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Probability Theory and Related Fields

Some dimension-free features of vector-valued martingales

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Summary. Given any local martingale M in $\mathbf{R}^d$ or l^2, there exists a local martingale N in $\mathbf{R}^2$, such that $|M|=|N|$, $[M]=[N]$, and $\langle M\rangle=\langle N\rangle$. It follows in particular that any inequality for martingales in $\mathbf{R}^2$ which involves only the processes $|M|$, $[M]$ and $\langle M\rangle$ remains true in arbitrary dimension. When M is continuous, the processes $|M|^2$ and $|M|$ satisfy certain SDE's which are independent of dimension and yield information about the growth rate of M. This leads in particular to tail estimates of the same order as in one dimension. The paper concludes with some new maximal inequalities in continuous time.

1. Introduction

The present work grew out from attempts to extend some martingale inequalities to higher dimensions. For most of the standard inequalities (notably Doob's classical inequalities, the Burkholder – Davis – Gundy (BDG) inequalities, certain exponential inequalities, etc.), this can be easily done by elementary methods. However, the inequalities obtained in this way will usually contain the dimension d as a parameter, and the estimates will often become very crude as d gets large.

To illustrate this point, consider the well-known exponential inequality

$$\mathsf{P}\{M^*>r\}\leqq 2\,\mathrm{e}^{-r^2/2}, \qquad r\geqq 0, \tag{1.1}$$

valid for any continuous real-valued martingale M with $M_0=0$ and with quadratic variation process $[M]$ bounded by 1 (cf. Rogers and Williams 1987, p. 77). Recall that M^* denotes $\sup|M_t|$. Applying (1.1) componentwise to a d-dimensional martingale $M=(M^1, \ldots, M^d)$ with $[M]=[M^1]+\ldots+[M^d]<1$, one gets the bound

$$\mathsf{P}\{M^*>r\}\leqq 2d\,\mathrm{e}^{-r^2/2d}, \qquad r\geqq 0, \tag{1.2}$$

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which has in fact been used by Stroock (1984) in the context of large deviations. Now the bound in (1.2) turns out (cf. Theorem 4.5 below) to be of completely wrong order, since (1.1) is in fact true in arbitrary (even infinite) dimension (possibly apart from a numerical factor *outside* the exponential).

In the present paper we shall prove that *any* martingale inequality in $\mathbf{R}^2$ which involves only the three basic processes $|M|$, $[M]$ and $\langle M\rangle$, extends automatically to any finite or infinite dimension d, with the same values of all constants. Here $|\cdot|$ denotes the Euclidean or Hilbert norm, while $[M]$ is the trace of the quadratic variation matrix or operator (see Sect. 2 for definitions), and $\langle M\rangle$ is the dual predictable projection or compensator of $[M]$ (considered only when M is locally square integrable). Note that all three processes are $\mathbf{R}_+$-valued.

The quoted result is an immediate consequence of our main Theorem 3.1, where we show that whenever M is a local martingale in $\mathbf{R}^d$ or l^2, there exists some local martingale N in $\mathbf{R}^2$ for which the three basic processes above are the same, i.e. such that a.s.

$$|M|=|N|, \qquad [M]=[N], \qquad \langle M\rangle=\langle N\rangle. \tag{1.3}$$

To get an idea about the construction, think of M as composed of a radial component $|M|$ and a tangential component described by the projection $X=M/|M|$ onto the unit sphere. Since we require $|N|$ to be equal to $|M|$, it suffices to construct a martingale Y on the unit circle in $\mathbf{R}^2$ with the same quadratic variation as X, and then define $N=|M|\,Y$.

This may seem easy and straightforward, but the actual construction and proof involve some rather subtle technical difficulties. Thus X is well-defined only if $M\neq 0$, and it may in fact explode at the beginning or end of every excursion interval for M from the origin. To get the martingale property of N at 0, it is necessary to randomize a "phase" individually for each excursion, and because of the explosions, the randomization has to be made at some interior point. Even the construction of V within the excursion intervals will clearly involve some randomizations. The construction will normally require an extension of the original probability space, and a new filtration will have to be introduced which is large enough to ensure adaptedness of N, yet small enough to preserve the martingale property of M and $[M]-\langle M\rangle$. There finally remains the non-trivial task of verifying, via the general version of Itô's formula, that N really becomes a martingale with respect to the new filtration. A detailed discussion of all those points will be given in Sect. 3.

The situation simplifies when M has continuous paths. This is not only because $[M]=\langle M\rangle$ in this case, but more importantly because $[M]$ can then be used as a natural time scale for M. Thus one may reduce by a random time change to the case when $[M]_t\equiv t$, so that only one of our three basic processes remains, namely $|M|$. For this case, we shall prove in Sect. 4 that $|M|^2$ and $|M|$ are solutions to the equivalent stochastic differential equations (SDE's)

$$\mathrm{d}|M_t|^2=2\alpha_t|M_t|\,\mathrm{d}B_t+\mathrm{d}t, \tag{1.4}$$

$$\mathrm{d}|M_t|=\alpha_t\,\mathrm{d}B_t+\frac{1}{2|M_t|}\,\mathbf{1}\{M_t\neq 0\}\,(1-\alpha_t^2)\,\mathrm{d}t+\mathrm{d}L_t, \tag{1.5}$$

where B is a Brownian motion while α is a predictable process taking values in $[0, 1]$, and L is a non-decreasing continuous process which increases only when $M=0$.

It is natural to think of α_t and $\sqrt{1-\alpha_t^2}$ as *stochastic controls* specifying the instantaneous rates of M in the radial and tangential directions, respectively. In the extreme case when $\alpha \equiv 1$, one gets a one-dimensional Brownian motion along a ray from the origin. If instead $\alpha \equiv 0$, then M becomes a "spiraling" Brownian motion such that $|M_t|^2 - t$ is constant and non-random. Intermediate constant values of α yield Bessel processes of different order. Curiously enough, not all random controls α are feasible in (1.4) and (1.5) (not even all Markovian ones), and the same control may yield different solutions $|M|$. In other words, weak existence or uniqueness may fail for the SDE's (1.4) and (1.5).

In the one-dimensional case, every continuous local martingale M with $[M]_t \equiv t$ is known, by a classical result of Lévy, to be a Brownian motion, so in this case α is always 1. This illustrates the fact that the processes $|M|$ arising in the real-valued case are much more special than those in higher dimensions. Hence there is no hope of extending our main result in Sect. 3 to a reduction down to dimension one. From the point of view of proving martingale inequalities, this limitation is not a serious problem, since the extension from $\mathbf{R}^1$ to $\mathbf{R}^2$ can usually be achieved by elementary methods (cf. Proposition 3.4).

Our remark in the last paragraph may be compared with a theorem of Gilat (1977) (see also Protter and Sharpe 1979), to the effect that every nonnegative submartingale is of the form $|N|$ for some real-valued martingale N. If M is a martingale in $\mathbf{R}^d$, then $|M|$ is certainly a submartingale, so by Gilat's theorem there exists some one-dimensional martingale N with $|M| = |N|$ a.s. However, $[M] \neq [N]$ in general. In fact, the construction of Protter and Sharpe involves the insertion of extra jumps between $|M|$ and $-|M|$ at some random times $\tau_1, \tau_2, \ldots$, so the quadratic variation of N becomes

$$[N]_t = [|M|]_t + 4 \sum_j |M_{\tau_j}|^2 \, \mathbf{1}\{\tau_j \leqq t\}, \qquad t \geqq 0.$$

The SDE's in (1.4) and (1.5) can be used to derive some interesting inequalities. Thus it is shown in Theorem 4.4 that, if M is a continuous martingale in $\mathbf{R}^d$ or l^2 with $M_0 = 0$ and $[M]_t \equiv t$, and if $A_t = \int_0^t \alpha_s^2 \, ds$ with α as above, then a.s.

$$M_t^* \leqq B^* \circ A_t + \sqrt{t - A_t}, \qquad t \geqq 0, \tag{1.6}$$

for some real-valued Brownian motion B. Note that the bound is sharp both for $\alpha \equiv 1$, when M is essentially a one-dimensional Brownian motion, and for $\alpha \equiv 0$, when $|M_t|^2 \equiv t$.

Another interesting consequence of (1.4) and (1.5), proved in Theorem 4.5, is the fact that for M and B as above,

$$\mathsf{P}\{M_t^* \geqq r\} \leqq c\, \mathsf{P}\{B_t^* \geqq r\}, \qquad r \geqq 0, \tag{1.7}$$

where c is an absolute constant independent of dimension. This improves the previously noted bound in (1.1), since the right-hand side of (1.7) is asymptotically of the order $r^{-1} e^{-r^2/2}$ as $r \to \infty$.

In Sect. 5 we return to our starting point, by applying the main results of Sect. 3 to deduce various martingale inequalities in arbitrary dimension. The extension from one to higher dimensions is usually trivial, given the previously mentioned results, so our main effort consists in obtaining good bounds in the real-valued case. In Theorem 5.1 we prove two Kolmogorov type exponential inequalities for continuous time martingales with bounded jumps, and in Theorem 5.3 we show that the simpler bound in (1.1) applies when the jumps are symmetric (corresponding to the property of conditional symmetry in discrete time). Finally we prove in Theorem 5.6 a general continuous time version of an inequality originally due to Dubins and Savage (1965).

It should be noted that our present versions of the mentioned inequalities are much more general than previously known discrete time results, in the sense that the latter reduce to simple special cases, often attainable by more elementary methods. Our present approach is typically to construct a suitable supermartingale, where the supermartingale property may be checked by computations involving the general Itô formula for semimartingales with jumps.

Our main results are potentially applicable even to other fields. Thus in the study of U-statistics one is led naturally to consider the asymptotics of certain Hilbert-space valued martingales. In this context, a.s. asymptotic bounds have been obtained in a number of papers (e.g., in Dehling et al. 1986). Now in view of our present results, it would have been enough in most cases to consider martingales in $\mathbf{R}^2$ only. It is not clear, however, whether that would have simplified the proofs or led to sharper estimates. Incidentally we do obtain some a.s. bounds in this paper, but only for martingales with continuous paths (cf. Theorem 4.6).

Though most readers may only be interested in finite-dimensional processes, we include the infinite-dimensional case in our statements for the sake of completeness. Actually all results remain valid in arbitrary Hilbert spaces, though for convenience they will be stated and proved only in l^2. There exists an extensive but rather inaccessible literature on Hilbert-space valued martingales, summarized in the two monographs by Métivier (1982) and Métivier and Pellaumail (1980). However, no previous knowledge of martingale theory in Hilbert space will be assumed in this paper, since the few facts we need are collected and will be proved directly from definitions in Theorem 2.1 below. Actually most of those facts would have to be proved in $\mathbf{R}^d$ anyway, so by inserting this theorem we hope to be gaining both in economy and in the reader's convenience.

On the other hand, notions and results from stochastic calculus and the "general theory of processes" will be used extensively throughout the paper, often without explicit references. Any one of the texts on those topics listed in the bibliography will be sufficient background. For less familiar results, specific references will be given.

A special convention used in this paper is to write $\mathbf{1}\{\cdot\}$ for the indicator function of the set within brackets. For positive functions f and g, we shall often write $f \lesssim g$, instead of $f = O(g)$ to indicate that $f \leq cg$ for some finite constant $c > 0$. Unless otherwise stated, the value of c is independent of any specific choices of processes, functions or parameters. A numerical value is usually implicit in our proofs. (Thus, for example, (1.7) holds with $c \approx 6.30$, though this value is probably far from sharp.) For vectors x and y in $\mathbf{R}^d$ or l^2, we shall write $x \cdot y$ for the inner product and $|x|$ and $|y|$ for the norms. All random objects in this paper are assumed to be defined on some fixed probability space

$(\Omega, \mathscr{A}, \mathsf{P})$ with associated expectation E, and whenever random processes are considered, we assume adaptedness to some discrete or continuous filtration $\mathscr{F}=(\mathscr{F}_t)$, which is always assumed to satisfy the "usual conditions" of right-continuity and completeness. Conventions relating to martingales will be explained in the next section.

2. Martingale preliminaries

The present section serves several purposes. An obvious one is to introduce some notation and conventions concerning martingales in higher dimensions that will be used in the rest of the paper. A second purpose is to prove some auxiliary martingale results that we shall need in subsequent section. Our final aim is to give a quick self-contained introduction to Hilbert-space valued martingales, and to show how some crucial finite-dimensional results carry over to a Hilbert-space setting. Readers who are only interested in the finite-dimensional case may skip the proof of the main Theorem 2.1, and ignore all future references to l^2-valued martingales.

Since every probability measure on a Hilbert space H is supported by some separable subspace, the corresponding statement is true for any H-valued martingale, so without loss of generality we may assume that H is separable. For convenience we may then take $H=l^2$, so that every H-valued process M may be represented by its coordinate processes $M^1, M^2, \ldots$. An obvious first problem is then to characterize the martingale property of M in terms of the M^i. To make this precise, let us first recall some definitions.

We shall say that M is *integrable*, if $\mathsf{E}|M_t|<\infty$ for each $t\geqq 0$. By a *martingale* in l^2 is meant an integrable process M, such that the inner product $v\cdot M$ is a real-valued martingale for every $v\in l^2$. In particular, $M^1, M^2, \ldots$ are then martingales, and since we require the underlying probability space to satisfy the 'usual conditions', we may assume that $M^1, M^2, \ldots$ are right-continuous with left-hand limits. It turns out that M will then have the same property.

Local martingales may next be defined in the usual way. For those the component processes M^i are again local martingales, so the covariation processes $[M^i, M^j]$ are well-defined. For convenience we shall often write $[M^i, M^i]=[M^i]$, and we define the *quadratic variation* of M as the trace process $[M]=\Sigma[M^i]$. Here $[M]$ turns out to be a.s. finite, so by dominated convergence it inherits the right-continuity from the $[M^i]$, and we have $\Delta[M]_t=|\Delta M_t|^2$ for all t. When $|M|$ is locally in L^2, we many further define the *conditional variation process* $\langle M\rangle$ as the compensator of $[M]$. Note that, by monotone convergence, $\langle M\rangle = \Sigma\langle M^i\rangle = \Sigma\langle M^i, M^i\rangle$.

For real-valued (semi)martingales M, it is well known that the quadratic variation $[M]_t$ may be obtained for fixed $t>0$ as a limit,

$$\sum_i |M_{t_i}-M_{t_{i-1}}|^2 \xrightarrow{\mathsf{P}} [M]_t, \tag{2.1}$$

where $0\leqq t_0<t_1<\ldots<t_n=t$ form an arbitrary partition of the interval $[0, t]$, and where convergence holds as the mesh size $\max|t_i-t_{i-1}|$ tends to zero. The

result remains true in l^2 and will be needed to prove that the process $[M]-[|M|]$ is non-decreasing, a crucial fact for our construction in Sect. 3.

Occasionally we shall need to refer to the *jump point process* of M, which we may consider alternatively as a counting random measure ξ on $\mathbf{R}_+ \times (l^2\setminus\{0\})$, or as the associated non-decreasing measure valued process $\xi.$ on $l^2\setminus\{0\}$, given by

$$\xi_t(B)=\xi([0,t]\times B)=\sum_{s\leqq t}\mathbf{1}\{\Delta M_s\in B\}, \qquad t\geqq 0.$$

Similarly, the compensator of ξ may be regarded interchangeably as a random measure $\hat{\xi}$ on $\mathbf{R}_+ \times (l^2\setminus\{0\})$ or as a measure valued process $\hat{\xi}.$ on $l^2\setminus\{0\}$, where the two are related by

$$\hat{\xi}_t(B)=\hat{\xi}([0,t]\times B), \qquad t\geqq 0.$$

Note that, for fixed B, the process $\hat{\xi}.(B)$ is the compensator of $\xi.(B)$ in the usual sense.

We may now state the main result of this section, which contains all the facts we need for l^2-valued martingales. Everything here is more or less implicit in the literature [e.g., in Métivier (1982) and Métivier and Pellaumail (1980)], except for the characterization of l^2-valued martingales which may be new.

Theorem 2.1. *Let $M^1, M^2, \ldots$ be real-valued right-continuous local martingales, and define $M=(M^1, M^2, \ldots)$ and $[M]=\Sigma[M^i]$. Then M is a local martingale in l^2, iff $|M_0|<\infty$* a.s. *while $[M]^{1/2}$ is locally integrable. In that case even M is* a.s. *right-continuous with left-hand limits. Moreover, M satisfies* (2.1), *and $|M|^2-[M]$ is a local martingale, while $|M|$ is a local submartingale such that $[M]-[|M|]$ is non-decreasing.*

Proof. Let us first assume that M is a local martingale in l^2. Then clearly $|M_0|<\infty$ a.s. To prove the remaining properties of M, $[M]$ and $[|M|]$, we may assume that M is a true martingale. First we note that, a.s. for fixed $s<t$ and $v\in l^2$,

$$|v\cdot M_s|=|\mathsf{E}[v\cdot M_t|\mathscr{F}_s]|\leqq \mathsf{E}[|v\cdot M_t||\mathscr{F}_s]\leqq |v|\,\mathsf{E}[|M_t||\mathscr{F}_s]. \tag{2.2}$$

Since l^2 is separable, the exceptional null-set in (2.2) may be taken to be independent of v, so we get $|M_s|\leqq \mathsf{E}[|M_t||\mathscr{F}_s]$ a.s., which shows that $|M|$ is a submartingale.

The same thing is true for the processes $|\bar{M}^m-\bar{M}^n|$, where

$$\bar{M}^n=(M^1,\ldots,M^n,0,0,\ldots),$$

so by a classical inequality

$$\mathsf{P}\{(\bar{M}^m-\bar{M}^n)_t^*\geqq r\}\leqq \frac{1}{r}\,\mathsf{E}|\bar{M}_t^m-\bar{M}_t^n|, \qquad r,t>0,$$

and since $\mathsf{E}|\bar{M}_t^n-M_t|\to 0$ by dominated convergence, we may conclude that $(\bar{M}^m-\bar{M}^n)_t^*\to 0$ a.s. as $m,n\to\infty$ along some subsequence. It follows easily that $(\bar{M}^n-M)_t^*\to 0$ a.s. along the same sequence, and since each process $\bar{M}^n$ is right-continuous with left hand limits, the same thing must be true for M.

To see that $[M]^{1/2}$ is locally integrable, we define

$$U_t = \sum_{s \leqq t} \Delta M_s 1\{|\Delta M_s| \geqq 1\}, \qquad t \geqq 0, \tag{2.3}$$

and let $\hat{U}$ denote the compensator of U. Thus if ξ denotes the jump point process of U, we have

$$\hat{U}_t = \int_0^t \int v \hat{\xi}(\mathrm{d}t\,\mathrm{d}v), \qquad t \geqq 0.$$

By a suitable localization argument, we may assume that U has integrable variation. The same thing is then true for $\hat{U}$, since

$$\mathsf{E}|\hat{U}_t| \leqq \mathsf{E} \int_0^t \int |v|\, \hat{\xi}(\mathrm{d}t\,\mathrm{d}v) = \mathsf{E} \int_0^t \int |v|\, \xi(\mathrm{d}t\,\mathrm{d}v) = \mathsf{E} \int_0^t |\mathrm{d}U_t| < \infty,$$

so the difference $D = U - \hat{U}$ is a martingale of integrable variation, while the process $N = M - D$ is a martingale. By an argument in Protter (1990), pp. 102ff, it is further seen that the jumps of $v \cdot N$ are a.s. bounded by 2 for every $v \in l^2$ with $|v| = 1$, and since l^2 is separable we get the same property for N. We may then assume by a further localization that N is bounded. It remains to write

$$\begin{aligned} \mathsf{E}[N]_t^{1/2} &\leqq (\mathsf{E}[N]_t)^{1/2} = (\sum \mathsf{E}[N^i]_t)^{1/2} = (\sum (\mathsf{E}|N_t^i|^2 - \mathsf{E}|N_0^i|^2))^{1/2} \\ &= (\mathsf{E}|N_t|^2 - \mathsf{E}|N_0|^2)^{1/2} < \infty, \end{aligned}$$

$$\mathsf{E}[D]_t^{1/2} = \mathsf{E}(\sum_{s \leqq t} |\Delta D_s|^2)^{1/2} \leqq \mathsf{E} \sum_{s \leqq t} |\Delta D_s| \leqq \mathsf{E} \int_0^t |\mathrm{d}D| < \infty,$$

$$\begin{aligned} [M] &= \sum [M^i] \leqq 2 \sum ([N^i] + [D^i]) = 2([N] + [D]) \\ &\leqq 2([N]^{1/2} + [D]^{1/2})^2. \end{aligned}$$

Conversely, assume that $|M_0| < \infty$ a.s. while $[M]^{1/2}$ is locally integrable. To show that M is a local martingale, we may assume that $M_0 = 0$. It suffices to show that M is locally integrable, since the desired martingale property will then follow by dominated convergence from that of the processes $\bar{M}^n$. Let us then define a process U as in (2.3) by the formula

$$U_t = \sum_{s \leqq t} (\Delta M_s^1, \Delta M_s^2, \ldots) 1\{|\Delta [M]_s| \geqq 1\}, \qquad t \geqq 0,$$

and put $D = U - \hat{U}$, where $\hat{U}$ denotes the compensator of U. As before, D is locally a martingale of bounded variation, so it remains to prove that even $N = M - D$ is locally integrable. As before it is seen that the processes $\bar{N}^n = (N^1, \ldots, N^n, 0, 0, \ldots)$ have only jumps bounded by 2, so the jumps of $[\bar{N}^n]$ are bounded by 4, and the same thing must be true for $[N]$. By suitable localization we may thus assume that $[N]$ is bounded, and obtain

$$\mathsf{E}|N_t| \leqq (\mathsf{E}|N_t|^2)^{1/2} = (\mathsf{E}[N]_t)^{1/2} < \infty.$$

To see that $|M|^2 - [M]$ is a local martingale, we may assume by suitable localization that $[M]^{1/2}$ is integrable, while $|M_-|$ is bounded by some constant

$c>0$ on the support of $[M]$. Again it is enough to prove that $|M|^2-[M]$ is integrable. To this aim, we first consider sums over finite index sets, and note that

$$\begin{aligned}[\sum_i \int M^i_- \, \mathrm{d}M^i] &= \sum_i \sum_j \int M^i_- M^j_- \, \mathrm{d}[M^i, M^j] \\ &\leqq \sum_i \int |M_-|^2 \, \mathrm{d}[M^i] \leqq c^2 \sum_i [M^i] \leqq c^2 [M],\end{aligned}$$

where the first inequality follows from the fact that $x \cdot Ax \leqq |x|^2 \operatorname{tr} A$ for any symmetric non-negative definite matrix A. We may now conclude by the BDG inequality (cf. Dellacherie and Meyer 1980) that, again for finite sums

$$\mathsf{E}\left|\sum_i \int M^i_- \, \mathrm{d}M^i\right| \lesssim \mathsf{E}[\sum_i \int M^i_- \, \mathrm{d}M^i]^{1/2} \leqq c\,\mathsf{E}\left\{\sum_i [M^i]\right\}^{1/2} \leqq c\,\mathsf{E}[M]^{1/2}.$$

By dominated convergence, the sequence

$$|\bar{M}^n|^2 - [\bar{M}^n] = \sum_{i=1}^n (|M^i|^2 - [M^i]) = 2 \sum_{i=1}^n \int M^i_- \, \mathrm{d}M^i$$

is then Cauchy convergent in L^1, so the limit $|M|^2-[M]$ must be integrable.

To prove (2.1), we may assume as before that $[M]^{1/2}$ is integrable while $|M_-|$ is bounded on the support of $[M]$. Fixing a partition $0=t_0<t_1<\ldots<t_n=t$ of $[0, t]$ we get

$$\sum_i |M_{t_i} - M_{t_{i-1}}|^2 - [M]_t = 2 \sum_i \sum_k \int_{t_{i-1}}^{t_i} (M^k_{s-} - M^k_{t_{i-1}}) \, \mathrm{d}M^k_s.$$

Using the BDG inequality as before, we obtain

$$\mathsf{E}\left|\sum_i |M_{t_i} - M_{t_{i-1}}|^2 - [M]_t\right| \lesssim \mathsf{E}\left\{\sum_i \int_{t_{i-1}}^{t_i} |M_{s-} - M_{t_{i-1}}| \, \mathrm{d}[M]_s\right\}^{1/2},$$

which tends to zero as $\max |t_i - t_{i-1}| \to 0$, by the left-continuity of M_- and dominated convergence.

It remains to show that $[M]-[|M|]$ is non-decreasing. But this is clear from the fact that, for any partition $0=t_0<t_1<\ldots<t_n=t$,

$$\sum_i |M_{t_i} - M_{t_{i-1}}|^2 \geqq \sum_i (|M_{t_i}| - |M_{t_{i-1}}|)^2,$$

by the triangle inequality in l^2. □

It is important to realize that every discrete time martingale $N_0, N_1, \ldots$ with associated filtration $\mathscr{G}_0, \mathscr{G}_1, \ldots$ can be embedded into continuous time as a step type martingale $M_t = N_{[t]}$, $t \geqq 0$, with respect to the filtration $\mathscr{F}_t = \mathscr{G}_{[t]}$, $t \geqq 0$. Here clearly

$$[M]_t = \sum_{k=1}^{[t]} |N_k - N_{k-1}|^2, \qquad t \geqq 0,$$

$$\langle M \rangle_t = \sum_{k=1}^{[t]} \mathsf{E}[|N_k - N_{k-1}|^2 | \mathscr{G}_{k-1}], \qquad t \geqq 0.$$

By this device, every result for continuous time martingales (notably those in Sect. 3 and 5 below) can be specialized to yield a corresponding result in discrete time. Usually the latter can also be obtained directly by elementary methods. This is true in particular for our main Theorem 3.1.

We conclude this section with three simple lemmas that will be needed for the proof of our main Theorem 3.1.

Lemma 2.2. *Let M and N be local martingales in $\mathbf{R}^d$ or l^2, and let σ be a stopping time such that* a.s. $|M| = |N|$ *and* $|\Delta M| = |\Delta N|$ *on* (σ, ∞). *Then on* (σ, ∞) *we have* a.s. $\mathrm{d}[M] = \mathrm{d}[N]$ *and* $\mathrm{d}\langle M \rangle = \mathrm{d}\langle N \rangle$ *(the latter when M is locally in L^2).*

Proof. Since $|M|^2 - [M]$ and $|N|^2 - [N]$ are local martingales, so is the difference, which equals $[M] - [N]$ on $[\sigma, \infty)$. Now the latter process is continuous on (σ, ∞), since

$$\Delta[M] = |\Delta M|^2 = |\Delta N|^2 = \Delta[N],$$

and it has locally bounded variation, so it reduces to a constant on $[\sigma, \infty)$, and we get $\mathrm{d}[M] = \mathrm{d}[N]$ a.s. on (σ, ∞). If M is locally in L^2, then so is N, so the compensators $\langle M \rangle$ of $[M]$ and $\langle N \rangle$ of $[N]$ exist, and we get $\mathrm{d}\langle M \rangle = \mathrm{d}\langle N \rangle$ a.s. on (σ, ∞). □

Lemma 2.3. *Consider a martingale M and a stopping time σ, and define*

$$\tau = \inf\{t > \sigma \colon |M_t| \wedge |M_{t-}| = 0\}.$$

Then the process $N_t = M_t \mathbf{1}\{t < \tau\}$ is also a martingale.

Proof. Introduce the stopping times

$$\tau_n = \inf\{t > \sigma \colon |M_t| < 1/n\}, \qquad n \in \mathbf{N},$$

and write M_t^n for the martingales $M(\tau_n \wedge t)$, $t \geqq 0$, $n \in \mathbf{N}$. Then $\tau_n \uparrow \tau$, so for $t < \tau$ we obtain $M_t^n \to M_t = N_t$. If instead $t \geqq \tau$, we may conclude from the right-continuity of M that

$$|M_t^n| = |M_{\tau_n}| \leqq \frac{1}{n} \to 0 = N_t.$$

Thus $M_t^n \to N_t$ for every t, and the assertion follows by uniform integrability. □

Lemma 2.4. *Consider a process X and a stopping time τ, such that $X_\tau \mathbf{1}\{\tau < \infty\}$ is an integrable and $\mathscr{F}_\tau$-measurable random variable, while the process $Y_t = X_{\tau \vee t}$*

$-X_\tau$ is a martingale. Let α be a zero mean random variable independent of $\mathscr{F}$, and define

$$M_t = \alpha X_t \mathbf{1}\{\tau \leqq t\}, \qquad t \geqq 0.$$

Then M is a martingale with respect to the filtration $\tilde{\mathscr{F}}$ generated by $\mathscr{F}$ and M.

Proof. First conclude from Fubini's theorem that M is integrable. To check the martingale property, fix any numbers $s<t$, and write

$$M_t = \alpha X_\tau \mathbf{1}\{\tau \leqq s\} + \alpha X_\tau \mathbf{1}\{s < \tau \leqq t\} + \alpha Y_t.$$

Here the first term is $\tilde{\mathscr{F}}_s$-measurable, since M is adapted to $\tilde{\mathscr{F}}$. As for the second term, we note that $\mathscr{F}_s$ and $\tilde{\mathscr{F}}_s$ agree on the set $\{s<\tau\}$, and write

$$\begin{aligned} \mathsf{E}[\alpha X_\tau; s<\tau \leqq t \mid \tilde{\mathscr{F}}_s] &= \mathsf{E}[\alpha X_\tau; \tau \leqq t \mid \tilde{\mathscr{F}}_s]\, \mathbf{1}\{s<\tau\} \\ &= \mathsf{E}[\alpha X_\tau; \tau \leqq t \mid \mathscr{F}_s]\, \mathbf{1}\{s<\tau\} \\ &= \mathsf{E}[\mathsf{E}[\alpha \mid \mathscr{F}_\infty]\, X_\tau; \tau \leqq t \mid \mathscr{F}_s]\, \mathbf{1}\{s<\tau\}, \end{aligned}$$

which equals 0 since $\mathsf{E}[\alpha \mid \mathscr{F}_\infty] = \mathsf{E}\alpha = 0$. Since Y is a martingale while M is adapted to $\tilde{\mathscr{F}}$, we finally obtain for the third term

$$\begin{aligned} \mathsf{E}[\alpha Y_t \mid \tilde{\mathscr{F}}_s] &= \mathsf{E}[\alpha \mathsf{E}[Y_t \mid \mathscr{F}_s, \alpha] \mid \tilde{\mathscr{F}}_s] \\ &= \mathsf{E}[\alpha \mathsf{E}[Y_t \mid \mathscr{F}_s] \mid \tilde{\mathscr{F}}_s] = \mathsf{E}[\alpha Y_s \mid \tilde{\mathscr{F}}_s] = \alpha Y_s. \end{aligned}$$

Summarizing, we get

$$\mathsf{E}[M_t \mid \tilde{\mathscr{F}}_s] = \alpha X_\tau \mathbf{1}\{\tau \leqq s\} + \alpha Y_s = M_s. \quad \Box$$

3. Reduction of dimension

Most of this section will be devoted to a precise statement and detailed proof of our main result, Theorem 3.1, which asserts the existence for any given martingale M in $\mathbf{R}^d$ or l^2 of some martingale N in $\mathbf{R}^2$ with the same norm and quadratic variation processes. For most purposes, this reduces the study of multidimensional martingales to that of martingales in $\mathbf{R}^2$. A couple of elementary results at the end of the section may be useful for a further reduction down to dimension one.

The construction of N from M will in general require some randomization, so it may be necessary to extend the original filtered probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Such an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$ will be called *standard* if it preserves martingales, in the sense that any martingale on Ω will remain a martingale when regarded as defined on $\tilde{\Omega}$. Note that our standard extensions correspond to *extensions* in the terminology of Ikeda and Watanabe (1981), p. 89, whereas their standard extensions are more special.

Theorem 3.1. *Let M be a local martingale in $\mathbf{R}^d$ or l^2. Then there exists some local martingale N in $\mathbf{R}^2$, defined on a suitable standard extension of the original filtered probability space, such that* a.s. $|M| = |N|$ *and* $[M] = [N]$. *If M is locally in L^2, then even $\langle M \rangle = \langle N \rangle$* a.s.

Because of some technical difficulties indicated already in the introduction, it will be convenient to begin our proof with the special case when M is stopped at the first time (if ever) it reaches the origin. The result in this case will then be applied to the individual excursions of M from the origin, to yield the general result.

With a slight abuse of terminology, we shall say that the random variables $\vartheta_1, \vartheta_2, \ldots$ form a *Rademacher sequence*, if the ϑ_k are i.i.d. with $\mathsf{P}\{\vartheta_k = \pm 1\} = 1/2$. Such sequences will be used to attach random signs to the jumps of M. We may then choose any enumeration $\tau_1, \tau_2, \ldots$ of the jump times of M and associate a random sign ϑ_k with the jump at τ_k. More precisely, we may take the τ_k to be any sequence of a.s. distinct finite stopping times, such that every jump of M will a.s. occur at some τ_k. If the ϑ_k are chosen to be independent of the filtration $\mathscr{F}$, then clearly the signs attached to the jumps will be conditionally Rademacher, given $\mathscr{F}$, and then also given M.

For notational convenience, we may think of the signs as obtained from a continuous time process $\vartheta = (\vartheta_t)$, so that our previous ϑ_k are the values of ϑ at the times τ_k. It then becomes necessary to specify the joint distribution (together with the required measurability) in terms of conditioning. It will further be convenient to use complex notation, and to define our new martingale N in $\mathbf{C}$ instead of $\mathbf{R}^2$. Note that the quadratic variation $[N]$ is then defined as $[\operatorname{Re} N] + [\operatorname{Im} N]$. We are now able to state our main lemma.

Lemma 3.2. *Consider a local martingale M in $\mathbf{R}^2$ or l^2 and a stopping time σ, put*

$$\zeta = \inf\{t > \sigma;\ |M_t| \wedge |M_{t-}| = 0\},$$

and assume that $M_\sigma \neq 0$ on $\{\sigma < \infty\}$, while $M_t = 0$ for $t \geqq \zeta$. Define

$$A_t = \int_\sigma^t \frac{\mathrm{d}[M]^c - \mathrm{d}[|M|]^c}{|M|^2}, \qquad t \in [\sigma, \zeta),$$

$$\alpha_t = \arccos \frac{M_{t-} \cdot M_t}{|M_{t-}|\,|M_t|}, \qquad t \in (\sigma, \zeta).$$

Further introduce a random variable γ independent of $\mathscr{F}$ and uniformly distributed over the unit circle in $\mathbf{C}$, a real-valued Brownian motion B independent of $\mathscr{F}$ and γ, and a process ϑ whose values on the set $\{t \in (\sigma, \zeta);\ \alpha_t \neq 0\}$ are conditionally Rademacher, given $\mathscr{F}$, γ and B. Let

$$V_t = B \circ A_t + \sum_{s \in (\sigma, t]} \vartheta_s \alpha_s, \qquad t \in [\sigma, \zeta).$$

Then the process

$$N_t = \gamma |M_t|\, \mathrm{e}^{\mathrm{i} V_t}\, 1\{t \geqq \sigma\}, \qquad t \geqq 0, \tag{3.1}$$

is a local martingale in $\mathbf{C}$ with respect to the filtration $\tilde{\mathscr{F}}$ generated by $\mathscr{F}$ and N, and on (σ, ∞) we have a.s. $|N| = |M|$, $\mathrm{d}[N] = \mathrm{d}[M]$ *and* $\mathrm{d}\langle N\rangle = \mathrm{d}\langle M\rangle$ *(the latter when M is locally in L^2).*

Here the introduction of γ, B and ϑ may require an extension of the original probability space. The easiest construction is to take $\tilde{\Omega} = \Omega \times [0, 1]$ and $\tilde{\mathsf{P}} = \mathsf{P} \times \lambda$, where λ denotes the Lebesgue measure on [0, 1]. Then $\mathscr{F}$ extends to a filtration $\mathscr{F}'=(\mathscr{F}_t\times[0,1])$ on $\tilde{\Omega}$, and the new filtration $\tilde{\mathscr{F}}$ will be the one generated by $\mathscr{F}'$ and N.

Proof. From the construction it is clear that the new filtration $\tilde{\mathscr{F}}$ (or the new filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathrm{P}})$) is a standard extension of the original filtration $\mathscr{F}$ (or space $(\Omega, \mathscr{F}, \mathrm{P})$), so M and (in the local L^2 case) $[M]-\langle M\rangle$ will remain local martingales with respect to $\tilde{\mathscr{F}}$. We further have $|M|=|N|$ on $[\sigma,\infty)$, and from the definitions it is then clear that, for $t\in(\sigma,\zeta)$,

$$\begin{aligned}|\Delta N_t|^2&=|N_t|^2+|N_{t-}|^2-2|N_t|\,|N_{t-}|\cos\alpha_t\\&=|M_t|^2+|M_{t-}|^2-2|M_t|\,|M_{t-}|\cos\alpha_t=|\Delta M_t|^2,\end{aligned}$$

while for $t=\zeta<\infty$,

$$|\Delta N_t|=|N_{t-}|=|M_{t-}|=|\Delta M_t|.$$

Thus $|\Delta N|=|\Delta M|$ on (σ,∞), so by Lemma 2.2 it remains only to show that N is a local martingale. By Lemma 2.4 it is then enough to show that $N_t'\equiv\bar{\gamma}N_t$ is a local martingale on (σ,∞), i.e. that $N'_{\sigma\vee t}-N'_\sigma$ is a local martingale on $\mathbf{R}_+$.

By a suitable localization argument, we may then assume that M is a uniformly integrable martingale. Fixing $K>0$, we may further assume that M remains constant after the stopping time

$$\tau=\inf\{t\geqq\sigma;\,|M_t|\vee\int_\sigma^t d[M]>K\}.$$

We may also assume that the local martingale $|M|^2-[M]$ mentioned in Theorem 2.1 is in fact a true martingale.

Next we introduce the stopping times

$$\tau_\varepsilon=\inf\{t\geqq\sigma\colon[M_t|<\varepsilon\},\qquad\varepsilon>0,$$

and note that the processes

$$M_t^\varepsilon=M(t\wedge\tau_\varepsilon),\qquad N_t^\varepsilon=N'(t\wedge\tau_\varepsilon),\qquad t\geqq0,$$

converge pointwise to M and N', respectively, as $\varepsilon\to0$. In fact, the convergence is obvious for $t<\zeta$ since $\tau_\varepsilon\uparrow\zeta$, and for $t\geqq\zeta$ we get

$$|M_t^\varepsilon|=|M_{\tau_\varepsilon}|\leqq\varepsilon\to0=|M_t|,$$

and similarly for N^ε. Now if N^ε is known to be a martingale for each $\varepsilon>0$, then the martingale property of N' will follow by uniform integrability. Thus we may henceforth assume that $|M_-|\geqq\varepsilon$ on the support of $[M]$ in (σ,∞).

In that case we get for $t\in(\sigma,\zeta)$,

$$A_t\leqq\varepsilon^{-2}\int_\sigma^t(d|M]^c-d[|M|]^c)\leqq K/\varepsilon^2,$$

and furthermore, by elementary geometry,

$$\sum_{s\leqq t}\alpha_s^2\leqq\pi^2\sum_{s\leqq t}\sin^2\frac{\alpha_s}{2}\leqq\pi^2+\frac{\pi^2}{4\varepsilon^2}\sum_{s<\tau}|\Delta M_s|^2\leqq\pi^2+\frac{\pi^2 K}{4\varepsilon^2}.$$

Thus we may now extend the definitions of A, α and V to $\mathbf{R}_+$, by taking $\mathrm{d}A_t=\mathrm{d}V_t=\alpha_t=0$ on $[0,\sigma]$ and $[\zeta,\infty)$. Then V clearly becomes a martingale on $\mathbf{R}_+$ with respect to the induced filtration, and moreover

$$[V]_\infty=A_\infty+\sum_t\alpha_t^2\leqq\frac{K}{\varepsilon^2}\left(1+\frac{\pi^2}{4}\right)+\pi^2=:K'.$$

To complete the proof, it is enough to show that N' is a martingale on (σ,∞) with respect to the filtration $\mathscr{F}'$ generated by $\mathscr{F}$ and V. Our first step is then to write, using Itô's formula,

$$\begin{aligned}(3.2)\qquad \mathrm{d}e^{iV_t}&=ie^{iV_{t-}}\mathrm{d}V_t-\tfrac12 e^{iV_{t-}}\mathrm{d}[V]_t^c+(\Delta e^{iV_t}-ie^{iV_{t-}}\Delta V_t)\\&=e^{iV_{t-}}\{i\,\mathrm{d}V_t-\tfrac12\mathrm{d}A_t+(e^{i\vartheta_t\alpha_t}-1-i\vartheta_t\alpha_t)\}.\end{aligned}$$

(Here and below, we are following the convenient practice of writing Itô's formula in differential form and without summation sign in the discrete correction term. No ambiguity should result from this.) Next it is seen from Theorem 2.1 that $|M|$ is a semimartingale, so by another application of Itô's formula,

$$\mathrm{d}|M_t|^2=2|M_{t-}|\,\mathrm{d}|M_t|+\mathrm{d}[|M|]_t.$$

Since $|M_-|\geqq\varepsilon$ on the support of $[M]$, we may solve for $\mathrm{d}|M_t|$ to get

$$\begin{aligned}(3.3)\qquad \mathrm{d}|M_t|&=\frac{1}{2|M_{t-}|}(\mathrm{d}|M_t|^2-\mathrm{d}[|M|]_t)\\&=\frac{1}{2|M_{t-}|}\{(\mathrm{d}|M_t|^2-\mathrm{d}[M]_t)+(\mathrm{d}[M]_t-\mathrm{d}[|M|]_t)\}.\end{aligned}$$

Next recall that $V^c=B\circ A$ while $D=|M|^2-[M]$ is a martingale. We need to show that the product $V^c D^c$ is a local martingale. To ensure integrability, we may then replace B by a bounded process $\tilde{B}$ obtained by suitable stopping, put $\tilde{V}^c=\tilde{B}\circ A$, and prove instead that $\tilde{V}^c D^c$ is a martingale. But this is clear, since for any $s\leqq t$,

$$\mathsf{E}[\tilde{V}_t^c D_t^c|\mathscr{F}_s']=\mathsf{E}[\mathsf{E}[\tilde{B}\circ A_t|\mathscr{F}_s',A,D]\cdot D_t^c|\mathscr{F}_s']=\tilde{V}_s^c D_s^c.$$

It follows that $[V,D]^c=[V^c,D^c]=0$, so by (3.2) and (3.3)

$$(3.4)\qquad \mathrm{d}[e^{iV},|M|]_t^c=ie^{iV_{t-}}\frac{1}{2|M_{t-}|}\mathrm{d}[V,D]_t^c=0.$$

Integrating by parts in (3.1), we obtain in view of (3.4),

$$\mathrm{d}N_t'=|M_{t-}|\,\mathrm{d}e^{iV_t}+e^{iV_{t-}}\,\mathrm{d}|M_t|+\Delta|M_t|\,\Delta e^{iV_t},$$

so by (3.2) and (3.3)

$$e^{-iV_{t-}}\,dN'_t = |M_{t-}|\{i\,dV_t - \tfrac{1}{2}dA_t + (e^{i\vartheta_t\alpha_t} - 1 - i\vartheta_t\alpha_t)\} + \frac{1}{2|M_{t-}|}\{(d|M_t|^2 - d[M]_t) + (d[M]_t - d[|M|]_t)\} + \Delta|M_t|(e^{i\vartheta_t\alpha_t} - 1). \tag{3.5}$$

But by the definitions of A and α,

$$d[M]_t^c - d[|M|]_t^c = |M_{t-}|^2\,dA_t,$$

while

$$|\Delta M_t|^2 - (\Delta|M_t|)^2 = |M_t - M_{t-}|^2 - (|M_t| - |M_{t-}|)^2 = 2|M_t|\,|M_{t-}|(1 - \cos\alpha_t),$$

so (3.5) simplifies to

$$e^{-iV_{t-}}\,dN'_t = i|M_{t-}|\,dV_t + \frac{1}{2|M_{t-}|}(d|M_t|^2 - d[M]_t) + i\vartheta_t\{|M_{t-}|(\sin\alpha_t - \alpha_t) + \Delta|M_t|\sin\alpha_t\}. \tag{3.6}$$

To see that this is a martingale, we note first that the summation process based on the last term on the right is of integrable variation. In fact,

$$\sum_t |M_{t-}|\,|\sin\alpha_t - \alpha_t| \lesssim K\sum_t \alpha_t^2 \leqq KK' < \infty,$$

while by Schwarz' inequality

$$\sum_t |\Delta M_t|\,|\sin\alpha_t| \leqq \{\sum_{t\in(\sigma,\tau)} (\Delta|M_t|)^2 \sum_{t<\tau} \alpha_t^2\}^{1/2} + |\Delta M_\tau| \leqq \sqrt{KK'} + K + |M_\tau|,$$

which is integrable, since $|M_\infty|$ is integrable while $|M_\tau| \leqq \mathsf{E}[|M_\infty|\,|\mathscr{F}_\tau]$ a.s. by Doob's optional sampling theorem.

Returning to Eq. (3.6), it is clear that the first two terms on the right define martingales on (σ, ∞), and for the last term the required martingale property follows from the assumed conditional independence and symmetry of the variables ϑ_t, given $\mathscr{F}$ and B. Thus the left-hand side of (3.6) is a martingale, so the process N' itself must be a local martingale. By uniform integrability, it follows that N' is in fact a true martingale on (σ, ∞). □

The next lemma will be needed to split the path of a martingale into disjoint excursions from the origin.

Lemma 3.3. *Let the function $f: \mathbf{R}_+ \to \mathbf{R}_+$ be right-continuous with left-hand limits. Then the set $\{t \geqq 0: f_t > 0\}$ can be decomposed uniquely into at most countably many intervals I_n of the form (a, b) or $[a, b)$, such that $f_{t-} > 0$ for $t \in I_n^\circ$, while $f_t \wedge f_{t-} = 0$ for $t \in \partial I_n$.*

Proof. For each $t \geqq 0$ with $f_t > 0$, we define

$$a_t = \sup\{s \in [0, t]: f_s \wedge f_{s-} = 0\}, \qquad b_t = \inf\{s > t: f_s \wedge f_{s-} = 0\}.$$

Then $b_t > t$ and $f_s \wedge f_{s-} > 0$ for all $s \in (a_t, b_t)$, while $f_s \wedge f_{s-} = 0$ for $s \in \{a_t, b_t\} \setminus \{0, \infty\}$, the latter because of the right continuity of f_s and the left continuity of f_{s-}. Define $I_t = (a_t, b_t)$ if $f(a_t) = 0$, and let $I_t = [a_t, b_t)$ otherwise. Then any two intersecting intervals I_s and I_t must be equal, so the I_t form a disjoint partition of the set $\{t \geq 0: f_t > 0\}$. Note also that there can be at most countably many distinct intervals I_t, since each one has positive length. □

Proof of Theorem 3.1. Let $\mathscr{I}$ denote the collection of excursion intervals of the path of $|M|$, as defined in the Lemma 3.3. Given any dense sequence $r_1, r_2, \ldots$ in $\mathbf{R}_+$, we may construct recursively an enumeration $I_1, I_2, \ldots$ of $\mathscr{I}$, by letting $I_n = \emptyset$ if $M_{r_n} = 0$ or $r_n \in I_1 \cup \ldots \cup I_{n-1}$, and otherwise by taking I_n to be the unique interval in $\mathscr{I}$ that contains r_n. With this construction, the endpoints of the intervals will clearly become random variables (provided we take both to be ∞ if $I_n = \emptyset$).

Independently of $\mathscr{F}$, we now introduce a sequence of independent real-valued Brownian motions $B^1, B^2, \ldots$ indexed by $\mathbf{R}$ (so that for each n the two processes B^n_t and B^n_{-t} are independent Brownian motions on $\mathbf{R}_+$), and further an independent sequence of independent random variables $\gamma_1, \gamma_2, \ldots$ in $\mathbf{C}$, each uniformly distributed over the unit circle. For every jump time t of M we next introduce a random variable ϑ_t, such that the family $\{\vartheta_t\}$ is conditionally Rademacher, given $\mathscr{F}$, $\{B^n\}$ and $\{\gamma_n\}$. Letting α_t, $t \geq 0$, be such as in Lemma 3.2 and writing

$$A_t = \int_{r_n}^{t} \frac{\mathrm{d}[M]^c - \mathrm{d}[|M|]^c}{|M|^2}, \qquad t \in I_n, \qquad n \in \mathbf{N}, \tag{3.7}$$

we may now define the process

$$V_t = B^n \circ A_t + \sum_{s \in (r_n, t]} \vartheta_s \alpha_s, \qquad t \in I_n, \qquad n \in \mathbf{N}. \tag{3.8}$$

Note that in (3.7) and (3.8), the integral or sum over an interval $(a, b]$ with $a > b$ is defined by

$$\int_{(a,b]} = -\int_{(b,a]} \quad \text{and} \quad \sum_{(a,b]} = -\sum_{(b,a]}.$$

Now put

$$N_t = \sum_{n=1}^{\infty} \gamma_n |M_t| \, \mathrm{e}^{\mathrm{i} V_t} \mathbf{1}\{t \in I_n\}, \qquad t \geq 0, \tag{3.9}$$

and let $\tilde{\mathscr{F}} = (\tilde{\mathscr{F}}_t)$ be the filtration generated by $\mathscr{F}$ and N. We claim that N is a complex-valued $\tilde{\mathscr{F}}$-martingale, such that a.s. $|M| = |N|$, $[M] = [N]$ and $\langle M \rangle = \langle N \rangle$, the latter when M is locally in L^2.

On every excursion interval I_n, we may choose a version of V which is right-continuous and such that the left-hand limits exist in the interior I_n°. We may further assume that the left limit exists at every finite right endpoint t where $M_{t-} \neq 0$. Since moreover $M_t = 0$ implies $N_{t+} = N_t = 0$ while $M_{t-} = 0$ implies $N_{t-} = 0$, it is clear that N has a right-continuous version with left-hand limits. From (3.9) it is further seen that $|N| = |M|$, and that $|\Delta N| = |\Delta M|$ in the interior of every excursion interval I_n. If $t \in (0, \infty)$ is instead an endpoint, we get $|N_t| \wedge |N_{t-}| = |M_t| \wedge |M_{t-}| = 0$, so in that case

$$|\Delta N_t| = |N_t| \vee |N_{t-}| = |M_t| \vee |M_{t-}| = |\Delta M_t|.$$

At every other point we have trivially $|\Delta N_t| = |\Delta M_t| = 0$. Thus the relation $|\Delta N| = |\Delta M|$ must be generally true.

Next we note that $\tilde{\mathscr{F}}$ is a standard extension of $\mathscr{F}$, so that the local martingale property of M and (in the local L^2 case) of $[M] - \langle M \rangle$ is preserved under the extension of the filtration. By Lemma 2.2 it remains only to show that N is a local $\tilde{\mathscr{F}}$-martingale. To this aim, we fix an $\varepsilon > 0$, put $\tau_0 = 0$, and define recursively the stopping times $\sigma_1 \leqq \tau_1 \leqq \sigma_2 \leqq \dots$ by

$$\begin{aligned} \sigma_k &= \inf\{t \geqq \tau_{k-1} : |M_t| \geqq \varepsilon\}, \\ \tau_k &= \inf\{t > \sigma_k : |M_t| \wedge |M_{t-}| = 0\}, \qquad k \in \mathbf{N}. \end{aligned}$$

Next we introduce the complex-valued process

$$N_t^\varepsilon = N_t \sum_{k=1}^{\infty} \mathbf{1}\{\sigma_k \leqq t < \tau_k\}, \qquad t \geqq 0, \tag{3.10}$$

and let $\mathscr{F}^\varepsilon = (\mathscr{F}_t^\varepsilon)$ be the filtration generated by N^ε and $\mathscr{F}$. Note that $|N^\varepsilon - N| \leqq \varepsilon$, and further that $\tilde{\mathscr{F}} = \vee \mathscr{F}^\varepsilon$, the filtration generated by all the $\mathscr{F}^\varepsilon$.

First we prove that the processes N_t^ε are local $\mathscr{F}^\varepsilon$-martingales. It is then enough to consider each term in (3.10) separately, and by suitable localization we may reduce to the case when M (and then also N) is uniformly integrable. To prove that the k-th term in (3.10) is a martingale, we may first assume that $\sigma_k < \infty$ a.s. Then $[\sigma_k, \tau_k) \subset I_\nu$ for some $\mathbf{N}$-valued random variable ν. By Lemmas 3.2 and 2.3 it is enough to show that the random variable

$$\gamma_k' \equiv N_{\sigma_k}/|N_{\sigma_k}| = \gamma_\nu \, e^{iV(\sigma_k)}$$

is uniformly distributed over the unit circle in $\mathbf{C}$, while the process

$$B_k'(t) \equiv B^\nu(t - A_{\sigma_k}) - B^\nu(-A_{\sigma_k}), \qquad t \geqq 0,$$

is an independent Brownian motion on $\mathbf{R}_+$, and that the pair (γ_k', B_k') is independent of $\mathscr{F}$. But this follows by a simple conditioning argument from the corresponding properties for the pairs (γ_n, B^n). If $P\{\sigma_k = \infty\} > 0$, then γ_k' and B_k' can only be defined as above on the set $\{\sigma_k < \infty\}$, and we may note instead that they are conditionally independent with the stated distributions, given $\mathscr{F}$ and on the set $\{\sigma_k < \infty\}$. They may then be extended to the set $\{\sigma_k = \infty\}$ with the same properties, and Lemma 3.2 applies as before.

To deduce the asserted martingale property of N, we fix an arbitrary $\mathscr{F}$-stopping time τ, such that the process $M_{\tau \wedge t} \mathbf{1}\{\tau > 0\}$ is uniformly integrable. Then the processes $N_{\tau \wedge t}^\varepsilon \mathbf{1}\{\tau > 0\}$ have the same property and are therefore $\mathscr{F}^\varepsilon$-martingales, so by the chain rule for conditional expectations, we get for any $s < t$ and $\varepsilon < \delta$,

$$\mathsf{E}[N_{\tau \wedge t}^\varepsilon; \tau > 0 \,|\, \mathscr{F}_s^\delta] = \mathsf{E}[N_{\tau \wedge s}^\varepsilon; \tau > 0 \,|\, \mathscr{F}_s^\delta].$$

Here we may conclude by letting $\varepsilon \to 0$ and then $\delta \to 0$ that

$$\mathsf{E}[N_{\tau \wedge t};\, \tau>0 \,|\, \tilde{\mathscr{F}}_s] = \mathsf{E}[N_{\tau \wedge s};\, \tau>0 \,|\, \tilde{\mathscr{F}}_s] = N_{\tau \wedge s} \mathbf{1}\{\tau>0\},$$

which means that $N_{\tau \wedge t}\, \mathbf{1}\{\tau>0\}$ is an $\tilde{\mathscr{F}}$-martingale. □

As already noted, Theorem 3.1 becomes false if we require the local martingale N to be real-valued. Thus in order to extend a one-dimensional martingale inequality to higher dimensions, we need to consider separately the extension to dimension two. The following inequality may then be a useful tool.

Proposition 3.4. *Let X be a measurable random process in $\mathbf{R}^2$. Then*

$$\mathsf{P}\{X^* \geqq r\} \leqq \frac{\pi}{2\varepsilon} \sup_{|v|=1} \mathsf{P}\{(v \cdot X)^* \geqq r(1-\varepsilon^2)\}, \qquad r \geqq 0,\ 0<\varepsilon<1.$$

Proof. First assume that $X_t = \xi$ is independent of t. Let the random vector η be independent of ξ and uniformly distributed over the unit circle. Writing $s = r(1-\varepsilon^2)$, we get by Fubini's theorem

$$\sup_{|v|=1} \mathsf{P}\{|v \cdot \xi| \geqq s\} \geqq \mathsf{P}\{|\eta \cdot \xi| \geqq s\} = \frac{1}{2\pi} \mathsf{E} \int_0^{2\pi} \mathbf{1}\{|\xi \cos \phi| \geqq s\}\, d\phi$$

$$= \frac{2}{\pi} \mathsf{E}\left[\arccos \frac{s}{|\xi|} ;\ |\xi| \geqq s\right]$$

$$\geqq \frac{2}{\pi} \arccos(1-\varepsilon^2) \cdot \mathsf{P}\{|\xi| \geqq r\},$$

and it remains to note that

$$\arccos(1-\varepsilon^2) \geqq \arccos \sqrt{1-\varepsilon^2} = \arcsin \varepsilon \geqq \varepsilon.$$

Next assume that X is an arbitrary measurable process. By the general section theorem (cf. Dellacherie and Meyer 1975), there exists some $\mathbf{R}_+$-valued random variable τ, such that $|X_\tau| \geqq r$ a.s. on the set where $X^* > r$. Using the result in the special case with $\xi = X_\tau$, we obtain

$$\mathsf{P}\{X^* > r\} \leqq \mathsf{P}\{|X_\tau| \geqq r\}$$

$$\leqq \frac{\pi}{2\varepsilon} \sup_{|v|=1} \mathsf{P}\{|v \cdot X_\tau| \geqq s\} \leqq \frac{\pi}{2\varepsilon} \sup_{|v|=1} \mathsf{P}\{(v \cdot X)^* \geqq s\}.$$

It remains to apply this for fixed s to a sequence $r_n \uparrow r$. □

In view of our intended applications to martingales, we shall need the last result only when X is right-continuous with left-hand limits. In that case it is easy to modify the preceding proof, so as to avoid any reference to the debut or section theorem.

In applying Proposition 3.4 one needs to find a good value for ε. The following specific result for a wide class of exponential bounds will be used repeatedly in Sect. 5. Recall that an ultimately non-decreasing and strictly positive function

f on $\mathbf{R}_+$ is said to be of *dominated variation* or *polynomial growth* (at infinity), if the ratio $f(2r)/f(r)$ remains bounded as $r\to\infty$.

Corollary 3.5. *Fix a convex function of dominated variation $f\colon \mathbf{R}_+\to(0,\infty)$, and let X be a measurable process in $\mathbf{R}^2$, such that*

$$\sup_{|v|=1} \mathsf{P}\{(v\cdot X)^*\geqq r\}\lesssim e^{-f(r)}, \qquad r\geqq 0.$$

Then

$$\mathsf{P}\{X^*\geqq r\}\lesssim (f(r)\vee 1)^{1/2}\, e^{-f(r)}, \qquad r\geqq 0.$$

Proof. Replacing f by $f\vee f(0)\vee 1$ if necessary, we may assume that f is non-decreasing with $f\geqq 1$. By Proposition 3.4 it is then enough to show that

$$f(r)-f\left(r-\frac{r}{f(r)}\right)\lesssim 1.$$

Since f is convex, the left-hand side is bounded by $rf'(r)/f(r)$, where f' denotes the right or left derivative of f. It remains to notice that, by the convexity and dominated variation of f,

$$\frac{rf'(r)}{f(r)}\leqq\frac{f(2r)-f(r)}{f(r)}\leqq\frac{f(2r)}{f(r)}\lesssim 1. \quad \square$$

4. Continuous paths

In this section we examine the special case of martingales M in $\mathbf{R}^d$ or l^2 with continuous paths. For most purposes we may then assume that $[M]_t\equiv t$ a.s., since we may easily reduce to that case by a random time change.

In fact, assuming first that $[M]_\infty=\infty$ a.s., we may introduce the stopping times

$$\tau_t=\inf\{s>0\colon [M]_s>t\}, \qquad t\geqq 0,$$

and define a new process N and filtration $\mathscr{G}$ by

$$N_t=M_{\tau_t}, \qquad \mathscr{G}=\mathscr{F}_{\tau_t}, \qquad t\geqq 0.$$

Since M and $[M]$ have a.s. the same intervals of constancy, even N is continuous (cf. Jacod (1979), p. 316), and from Doob's optional sampling theorem it is further seen that the processes N_t-N_0 and $|N_t|^2-|N_0|^2-t$ are $\mathscr{G}$-martingales. Thus N is a continuous local $\mathscr{G}$-martingale with $[N]_t\equiv t$ a.s. Note also that $M_t=N\circ[M]_t$. It is further clear that the graph of N on $[0,[M]_t]$ coincides with the graph of M on $[0,t]$, so in particular $M_t^*=N^*\circ[M]_t$ a.s.

The situation is slightly more complicated when $\mathsf{P}\{[M]_\infty<\infty\}>0$, since in that case τ_t may be infinite. To preserve the linear rate $[N]_t\equiv t$ in this case, we may proceed as in Ikeda and Watanabe (1981), pp. 91f, to randomize a suitable continuation of N on the time interval $([M]_\infty,\infty)$. Note that the a.s. relations $M_t\equiv N\circ[M]_t$ and $M_t^*\equiv N^*\circ[M]_t$ remain true in this case.

In view of these remarks and in order to simplify the statements of our results, we shall assume for the remainder of this section that M is a continuous local martingale in $\mathbf{R}^d$ or l^2 with $[M]_t \equiv t$ a.s. Our first aim is to derive the two equivalent SDE's in (1.4) and (1.5).

Theorem 4.1. *Let M be a continuous local martingale in $\mathbf{R}^d$ or l^2 with $[M]_t \equiv t$. Then there exist some real-valued Brownian motion B, some measurable and adapted $[0, 1]$-valued process α, and some non-decreasing continuous process L with $L_0 = 0$ which grows only when $M = 0$, such that*

$$\mathrm{d}|M_t|^2 = 2\,|M_t|\,\alpha_t\,\mathrm{d}B_t + \mathrm{d}t, \tag{4.1}$$

$$\mathrm{d}|M_t| = \alpha_t\,\mathrm{d}B_t + \frac{1}{2|M_t|}\,\mathbf{1}\{M_t \neq 0\}\,(1-\alpha_t^2)\,\mathrm{d}t + \mathrm{d}L_t, \tag{4.2}$$

$$\int_0^t \alpha_s^2\,\mathrm{d}s = [|M|]_t, \qquad t \geqq 0. \tag{4.3}$$

$$L_t = \sup_{s \leqq t}(L_s - |M_s|) \vee 0, \qquad t \geqq 0. \tag{4.4}$$

Here and below, 'existence' of random processes should always be understood in the *weak* sense. Thus our auxiliary processes may have to be defined on some extension of the original probability space.

Proof. By Theorem 2.1 the process

$$N_t = |M_t|^2 - [M]_t = |M_t|^2 - t, \qquad t \geqq 0, \tag{4.5}$$

is a local martingale while $|M_t|$ is a semimartingale. By Itô's formula,

$$\mathrm{d}|M_t|^2 = 2|M_t|\,\mathrm{d}|M_t| + \mathrm{d}[|M|]_t, \tag{4.6}$$

so

$$\mathrm{d}[N]_t = \mathrm{d}[|M|^2]_t = 4|M_t|^2\,\mathrm{d}[|M|]_t.$$

Now

$$\mathrm{d}[|M|]_t \leqq \mathrm{d}[M]_t = \mathrm{d}t, \tag{4.7}$$

so by a theorem of Doob (cf. Karatzas and Shreve 1988, p. 170) we may write

$$\mathrm{d}N_t = 2|M_t|\,\alpha_t\,\mathrm{d}B_t, \tag{4.8}$$

for some Brownian motion B and some measurable adapted process $\alpha \geqq 0$ satisfying (4.3). By (4.7) we may assume that $\alpha \leqq 1$. Now (4.1) follows by combination of (4.5) and (4.8).

Next, we get from (4.1), (4.3) and (4.6),

$$2|M_t|\,\alpha_t\,\mathrm{d}B_t + \mathrm{d}t = \mathrm{d}|M_t|^2 = 2|M_t|\,\mathrm{d}|M_t| + \alpha_t^2\,\mathrm{d}t,$$

so for $M_t \neq 0$,

$$\mathrm{d}|M_t| = \alpha_t\,\mathrm{d}B_t + \frac{1}{2|M_t|}\,(1-\alpha_t^2)\,\mathrm{d}t.$$

Thus, if $2L$ denotes the local time of $|M|$ at zero, then (4.2) follows with α replaced by $\alpha' = \alpha \mathbf{1}\{M \neq 0\}$, by Tanaka's formula (cf. Rogers and Williams 1987, p. 96). But then (4.3) remains true with α' in the place of α, and we may also replace α by α' in (4.2). Finally, (4.4) follows from (4.2) by Skorohod's lemma (cf. Karatzas and Shreve 1988, p. 210). □

Corollary 4.2. *Let M be a continuous local martingale in* $\mathbf{R}^d$ or l^2 *with* $[M]_t \equiv t$ a.s. *Then*

$$\int_0^\infty \mathbf{1}\{M_t = 0\}\,\mathrm{d}t = 0 \quad \text{a.s.}$$

Proof. By Tanaka's formula, (4.2) remains true with α replaced by $\alpha \mathbf{1}\{M \neq 0\}$, so by Itô's formula,

$$\begin{aligned} \mathrm{d}|M_t|^2 &= 2|M_t|\,\mathrm{d}|M_t| + \mathrm{d}[|M|]_t \\ &= 2|M_t|\,\alpha_t\,\mathrm{d}B_t + \mathbf{1}\{M_t \neq 0\}(1-\alpha_t^2)\,\mathrm{d}t + \mathbf{1}\{M_t \neq 0\}\,\alpha_t^2\,\mathrm{d}t \\ &= 2|M_t|\,\alpha_t\,\mathrm{d}B_t + \mathbf{1}\{M_t \neq 0\}\,\mathrm{d}t, \end{aligned}$$

and the assertion follows by comparison with (4.1). □

The next result is a kind of converse to Theorem 4.1, which shows that the SDE's (4.1) and (4.2) are not only necessary, but also essentially sufficient for the solution process to be of the form $|M|$ for some continuous local martingale M in $\mathbf{R}^d$ with $d \geqq 2$. Note that the corresponding statement for $d=1$ is true only when $\alpha \equiv 1$. The extra condition required in this direction is precisely the property in Corollary 4.2 of a "non-sticky boundary".

Theorem 4.3. *Let X be a continuous $\mathbf{R}_+$-valued process such that*

$$\int_0^\infty \mathbf{1}\{X_t = 0\}\,\mathrm{d}t = 0 \quad \text{a.s.,} \tag{4.9}$$

and assume that X satisfies either one of the SDE's

$$\mathrm{d}X_t^2 = 2X_t\,\alpha_t\,\mathrm{d}B_t + \mathrm{d}t, \tag{4.10}$$

$$\mathrm{d}X_t = \alpha_t\,\mathrm{d}B_t + \frac{1}{2X_t}\mathbf{1}\{X_t > 0\}(1-\alpha_t^2)\,\mathrm{d}t + \mathrm{d}L_t, \tag{4.11}$$

for some real-valued Brownian motion B, some measurable and adapted $[0, 1]$-valued process α, and some non-decreasing continuous process L which grows only when $X = 0$. Then X satisfies both equations, and there exists some continuous local martingale M in $\mathbf{R}^2$, such that a.s.

$$|M_t| = X_t, \quad [M]_t \equiv t, \quad [|M|]_t = \int_0^t \alpha_s^2\,\mathrm{d}s, \quad t \geqq 0.$$

Proof. First assume (4.10). Applying Tanaka's formula to the semimartingale $X \vee \varepsilon$ with $\varepsilon > 0$, we get

$$(4.12) \qquad \mathrm{d}(X_t \vee \varepsilon) = 1\{X_t > \varepsilon\}\, \alpha_t\, \mathrm{d}B_t + \frac{1}{2X_t}\, 1\{X_t > \varepsilon\}\, (1 - \alpha_t^2)\, \mathrm{d}t + \mathrm{d}L_t^\varepsilon,$$

where L^ε is a non-decreasing continuous process which grows only when $X_t = \varepsilon$. This shows in particular that $X \vee \varepsilon$ is a local submartingale, so by uniform convergence the same thing is true for X. Thus X is a semimartingale, so by Tanaka's formula and (4.12),

$$\begin{aligned}\mathrm{d}X_t &= 1\{X_t > 0\}\, \mathrm{d}X_t + \mathrm{d}L_t \\ &= 1\{X_t > 0\}\, \alpha_t\, \mathrm{d}B_t + \frac{1}{2X_t}\, 1\{X_t > 0\}\, (1 - \alpha_t^2)\, \mathrm{d}t + \mathrm{d}L_t,\end{aligned}$$

for some non-decreasing continuous process L which grows only when $X_t = 0$. To get (4.11), it remains to note that $\int 1\{X_t = 0\}\, \alpha_t\, \mathrm{d}B_t = 0$, since by (4.9) $\int 1\{X_t = 0\}\, \alpha_t^2\, \mathrm{d}t = 0$.

Assuming instead that (4.11) holds, we get by Itô's formula and (4.9)

$$\begin{aligned}\mathrm{d}X_t^2 &= 2X_t\, \mathrm{d}X_t + \mathrm{d}[X]_t = 2X_t\, \alpha_t\, \mathrm{d}B_t + 1\{X_t > 0\}\, (1 - \alpha_t^2)\, \mathrm{d}t + \alpha_t^2\, \mathrm{d}t \\ &= 2X_t\, \alpha_t\, \mathrm{d}B_t + \mathrm{d}t.\end{aligned}$$

Thus even (4.10) holds, so Eqs. (4.10) and (4.11) are equivalent under condition (4.9).

Now assume that X satisfies (4.9)–(4.11). Proceeding as in the proof of Theorem 3.1, we may then construct, on each interval where $X > 0$, some continuous local martingale V in $\mathbf{R}/2\pi\mathbf{Z}$ with

$$\mathrm{d}[V]_t = \frac{1}{X_t^2}\, (1 - \alpha_t^2)\, \mathrm{d}t,$$

such that the process $M_t = X_t e^{iV_t}$ is a continuous local martingale in $\mathbf{C}$ with $|M_t| = X_t$ and $[M]_t = t$ a.s. for all $t \geqq 0$. By (4.11) it follows that

$$\mathrm{d}[|M|]_t = \mathrm{d}[X]_t = \alpha_t^2\, \mathrm{d}t. \quad \square$$

We shall now use the SDE's in Theorem 4.1 to derive some pathwise and distributional bounds for M.

Theorem 4.4. *Let M be a continuous martingale in $\mathbf{R}^d$ or l^2 with $M_0 = 0$ and* $[M]_t \equiv t$ a.s., *and define $A_t = [|M|]_t$. Then there exists some real-valued Brownian motion B, such that* a.s.

$$M_t^* \leqq B^* \circ A_t + \sqrt{t - A_t}, \qquad t \geqq 0.$$

and

$$M_t^* \leqq \sup_{s \leqq A_t} (|B_s| + \sqrt{t - s}), \qquad t \geqq 0.$$

Proof. Define $X_t = |M_t| - \sqrt{t - A_t}$, and conclude from Theorem 4.1 and Tanaka's formula that

$$\begin{aligned} \mathrm{d} X_t^+ &= \mathbf{1}\{X_t > 0\}\, \alpha_t \, \mathrm{d} B'_t \\ &\quad - \mathbf{1}\{X_t > 0\} \frac{(1-\alpha_t^2)}{2} \left(\frac{1}{\sqrt{t - A_t}} - \frac{1}{|M_t|} \right) \mathrm{d} t + \mathrm{d} L_t \\ &= \mathrm{d} Y_t - \mathrm{d} V_t + \mathrm{d} L_t, \end{aligned}$$

where B' is a Brownian motion while Y is a continuous martingale with $[Y]_t \leqq A_t$, and where both V and L are continuous and non-decreasing, the latter process growing only when $X_t = 0$. Assuming $Y_0 = V_0 = L_0 = 0$, we get by Skorohod's lemma,

$$L_t = -\inf_{s \leqq t} (Y_s - V_s), \qquad t \geqq 0,$$

so

$$X_t^+ = Y_t - V_t - \inf_{s \leqq t} (Y_s - V_s) \leqq Y_t - \inf_{s \leqq t} Y_s.$$

Now $Y_t = B'' \circ [Y]_t$ for another Brownian motion B'', and for a third Brownian motion B we have

$$B''_t - \inf_{s \leqq t} B''_s = |B_t|, \qquad t \geqq 0.$$

Thus

$$X_t^+ \leqq B'' \circ [Y]_t - \inf_{s \leqq [Y]_t} B''_s = |B \circ [Y]_t| \leqq B^* \circ A_t,$$

and we get for $0 \leqq s \leqq t$,

$$|M_s| \leqq X_s^+ + \sqrt{s - A_s} \leqq B^* \circ A_t + \sqrt{t - A_t},$$

or alternatively

$$|M_s| \leqq X_s^+ + \sqrt{s - A_s} \leqq |B \circ [Y]_s| + \sqrt{t - [Y]_s}. \quad \square$$

Theorem 4.5. *Let M be a continuous martingale in $\mathbf{R}^d$ or l^2 with $M_0 = 0$ and $[M]_t \equiv t$ a.s., and let B be a real-valued Brownian motion. Then*

$$\mathsf{P}\{M_t^* \geqq r\} \lesssim \mathsf{P}\{B_t^* \geqq r\}, \qquad r, t \geqq 0. \tag{4.13}$$

This improves a result in Chow and Menaldi (1989), where exponential bounds for the tail probabilities on the left are obtained under special conditions on M. Recall that the right-hand side of (4.13) is bounded by $4\mathsf{P}\{B_t \geqq r\}$, by the reflection principle for Brownian motion.

Proof. By scaling we can take $t = 1$. By Theorem 4.4,

$$M_1^* \leqq \sup_{t \leqq 1} (|B_t| + \sqrt{1 - t}) \quad \text{a.s.} \tag{4.14}$$

for some Brownian motion B, so writing

$$\tau = \inf\{t \in [0, 1]: B_t + \sqrt{1 - t} \geqq r\}$$

for fixed $r \geqq 0$ and using the symmetry of B, we get

$$\mathsf{P}\{M_1^* \geqq r\} \leqq 2\mathsf{P}\{\sup_{t \leqq 1}(B_t + \sqrt{1-t}) \geqq r\} = 2\mathsf{P}\{B_\tau + \sqrt{1-\tau} \geqq r\}.$$

By the strong Markov property of B at τ, we hence obtain

$$\begin{aligned}\mathsf{P}\{B_1^* \geqq r\} &\geqq 2\mathsf{P}\{B_1 \geqq r\} \geqq 2\mathsf{P}\{B_\tau + \sqrt{1-\tau} \geqq r,\ B_1 - B_\tau \geqq \sqrt{1-\tau}\} \\ &= 2\mathsf{E}[\mathsf{P}[B_1 - B_\tau \geqq \sqrt{1-\tau} \mid \mathscr{F}_\tau];\ B_\tau + \sqrt{1-\tau} \geqq r] \\ &\geqq 2\mathsf{P}\{B_1 \geqq 1\}\ \mathsf{P}\{B_\tau + \sqrt{1-\tau} \geqq r\} \\ &\geqq \mathsf{P}\{B_1 \geqq 1\}\ \mathsf{P}\{M_1^* \geqq r\}. \quad \square\end{aligned}$$

The next result gives some sharp a.s. bounds for the asymptotic rate of increase of $|M_t|$ as $t \to \infty$ or 0.

Theorem 4.6. *Let M be a continuous martingale in* $\mathbf{R}^d$ *or* l^2 *with* $M_0 = 0$ *and* $[M]_t \equiv t$ a.s. *Then, as* $t \to 0$ *or* ∞,

$$\limsup \frac{|M_t|}{\sqrt{2t \log|\log t|}} \leqq 1 \quad \text{a.s.,} \tag{4.15}$$

while

$$\liminf \frac{|M_t|}{\sqrt{t}} \leqq 1 \leqq \limsup \frac{|M_t|}{\sqrt{t}} \quad \text{a.s.} \tag{4.16}$$

Moreover, these bounds are sharp, except for those in (4.16) *when* $d = 1$.

Proof. Write

$$L(t) = \sqrt{2t|\log|\log t||}, \qquad t > 0.$$

and conclude from the law of the iterated logarithm for one-dimensional Brownian motion B that, as $t \to 0$ or ∞,

$$\limsup \frac{B_t^*}{L_t} = 1 \quad \text{a.s.}$$

Thus (4.15) holds by Theorem 4.4. Note also that the bound is attained when $\alpha \equiv 1$.

To prove (4.16), conclude from Theorem 4.1 that a.s.

$$|M_t|^2 = B \circ [|M|^2]_t + t, \qquad t \geqq 0, \tag{4.17}$$

for some one-dimensional Brownian motion B. Here the first term is bounded when $[|M|^2]_\infty < \infty$, and otherwise its zero set is a.s. unbounded. Hence in both cases (4.16) holds as $t \to \infty$. Similarly, either $[|M|^2]_t = 0$ near zero, so that the first term in (4.17) vanishes, or else $[|M|^2]_t > 0$ for all $t > 0$, in which case the zero set of the first term clusters at zero. Thus (4.16) holds in both cases as $t \to 0$. Note that even the bounds in (4.16) are sharp (except when $d = 1$), since we get $|M_t| \equiv \sqrt{t}$ a.s. by taking $\alpha \equiv 0$. $\square$

Let us now return to the basic SDE's governing a continuous martingale in higher dimensions. Thinking of the process α in Theorem 4.1 as a *stochastic control*, it is natural to ask which non-anticipating choices of α are possible, and when the associated norm processes $|M|$ are unique. Restricting ourselves to *Markovian* controls, where α_t is of the form $\sigma(|M_t|, t)$ for some measurable function $\sigma\colon \mathbf{R}_+^2 \to [0, 1]$, it is equivalent in view of Theorem 4.3 to consider the SDE

$$(4.18) \qquad \mathrm{d}X_t^2 = 2X_t\sigma(X_t, t)\,\mathrm{d}B_t + dt,$$

and to ask for conditions ensuring the existence and uniqueness of weak solutions. Note that the drift may be removed by the substitution $X_t^2 = Y_t + t$, which yields the simpler equation

$$(4.19) \qquad \mathrm{d}Y_t = 2(Y_t + t)^{1/2}\,\sigma((Y_t + t)^{1/2}, t)\,\mathrm{d}B_t.$$

Following the discussion of the Engelbert-Schmidt theory in Karatzas and Shreve (1988), pp. 329ff, it is now easy to give examples where the existence or uniqueness fails. Let us first assume that

$$\sigma(x, t) = \mathbf{1}\{x^2 \geqq t\}, \qquad x, t \geqq 0.$$

Then (4.18) has no solution for any starting distribution $\mu \neq \delta_0$, since if X starts at some $x > 0$, it will eventually get absorbed in the orbit where $x^2 = t$, which is impossible since $\sigma = 1$ there.

Let us next assume that

$$\sigma(x, t) = \mathbf{1}\{x^2 \neq t\}, \qquad x, t \geqq 0.$$

Then (4.18) has the weak solution $|B^\mu|$, where B^μ is a one-dimensional Brownian motion with arbitrary starting distribution μ. But (4.18) is also satisfied by the process

$$X_t = \{(B^\mu_{t\wedge\tau})^2 + (t-\tau)^+\}^{1/2}, \qquad t \geqq 0,$$

where

$$\tau = \inf\{t \geqq 0\colon |B_t^\mu| = \sqrt{t}\},$$

and it is clear that X and $|B^\mu|$ have different distributions, since $\tau < \infty$ a.s.

Most interesting, however, is the *autonomous* case, when $\sigma(x, t) = \sigma(x)$ is independent of t. This corresponds to the case when

$$\mathrm{d}|M_t|^2 = 2|M_t|\,\sigma(|M_t|)\,\mathrm{d}B_t + \mathrm{d}t.$$

Although current theories for weak solutions do not seem to apply here, unless special conditions are imposed on σ, we conjecture that in this case equation (4.18) has always a unique weak solution.

5. Some maximal inequalities

The main purpose of this section is to show how the results of Sect. 3 can be used to obtain martingale inequalities in $\mathbf{R}^d$ or l^2. Recall that, by Theorem

3.1, any martingale inequality in $\mathbf{R}^2$ which only involves the processes $|M|$, $[M]$, and $\langle M \rangle$ remains valid in higher dimensions with the same constants. Thus in order to extend a one-dimensional inequality to $\mathbf{R}^d$ or l^2, all we need to do is to prove an extension to $\mathbf{R}^2$, which is usually trivial (at least if we do not care about best constants). In certain cases, it is even possible to give a proof directly in l^2 (cf. Burkholder 1988).

For those reasons, we shall focus our attention on inequalities which are new even in one dimension, or where the extension to higher dimensions is nontrivial. Our main results are then exponential inequalities for martingales with bounded or symmetric jumps, and a continuous time version of the Dubins-Savage inequality. Let us begin with the former.

Theorem 5.1. *Let M be a martingale in $\mathbf{R}^d$ or l^2 with $M_0=0$ and $|\Delta M| \leqq c \leqq 1$* a.s. *Then $[M] \leqq 1$* a.s. *implies*

$$\mathsf{P}\{M^* \geqq r\} \lesssim \frac{1+r}{\sqrt{1+rc}} \exp\left(-\frac{r^2}{2(1+rc)}\right), \quad r \geqq 0, \tag{5.1}$$

while $\langle M \rangle \leqq 1$ a.s. *implies*

$$\mathsf{P}\{M^* \geqq r\} \lesssim \frac{1+r}{1+rc} \exp\left(-\frac{r}{2c} \log(1+rc)\right), \quad r \geqq 0. \tag{5.2}$$

For $d=1$ the factors outside the exponential may be replaced by 1.

For sequences of independent real-valued random variables with zero mean and finite variance, (5.2) is essentially due to Kolmogorov, though with the exponent in (5.1) (cf. Loève 1977, p. 266). The sharper form with the logarithm in the exponent is due to Prohorov (cf. Stout 1974, p. 262). Martingale versions in discrete time and one dimension of Prohorov's result have been proved by Johnson et al. (1985), and by Hitczenko (1990). The inequality in (5.1) seems to be new, and no continuous time or higher dimensional version of either inequality seems to be previously known.

For the proof of Theorem 5.1, we shall need a lemma.

Lemma 5.2. *Let M be a real-valued martingale with $M_0=0$ and $|\Delta M| \leqq c < \infty$, and define*

$$f(x) = -\frac{x+\log(1-x)_+}{x^2}, \qquad g(x) = \frac{e^x-1-x}{x^2}, \qquad x \geqq 0.$$

Then the processes

$$X_t = \exp(M_t - f(c)[M]_t), \qquad Y_t = \exp(M_t - g(c)\langle M \rangle_t), \qquad t \geqq 0,$$

are supermartingales.

Proof. To prove that X is a supermartingale, we may assume that $c<1$. Writing $a=f(c)$ we get by Itô's formula,

$$\begin{aligned} X_{t-}^{-1}\, dX_t = {} & dM_t - a\, d[M]_t + \tfrac{1}{2} d[M^c]_t \\ & + \{\exp(\Delta M_t - a(\Delta M_t)^2) - 1 - \Delta M_t + a(\Delta M_t)^2\}. \end{aligned} \tag{5.3}$$

Here the first term on the right gives rise to a local martingale $X_{t-}\,\mathrm{d}M_t$, and we shall prove that the sum of all remaining terms is non-increasing, so that the same thing is true for the corresponding integral $\mathrm{d}X_t - X_{t-}\,\mathrm{d}M_t$. It then follows that X is a local supermartingale, and by Fatou's lemma it is also a true supermartingale.

To prove the desired monotonicity of the bounded variation component in (5.3), we note first that its continuous part equals $(\frac{1}{2}-a)\,[M^c]_t$, which is non-increasing since $a \geqq \frac{1}{2}$. Turning to the contribution of the jump $\xi = \Delta M_t$, we need to show that $\exp(\xi - a\xi^2) \leqq 1+\xi$, or equivalently that $f(-\xi) \leqq f(c)$ for $|\xi| \leqq c$. But this is clear by a Taylor expansion of each side.

Next we apply Itô's formula to Y with $g(c)=a$, to obtain

(5.4) $$Y_{t-}^{-1}\,\mathrm{d}Y_t = \mathrm{d}M_t - a\,\mathrm{d}\langle M\rangle_t + \tfrac{1}{2}\mathrm{d}[M^c]_t + \{\exp(\Delta M_t - a\Delta\langle M\rangle_t) - 1 - \Delta M_t + a\Delta\langle M\rangle_t\}.$$

Here M_t is a martingale while $(\frac{1}{2}-a)\,[M^c]_t$ is non-increasing, so it remains to consider the contribution of $M^d = M - M^c$ to the bounded variation terms in (5.4). To this aim, consider any bounded predictable stopping time τ, and note that the jump at τ of the process $Y_-^{-1}\,\mathrm{d}Y - \mathrm{d}M$ equals $\zeta \equiv e^{\xi - a\eta} - 1 - \xi$, where $\xi = \Delta M_\tau$ and $\eta = \Delta\langle M\rangle_\tau$. Since $\mathsf{E}[\xi|\mathscr{F}_{\tau-}]=0$ and $\mathsf{E}[\xi^2|\mathscr{F}_{\tau-}]=\eta$ a.s., we get by a Taylor expansion of e^ξ,

$$\mathsf{E}[\zeta|\mathscr{F}_{\tau-}] \leqq (1+a\eta)\,e^{-a\eta} - 1 \leqq 0.$$

Thus the accessible jumps of Y form a supermartingale, so subtracting this part, we may henceforth assume that M is quasi-leftcontinuous. But then $\langle M\rangle$ is continuous, so writing $\xi = \Delta M_t$ as before, the remaining terms of (5.4) become

$$-a\,\mathrm{d}\langle M^d\rangle_t + (e^\xi - 1 - \xi) \leqq -a\,\mathrm{d}\langle M^d\rangle_t + a\xi^2 = a(\mathrm{d}[M^d]_t - \mathrm{d}\langle M^d\rangle_t),$$

where the right-hand side is a martingale. Thus the contribution to Y is again a supermartingale. □

Proof of Theorem 5.1. Assuming M to be real-valued with $[M] \leqq 1$, and applying Lemma 5.2 to the martingale uM for fixed $u>0$, we get by a classical supermartingale inequality,

$$\mathsf{P}\{\sup M_t \geqq r\} \leqq \exp(-ur + u^2 f(uc)), \qquad r \geqq 0.$$

Now the function $F(x) = 2xf(x)$ is continuous and strictly increasing on $[0,1)$ with range $\mathbf{R}_+$, so it has a unique inverse $F^{-1}: \mathbf{R}_+ \to [0,1)$, and we may choose $u = F^{-1}(rc)/c$. This yields

(5.5) $$\mathsf{P}\{\sup M_t \geqq r\} \leqq \exp\left(-\frac{r}{2c}\,F^{-1}(rc)\right) \leqq \exp\left(-\frac{r^2}{2(1+rc)}\right), \qquad r \geqq 0,$$

where the last inequality stems from the fact that $F(x) \leqq x/(1-x)$ and hence $F^{-1}(y) \geqq y/(1+y)$. Combining (5.5) with the same inequality for $-M$, we get twice the same bound for $\mathsf{P}\{M^* \geqq r\}$, and the higher dimensional estimate now follows by Theorem 3.1 and Corollary 3.5.

Assuming instead that M is real-valued with $\langle M\rangle\leqq 1$, we get as in (5.5)

$$\mathsf{P}\{\sup M_t\geqq r\}\leqq\exp\left(-\frac{r}{2c}G^{-1}(rc)\right)\leqq\exp\left(-\frac{r}{2c}\log(1+rc)\right),\qquad r\geqq 0,$$

where G denotes the continuous and strictly increasing function $xg(x)\leqq e^x-1$, which has a unique inverse $G^{-1}(y)\geqq\log(1+y)$. Using Theorem 3.1 and Corollary 3.5 as before, we get the higher dimensional version

$$\text{(5.6)}\qquad \mathsf{P}\{M^*\geqq r\}\lesssim\left(1+\frac{r}{c}\log(1+rc)\right)^{1/2}\exp\left(-\frac{r}{2c}\log(1+rc)\right),\qquad r\geqq 0,$$

which is equivalent to (5.2) for $rc\leqq 1$. For large rc, however, the bound in (5.6) exceeds the one in (5.2) by a factor $(rc\log(1+rc))^{1/2}$, so to get (5.2) it is necessary to replace the estimate $G^{-1}(y)\geqq\log(1+y)$ by a more accurate bound. Writing $x=G(y)$ and using the fact that $x\geqq\log(1+y)$, we get

$$e^x=1+x+\frac{xy}{2}\geqq 1+\left(1+\frac{y}{2}\right)\log(1+y)\geqq(1+y)\log\sqrt{y},$$

and for large y we also have $y\geqq x$, so

$$\log(1+y)+\log\log\sqrt{y}\leqq G^{-1}(y)\leqq y.$$

Thus we get in place of (5.6), for large rc,

$$\mathsf{P}\{M^*\geqq r\}\lesssim\exp\left\{-\frac{r}{2c}\log(1+rc)-\frac{r}{2c}\log\log\sqrt{rc}+\tfrac{1}{2}\log(1+r)\right\},$$

and it remains to notice that $c^{-1}\geqq 1$ while $r/c\geqq r\geqq\log(1+r)$. □

Sharper inequalities may be obtained under stronger conditions. Thus it has been shown by Azuma (cf. Pisier 1986, Lemma 2.6) that if M is a real-valued martingale in discrete time with $M_0=0$ and $|\Delta M_k|\leqq c_k$ a.s. for all k, where the c_k are constants satisfying $\Sigma c_k^2\leqq 1$, then $\mathsf{P}\{M^*\geqq r\}\leqq 2\exp(-r^2/2)$ for all $r\geqq 0$. Such inequalities are useful in the local theory of Banach spaces (cf. Milman and Schechtman 1986). Using our results from Sect. 3, we get immediately the higher-dimensional version

$$\mathsf{P}\{M^*\geqq r\}\lesssim(r+1)\exp(-r^2/2),\qquad r\geqq 0.$$

Another case where the bounds in Theorem 5.1 can be improved is when the jumps of M are *symmetric*. By this we mean that the compensating random measure $\hat{\xi}$ of the jump point process

$$\xi_t(B)=\sum_{s\leqq t}\mathbf{1}\{\Delta M_s\in B\},\qquad B\in\mathscr{B},\qquad t\geqq 0,$$

is symmetric under a change of sign, in the sense that $\hat{\xi}(-B)=\hat{\xi}(B)$ a.s. for all $B\in\mathscr{B}$ and $t\geqq 0$. Here $\mathscr{B}$ denotes the Borel σ-field in $\mathbf{R}^d$ or l^2 respectively.

In discrete time, symmetry of the jumps is clearly equivalent to *conditional symmetry*, in the sense that

$$\mathsf{P}[-\Delta M_n \in B \mid \mathcal{F}_{n-1}] = \mathsf{P}[\Delta M_n \in B \mid \mathcal{F}_{n-1}] \quad \text{a.s.}, \quad B \in \mathcal{B},\ n \in \mathbf{N}.$$

Theorem 5.3. *Let M be a martingale in $\mathbf{R}^d$ or l^2 with symmetric jumps and such that $M_0 = 0$ and $[M] \leqq 1$. Then*

$$\mathsf{P}\{M^* \geqq r\} \lesssim (r+1)\exp(-r^2/2), \qquad r \geqq 0. \tag{5.7}$$

We shall need two lemmas.

Lemma 5.4. *Let M be a martingale in $\mathbf{R}^d$ or l^2 with symmetric jumps, and let N be the corresponding martingale in $\mathbf{R}^2$, as constructed in Theorem 3.1. Then even N has symmetric jumps.*

Proof. By the definition of symmetry, it is enough to show that all jumps with size in some interval (a, b) with $0 < a < b < \infty$ are symmetric, and we may then treat separately the cases of jump times τ with $M_{\tau-} = 0$ or $M_{\tau-} \neq 0$.

In the former case, let N be such as in the proof of the Theorem 3.1, and note that the jump at τ occurs in the approximating processes N^ε for any $\varepsilon < a$. From the mentioned proof it is clear that arg N_τ^ε is uniformly distributed over $[0, 2\pi)$ and is independent of $\mathcal{F}$ and $[N_\tau^\varepsilon|$, given that $\tau < \infty$. It follows easily that the one-jump process

$$J_t = N_\tau \mathbf{1}\{\tau \leqq t\}, \qquad t \geqq 0, \tag{5.8}$$

is symmetric with respect to the filtration $\mathcal{F}^\varepsilon$ generated by $\mathcal{F}$ and N^ε. Letting $\varepsilon \to 0$, we get the desired symmetry with respect to $\tilde{\mathcal{F}} = \bigvee \mathcal{F}^\varepsilon$.

Turning to the case when $M_{\tau-} \neq 0$, let $\vartheta = \vartheta_\tau$ be the associated random sign, as defined in the proof of Theorem 3.1, and introduce the process

$$X_t = (\Delta M_\tau, \vartheta)\mathbf{1}\{\tau \leqq t\}, \qquad t \geqq 0,$$

in $H = \mathbf{R}^{d+1}$ or $l^2 \times \mathbf{R}$. Then X is again symmetric with respect to the extended filtration $\tilde{\mathcal{F}}$. Moreover,

$$\Delta N_\tau = \frac{(M_{\tau-} \cdot \Delta M_\tau) N_{\tau-}}{|M_{\tau-}|^2} + \left(|\Delta M_\tau|^2 - \frac{|M_{\tau-} \cdot \Delta M_\tau|^2}{|M_{\tau-}|^2}\right)^{1/2} \cdot \frac{\vartheta \mathrm{i} N_{\tau-}}{|N_{\tau-}|},$$

which is clearly of the form

$$\Delta N_\tau = F(M_{\tau-}, N_{\tau-}, X_\tau) = V_\tau(X_\tau),$$

for a predictable process V on $\mathbf{R}_+ \times H$. Thus the compensating random measure of the process J in (5.8) is the image under V of the compensator of X, and it remains to show that V preserves the symmetry. But this is clear from the fact that $V_t(-x) = V_t(x)$ for all $t \geqq 0$ and $x \in H$. □

Lemma 5.5. *Let M be a real-valued martingale with symmetric jumps and bounded quadratic variation $[M]$. Then the process*

$$Z_t = \exp(M_t - \tfrac{1}{2}[M]_t), \qquad t \geqq 0,$$

is a supermartingale.

Proof. Writing $X_t = M_t - \frac{1}{2}[M]_t$, we get by Itô's formula

$$\mathrm{d}Z_t = Z_{t-}\,\mathrm{d}X_t + \tfrac{1}{2}Z_{t-}\,\mathrm{d}[X^c]_t + \{\Delta Z_t - Z_{t-}\,\Delta X_t\},$$

so

$$\begin{aligned}(5.9)\qquad Z_{t-}^{-1}\,\mathrm{d}Z_t &= \mathrm{d}M_t - \tfrac{1}{2}\mathrm{d}[M]_t + \tfrac{1}{2}\mathrm{d}[M^c]_t \\ &\quad + \{\exp(\Delta M_t - \tfrac{1}{2}(\Delta M_t)^2) - 1 - \Delta M_t + \tfrac{1}{2}(\Delta M_t)^2\} \\ &= \mathrm{d}M_t + \{\exp(\Delta M_t - \tfrac{1}{2}(\Delta M_t)^2) - 1 - \Delta M_t\}.\end{aligned}$$

Now $e^x + e^{-x} \leqq 2e^{x^2/2}$, so defining $f(x) = \exp(x - \frac{1}{2}x^2) - 1 - x$, we get $f(x) + f(-x) \leqq 0$ for all x. Thus the functions

$$g(x) = f(|x|)\,\mathrm{sgn}\,x, \qquad h(x) = g(x) - f(x), \qquad x \in \mathbf{R}$$

satisfy $g(-x) = -g(x)$ and $h(x) \geqq 0$ for all x, and by (5.9) we have

$$(5.10)\qquad Z_{t-}^{-1}\,\mathrm{d}Z_t = \mathrm{d}M_t + g(\Delta M_t) - h(\Delta M_t).$$

Noting that $g(x) \lesssim x^2$ near the origin, it is clear that the jumps $g(\Delta M_t)$ add up to a martingale, so the right-hand side of (5.10) is a supermartingale, and the same must be true for Z. □

Proof of Theorem 5.3. If M is real-valued, it is clear from Lemma 5.5 applied to the martingales $\pm rM$ that

$$(5.11)\qquad \mathsf{P}\{M^* > r\} \leqq 2\exp(-r^2/2), \qquad r \geqq 0.$$

If M is instead $\mathbf{R}^2$-valued, then the one-dimensional projections of M have symmetric jumps, so (5.7) holds by (5.11) and Corollary 3.5. The further extension to $\mathbf{R}^d$ or l^2 is accomplished by means of Theorem 3.1 and Lemma 5.4. □

To state the next result, let $\mathscr{A}_1$ denote the class of continuously differentiable functions $f: \mathbf{R}_+ \to [0, \infty]$ with $f(0) = f'(0) = 0$, and such that f' is concave. Let us further denote by $\mathscr{A}_2$ the class of continuous, non-increasing and integrable functions $g: \mathbf{R}_+ \to [0, \infty]$. For any $f \in \mathscr{A}_1$ and any local martingale M in $\mathbf{R}^d$ or l^2, we define the *f-variation* of M as the process

$$(5.12)\qquad \int_0^t f(|\mathrm{d}M|) = \tfrac{1}{2}f''(0)[M^c]_t + \sum_{s \leqq t} f(|\Delta M_s|), \qquad t \geqq 0.$$

In the sequel, the product $0 \cdot \infty$ should be interpreted as 0.

Theorem 5.6. *Let M be a local martingale in $\mathbf{R}^d$ or l^2 with $M_0=0$, let $f\in\mathscr{A}_1$ and $g\in\mathscr{A}_2$ be arbitrary, and assume that the process $A=\int f(|\mathrm{d}M|)$ is locally integrable with compensator $\hat{A}$. Then*

$$\mathsf{P}\{\sup_t f(|M_t|)\, g(\hat{A}_t)\geqq r\}\lesssim\frac{1}{r}\,\|g\|_1,\qquad r>0. \tag{5.13}$$

Specializing to $f(x)=x^2$ and $g(x)\sim x^{-2}$, one may easily deduce the inequality

$$\mathsf{P}\{\sup_t(|M_t|-a\langle M\rangle_t)\geqq b\}\lesssim\frac{1}{1+ab},\qquad a,b>0,$$

due for real-valued martingales in discrete time to Dubins and Savage (1965). Our present version was proved in Kallenberg (1975) for real-valued discrete-time martingales and for stochastic integrals with respect to centered Lévy processes. The inequality leads to amazingly sharp asymptotic results, both at 0 and ∞. Interesting choices of f and g might be to take (for any $\varepsilon>0$)

$$f(x)=\frac{x^2}{1+x},\qquad g(x)=(1+x)^{-1-\varepsilon},\qquad x\geqq 0.$$

The proof of Theorem 5.6 relies on a supermartingale employed in discrete time by Kallenberg (1975), and first considered in a special case by Dubins and Freedman (1965).

Lemma 5.7. *Let M be a real-valued local martingale with $M_0=0$, and define $G=\int g(u)\,\mathrm{d}u$. Then the process*

$$X_t=f(|M_t|)\,g(\hat{A}_t)-2G(\hat{A}_t),\qquad t\geqq 0, \tag{5.14}$$

is a supermartingale.

Proof. We may assume that g is bounded, since if the statement is true in that case, it extends to unbounded g by monotone convergence, separately for each term in (5.14). Note also that X is continuous at 0. In fact, X_{0+} exists by martingale theory, and $X_{0+}\geqq 0$ by (5.14). On the other hand, the supermartingale property and Fatou's lemma yield $\mathsf{E}X_{0+}\leqq\mathsf{E}X_0=0$. Thus $X_{0+}=0=X_0$.

We may next reduce to the case when $g\in C^1$ while $f\in C^2$. For g this is easy, since any continuous function on $\mathbf{R}_+$ can be approximated uniformly by functions in C^1. In case of f we choose continuous and non-increasing functions $\phi_1,\phi_2,\ldots:\mathbf{R}_+\to\mathbf{R}_+$, such that $\phi_n\uparrow f''$ at all continuity points of f''. Integrating the ϕ_n twice yields a sequence $f_1,f_2,\ldots\in\mathscr{A}_1\cap C^2$, such that $f_n\uparrow f$ and $f_n''\uparrow f''$. Then $A_n\equiv\int f_n(|\mathrm{d}M|)\uparrow A$ by monotone convergence, so $\hat{A}_n\uparrow\hat{A}$, and we get $X_n\equiv f_n(|M|)\;g(\hat{A}_n)-2G(\hat{A}_n)\to X$. Thus if X_n is a supermartingale for each n, then so is X by Fatou's lemma.

Let us now extend f to an even function on $\mathbf{R}$, and note that even the extension lies in C^2. From the proof of Lemma 2.1 in Kallenberg (1975), we conclude that

$$f(x+y)\leqq f(x)+yf'(x)+2f(y),\qquad x,y\in\mathbf{R}. \tag{5.15}$$

By localization we may assume that A and $f(M)$ are uniformly integrable, while M is a true martingale such that the process $M_{t-}\mathbf{1}\{\Delta M_t \neq 0\}$ is bounded by a constant. For any bounded predictable stopping time τ, we then get by (5.15),

$$\begin{aligned}\mathsf{E}[f(M_\tau)|\mathscr{F}_{\tau-}] &\leqq f(M_{\tau-}) + f'(M_{\tau-})\,\mathsf{E}[\Delta M_\tau|\mathscr{F}_{\tau-}] + 2\,\mathsf{E}[f(\Delta M_\tau)|\mathscr{F}_{\tau-}]\\ &= f(M_{\tau-}) + 2\Delta\hat{A}_\tau,\end{aligned}$$

so

$$\begin{aligned}\mathsf{E}[X_\tau|\mathscr{F}_{\tau-}] &\leqq (f(M_{\tau-}) + 2\Delta A_\tau)\,g(\hat{A}_\tau) - 2G(\hat{A}_\tau)\\ &\leqq \{f(M_{\tau-})\,g(\hat{A}_{\tau-}) - 2G(\hat{A}_{\tau-})\} + 2\{g(\hat{A}_\tau)\,\Delta A_\tau - \Delta G(\hat{A}_\tau)\}\\ &\leqq X_{\tau-}.\end{aligned}$$

The computation shows that the jump $\Delta X_\tau \mathbf{1}\{\tau \leqq t\}$ of X at τ is the sum of a non-increasing process and the jump at τ of the local martingale

$$\text{(5.16)} \qquad \mathrm{d}M'_t = g(\hat{A}_t)\,f'(M_{t-})\,\mathrm{d}M_t + g(\hat{A}_t)\,\mathrm{d}(A_t - \hat{A}_t).$$

By dominated convergence for stochastic integrals, the accessible jumps of X combine into the sum of a non-increasing process and a stochastic integral with respect to M' in (5.16) over some predictable set.

On the remaining set the process $\hat{A}$ is continuous, and we get by Itô's formula and (5.15)

$$\begin{aligned}\mathrm{d}X_t &= f(M_t)\,\mathrm{d}g(\hat{A}_t) + g(\hat{A}_t)\,\mathrm{d}f(M_t) - 2\,\mathrm{d}G(\hat{A}_t)\\ &= \{f(M_t)\,g'(\hat{A}_t) - 2g(\hat{A}_t)\}\,\mathrm{d}\hat{A}_t + g(A_t)\,\{f'(M_{t-})\,\mathrm{d}M_t\\ &\quad + \tfrac{1}{2}f''(M_{t-})\,\mathrm{d}[M^c]_t + (\Delta f(M_t) - f'(M_{t-})\,\Delta M_t)\}\\ &\leqq g(A_t)\,\{-2\,\mathrm{d}\hat{A}_t + f'(M_{t-})\,\mathrm{d}M_t + \tfrac{1}{2}f''(0)\,\mathrm{d}[M^c]_t + 2\Delta f(M_t)\}\\ &= g(A_t)\,\{f'(M_{t-})\,\mathrm{d}M_t + 2\,\mathrm{d}(A_t - \hat{A}_t)\} = \mathrm{d}M'_t,\end{aligned}$$

the local martingale in (5.16). Thus X is the sum of a local martingale and a non-increasing process, hence locally a supermartingale. Since X is bounded from below, we may conclude by Fatou's lemma that X is a true supermartingale. □

Proof of Theorem 5.6. Let us first assume that M is real-valued, and let X be the supermartingale in Lemma 5.7. By a classical inequality for positive supermartingales, we get

$$\begin{aligned}\mathsf{P}\{\sup_t f(|M_t|)\,g(A_t) \geqq r\} &\leqq \mathsf{P}\{X^* + 2\,\|g\|_1 \geqq r\}\\ &\leqq \frac{1}{r}\,\mathsf{E}(X_0 + 2\,\|g\|_1) = \frac{2}{r}\,\|g\|_1.\end{aligned}$$

If M is instead $\mathbf{R}^2$-valued, say $M = (M^1, M^2)$, we may conclude from the concavity of f' that even $F(x) = f(\sqrt{x})$ is concave and therefore subadditive. Hence

$$f(|M|) = F(|M|^2) \leqq F((M^1)^2 + F((M^2)^2) = f(|M^1|) + f(|M^2|).$$

Defining $A^i=\int f(|dM^i|)$ for $i=1,2$, it is further seen that $dA^i\leqq dA$, so $\hat{A}^i\leqq\hat{A}$ and therefore $g(\hat{A})\leqq g(\hat{A}^i)$, $i=1,2$. Hence

$$f(|M|)\,g(\hat{A})\leqq f(|M^1|)\,g(\hat{A}^1)+f(|M^2|)\,g(\hat{A}^2),$$

so by the one-dimensional result,

$$\mathsf{P}\{\sup_t f(|M_t|)\,g(\hat{A}_t)\geqq r\}\leqq\sum_{i=1}^{2}\mathsf{P}\left\{\sup_t f(|M_t^i|)\,g(\hat{A}_t^i)\geqq\frac{r}{2}\right\}\leqq\frac{8}{r}\,\|g\|_1.$$

The further extension to $\mathbf{R}^d$ or l^2 is immediate by Theorem 3.1. □

Remark. In conclusion we indicate how certain exponential inequalities can be obtained in higher dimension by an elementary method which does not require the theory of Sect. 3 or 4. As we have seen, the key step in one dimension is typically to prove that a certain process is a positive supermartingale. Now assume that M is a martingale in $\mathbf{R}^d$ with $M_0=0$ and $[M]\leqq 1$, and such that the process

$$Z_a(t)=\exp(a\cdot M_t-c[a\cdot M]_t),\qquad t\geqq 0,$$

is a supermartingale for every $a\in\mathbf{R}^d$, where $c>0$ is some constant. Let $\xi=(\xi_1,\ldots,\xi_d)$ be a random vector independent of M and uniformly distributed over the unit sphere, and note that

$$\mathsf{E}|\xi_1|\geqq\sqrt{\frac{2}{\pi d}},\qquad \mathsf{E}\xi_1^2=\frac{1}{d}. \tag{5.17}$$

Using Jensen's inequality, Fubini's theorem, formulas (5.17) and the inequality $e^{|x|}\leqq e^x+e^{-x}$, we get for any $u\in\mathbf{R}$

$$\begin{aligned}\mathsf{E}\exp\left(u\sqrt{\frac{2}{\pi d}}|M_t|-\frac{u^2c}{d}\right)&\leqq\mathsf{E}\exp\mathsf{E}[u|\xi\cdot M_t|-u^2c[\xi\cdot M]_t\,|\,M]\\&\leqq\mathsf{E}\exp\left(u|\xi\cdot M_t|-u^2c[\xi\cdot M]_t\right)\\&\leqq\mathsf{E}Z_{u\xi}(t)+\mathsf{E}Z_{-u\xi}(t)\leqq 2.\end{aligned}$$

Here the process on the left is a submartingale for $u>0$, so by a standard inequality we get with $u=(r/c)\sqrt{d/2\pi}$,

$$\mathsf{P}\{M^*\geqq r\}\leqq 2\exp\left(-ur\sqrt{\frac{2}{\pi d}}+\frac{u^2c}{d}\right)=2\exp\left(-\frac{r^2}{2\pi c}\right),\qquad r\geqq 0. \tag{5.18}$$

Proceeding instead as in the proof of Theorem 5.3, we get the bound

$$\mathsf{P}\{M^*\geqq r\}\lesssim(r+1)\exp(-r^2/4c),\qquad r\geqq 0,$$

which differs from (5.18) by a factor $\pi/2$ in the exponent. For many purposes, the cruder but more elementary bound in (5.18) may be sufficient.

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