}\right)^2 \right]$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}\left(f_1(\tau)\, \phi_{,y}\right)^2 \right] d\tau \Bigg\}$$

$$- 3\nu\gamma_y I_1 \varphi_{,yy} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_3(t)\, \phi_{,y} + \frac{1}{2}\left(f_1(t)\, \phi_{,y}\right)^2 \right]^2$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}\left(f_1(\tau)\, \phi_{,y}\right)^2 \right]^2 d\tau \Bigg\} \tag{15}$$

$$M_y = -I_1(\nu\varphi_{,xx} + \varphi_{,yy}) \left(Q_{11}(0)\, f_1(t) + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)\, d\tau \right)$$

$$- 2\nu\beta_x I_1 \varphi_{,xx} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}\left(f_1(t)\, \varphi_{,x}\right)^2 \right]$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}\left(f_1(t)\, \varphi_{,x}\right)^2 \right] d\tau \Bigg\}$$

$$- 3\nu\gamma_x I_1 \varphi_{,xx} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}\left(f_1(t)\, \varphi_{,x}\right)^2 \right]^2$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_2(t)\, \psi_{,x} + \frac{1}{2}\left(f_1(t)\, \varphi_{,x}\right)^2 \right]^2 d\tau \Bigg\}$$

$$- I_2(\nu\gamma_x \varphi_{,xx}^3 + \gamma_y \varphi_{,yy}^3) \left(Q_{11}(0)\, f_1(t)^3 + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)^3\, d\tau \right)$$

$$- 2\beta_y I_1 \varphi_{,yy} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_3(t)\, \phi_{,y} + \frac{1}{2}\left(f_1(t)\, \phi_{,y}\right)^2 \right]$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}\left(f_1(\tau)\, \phi_{,y}\right)^2 \right] d\tau \Bigg\}$$

$$- 3\gamma_y I_1 \varphi_{,yy} \Bigg\{ Q_{11}(0)\, f_1(t) \left[f_3(t)\, \phi_{,y} + \frac{1}{2}\left(f_1(t)\, \phi_{,y}\right)^2 \right]^2$$

$$+ \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau) \left[f_3(\tau)\, \phi_{,y} + \frac{1}{2}\left(f_1(\tau)\, \phi_{,y}\right)^2 \right]^2 d\tau \Bigg\} \tag{16}$$

$$M_{xy} = -[1 - \nu(t)]\, I_1 \varphi_{,xy} \left(Q_{11}(0)\, f_1(t) + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)\, d\tau \right)$$
$$-2[1 - \nu(t)]\, \beta_{xy} I_1 \varphi_{,xy} \left[Q_{11}(0)\, f_1(t)\, (f_3(t)\, \phi_{,x} + f_2(t)\, \psi_{,y} + f_1(t)^2\, \varphi_{,x}\varphi_{,y}) \right.$$
$$\left. + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)\, (f_3(\tau)\, \phi_{,x} + f_2(\tau)\, \psi_{,y} + f_1(\tau)^2\, \varphi_{,x}\varphi_{,y})\, d\tau \right]$$
$$-3[1 - \nu(t)]\, \gamma_{xy} I_1 \varphi_{,xy} \left[Q_{11}(0)\, f_1(t)\, (f_3(t)\, \phi_{,x} + f_2(t)\, \psi_{,y} + f_1(t)^2\, \varphi_{,x}\varphi_{,y})^2 \right.$$
$$\left. + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)\, (f_3(\tau)\, \phi_{,x} + f_2(\tau)\, \psi_{,y} + f_1(\tau)^2\, \varphi_{,x}\varphi_{,y})^2\, d\tau \right]$$
$$-4[1 - \nu(t)]\, \gamma_{xy} I_2 \varphi_{,xy}^3 \left(Q_{11}(0)\, f_1(t)^3 + \int_{0^+}^{t} \dot{Q}_{11}(t-\tau)\, f_1(\tau)^3\, d\tau \right) \tag{17}$$

where $I_1 = h^3/12$ and $I_2 = h^5/80$. The nonlinear viscoelastic constants $\gamma_x, \gamma_y, \beta_x, \beta_y, \gamma_{xy}$ and β_{xy} are related to those in Eq. (7).

In this research we consider the following in-plane loads ($N_{xy} = 0$):

$$\bar{N}_x = -N_{xs} - N_{xd} \cos(\theta t)$$
$$\bar{N}_y = -N_{ys} - N_{yd} \cos(\theta t) \tag{18}$$

where t is time and θ is the loading frequency.

Introducing now Eqs. (12)–(18) in Eqs. (1)–(3), the equations of motion are derived in the form of a system of nonlinear differential equations with time-dependent coefficients for which an exact solution is generally not available. Thus, the unknown functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ can be obtained by using the Galerkin method (see e.g. [1] and [19]).

Consider the case of a simply-supported plate, for which the boundary conditions can be satisfied when the spatial parts of the solution functions are given by

$$\varphi(x, y) = \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$
$$\psi(x, y) = \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{19}$$
$$\phi(x, y) = \sin\frac{\pi x}{a} \cos\frac{\pi y}{a},$$

and where a and b are the side-lengths of the plate.

By substituting Eqs. (12)–(18) into Eqs. (1)–(3), and considering terms up to order three, the following equations of motion are derived:

$$\ddot{f}_1(t) + \Omega^2[1 - 2\eta \cos(\theta t)]\, f_1(t) + [k_6 \ddot{f}_2(t) + k_7 \ddot{f}_3(t) + k_3 f_2(t)$$
$$+ k_4 f_3(t) + (k + k_5)\, f_1(t)^2 + k_8 f_2(t)^2 + k_9 f_3(t)^2 + k_{10} f_2(t)\, f_3(t)]\, f_1(t)$$
$$= -\int_{0^+}^{t} \dot{D}(t-\tau)\, f_1(\tau)\, [\omega^2 + k_3 f_2(\tau) + k_4 f_3(\tau) + (k + k_5)\, f_1(\tau)^2$$
$$+ k_8 f_2(\tau)^2 + k_9 f_3(\tau)^2 + k_{10} f_2(\tau)\, f_3(\tau)]\, d\tau \tag{20}$$

$$\ddot{f}_2(t) + A_1 f_2(t) + A_2 f_3(t) + k_{11} f_1(t)^2 + k_{13} f_2(t)^2 + k_{14} f_2(t)^3 + k_{15} f_2(t)\, f_1(t)^2 + k_{16} f_3(t)^2$$
$$+ k_{17} f_3(t)\, f_1(t)^2 + k_{18} f_3(t)^3 + k_{19}\big(f_2(t)\, f_3(t)^2 + f_3(t)\, f_2(t)^2\big)$$
$$= -\int_{0^+}^{t} \dot{D}(t-\tau)\, \big[A_1 f_2(\tau) + A_2 f_3(\tau) + k_{11} f_1(\tau)^2 + k_{13} f_2(\tau)^2 + k_{14} f_2(\tau)^3 + k_{15} f_2(t)\, f_1(\tau)^2$$
$$+ k_{16} f_3(\tau)^2 + k_{17} f_3(\tau)\, f_1(t)^2 + k_{18} f_3(\tau)^3 + k_{19}\big(f_2(\tau)\, f_3(\tau)^2 + f_3(\tau)\, f_2(\tau)^2\big)\big]\, d\tau \qquad (21)$$

$$\ddot{f}_3(t) + A_2 f_2(t) + A_3 f_3(t) + k_{12} f_1(t)^2 + k_{13} f_3(t)^2 + k_{14} f_3(t)^3 + k_{15} f_3(t)\, f_1(t)^2 + k_{16} f_2(t)^2$$
$$+ k_{17} f_2(t)\, f_1(t)^2 + k_{18} f_2(t)^3 + k_{19}\big(f_2(t)\, f_3(t)^2 + f_3(t)\, f_2(t)^2\big)$$
$$= -\int_{0^+}^{t} \dot{D}(t-\tau)\, \big[A_2 f_2(\tau) + A_3 f_3(\tau) + k_{12} f_1(\tau)^2 + k_{13} f_3(\tau)^2 + k_{14} f_3(\tau)^3 + k_{15} f_3(t)\, f_1(\tau)^2$$
$$+ k_{16} f_2(\tau)^2 + k_{17} f_2(\tau)\, f_1(t)^2 + k_{18} f_2(\tau)^3 + k_{19}\big(f_2(\tau)\, f_3(\tau)^2 + f_3(t)\, f_2(t)^2\big)\big]\, d\tau \qquad (22)$$

where $a = b = l$, $\gamma_x = \gamma_y = \gamma$, $\beta_x = \beta_y = \beta$ and

$$\omega^2 = \frac{4I_1 Q_{11}(0)}{\varrho h}\left(\frac{\pi}{l}\right)^4 \qquad N = \frac{4\pi^2 I_1 Q_{11}(0)}{l^2} \qquad D(t) = \frac{Q_{11}(t)}{Q_{11}(0)}$$

$$\Omega^2 = \omega^2\left[1 - \frac{N_{xs} + N_{ys}}{N}\right] \qquad \eta = \frac{N_{xd} + N_{yd}}{2[N - (N_{xs} + N_{ys})]}$$

$$A_1 = A_3 = \frac{3(3-\nu)}{2\pi^2}\left(\frac{l}{h}\right)^2 \omega^2 \qquad A_2 = \frac{3(1+\nu)}{2\pi^2}\left(\frac{l}{h}\right)^2 \omega^2$$

$$k = \frac{27\pi^4 h^2}{640 l^4}\,[\gamma(1+\nu) + 4\gamma_{xy}(1-\nu)]\,\omega^2$$

$$k_3 = \left\{\frac{32}{9\pi l}\,[\beta_{xy}(1-\nu) - \beta(1+\nu)] - \frac{16(5+3\nu)\,l}{3\pi^3 h^2}\right\}\omega^2$$

$$k_5 = \left\{\frac{3\pi^2}{32 l^2}\left[\beta(1+\nu) + \frac{4}{3}\,\beta_{xy}(1-\nu)\right] + \frac{3(2\nu+1)}{8h^2}\right\}\omega^2$$

$$k_8 = \left\{\frac{27\pi^2}{64 l^2}\,[\gamma(1+\nu) + 2\gamma_{xy}(1-\nu)] + \frac{27\beta}{16h^2}\,(1+\nu)\right\}\omega^2$$

$$k_{10} = \frac{729\pi^2(1-\nu)\,\gamma_{xy}}{48 l^2}\,\omega^2 \qquad k_{11} = k_{12} = \frac{24 l(5-3\nu)}{9\pi^3 h^2}\,\omega^2$$

$$k_{13} = -\frac{64\beta l}{3\pi^3 h^2}\,\omega^2 \qquad k_{14} = \frac{27}{16h^2}\,[\gamma + \gamma_{xy}(1-\nu)/2]\,\omega^2$$

$$k_{15} = \frac{9\beta}{16h^2}\,\omega^2 \qquad k_{16} = \nu k_{13} \qquad k_{17} = \nu k_{15}$$

$$k_{18} = \frac{27}{16h^2}\,[\nu\gamma + \gamma_{xy}(1-\nu)/2]\,\omega^2 \qquad k_{19} = \frac{81(1-\nu)\,\gamma_{xy}}{32h^2}\,\omega^2$$

$$k_4 = k_3 \qquad k_6 = k_7 = -\frac{32}{9l\pi} \qquad k_9 = k_8 .$$

Here, ω and Ω represent the fundamental natural frequencies of the unloaded and loaded plate, respectively, N is the Euler critical load, η is the excitation parameter and the k_{ij} and k are the cofficients of non-linearity.

Equations (20)–(22) are the nonlinear integro-differential equations, which govern the motion of the nonlinear viscoelastic plate subjected to in-plane parametric loading.

3 Method of solution

Here we are interested in the stability of the unperturbed equilibrium of the nonlinear viscoelastic plate. To this end the integro-differential equations (20)–(22) are investigated. For the handling of non-linear differential equations with time-dependent coefficients, Lyapunov introduced the concept of characteristic numbers, the sign of which determines whether or not the unperturbed motion is stable [20]. The negative values of these chracteristic numbers are referred to as the Lyapunov exponents.

According to Lyapunov, if all these exponents are negative, the unperturbed motion is asymptotically stable. In addition, Chetaev [21], [22] showed that if one of the Lyapunov exponents is positive then the unperturbed motion is unstable. Thus, it suffices to compute the largest Lyapunov exponent in order to determine the stability of the unperturbed motion of the non-linear viscoelastic plate in question. The largest Lyapunov exponent of the system is derived within the following procedure [23]:

Consider the system of ordinary differential equations

$$\dot{X} = F(x, t). \tag{23}$$

For a given solution of Eq. (23), $x(t)$, define the matrix

$$G_{ij}[x(t)] = \left.\frac{\partial F_i}{\partial x_j}\right|_{x=x(t)}. \tag{24}$$

The largest Lyapunov exponent is then determined by solving the equations

$$\dot{y} = Gy \tag{25}$$

and performing the following steps:

(i) For the first time interval, Δt, solve Eq. (25) by considering initial conditions, $y(0)$, normalized such that $\|y(0)\| = 1$, where $\|\cdot\|$ is the Euclidean norm.

(ii) Compute $\mu_1 = \ln\ \|y(\Delta t)\|$

(iii) Let $z(\Delta t) = y(\Delta t)/\|y(\Delta t)\|$

(iv) For the second time interval, $2\Delta t$, solve Eq. (25) with $z(\Delta t)$ as the initial condition (G has to be changed according to Eq. (24)) and determine $\mu_2 = \ln\ \|y(2\Delta t)\|$.

(v) Repeat the process for n iterations.

One defines then

$$\lambda_1 = \frac{1}{n\Delta t} \sum_{m=1}^{n} \mu_m \tag{26}$$

which, for $n \to \infty$, is the largest Lyapunov exponent.

In order to compute λ_1, the governing equations (20)–(22) must be reduced to a system of first-order equations of the form (23).
Declaring the variables $x_1(t) = f_1(t)$, $x_4(t) = f_2(t)$, $x_7(t) = f_3(t)$, and $x_3(t)$, $x_6(t)$, and $x_9(t)$ the integrals in Eqs. (20), (21), (22) respectively, the following ordinary integro-differential equations are derived:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1[\Omega^2(1 - 2\eta\cos(\theta t)) + k_3 x_4 + k_4 x_7 + k_6 \dot{x}_5 + k_7 \dot{x}_8 + k_8 x_4^2 + k_9 x_7^2 + k_{10} x_4 x_7 + (k + k_5)\, x_1^2] - x_3$$

$$\dot{x}_3 = \frac{\partial}{\partial t}\int_{0^+}^{t} \dot{D}(t-\tau)\, x_1[\omega^2 + k_3 x_4(\tau) + k_4 x_7(\tau) + k_8 x_4(\tau)^2 + k_9 x_7(\tau)^2 + k_{10} x_4(\tau)\, x_7(\tau) + (k + k_5)\, x_1(\tau)^2]\, d\tau$$

$$\dot{x}_4 = x_5$$

$$\dot{x}_5 = -[A_1 x_4 + A_2 x_7 + x_1^2(k_{11} + k_{15} x_4 + k_{17} x_7) + x_4^2(k_{13} + k_{19} x_7) + x_7^2(k_{16} + k_{19} x_4) + k_{14} x_4^3 + k_{18} x_7^3] - x_6$$

$$\dot{x}_6 = \frac{\partial}{\partial t}\int_{0^+}^{t} \dot{D}(t-\tau)[A_1 x_4(\tau) + A_2 x_7 + x_1(\tau)^2\,(k_{11} + k_{15} x_4(\tau) + k_{17} x_7(\tau)) + x_4(\tau)^2\,(k_{13} + k_{19} x_7(\tau)) + x_7(\tau)^2\,(k_{16} + k_{19} x_4(\tau)) + k_{14} x_4(\tau)^3 + k_{18} x_7(\tau)^3]\, d\tau$$

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = -[A_2 x_4 + A_3 x_7 + x_1^2(k_{12} + k_{15} x_7 + k_{17} x_4) + x_7^2(k_{13} + k_{19} x_4) + x_4^2(k_{16} + k_{19} x_7) + k_{14} x_7^3 + k_{18} x_4^3] - x_9$$

$$\dot{x}_9 = \frac{\partial}{\partial t}\int_{0^+}^{t} \dot{D}(t-\tau)[A_2 x_4(\tau) + A_3 x_7 + x_1(\tau)^2\,(k_{12} + k_{15} x_7(\tau) + k_{17} x_4(\tau)) + x_7(\tau)^2\,(k_{13} + k_{19} x_4(\tau)) + x_4(\tau)^2\,(k_{16} + k_{19} x_7(\tau)) + k_{14} x_7(\tau)^3 + k_{18} x_4(\tau)^3]\, d\tau. \quad (27)$$

As for the material relaxation function, the standard linear solid model

$$E(t) = a + be^{-\alpha t} \quad (28)$$

is considered where a, b and α are appropriate parameters. Thus, for a material with time independent Poisson's ratio one obtains

$$Q_{11}(t) = \frac{E(t)}{1 - \nu^2} = A + Be^{-\alpha t} \quad (29)$$

so that

$$D(t) = \frac{Q_{11}(t)}{Q_{11}(0)} = \frac{A + Be^{-\alpha t}}{A + B}. \quad (30)$$

Introducing the above model into (27) will affect only the equations for $\dot{x}_3$, $\dot{x}_6$, and $\dot{x}_9$. Their final form is obtained by differentiating via Leibniz's rule,

$$\dot{x}_3 = -\alpha \left\{ x_3 + \frac{Bx_1}{A+B} [\omega^2 + k_3x_4 + k_4x_7 + k_8x_4^2 + k_9x_7^2 + k_{10}x_4x_7 + (k + k_5)\, x_1^2] \right\}$$

$$\dot{x}_6 = -\alpha \left\{ x_6 + \frac{B}{A+B} [A_1x_4 + A_2x_7 + x_1^2(k_{11} + k_{15}x_4 + k_{17}x_7) + x_4^2(k_{13} + k_{19}x_7) + x_7^2(k_{16} + k_{19}x_4) + k_{14}x_4^3 + k_{18}x_7^3] \right\} \tag{31}$$

$$\dot{x}_9 = -\alpha \left\{ x_9 + \frac{B}{A+B} [A_2x_4 + A_3x_7 + x_1^2(k_{12} + k_{15}x_7 + k_{17}x_4) + x_7^2(k_{13} + k_{19}x_4) + x_4^2(k_{16} + k_{19}x_7) + k_{14}x_7^3 + k_{18}x_4^3] \right\}.$$

With (31), the system of equations (25) is given by

$$\dot{y}_1 = y_2$$

$$\begin{aligned}\dot{y}_2 = {} & y_1\{-[\Omega^2(1 - 2\eta\cos(\theta t)) + k_3x_4 + k_4x_7 + k_6\dot{x}_5 + k_7\dot{x}_8 + k_8x_4^2 + k_9x_7^2 + k_{10}x_4x_7 \\ & + (k + k_5)\, x_1^2] + 2x_1^2[k_6(k_{11} + k_{15}x_4 + k_{17}x_7) + k_7(k_{12} + k_{15}x_7 + k_{17}x_4) - (k + k_5)]\} \\ & - y_3 - y_4x_1\{k_3 - k_6[A_1 + k_{15}x_1^2 + 2x_4(k_{13} + k_{19}x_7) + k_{19}x_7^2 + 3k_{14}x_4^2] \\ & - k_7[A_2 + k_{17}x_1^2 + 2x_4(k_{16} + k_{19}x_7) + k_{19}x_7^2 + 3k_{18}x_4^2] + 2k_8x_4 + k_{10}x_7\} \\ & + k_6y_6x_1 - y_7x_1\{k_4 - k_6[A_2 + k_{17}x_1^2 + k_{19}x_7^2 + 2x_7(k_{16} + k_{19}x_4) + k_{19}x_4^2 + 3k_{14}x_7^2] \\ & + 2k_9x_7 + k_{10}x_4\} + k_7y_9x_1\end{aligned}$$

$$\begin{aligned}\dot{y}_3 = {} & -\frac{\alpha B}{A+B} y_1[\omega^2 + k_3x_4 + k_4x_7 + k_8x_4^2 + k_9x_7^2 + k_{10}x_4x_7 + 3x_1^2(k + k_5)] \\ & - \alpha y_3 - \frac{\alpha B}{A+B} y_4x_1(k_3 + k_8x_4 + k_{10}x_7) - \frac{\alpha B}{A+B} y_7x_1(k_4 + k_9x_7 + k_{10}x_4)\end{aligned}$$

$$\dot{y}_4 = y_5$$

$$\begin{aligned}\dot{y}_5 = {} & -2y_1x_1(k_{11} + k_{15}x_4 + k_{17}x_7) - y_4[A_1 + k_{15}x_1^2 + 2x_4(k_{13} + k_{19}x_7) \\ & + k_{19}x_7^2 + 3k_{14}x_4^2] - y_6 - y_7[A_2 + k_{17}x_1^2 + 2x_7(k_{16} + k_{19}x_4) + k_{19}x_4^2 + 3k_{18}x_7^2]\end{aligned}$$

$$\begin{aligned}\dot{y}_6 = {} & -\frac{2\alpha B}{A+B} y_1x_1(k_{11} + k_{15}x_4 + k_{17}x_7) \\ & - \frac{\alpha B}{A+B} y_4[A_1 + k_{15}x_1^2 + 2x_4(k_{13} + k_{19}x_7) + k_{19}x_7^2 + 3k_{14}x_4^2] - \alpha y_6 \\ & - \frac{\alpha B}{A+B} y_7[A_2 + k_{17}x_1^2 + 2x_7(k_{16} + k_{19}x_4) + k_{19}x_4^2 + 3k_{18}x_7^2]\end{aligned}$$

$$\dot{y}_7 = y_8$$

$$\begin{aligned}\dot{y}_8 = {} & -2y_1x_1(k_{12} + k_{15}x_7 + k_{17}x_4) - y_4[A_2 + k_{17}x_1^2 + 2x_4(k_{16} + k_{19}x_7) \\ & + k_{19}x_7^2 + 3k_{18}x_4^2] - y_9 - y_7[A_3 + k_{15}x_1^2 + 2x_7(k_{13} + k_{19}x_4) + k_{19}x_4^2 + 3k_{14}x_7^2]\end{aligned}$$

$$\dot{y}_9 = -\frac{2\alpha B}{A+B}\, y_1 x_1 (k_{12} + k_{15}x_7 + k_{17}x_4)$$
$$- \frac{\alpha B}{A+B}\, y_4[A_2 + k_{17}x_1{}^2 + 2x_4(k_{16} + k_{19}x_7) + k_{19}x_7{}^2 + 3k_{18}x_4{}^2] - \alpha y_6$$
$$- \frac{\alpha B}{A+B}\, y_7[A_3 + k_{15}x_1{}^2 + 2x_7(k_{13} + k_{19}x_4) + k_{19}x_4{}^2 + 3k_{14}x_7{}^2]. \qquad (32)$$

The system presented in (32) is for the general case when $a \neq b$. When $a = b$, $f_2 = f_3$ and Eqs. (20)−(22) are reduced to two equations of motion and consequently the system (32) is reduced to six equations only.

4 Numerical results and discussion

In this section the stability of Eqs. (20)−(22) is analyzed with respect to the various parameters involved. The solution of those equations and of Eq. (25) is obtained within the Runge-Kutta method [24]. First, it is recognized that for the case where $\alpha = k = k_{ij} = 0$ $(i, j = 3, \ldots, 10)$ in Eq. (20), one obtains the well-known linear Mathieu equation, which was extensively investigated, e.g., by McLachlan [25].

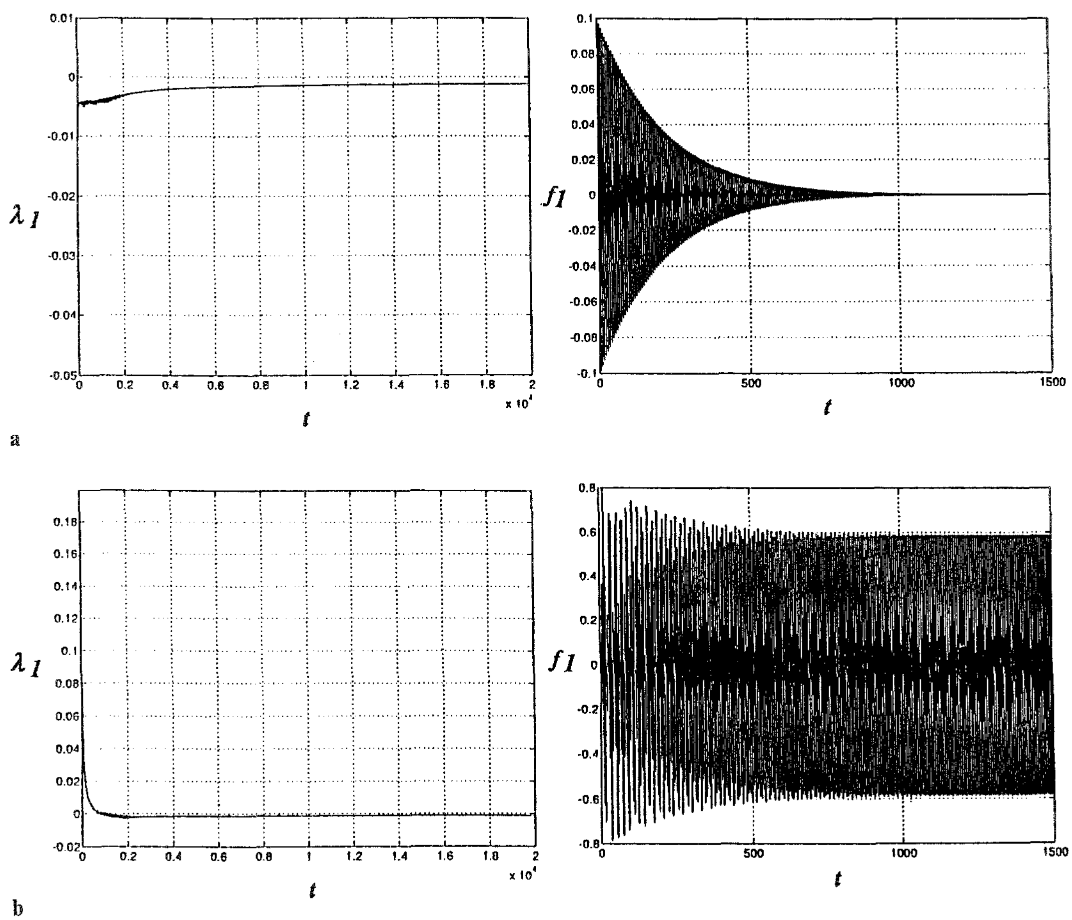

Fig. 2. The response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $\omega = 1$, $l/h = 50$ and **a** $\eta = 0.0005$, **b** $\eta = 0.5$

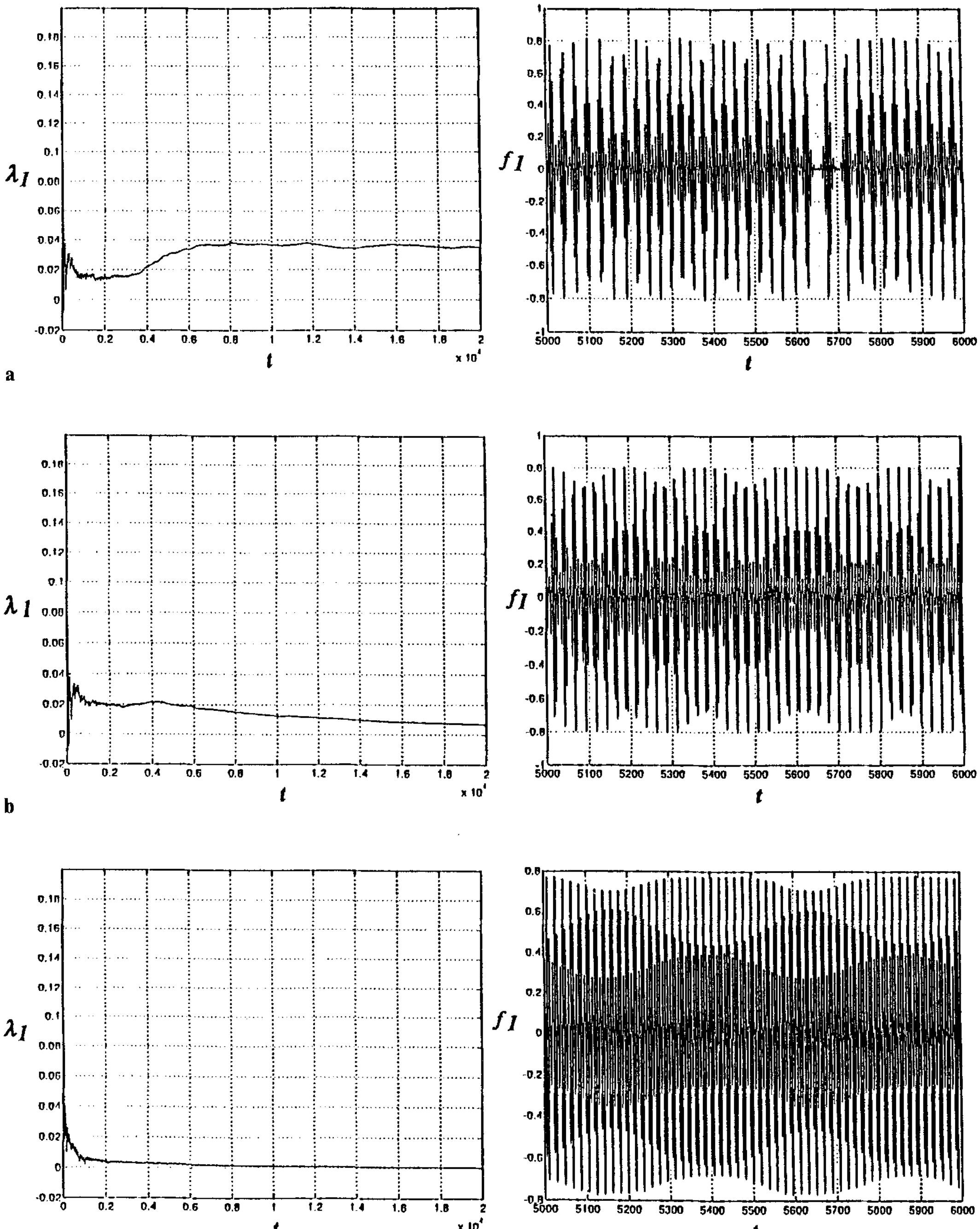

Fig. 3. The response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, for $\eta = 0.5$, $\omega = 1$, $l/h = 50$ and **a** $\alpha = 0$, **b** $\alpha = 0.000001$, **c** $\alpha = 0.0001$

When $k = k_{ij} = 0$ and $\alpha \neq 0$ we have

$$\ddot{f}_1(t) + \Omega^2[1 - 2\eta \cos(\theta t)]\, f_1(t) = -\omega^2 \int_{0^+}^{t} \dot{D}(t - \tau)\, f_1(\tau)\, d\tau \tag{33}$$

describing the motion of a linear viscoelastic structure. The stability of this equation was investigated in [11] by using the concept of Lyapunov exponents, and later on analytically in [26], [27], where an the expression for the critical (minimum) value of the excitation parameter, η_c, at which instability may occur, was obtained. For the case of the standard linear solid model it is

$$\eta_c = \frac{2}{\theta} |\dot{D}(0)| = \frac{2\alpha B}{\theta(A + B)} \tag{34}$$

and will be used later on. When $k_{ij} = 0$ $(i, j = 3, \ldots, 10)$, $k \neq 0$ and $\alpha \neq 0$ we have

$$\ddot{f}_1(t) + \Omega^2[1 - 2\eta \cos(\theta t)] f_1(t) + k f_1(t)^3 = - \int_{0^+}^{t} \dot{D}(t - \tau) [\omega^2 f_1(\tau) + k f_1(\tau)^3] \, d\tau \tag{35}$$

which describes the motion of a nonlinear viscoelastic plate under small deflections. The stability of this equation was investigated numerically by the authors of the present work [18], where it was also shown that in some cases the system turns out to be chaotic. For the case where $\alpha = k_{ij} = 0$, $(i, j = 3, \ldots, 10)$ and $k \neq 0$, one obtains

$$\ddot{f}_1(t) + \Omega^2[1 - 2\eta \cos(\theta t)] f_1(t) + k f_1(t)^3 = 0 \tag{36}$$

representing a non-linear version of the Mathieu equation, and was examined in [1].

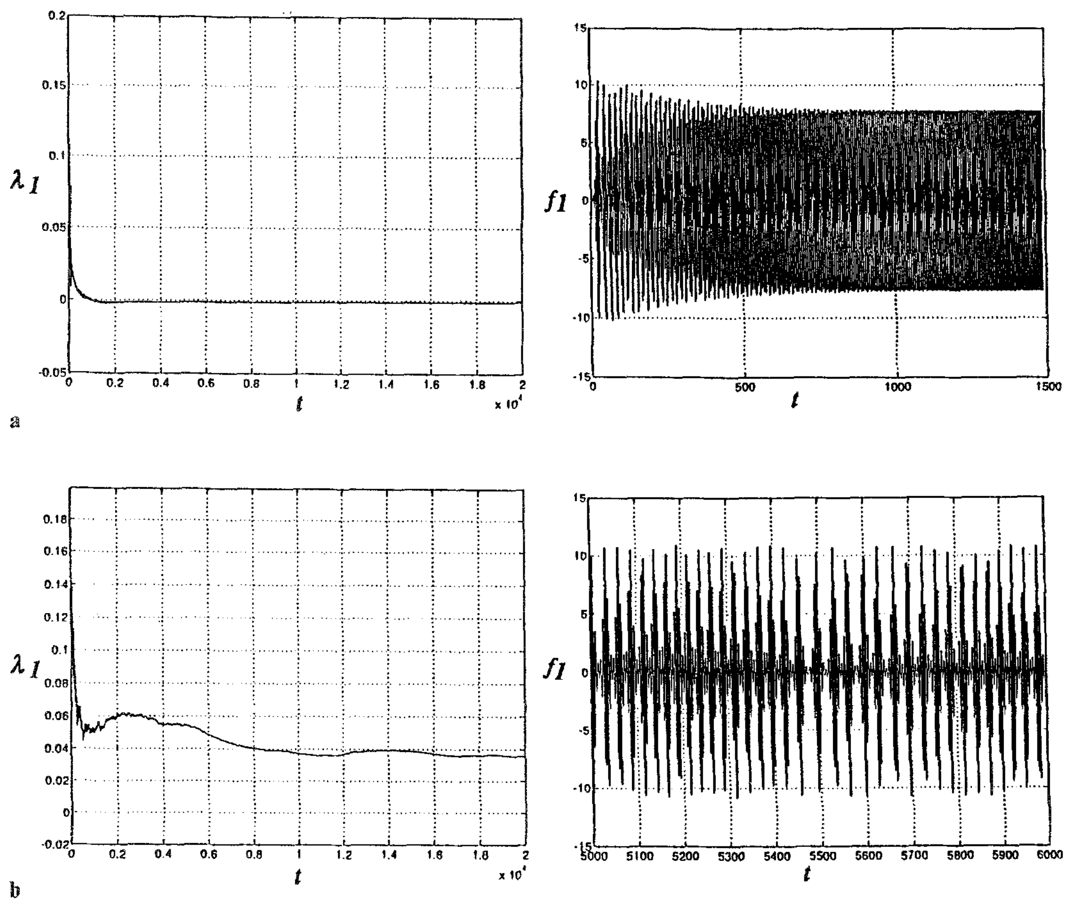

Fig. 4. The response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $\omega = 1$, $l/h = 50$, $k_{ij} = 0$, $i, j = 3, \ldots, 10$ and **a** the case of Fig. 2b, **b** the case of Fig. 3b

The numerical results herein were obtained by using $A = 0.1$, $B = 0.9$, $N_{xs} = N_{ys} = 0$, $\Omega = \omega$ and $\theta = 2\omega$. In addition, the following values for the nonlinear viscoelastic parameters were considered: $\gamma = 2000$, $\gamma_{xy} = 4500$, $\beta = -45$ and $\beta_{xy} = -67$ as derived for a high density polyethylene from [28].

Figures 2 and 3 show the response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, drived for the case where $\omega = 1$, and $l/h = 50$.

In Fig. 2 $\alpha = 0.01$ and η is equal to a) 0.0005 ($< \eta_c$) and b) 0.5 ($> \eta_c$). In Fig. 2a the system is asymptotically stable, that is λ_1 is negative and the response is approaching zero. In Fig. 2b the system is stable with limit cycle and $\lambda_1 \to 0$.

In Fig. 3 $\eta = 0.5$ and the following cases for α are considered: a) 0, b) 0.000001 and c) 0.0001. In Figs. 3a and 3b λ_1 is positive, indicating instability. For relatively large α (case c) $\lambda_1 \to 0$ and the system is stable.

Figures 4a and 4b exhibit the results for the cases of Figs. 2b and 3b respectively, as obtained within the small deflections theory (by substituting $k_{ij} = 0$, $(i, j = 3, \ldots, 10)$ in Eq. (20)). In Fig. 4a the system is stable, $\lambda_1 \to 0$, while in Fig. 4b λ_1 is positive, indicating instability.

Figures 5 and 6 show the response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, as derived within the large and small deflections theories, respectively, for the case where

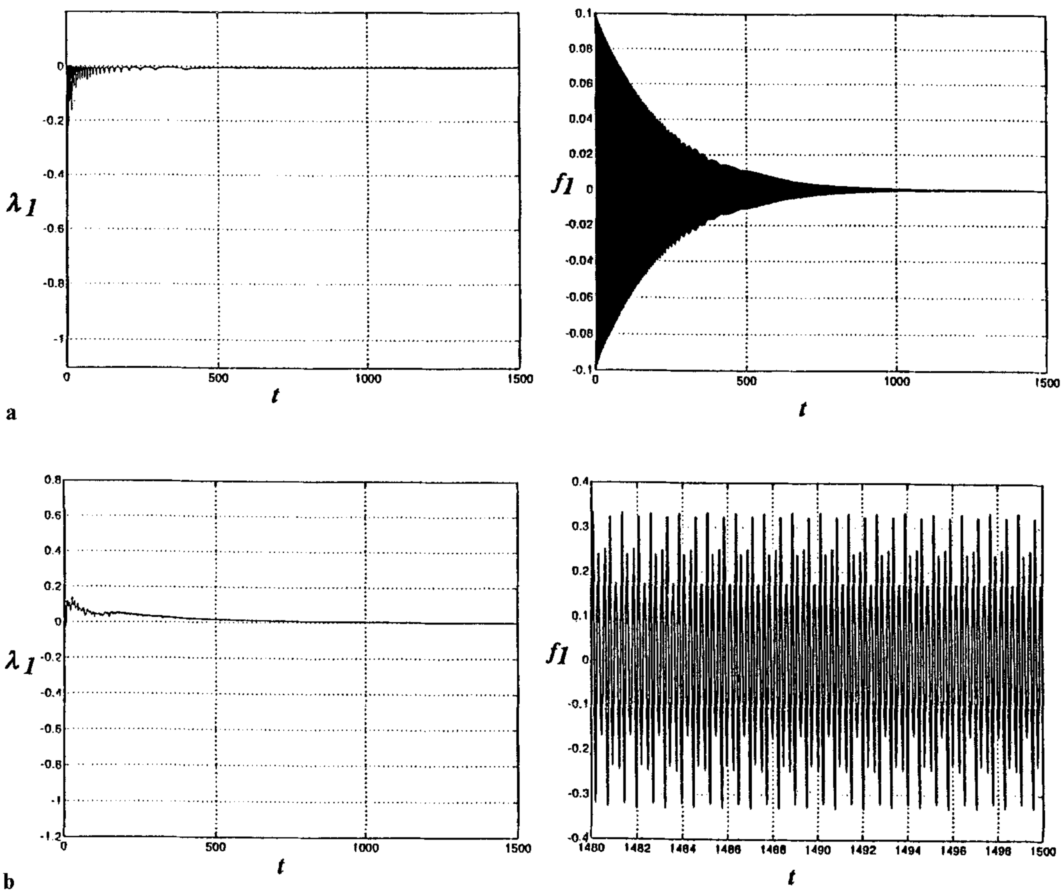

Fig. 5. The response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $\omega = 25$, $l/h = 10$ and **a** $\eta = 0.0001$, **b** $\eta = 0.5$

$l/h = 10$ ($\omega = 25$), $\alpha = 0.01$ and η equal to a) 0.0001, b) 0.5. In Figs. 5a and 6a λ_1 is negative and the response is approaching zero. In Figs. 5b and 6b the system is stable, $\lambda_1 \to 0$.

From the above we conclude the following:

(i) Due to the stretching of the middle plane, the response amplitude is smaller than that predicted by using the small deflections theory for relatively large l/h (see Figs. 2–4).

(ii) At small ratios of l/h, the small and large deflections theories give practically the same result.

(iii) From the stability point of view, both theories yield the same behavior when analyzed within the Lyapunov exponents. However, the stable response within the large deflections theory can be of multiple frequencies (see Figs. 5–6).

(iv) The material coefficient, α, has a great influence on the system in the sense that an unstable system may become stable at large values of α (see Fig. 3c). The above is correct at $\eta > \eta_c$. But α is one of the parameters by which η_c is determined (see Eq. (34)) in a way that at large α, η_c is increased, so that α stabilizes the system too.

(v) At $\eta < \eta_c$, the system is asymptotically stable regardless of the values of the viscoelastic parameters.

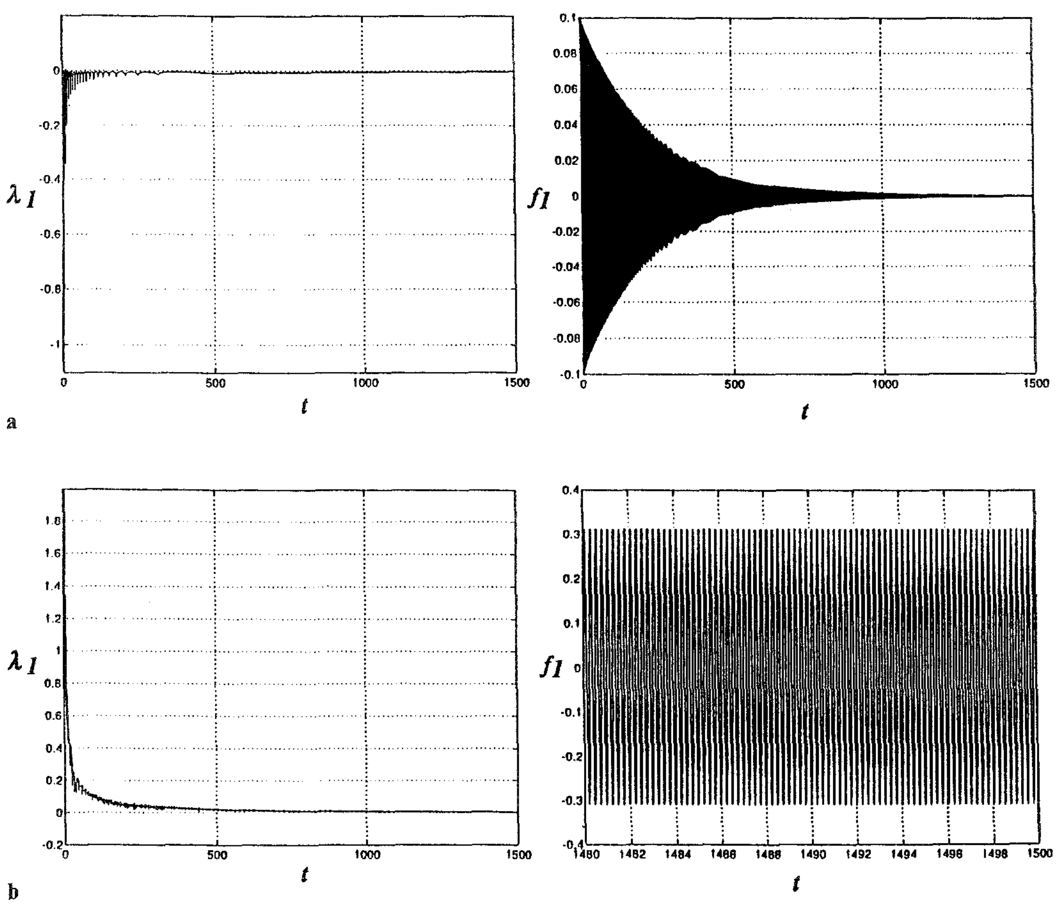

Fig. 6. The response, $f_1(t)$, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $\omega = 25$, $l/h = 10$, $k_{ij} = 0$, $i, j = 3, \ldots, 10$ and **a** $\eta = 0.0001$, **b** $\eta = 0.5$

Finally, it is noted that the Lyapunov exponents serve also as a powerful tool in the study of a chaotic motion, and actually, the existence of at least one positive Lyapunov exponent indicates a chaotic state (see e.g. [23], [29], [30]). However, there are other ways to examine the response nature. Figure 7 a exhibits the Fourier power spectrum, phase plane and Poincare map plots of the instability case given in Fig. 3 a, while Fig. 7 b shows the same but for the stable case shown in Fig. 2 b. Thus, we believe that more attention should be given to the chaotic behavior possible in this problem.

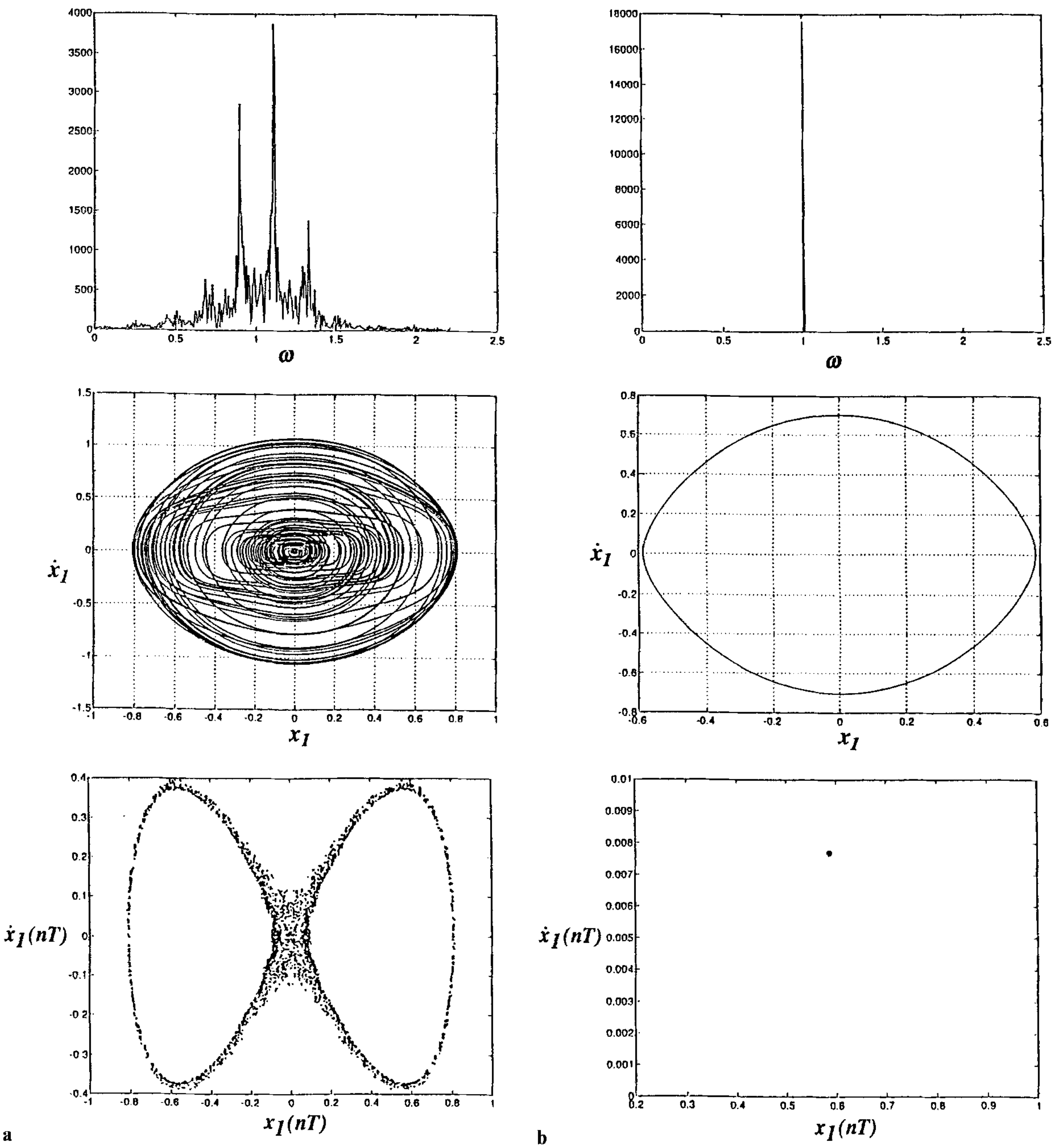

Fig. 7. The Fourier power spectrum, phase plane and Poincare map of **a** the case of Fig. 3 a, **b** the case of Fig. 2 b

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