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Effect of pressure gradient on MHD boundary layer over a flat plate

P. Sam Lawrence and **B. Nageswara Rao,** Trivandrum, India

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Summary. The magnetohydrodynamic (MHD) boundary layer flow over a flat plate is examined here for two cases, viz. a uniform free-stream velocity and a uniform hydrostatic pressure. The nonlinear boundary layer equations are solved using a reliable finite-difference method. The boundary layer physical parameters such as skin-friction coefficient, displacement, momentum and energy thicknesses of the boundary layer are determined. It is found that the normal surface velocity gradient decreases with the local magnetic interaction parameter for the cases of a uniform hydrostatic pressure, whereas in the case of a uniform free-stream velocity it increases with the interaction parameter.

1 Introduction

Within the boundary layer, the velocity increases from zero at the surface to the free-stream velocity at the edge of the boundary layer and, therefore, velocity gradients may be appreciable, even if the viscosity is small. Determination of the wall-shearing stress is one of the important objectives in the solution of the boundary layer equations. The equations governing the boundary layer flow in general become nonsimilar due to the presence of a magnetic field or variable fluid properties. Wu [1] has studied the effects of suction or injection on a steady two dimensional magnetohydrodynamic (MHD) boundary layer flow on a flat plate. He assumed that both the free-stream velocity and the hydrostatic pressure were constant as in the case of boundary layers wherein the magnetic force term or Lorentz force term is absent in the equation of motion. Chuang [2] has pointed out the shortcomings of Wu's model and suggested to assume either free-stream velocity, or the hydrostatic pressure as constant in the solution of the boundary layer equations. The pressure gradient across the boundary layer is of the order of the boundary layer thickness and the pressure can be assumed constant across this thin layer. The pressure gradient along the flow direction may, in certain specific cases, be small or even zero; but, in general, it is determined by the external flow. Since the effects of viscosity are confined to a thin layer of fluid adjacent to the boundary, the pressure may be calculated on the basis of potential flow past the surface. This approach yields a reasonably accurate prediction of the pressure gradient when the boundary layer is not near to separation. If the free-stream velocity is constant, then the hydrostatic and magnetic pressure gradients are counter balancing with each other. For the case of zero hydrostatic pressure gradient, the free-stream velocity decreases along the flat plate due to the presence of a magnetic force.

Motivated by the work of the above-mentioned authors, the effect of the pressure gradient on the MHD boundary layer over a flat plate is examined here. The nonlinear boundary layer equations were solved numerically by the finite-difference method and obtained the boundary layer physical parameters.

2 Basic equations and numerical analysis

A steady two-dimensional laminar flow of an incompressible electrically conducting fluid over a flat-plate (Fig. 1) is considered. The plate is located in the plane $y = 0$. The x-axis corresponds to the direction of the flow and the y-axis is normal to the flow direction. $u_\infty(x)$ is a nonuniform free-stream flow velocity and B_y is a uniform magnetic field applied along the y direction. u and v are the axial and transverse velocity components in the boundary layer flow. u_0 and L are chosen as the characteristic velocity and length for nondimensionalising the flow variables. The Reynolds number, $Re = (Lu_0)/\nu$, where ν is the kinematic viscosity, plays an important role in the solution of viscous boundary layer equations of motion, as the order of the viscous boundary layer thickness is $1/\sqrt{Re}$.

The magnetic Reynolds number, $Rem = \sigma\mu_e u_\infty x \leqq \sigma\mu_e u_0 L \ll 1$, where $1/(\sigma\mu_e)$ can be thought of as a magnetic "kinematic viscosity", μ_e is the magnetic permeability, and σ is the electrical conductivity. Rem acts like Reynolds number (Re) in the magnetic boundary layer flow which is governed by the magnetic induction equation derived from Ohm's law and Maxwell's equations. When $Rem \ll 1$, the field lines are undisturbed by the fluid flow and the strength of the induced magnetic field is negligible in comparison with the applied magnetic field. This corresponds to the weak interaction of viscous boundary layer equations of motion and Maxwell's equations, which is often the case in engineering MHD. Then the Lorentz force term in the equations of motion is only given by the applied magnetic field, B_y. Hence the equations of motion are decoupled from Maxwell's equations.

The governing equations for a steady two-dimensional laminar MHD boundary layer flow on a flat plate with a uniform magnetic field and without an applied electric field are [3]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{dp}{dx} - \frac{\sigma B_y{}^2 u}{\varrho} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{3}$$

$$u \to u_\infty(x) \quad \text{as} \quad y \to \infty, \tag{4}$$

$$u(x = x_0, y) = u_i(y), \tag{5}$$

where the kinematic viscosity $\nu = \mu/\varrho$, ϱ is the density, μ is the coefficient of viscosity and u_i is the initial velocity profile at any point x_0 along the flow direction.

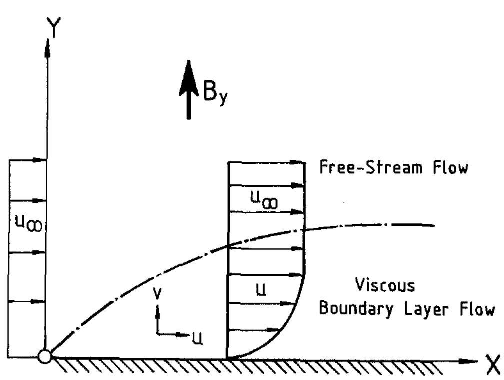

Fig. 1. MHD boundary layer along a flat plate

The pressure gradient obtained from the oncoming flow is

$$-\frac{1}{\varrho}\frac{dp}{dx} = u_\infty \frac{du_\infty}{dx} + \frac{\sigma B_y{}^2 u_\infty}{\varrho}. \tag{6}$$

For the specified free-stream flow velocity, $u_\infty(x)$, the pressure gradient is determined from the Eq. (6) of the oncoming flow, whereas for the specified pressure distribution, $u_\infty(x)$ is obtained by solving Eq. (6). In both cases, $u_\infty(x)$ is the basic input for the boundary layer equations (1)–(5). The boundary layer flow problem is formulated assuming the free-stream velocity, u_∞, as a function of x.

Defining

$$\bar{x} = x/L, \quad \bar{y} = y\sqrt{Re}/L, \quad \bar{u} = u/u_0, \quad \bar{v} = v\sqrt{Re}/u_0, \quad \bar{u}_\infty = u_\infty/u_0$$

and introducing

$$\xi = \int_{\bar{x}_0}^{\bar{x}} \bar{u}_\infty\, dx, \quad \eta = \bar{u}_\infty \bar{y}/\sqrt{2\xi}, \quad \bar{u} = \bar{u}_\infty f'(\xi, \eta) \quad \text{and} \quad W = \bar{v}\sqrt{2\xi}/\bar{u}_\infty,$$

which give

$$\frac{d\xi}{dx} = \frac{1}{L}\bar{u}_\infty, \quad \frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial\xi}\frac{d\xi}{dx} = \frac{1}{L}\frac{\bar{u}_\infty}{2\xi}\eta(\beta - 1), \quad \frac{\partial\eta}{\partial y} = \frac{1}{L}\bar{u}_\infty\sqrt{Re/(2\xi)},$$

$$\frac{\partial}{\partial x}() = \frac{d\xi}{dx}\frac{\partial}{\partial\xi}() + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}() = \frac{1}{L}\frac{\bar{u}_\infty}{2\xi}\left(2\xi\frac{\partial}{\partial\xi}() + \eta(\beta - 1)\frac{\partial}{\partial\eta}()\right),$$

$$\frac{\partial}{\partial y}() = \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}() = \frac{1}{L}\bar{u}_\infty\sqrt{Re/(2\xi)}\frac{\partial}{\partial\eta}(),$$

$$\frac{\partial^2}{\partial y^2}() = \left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial^2}{\partial\eta^2}() = \frac{1}{L^2}\frac{Re}{2\xi}\bar{u}_\infty^2\frac{\partial^2}{\partial\eta^2}(),$$

Eqs. (1)–(6) are transformed to

$$2\xi\frac{\partial f'}{\partial\xi} + \beta f' + \eta(\beta - 1)\frac{\partial f'}{\partial\eta} + \frac{\partial W}{\partial\eta} = 0, \tag{7}$$

$$2\xi f'\frac{\partial f'}{\partial\xi} + \bar{W}\frac{\partial f'}{\partial\eta} = \beta(1 - f'^2) + \gamma(1 - f') + \frac{\partial^2 f'}{\partial\eta^2}, \tag{8}$$

$$f' = W = 0 \quad \text{at} \quad \eta = 0, \tag{9}$$

$$f' \to 1 \quad \text{as} \quad \eta \to \infty, \tag{10}$$

$$f' = f_i'(\eta) \quad \text{at} \quad \xi = 0. \tag{11}$$

Here $\bar{W} = W + (\beta - 1)\eta f'$; $\beta = \frac{2\xi}{\bar{u}_\infty}\frac{d\bar{u}_\infty}{d\xi}$; $\gamma = \frac{2I\xi}{\bar{u}_\infty^2}$; $I = \sigma B_y{}^2 L/\varrho u_0$ is the magnetic interaction parameter; and primes denote differentiation with respect to η.

The nonlinear boundary layer equations (7)–(11) are the parabolic type partial differential equations amenable to numerical integration, which can be solved numerically by finite-difference method. The nonlinear equation (8) is linearised as in [4]. The derivatives in the η-direction are then expressed by three-point difference formulae, whereas the derivatives in the ξ-direction are approximated by a forward-difference scheme. The finite-difference equations obtained from Eqs. (7) and (8) are

$$A_{mn} f'_{m+1,n+1} + B_{mn} f'_{m+1,n} + C_{mn} f'_{m+1,n-1} = D_{mn}, \tag{12}$$

$$\begin{aligned} W_{m+1,n} = {} & W_{m+1,n-1} + W_{m,n-1} - W_{m,n} \\ & + 2\Delta\eta [a_{mn} f'_{m+1,n} + b_{mn} f'_{m+1,n-1} + c_{mn} f'_{m,n} + d_{mn} f'_{m,n-1}] \end{aligned} \tag{13}$$

where

$$A_{mn} = -\frac{1}{2(\Delta\eta)^2} + \frac{\bar{W}_{m,n}}{4(\Delta\eta)},$$

$$B_{mn} = \frac{1}{(\Delta\eta)^2} + \frac{2\xi^*}{\Delta\xi} f'_{m,n} + \beta^* f'_{m,n} + \frac{1}{2}\gamma^*,$$

$$C_{mn} = -\frac{1}{2(\Delta\eta)^2} - \frac{\bar{W}_{m,n}}{4(\Delta\eta)},$$

$$D_{mn} = \frac{f'_{m,n+1} - 2f'_{m,n} + f'_{m,n-1}}{2(\Delta\eta)^2} - \bar{W}_{m,n} \frac{f'_{m,n+1} - f'_{m,n-1}}{4(\Delta\eta)} + \beta^* + 2\frac{\xi^*}{\Delta\xi}(f'_{m,n})^2 + \gamma^*\left(1 - \frac{1}{2} f'_{m,n}\right),$$

$$a_{mn} = -R - S - T, \quad b_{mn} = R - S - T,$$

$$c_{mn} = -R - S + T, \quad d_{mn} = R - S + T,$$

$$R = \frac{1}{2(\Delta\eta)}\eta^*(\beta^* - 1), \quad S = \frac{1}{4}\beta^*, \quad T = \frac{1}{(\Delta\xi)}\xi^*, \quad \beta^* = \beta(\xi^*),$$

$$\gamma^* = \gamma(\xi^*), \quad \xi^* = \xi_{m+1/2} = \frac{1}{2}(\xi_{m+1} + \xi_m),$$

$$\eta^* = \eta_{n-1/2} = \frac{1}{2}(\eta_n + \eta_{n-1}),$$

m and n are the grid points along ξ and η directions. The wall $\eta = 0$ is the grid point $n = 1$, and $\eta = \infty$ is taken to be the finite position $\eta = 8$, where $n = 81$. Thus the mesh spacing is $\Delta\eta = 0.1$ and $\eta = 0.1(n - 1)$. In the ξ-direction, the grid points are denoted as $m = 1, 2, \ldots$ with an increment $\Delta\xi$, which can be specified arbitrarily (say, 0.001). Since the boundary layer problem is parabolic, the solution marches forward in ξ from the known profiles $f_i'(\eta)$ at $m = 1$. For the grid points $n = 2, 3, \ldots, 80$, across the boundary layer, Eq. (12) produces 79 equations in 81 unknowns for $f'_{2,n}$. Boundary conditions on f' supply the fact that $f'_{2,1} = 0$ and $f'_{2,81} = 1$, thus eliminate two unknowns. These 79 equations are now a tridiagonal system of linear equations for $f'_{2,2}$ through $f'_{2,80}$ which are solved by the Thomas method. Using the f' distribution in the Eq. (13), the transverse velocity $W_{2,n}$ across the boundary layer is obtained. When the solution for $m = 2$ is found, this acts as an initial condition for $m = 3$. This process continues for as long as the boundary data $\beta(\xi)$ and $\gamma(\xi)$ are specified. A check is made after each step to see if the flow has separated (i.e., the velocity gradient at the wall is zero). If the separation occurs, the calculation stops at once. If no separation occurs, the calculation continues until the last position $\xi = \xi_{max.}$ is reached.

The wall shearing stress τ_w can be obtained from

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\frac{\partial\eta}{\partial y} u_0 \left(\frac{\partial\bar{u}}{\partial\eta}\right)_{\eta=0} = \frac{\mu u_0}{L}\sqrt{Re/(2\xi)}\,\bar{u}_\infty^2\left(\frac{\partial f'}{\partial\eta}\right)_{\eta=0}.$$

The coefficient of skin-friction, $C_f\left[\equiv \tau_w \Big/ \frac{1}{2}\varrho u_\infty^2\right]$ becomes

$$C_f\sqrt{\frac{Re\,\xi}{2}} = f''(\xi,0) = \frac{1}{2(\Delta\eta)}\{3f'_{m,3} - 4f'_{m,2}\}. \tag{14}$$

The unknown normal surface velocity gradient, $f''(\xi, 0)$ is obtained from the solution of $f'(\xi, \eta)$ at each grid point m along the ξ-direction.

The displacement, momentum and energy thicknesses (viz., δ_1, δ_2, and δ_3) of the boundary layer are

$$\begin{aligned}(\delta_1, \delta_2, \delta_3) &= \int_0^\infty \left(\left(1-\frac{u}{u_\infty}\right), \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right), \frac{u}{u_\infty}\left\{1-\left(\frac{u}{u_\infty}\right)^2\right\}\right) dy \\ &= \frac{\partial y}{\partial \eta}\int_0^\infty ((1-f'), f'(1-f'), f'(1-f'^2))\, d\eta \\ &= (L/\bar{u}_\infty)\sqrt{2\xi/Re}\,(I_1, I_2, I_3).\end{aligned} \tag{15}$$

The integrals I_1, I_2 and I_3 in Eq. (15) are evaluated numerically from the solution of $f'(\xi, \eta)$ by using Simpson's 1/3 rule.

The accuracy of the present numerical scheme is verified with Görtler's rigorous analytical solution of laminar boundary layer separation [5] for the free-stream velocity, $\bar{u}_\infty = 1 - \bar{x}$, for $\bar{x} \in [0, 1]$. The parameters β and γ for this case in Eqs. (7) and (9) become $\beta(\xi) = 1 - (1 - 2\xi)^{-1}$ and $\gamma(\xi) = 0$. The initial velocity profiles $f_i'(\eta)$ and $W = \eta f_i'(\eta) - f_i(\eta)$ at $\xi = 0$ are obtained by integrating the following nonlinear ordinary differential equations:

$$f_i''' + f_i f_i'' = 0, \tag{16.1}$$

$$f_i = f_i' = 0, \quad f_i'' = 0.4696 \quad \text{at} \quad \eta = 0, \tag{16.2}$$

using a fourth-order Runge-Kutta integration scheme with a fixed step-size $\Delta\eta$ of 0.01. Using these profiles and following the above described marching procedure, the solution of the boundary layer equations (7) to (11), is obtained by solving the finite-difference equations (12) and (13), for $\xi > 0$. It is found that the normal surface velocity gradient, $f''(\xi, 0)$ approaches zero at $\xi = 0.119$, which corresponds to the value of $\bar{x} = 0.127074$. This value compares well with Görtler's analytical solution [5], $\bar{x}_{sep} = 0.126$.

3 Discussion

The magnetohydrodynamic boundary layer flow over a flat plate is examined for two cases, viz. (i) a uniform free-stream velocity and (ii) a uniform hydrostatic pressure. In the case of a uniform free-stream velocity,

$$u_\infty(x) = u_0 \quad \text{for} \quad x \in [0, L], \tag{17.1}$$

the pressure distribution is obtained from Eq. (6) as

$$p(x) = p_0 - \frac{1}{2}\varrho u_0{}^2(I\bar{x}), \tag{17.2}$$

Table 1. Boundary layer physical parameters

$I\bar{x}$	Uniform free-stream velocity				Uniform hydrostatic pressure			
	$f''(0)$	I_1	I_2	I_3	$f''(0)$	I_1	I_2	I_3
0.0	0.4696	1.2168	.4696	.7385	0.4696	1.2168	.4696	.7385
0.1	0.6153	1.0887	.4445	.7054	0.4223	1.2871	.4818	.7523
0.2	0.7393	0.9965	.4228	.6759	0.3573	1.4053	.5046	.7818
0.3	0.8489	0.9249	.4037	.6491	0.2716	1.5947	.5346	.8189
0.4	0.9482	0.8668	.3867	.6248	0.1595	1.9749	.5770	.8682
0.5	1.0390	0.8185	.3715	.6026	0.0341	3.1708	.6297	.9227

where p_0 is the pressure at $x = 0$. The parameters in Eqs. (7) and (8) become: $\bar{u}_\infty = 1$, $\xi = \bar{x}$, $\beta(\xi) = 0$ and $\gamma(\xi) = 2I\xi$.

For the case of a uniform hydrostatic pressure,

$$p(x) = p_0 \quad \text{for} \quad x \in [0, L], \tag{18.1}$$

the free-stream velocity is obtained from Eq. (6) as

$$u_\infty(x) = u_0(1 - I\bar{x}). \tag{18.2}$$

The free-stream velocity, $u_\infty(x)$ decreases along the flat plate with the magnetic interaction parameter, I. The parameters in Eqs. (7) and (8) for this case become: $\bar{u}_\infty = \sqrt{1 - 2I\xi}$, $\xi = \bar{x}\left(1 - \frac{1}{2} I\bar{x}\right)$, $\beta = 1 - (1 - 2I\xi)^{-1}$, and $\gamma = -\beta$. For both cases, the parameters β and γ in Eqs. (7) and (8) are found to be functions of $I\xi$, in turn functions of $I\bar{x}$. Following [6], the singularity at the leading edge is eliminated by using the transformation $\zeta = I\xi$ in the boundary layer equations (7)–(11). By replacing ξ as ζ and $I = 1$, in the Eqs. (7)–(11), one can get the transformed boundary layer equations. With this transformation, the boundary layer problem becomes locally nonsimilar with respect to the local magnetic interaction parameter, $I\bar{x}$. The solution of the boundary layer equations can also be obtained for $\bar{x} \in [0, 1]$ by specifying the values for the interaction parameter, I. In order to examine the effect of the pressure gradient on the boundary layer physical parameters, locally nonsimilar solutions are obtained for the boundary layer equations. The initial value profiles for the above two cases are obtained from Eqs. (16). Using these in the finite-difference equations and following the forward-marching procedure, the solution of the boundary layer equations is obtained for the two cases. The nondimensional boundary layer physical parameters such as skin-friction coefficient $(f''(0))$, displacement thickness (I_1) momentum thickness (I_2) and energy thickness (I_3) of the boundary layer for different values of the local magnetic interaction parameter, $I\bar{x}$, are presented in Table 1. It is found that the normal surface velocity gradient decreases with the local magnetic interaction parameter for the case of a uniform hydrostatic pressure. In the case of a uniform free-stream velocity, the normal surface velocity gradient increases with the local magnetic interaction parameter. Since the formulation of the problem is general, the solution of the boundary layer equations can be obtained for any specified free-stream velocity.

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Authors' addresses: P. Sam Lawrence, VSSC Central School, Trivandrum 695 586, India, and B. Nageswara Rao, Structural Engineering Group, Vikram Sarabhai Space Centre, Trivandrum 695 022, India