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Existence of Solutions for Schrödinger Evolution Equations

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Abstract. We study the existence, uniqueness and regularity of the solution of the initial value problem for the time dependent Schrödinger equation $i\partial u/\partial t = (-1/2)\Delta u + V(t,x)u$, $u(0) = u_0$. We provide sufficient conditions on $V(t,x)$ such that the equation generates a unique unitary propagator $U(t,s)$ and such that $U(t,s)u_0 \in C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^2(\mathbb{R}^n))$ for $u_0 \in H^2(\mathbb{R}^n)$. The conditions are general enough to accommodate moving singularities of type $|x|^{-2+\varepsilon}(n \geqq 4)$ or $|x|^{-n/2+\varepsilon}(n \leqq 3)$.

1. Introduction, Assumptions and Theorems

In this paper, we study the existence, uniqueness and regularity of the solution of the initial value problem for the time dependent Schrödinger equation in $\mathbb{R}^n$:

$$i\partial u/\partial t = -(1/2)\Delta u + V(t,x)u, \quad t \in [-T, T] = I_T, \quad x \in \mathbb{R}^n,$$
$$u(s,x) = u_0(x), \tag{1.1}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ and $V(t,x)$ is a real valued function. We regard Eq. (1.1) as an evolution equation in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$:

$$idu/dt = H(t)u, \quad H(t) = -(1/2)\Delta + V(t,x), \quad u(s) = u_0, \tag{1.2}$$

and treat the problem by using the perturbation technique and the well-known $L^p - L^q$-type estimates for the free propagator $\exp(it\Delta/2)$. We shall give sufficient conditions on $V(t,x)$ such that Eq. (1.2) uniquely generates a strongly continuous unitary propagator $\{U(t,s)\}$ on $\mathscr{H}$, and such that $U(t,s)u_0 \in C(I_T, H^2(\mathbb{R}^n)) \cap C^1(I_T, \mathscr{H})$ for every $u_0 \in H^2(\mathbb{R}^n)$. The conditions are general enough to accommodate potentials which have moving singularities of type $|x|^{-2+\varepsilon}$ for $n \geqq 4$ and $|x|^{-n/2+\varepsilon}$ for $n \leqq 3$, $\varepsilon > 0$.

We consider, along with Eq. (1.2), the integral equation

$$u(t) = U_0(t-s)u_0 - i\int_s^t U_0(t-\tau)V(\tau)u(\tau)d\tau, \tag{1.3}$$

where $U_0(t) = \exp(it\Delta/2)$ and $V(t)$ is the multiplication operator by $V(t,x)$. For an interval I and m, $\rho \geqq 1$, $L^{m,\rho}(I)$ is the Banach space of $L^m(\mathbb{R}^n)$-valued ρ-summable functions over I:

$$L^{m,\rho}(I)=\{u:[\int_I\{\int_{\mathbb{R}^n}|u(t,x)|^m dx\}^{\rho/m}dt]^{1/\rho}=\|u\|_{m,\rho}<\infty\}. \tag{1.4}$$

If $\mathscr{X}$ is a Banach space $C^k(I,\mathscr{X})$ is the space of all $\mathscr{X}$-valued C^k-functions, $C(I,\mathscr{X})=C^0(I,\mathscr{X})$.

Assumption (A.1). For some $p\geqq 1$, $\alpha\geqq 1$, $\beta>1$ with $0\leqq 1/\alpha<1-n/2p$, $V\in L^{p,\alpha}(I_T)+L^{\infty,\beta}(I_T)$, that is, there exist $V_1\in L^{p,\alpha}(I_T)$ and $V_2\in L^{\infty,\beta}(I_T)$ such that

$$V(t,x)=V_1(t,x)+V_2(t,x)\quad \text{a.e.}\,(t,x)\in I_T\times\mathbb{R}^n. \tag{1.5}$$

Under the assumption (A.1), we shall prove in Sect. 3 that the integral equation (1.3) admits a unique L^2-solution for any $u_0\in\mathscr{H}$. We set $\theta(l)=4l/n(l-2)$, or $2/\theta(l)=n(1/2-1/l)$ for $2\leqq l\leqq\infty$.

Theorem 1.1. *Let Assumption (A.1) be satisfied and $q=2p/p-1$. Then:*
(1) *Eq. (1.3) has a unique solution $u\in C(I_T,\mathscr{H})\cap L^{q,\theta}(I_T)$, $\theta=\theta(q)=4p/n$, for every $u_0\in\mathscr{H}$* and *$s\in I_T$.*
(2) $\|u(t)\|=\|u_0\|$, $t\in I_T$.

Corollary 1.2. *There uniquely exists a family of unitary operators $\{U(t,s),t,s\in I_T\}$ on $\mathscr{H}$ which satisfies the following properties:*
(1) $U(t,s)U(s,r)=U(t,r)$, $U(t,t)=\mathbb{1}$, t, s, $r\in I_T$.
(2) *$U(t,s)$ is strongly continuous in $\mathscr{H}$ with respect to (t,s).*
(3) $\left(\int_{-T}^{T}\|U(t,s)u\|_q^\theta dt\right)^{1/\theta}\leqq C_{q,T}\|u\|,\quad u\in\mathscr{H},\quad \theta=\theta(q).$
(4) *$u(t)=U(t,s)u_0$ is a solution of Eq. (1.3) for every $u_0\in\mathscr{H}$.*

When $n\geqq 4$, the assumption (A.1) implies that for each t the operator $H(t)$ is selfadjoint in $\mathscr{H}$ with t-independent domain $\mathscr{D}(H(t))=H^2(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is its core (cf. Reed–Simon [10]). This is also the case when $n\leqq 3$ and $p\geqq 2$. In such cases, it is easy to see that the solution $u(t)=U(t,s)u_0$, $u_0\in\mathscr{H}$ satisfies Eq. (1.2) in $H^{-2}(\mathbb{R}^n)$, a.e. Then it is natural to ask, for $u_0\in H^2(\mathbb{R}^n)$, whether $u(t)=U(t,s)u_0$ is strongly differentiable in $\mathscr{H}$, strongly continuous in $H^2(\mathbb{R}^n)$ and satisfies Eq. (1.2) in $\mathscr{H}$. The following theorem provides a sufficient condition for this to be the case. The exponents p, α and β are hereafter fixed as in Assumption (A.1) and $q=2p/p-1$.

Assumption (A.2). The function $V\in C(I_T,L^{\tilde{p}}(\mathbb{R}^n))+C(I_T,L^\infty(\mathbb{R}^n))$ and $\partial V/\partial t\in L^{p_1,\alpha_1}(I_T)+L^{\infty,\beta}(I_T)$, where $\tilde{p}=\max(p,2)$; $p_1=2np/n+4p$ if $n\geqq 5$, $p_1>2p/p+1$ if $n=4$ and $p_1=2p/p+1$ if $n\leqq 3$; $\alpha_1>4p/4p-n$.

Note that Assumption (A.2) implies (A.1), hence, Theorem 1.1 and Corollary 1.2.

Theorem 1.3. *Let Assumption (A.2) be satisfied. Then the strongly continuous unitary propagator $\{U(t,s)\}$ of Corollary 1.2 satisfies the properties (1) ~ (4), and*
(5) *$U(t,s)H^2(\mathbb{R}^n)\subset H^2(\mathbb{R}^n)$ for every t, $s\in I_T$ and $U(t,s)$ is strongly continuous in $H^2(\mathbb{R}^n)$ with respect to (t,s).*
(6) *For every $u_0\in H^2(\mathbb{R}^n)$, $U(t,s)u_0\in C^1(I_T\times I_T,\mathscr{H})$ and*

$$i(\partial/\partial t)U(t,s)u_0=H(t)U(t,s)u_0, \tag{1.6}$$

$$-i(\partial/\partial s)U(t,s)u_0 = U(t,s)H(s)u_0. \tag{1.7}$$

(7) *For every* $u_0 \in H^2(\mathbb{R}^n)$ *and* $s \in I_T$, $(\partial/\partial t)U(t,s)u_0 \in C(I_T, \mathscr{H}) \cap L^{q,\theta}(I_T)$, $\theta = \theta(q) = 4q/n$.

Moreover the family $\{U(t,s)\}$ *which satisfies the properties* (1), (2), (5), (6) *is unique.*

Remark 1.4. In the assumptions (A.1) and (A.2), $1/\alpha + n/2p < 1$ and $1/\alpha_1 + n/4p < 1$ can be taken as close to 1 as pleased and the potential $V(t,x)$ can be less regular in the time variable if it is more regular in the space variables. In the case when $p > n/2$ is close to $n/2$, $p_1 \sim n/3 (n \geqq 4)$ or $p_1 \sim 2n/n+2 (n \leqq 3)$, $\alpha \sim \infty$ and $\alpha_1 \sim 2$. Thus for (1.2) to have a unique H^2-solution for $u_0 \in H^2$, V_1 (or $\partial V/\partial t$) can carry $|x|^{-2+\varepsilon}$ (or $|x|^{-3+\varepsilon}$)-type singularities for $n \geqq 4$ and $|x|^{-n/2+\varepsilon}$ (or $|x|^{-n/2-1+\varepsilon}$)-type singularities for $n \leqq 3$.

In the case when $H(t) = H$ is independent of t, the celebrated Stone theorem (or the functional calculus of selfadjoint operators) guarantees the existence of the unitary group $\exp(-itH)$ such that $\exp(-i(t-s)H) = U(t,s)$ satisfies the properties (1), (2), (4) $\sim$ (6) of Theorems 1.1 $\sim$ 1.3 under the assumption (A.2). On the other hand, in spite of the increasing interest in time dependent Schrödinger equations ([1, 2, 14, 15]), the initial value problem for Eq. (1.2) is not yet fully studied when the potentials are genuinely t-dependent, and the authors usually rely upon the abstract theory of evolution equations for obtaining their solutions. In the abstract theory of evolution equations (cf. Tanabe [12], Masuda [8], Pazy [9], Goldstein [4]) authors classify evolution equations into two types, the parabolic and the hyperbolic types, putting Schrödinger equations into the latter. However, unfortunately, direct applications of the existing abstract theories generally lead to rather strong smoothness conditions on $V(t,x)$. For example, if one would naively apply Kato's theorem on evolution equations [6] to (1.1) with $V(t,x) = W(x - vt)(v \in \mathbb{R}^n, v \neq 0)$, which incidentally may be reduced to the equation with t-independent $W(x)$ via a simple unitary transformation, one would have to impose on $W(x)$ obviously superfluous smoothness conditions which even exclude Coulomb potentials $W(x) = Z/|x| (n=3)$ (see [16 and 17] for time translation potentials where suitable reductions were made before applying Kato's theorem).

Thus we feel it appropriate to study Eq. (1.1), taking the characteristic features of Schrödinger equations into account and establish a theorem which is directly applicable for obtaining the solution of (1.1) for a larger class of potentials than in existing abstract theories. The advantageous character of Schrödinger equations which we shall exploit in this paper is the smoothing property of the free propagator $U_0(t) = \exp(it\Delta/2)$, which is spelled out in Lemma 2.1, and is a simple consequence of Kato's inequality (Lemma 2.2), the estimate of common use in the scattering theory for Schrödinger equations. In fact, Kato's inequality was first used by Howland [5] to prove the existence of the strongly continuous unitary propagator for Eq. (1.1) with singular $V(t,x)$ in a slightly different framework. The inequality is also an indispensable tool in recent studies of non-linear Schrödinger equations, and we refer to Kato [18] for this, which we were informed of after the submission of the paper (see also Ginibre–Velo [3], Tsutsumi [12]).

The following notations and conventions are used throughout the paper. $L^l(\mathbb{R}^n)$ is the Banach space of l-summable (complex-valued) functions on $\mathbb{R}^n$ with the norm

$\|u\|_l = (\int |u(x)|^l dx)^{1/l}$ $\|u\| = \|u\|_2$. For $s \in \mathbb{R}^1$, $H^s(\mathbb{R}^n)$ is the Sobolev space or order s and its norm is denoted by $\|u\|_{H^s}$. We often write $H_0 = -\frac{1}{2}\Delta$. The free propagator $U_0(t) = \exp(-itH_0) = \exp(it\Delta/2)$ is defined by the oscillatory integral

$$U_0(t)f(x) = (2\pi i|t|)^{-n/2} \int_{\mathbb{R}^n} \exp\left(\frac{|x-y|^2}{2t} i\right) f(y)dy.$$

The same symbol may represent different operators in the sense that their domains and images are considered in different spaces. For example, $U_0(t)$ may be considered as a unitary operator in $\mathscr{H}$ as well as bounded operator from $L^l(\mathbb{R}^n)$ to $L^{l'}(\mathbb{R}^n)$, $l^{-1} + l'^{-1} = 1$, $1 \leqq l \leqq 2$. Likewise the integrals of vector-valued functions which appear in what follows may be understood in various senses, although all of them make sense at least as the weak integrals in $\mathscr{S}'$. It should be clear from the context in which sense they should be understood and we often do not mention it explicitly. For a Banach space $\mathscr{X}$, $\mathfrak{B}(\mathscr{X})$ is the Banach algebra of bounded operators on $\mathscr{X}$ and its norm is denoted by $\|\cdot\|_{\mathfrak{B}(\mathscr{X})}$. For a function $W(t,x)$, $W(t)$ and W stand both for the multiplication operator by $W(t,x)$ and the function $W(t,\cdot)$. For $1 \leqq \rho \leqq \infty$, $\rho' = \rho/\rho - 1$ is its dual exponent. In this paper the exponents p and q are reserved to denote those p and q in Assumptions (A.1) $\sim$ (A.2) and Theorems 1.1 $\sim$ 1.3.

2. Preliminary Estimates

In this section, we collect some basic estimates which will be needed in the sequel.

By Q and S we denote the integral operators

$$(Qu)(t) = \int_0^t U_0(t-s)V(s)u(s)ds, \tag{2.1}$$

$$(Su)(t) = \int_0^t U_0(t-s)u(s)ds, \tag{2.2}$$

$$(Qu)(t) = (SVu)(t). \tag{2.3}$$

For studying these operators we introduce two sets of Banach spaces over $I \times \mathbb{R}^n$, $I = [-a, a]$, for the parameter l, $0 \leqq n(1/2 - 1/l) < 1$:

$$\begin{aligned} &\mathscr{X}(a,l) = C(I,\mathscr{H}) \cap L^{l,\theta}, \quad \theta = \theta(l) = 4l/n(l-2), \\ &\mathscr{X}^*(a,l) = L^1(I,\mathscr{H}) + L^{l',\theta'}, \quad l' = l/l-1, \quad \theta' = \theta/\theta - 1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\mathscr{Y}(a,l) = \{u : u \in C(I,H^2), \quad \dot{u} \in \mathscr{X}(a,l)\}, \quad \dot{u} = \partial u/\partial t, \\ &\mathscr{Y}^*(a,l) = \{u : u \in C(I,\mathscr{H}), \quad \dot{u} \in \mathscr{X}^*(a,l)\}, \end{aligned} \tag{2.5}$$

with the norms defined respectively as

$$\|u\|_{\mathscr{X}(a,l)} = \|u\|_{2,\infty} + \|u\|_{l,\theta'}$$

$$\|u\|_{\mathscr{X}^*(a,l)} = \inf\{\|u_1\|_{2,1} + \|u_2\|_{l',\theta'} : u = u_1 + u_2\},$$

$$\|u\|_{\mathscr{Y}(a,l)} = \sup_{t\in I} \|u(t)\|_{H^2} + \|\dot{u}\|_{\mathscr{X}(a,l)},$$

$$\|u\|_{\mathscr{Y}^*(a,l)} = \|u\|_{2,\infty} + \|\dot{u}\|_{\mathscr{X}^*(a,l)}.$$

The following integrability property of the free propagator $U_0(t)$, which is mostly known, is fundamental in the following discussions.

Lemma 2.1. *Let* $0 \leqq n(1/2 - 1/l) < 1$. *Then*

$$\| U_0(\cdot) f \|_{\mathscr{X}(a,l)} \leqq C \| f \|, \quad f \in \mathscr{H}. \tag{2.6}$$

$$\| Su \|_{\mathscr{X}(a,l)} \leqq C \| u \|_{\mathscr{X}^*(a,l)}, \quad u \in \mathscr{X}^*(a,l). \tag{2.7}$$

$$\| Su \|_{\mathscr{Y}(a,l)} \leqq C(1+a) \| u \|_{\mathscr{Y}^*(a,l)}, \quad u \in \mathscr{Y}^*(a,l). \tag{2.8}$$

Here the constants $C > 0$ *are independent of* a *and* u.

For proving the lemma we need the following well-known

Lemma 2.2. (*Kato* [6]). *Let* $2 \leqq m \leqq \infty$ *and* $m' = m/m-1$ *be its dual exponent. Then*

$$\| U_0(t) f \|_m \leqq (2\pi |t|)^{-2/\theta(m)} \| f \|_{m'}, \quad 2/\theta(m) = n(1/2 - 1/m), \tag{2.9}$$

and $U_0(t)u \in C(\mathbb{R}^1 \setminus \{0\}, L^m(\mathbb{R}^n))$ *for every* $u \in L^{m'}(\mathbb{R}^n)$.

Proof. Let $G(t) = e^{ix^2/2t}$ and $\mathscr{F}$ be the Fourier transform. Then

$$U_0(t) f(x) = G(t)(it)^{-n/2} (\mathscr{F} G(t) f)(x/t). \tag{2.10}$$

We apply Young's inequality to (2.10) and obtain (2.9).

Proof of Lemma 2.1. For proving (2.6) and (2.7) it is convenient to consider a slightly more general integral operator,

$$Fu(t) = \int_{-\infty}^{\infty} K(t,s) U_0(t-s) u(s) ds, \tag{2.11}$$

where $K(t,s) \in L^\infty(\mathbb{R}^2)$ is a piecewise continuous complex-valued function. If $K(t,s) = 1$ for $0 \leqq \pm s \leqq \pm t \leqq \pm a$ and $K(t,s) = 0$ otherwise, we have $F = S$. Since $0 \leqq n(1/2 - 1/l) = 2/\theta < 1$, Lemma 2.2 and Sobolev's inequality imply

$$\begin{aligned} \| Fu \|_{l,\theta} &\leqq (2\pi)^{-2/\theta} L \left\{ \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{\infty} (t-s)^{-2/\theta} \| u(s) \|_{l'} ds \right)^\theta \right\}^{1/\theta} \\ &\leqq C_\theta L \| u \|_{l',\theta'}, \quad u \in \mathscr{S}(\mathbb{R}^{n+1}), \quad L = \| K \|_\infty . \end{aligned} \tag{2.12}$$

Applying (2.12) to the case $K_r(t,s) = \overline{K(r,t)} K(r,s)$, we have

$$\begin{aligned} \| Fu \|_{2,\infty}^2 &= \sup_r \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{\infty} K_r(t,s) U_0(t-s) u(s) ds, u(t) \right) \\ &\leqq C_\theta L^2 \| u \|_{l',\theta'}^2, \quad u \in \mathscr{S}(\mathbb{R}^{n+1}). \end{aligned} \tag{2.13}$$

By (2.13) and Hölder's inequality,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} (Fu(t), v(t)) dt \right| &= \left| \int_{-\infty}^{\infty} \left(u(s), \int_{-\infty}^{\infty} \overline{K(t,s)}\, U_0(s-t) v(t) dt \right) ds \right| \\ &\leqq \sqrt{C_\theta} L \| u \|_{2,1} \| v \|_{l',\theta'}, \; u, v \in \mathscr{S}, \end{aligned} \tag{2.14}$$

$$\left| \int_{-\infty}^{\infty} (u(t), \overline{K(0,t)} U_0(t) f) dt \right| = |(Fu(0), f)|$$

$$\leqq C_\theta^{1/2} L \| u \|_{l',\theta'} \| f \|, \quad u \in \mathscr{S}(\mathbb{R}^{n+1}), \quad f \in \mathscr{S}(\mathbb{R}^n). \tag{2.15}$$

It follows by the standard approximation procedure and the duality argument that (2.12) ~ (2.15) hold for general u, $v \in L^{l',\theta'}$, $f \in L^2(\mathbb{R}^n)$ and that

$$\| Fu \|_{l,\theta} \leqq C_\theta^{1/2} L \| u \|_{2,1}, \quad u \in L^{2,1}, \tag{2.16}$$

$$\| \overline{K(0,\cdot)} U_0(\cdot) f \|_{l,\theta} \leqq C_\theta^{1/2} L \| f \|, \quad f \in L^2. \tag{2.17}$$

Taking $K(t,s) \equiv 1$ in (2.17), we obtain (2.6). Combining (2.12), (2.13), (2.16) with the obvious estimate,

$$\| Fu \|_{2,\infty} \leqq L \| u \|_{2,1}, \tag{2.18}$$

we have (2.7) for F, hence for S. By the density argument $Su \in C(I, \mathscr{H})$ for $u \in \mathscr{X}^*(a,l)$.

For proving (2.8), we note, for $t \in I_1 = (-a, a)$,

$$(-id/dt + H_0) Su(t) = -iu(t), \quad u \in C^\infty(I_1, \mathscr{S}(\mathbb{R}^n)), \tag{2.19}$$

and

$$(d/dt) Su(t) = (S\dot{u})(t) + U_0(t) u(0). \tag{2.20}$$

By (2.6) and (2.7), it follows from (2.20) that

$$\| (d/dt) Su(t) \|_{\mathscr{X}(a,l)} \leqq C(\| \dot{u} \|_{\mathscr{X}^*(a,l)} + \| u(0) \|_2) \leqq C \| u \|_{\mathscr{Y}^*(a,l)}, \tag{2.21}$$

and from (2.19), (2.21) and (2.18) that

$$\begin{aligned} \sup \| Su(t) \|_{H^2} &\leqq 2(\| Su(t) \|_{2,\infty} + \| H_0 Su(t) \|_{2,\infty}) \\ &\leqq 2(\| u \|_{2,1} + \| u \|_{2,\infty} + \| (d/dt) Su \|_{\mathscr{X}(a,l)}) \\ &\leqq 4(a+1) \| u \|_{2,\infty} + C \| u \|_{\mathscr{Y}^*(a,l)}. \end{aligned} \tag{2.22}$$

Estimates (2.21) and (2.22) imply the desired (2.8) for $u \in C^\infty(I_1, \mathscr{S}(\mathbb{R}^n))$. By approximation, it is easy to see that Eqs. (2.19) and (2.20) and estimate (2.8) extend to all $u \in \mathscr{Y}^*(a,l)$.

As for the multiplication by the function $V(t,x)$, we need the following lemmas. We denote

$$\| V \|_{\mathscr{M}} = \inf \{ \| V_1 \|_{p,\alpha} + \| V_2 \|_{\infty,\beta}, V = V_1 + V_2 \}, \tag{2.23}$$

$$\| V \|_{\tilde{\mathscr{M}}} = \inf \{ \| V_1 \|_{\tilde{p},\beta} + \| V_2 \|_{\infty,\infty}, V = V_1 + V_2 \}, \tag{2.24}$$

$$\| \dot{V} \|_{\mathscr{N}} = \inf \{ \| W_1 \|_{p_1,\alpha_1} + \| W_2 \|_{\infty,\beta}, \partial V/\partial t = W_1 + W_2 \}, \tag{2.25}$$

$$\mathscr{X}(a) = \mathscr{X}(a,q), \quad \mathscr{X}^*(a) = \mathscr{X}^*(a,q), \tag{2.26}$$

$$\mathscr{Y}(a) = \mathscr{Y}(a,q), \quad \mathscr{Y}^*(a) = \mathscr{Y}(a,q), \quad q = 2p/p-1. \tag{2.27}$$

Note $\| V \|_{\mathscr{M}} \leqq C \| V \|_{\tilde{\mathscr{M}}}$.

Lemma 2.3. *Let Assumption (A.1) be satisfied. Then*

$$\| Vu \|_{\mathscr{X}^*(a)} \leqq (2a)^\gamma \| V \|_{\mathscr{M}} \| u \|_{\mathscr{X}(a)}, \quad 2a < 1 \tag{2.28}$$

with $\gamma = \min(1 - 1/\beta, 1 - n/2p - 1/\alpha)$.

Proof. Recall the relation of exponents: $0 \leqq 1/\beta < 1$, $0 \leqq 1/\alpha < 1 - n/2p =$

$1/\theta' - 1/\theta$, $\theta = \theta(q)$ and $1/q' - 1/q = 1/p$. Then by Hölder's inequality,

$$\begin{aligned} \| V_1 u \|_{q',\theta'} &\leqq \| V_1 \|_{p,2p/2p-n} \| u \|_{q,\theta} \\ &\leqq |I|^{(1-n/2p-1/\alpha)} \| V_1 \|_{p,\alpha} \| u \|_{q,\theta}, \quad |I| = 2a, \\ \| V_2 u \|_{2,1} &\leqq \| V_2 \|_{\infty,1} \| u \|_{2,\infty} \leqq |I|^{1-1/\beta} \| V_2 \|_{\infty,\beta} \| u \|_{2,\infty}. \end{aligned}$$

This proves (2.28).

Lemma 2.4. *Let Assumption (A.2) be satisfied. Then*
(1) *V maps $\mathscr{Y}(a)$ into $\mathscr{Y}^*(a)$ continuously.*
(2) *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for $|2a| < 1$,*

$$\| Vu \|_{\mathscr{Y}^*(a)} \leqq (\varepsilon \| V \|_{\tilde{\mathscr{M}}} + (2a)^\kappa \| \dot{V} \|_{\mathscr{N}}) \| u \|_{\mathscr{Y}(a)} + C_\varepsilon \| V \|_{\tilde{\mathscr{M}}} \| u \|_{2,\infty} \tag{2.29}$$

for all $u \in \mathscr{Y}(a)$, where $\kappa = \min(1/\theta' - 1/\alpha_1, \gamma)$.

Proof. We prove the lemma for the case $n \geqq 5$ only. Other cases may be proved similarly. Decompose $V = V_1 + V_2$, $V_1 \in C(I, L^{\tilde{p}})$, $V_2 \in C(I, L^\infty)$ and $\dot{V} = W_1 + W_2$, $W_1 \in L^{p_1,\alpha_1}$, $W_2 \in L^{\infty,\beta}$. By the Sobolev embedding theorem, we have

$$\| u \|_{2,\infty} + \| u \|_{q_1,\infty} \leqq C \| u \|_{\mathscr{Y}(a)}, \quad 1/q_1 = 1/2 - 2/n. \tag{2.30}$$

Since $p > n/2$, $1/p_1 + 1/q_1 = 1/q'$ and $1/\alpha_1 < 1/\theta' = 1 - n/4p$, we see that

$$\| Vu \|_{2,\infty} \leqq (\varepsilon \| u \|_{\mathscr{Y}} + C_\varepsilon \| u \|_{2,\infty}) \| V \|_{\tilde{\mathscr{M}}}, \tag{2.31}$$

$$\begin{aligned} \| W_1 u \|_{q',\theta'} &\leqq \| W_1 \|_{p_1,\theta'} \| u \|_{q_1,\infty} \\ &\leqq C |I|^{1/\theta' - 1/\alpha_1} \| W_1 \|_{p_1,\alpha_1} \| u \|_{\mathscr{Y}(a)}, \quad |I| = 2a, \end{aligned} \tag{2.32}$$

$$\| W_2 u \|_{2,1} \leqq \| W_2 \|_{\infty,1} \| u \|_{2,\infty} \leqq |I|^{1-1/\beta} \| W_2 \|_{\infty,\beta} \| u \|_{\mathscr{Y}(a)}. \tag{2.33}$$

On the other hand, we have by Lemma 2.3

$$\| V\dot{u} \|_{\mathscr{X}(a)} \leqq |I|^\gamma \| V \|_{\mathscr{M}} \| \dot{u} \|_{\mathscr{X}(a)}. \tag{2.34}$$

Combining estimates (2.31) ~ (2.34), we obtain the statements of the lemma.

Lemma 2.5. *Suppose that $u \in \mathscr{X}(T)$ satisfies*

$$u(t) = U_0(t)u_0 - iSVu(t), \quad u_0 \in H^2(\mathbb{R}^n), \tag{2.35}$$

and $Vu \in \mathscr{Y}^*(T)$. Then $u \in \mathscr{Y}(T)$ and it satisfies (1.2), $idu/dt = H(t)u$, and

$$(d/dt)Sf(t) = U_0(t)f(0) + S\dot{f}(t), \quad f = Vu. \tag{2.36}$$

Proof. Since $U_0(t)u_0 \in \mathscr{Y}(T)$, for $u_0 \in H^2(\mathbb{R}^n)$, the lemma follows by Lemma 2.1 and the remark at the end of its proof.

3. Existence of L^2-Solutions, Proof of Theorem 1.1

Recall for $0 < a \leqq T$,

$$\mathscr{X}(a) = \mathscr{X}(a, q), \quad q = 2p/p - 1. \tag{3.1}$$

It follows by Lemma 2.1 and 2.3 that the integral operator

$$(Qu)(t)=\int_0^t U_0(t-s)V(s)u(s)ds \tag{3.2}$$

is bounded on $\mathscr{X}(a)$ and

$$\|Qu\|_{\mathscr{X}(a)} \leqq C_{n,l}a^{\gamma}\|V\|_{\mathscr{M}}\|u\|_{\mathscr{X}(a)}, \quad \gamma>0. \tag{3.3}$$

Hence if a is sufficiently small so that

$$C_{n,l}a^{\gamma}\|V\|_{\mathscr{M}} \leqq 1/2, \tag{3.4}$$

the operator Q is a contraction on $\mathscr{X}(a)$:

$$\|Qu\|_{\mathscr{X}(a)} \leqq (1/2)\|u\|_{\mathscr{X}(a)}. \tag{3.5}$$

Since, by Lemma 2.1, $u_0(t)=U_0(t)u_0 \in \mathscr{X}(T)$, for any $u_0 \in \mathscr{H}$, it follows that the integral equation

$$u(t)=u_0(t)-i(Qu)(t) \tag{3.6}$$

has a unique solution $u(t)=(\mathbb{1}+iQ)^{-1}u_0(t) \in \mathscr{X}(a)$. Considering Eq. (3.6) with $U_0(-s)u_0$ and $V(t+s)$ in place of u_0 and $V(t)$, respectively, we see that the integral equation (1.3),

$$u(t)=U_0(t-s)u_0-i\int_s^t U_0(t-\tau)V(\tau)u(\tau)d\tau,$$

has a unique solution $u(t) \in C([-a+s,a+s],\mathscr{H}) \cap L^{q,\theta}([-a+s;a+s])$ for any s and $u_0 \in \mathscr{H}$. Thus the standard continuation procedure for the solution of linear integral equations yields a global unique solution $u \in \mathscr{X}(T)$.

For proving the equation $\|u(t)\|=\|u_0\|$ and also for later use, we need the following lemmas.

Lemma 3.1. *Suppose that V_ε $(\varepsilon>0)$ satisfies the assumption (A.1) and that*

$$\lim_{\varepsilon\to 0}\|V_\varepsilon - V\|_{\mathscr{M}}=0. \tag{3.7}$$

Suppose also that

$$\lim_{\varepsilon\to 0}\|u_{0\varepsilon}-u_0\|=0. \tag{3.8}$$

Let $u_\varepsilon \in \mathscr{X}(T)$ and $u \in \mathscr{X}(T)$ be the solutions of

$$u_\varepsilon(t)=U_0(t-s)u_{0\varepsilon}-i\int_s^t U_0(t-\tau)V_\varepsilon(\tau)u_\varepsilon(\tau)d\tau \tag{3.9}$$

and (1.3), respectively. Then

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{\mathscr{X}(T)}=0. \tag{3.10}$$

Proof. By the argument as above, it suffices to show (3.10) for the case $s=0$ and for $T>0$ small. If we denote as $u_{0\varepsilon}(t)=U_0(t)u_{0\varepsilon}$ and $Q_\varepsilon = SV_\varepsilon$, we see by Lemma 2.1 and

(3.3) that

$$\lim_{\varepsilon\to 0} \| u_0(t) - u_{0\varepsilon}(t) \|_{\mathscr{X}(T)} = 0, \tag{3.11}$$

$$\lim_{\varepsilon\to 0} \| Q - Q_\varepsilon \|_{\mathfrak{B}(\mathscr{X}(T))} = 0. \tag{3.12}$$

Thus $u_\varepsilon(t) = (\mathbb{1} + iQ_\varepsilon)^{-1} u_{0\varepsilon}(t) \to u(t) = (\mathbb{1} + iQ)^{-1} u_0(t)$ in $\mathscr{X}(T)$.

Lemma 3.2. *Let $V \in C^1(I_T, L^\infty(\mathbb{R}^n))$ and $u_0 \in H^2(\mathbb{R}^n)$. Then the solution $u \in \mathscr{X}(T)$ of (1.3) belongs to $\mathscr{Y}(T)$ and it satisfies (1.2),*

$$i\dot{u}(t) = H(t)u(t), \quad H(t) = H_0 + V(t), \quad \dot{u} = du/dt.$$

Proof. As before it suffices to show the lemma for $T > 0$ small. Set for $h \neq 0$ small, $v_h(t) = (u(t+h) - u(t))/h$, $|t| < T$. Then $v_h(t)$ satisfies the equation

$$v_h(t) = f_h(t) - i(Qv_h)(t), \tag{3.13}$$

with

$$f_h(t) = U_0(t)(U_0(h) - \mathbb{1})u_0/h - ih^{-1} U_0(t+h) \int_0^h U_0(-\tau)(Vu)(\tau) d\tau$$
$$- i \int_0^t U_0(t-\tau)\{(V(\tau+h) - V(\tau))/h\} u(\tau+h) d\tau. \tag{3.14}$$

Since $u_0 \in H^2(\mathbb{R}^n)$ and $u \in \mathscr{X}(T)$, we see by using Lemma 2.1 that

$$f_h(t) \to f(t) = -i[U_0(t)H(0)u_0 + S\dot{V}u(t)] \quad \text{in } \mathscr{X}_{\text{loc}}(T). \tag{3.15}$$

Hence by the contraction property of the operator $Q = SV$ in $\mathscr{X}(T)$ for $T > 0$ small, we have

$$v_h = (\mathbb{1} + iQ)^{-1} f_h \to (\mathbb{1} + iQ)^{-1} f \quad \text{in } \mathscr{X}_{\text{loc}}(T). \tag{3.16}$$

This proves that $du/dt = (\mathbb{1} + iQ)^{-1} f \in \mathscr{X}(T)$ and

$$i\dot{u}(t) = U_0(t)H(0)u_0 + S\dot{V}u(t) + SV\dot{u}(t).$$

Thus $Vu \in \mathscr{Y}^*(T)$ and Lemma 2.5 implies the desired result.

Now it is easy to prove the statement (2) of the theorem: $\|u(t)\| = \|u_0\|$. We approximate $V = V_1 + V_2 \in L^{p,\alpha} + L^{\infty,\beta}$ by $V_\varepsilon \in C^1(I_T, L^\infty(\mathbb{R}^n))$. For this we take $\rho(x) \in C_0^\infty(\mathbb{R}^n)$ and $\chi \in C_0^\infty(\mathbb{R}^1)$ such that

$$\rho(x), \chi(t) \geqq 0 \quad \text{and} \quad \int \rho(x) dx = \int \chi(t) dt = 1, \tag{3.17}$$

and define

$$V_\varepsilon(t,x) = \int V_1(t + \varepsilon s, x + \varepsilon y)\chi(s)\rho(y) dt dy + \int V_2(t + \varepsilon s, x)\chi(s) ds, \tag{3.18}$$

$$u_{0\varepsilon}(x) = \int u_0(x + \varepsilon y)\rho(y) dy, \tag{3.19}$$

where we extended $V_j(t,x)$ outside $I_T \times \mathbb{R}^n$ as $V_j(t,x) \equiv 0$, $j = 1, 2$. Then it is well-known that

$$V_\varepsilon \in C^1(I_T, L^\infty(\mathbb{R}^n)), \quad \| V_\varepsilon - V \|_{\mathscr{M}} \to 0 \ (\varepsilon \to 0), \tag{3.20}$$

$$u_{0\varepsilon}\in H^2(\mathbb{R}^n), \quad \| u_{0\varepsilon} - u_0 \| \to 0. \tag{3.21}$$

Then the application of Lemma 3.1 and 3.2 shows the solution $u_\varepsilon \in \mathscr{Y}(T)$ of (3.9) satisfies

$$i du_\varepsilon/dt = (H_0 + V_\varepsilon(t))u_\varepsilon, \tag{3.22}$$

$$\| u_\varepsilon - u \|_{\mathscr{X}(T)} \to 0. \tag{3.23}$$

Since $H_0 + V_\varepsilon(t)$ is selfadjoint with the domain $H^2(\mathbb{R}^n)$, we immediately see from (3.22) that $d\| u_\varepsilon(t) \|^2/dt = 0$, and hence, $\| u_\varepsilon(t) \| = \| u_{0\varepsilon} \|$. Then by (3.21) and (3.23), we obtain the desired equation $\| u(t) \| = \| u_0 \|$ for $t \in I_T$.

4. Regularity of Solutions, Proof of Theorem 1.3

As in the preceding section, it suffices to prove the theorem for the case $s = 0$ and $T > 0$ is small. We have only to prove that if $u_0 \in H^2(\mathbb{R}^n)$, the solution $u(t) \in \mathscr{X}(T)$ of (1.3) satisfies

$$u \in \mathscr{Y}(T) \tag{4.1}$$

and

$$i du/dt = H(t)u(t). \tag{4.2}$$

The statement of Theorem 1.3 then follows by the standard technique of semi-group theory ([4, 8, 9, 12]).

We begin with the following

Lemma 4.1. *Let* $V \in C^1(I_T, L^{\tilde{p}}(\mathbb{R}^n)) + C^1(I_T, L^\infty(\mathbb{R}^n))$ *and* $u_0 \in H^2(\mathbb{R}^n)$. *Then the solution* $u \in \mathscr{X}(T)$ *of (1.3) satisfies (4.1) and (4.2).*

Proof. We write as $V = V_1 + V_2$, $V_1 \in C^1(I_T, L^{\tilde{p}})$ and $V_2 \in C^1(I_T, L^\infty)$. Take $\rho \in C_0^\infty(\mathbb{R}^n)$ of (3.17) and set for $\varepsilon > 0$,

$$V_{1\varepsilon}(t,x) = \int V_1(t, x + \varepsilon y)\rho(y)dy, \quad t \in I_T,$$
$$V_\varepsilon(t,x) = V_{1\varepsilon}(t,x) + V_2(t,x).$$

It is easy to see that $V_\varepsilon \in C^1(I_T, L^\infty(\mathbb{R}^n))$ and

$$\lim_{\varepsilon\to 0} (\| V_\varepsilon - V \|_{\tilde{\mathscr{M}}} + \| \dot{V}_\varepsilon - \dot{V} \|_{\mathscr{N}}) = 0. \tag{4.3}$$

Therefore, by Lemma 3.1 $\sim$ 3.2, the solution u_ε of the integral equation,

$$u_\varepsilon(t) = U_0(t)u_0 - iSV_\varepsilon u_\varepsilon(t), \quad u_0 \in H^2(\mathbb{R}^n), \tag{4.4}$$

belongs to $\mathscr{Y}(T)$ and satisfies

$$\| u_\varepsilon(t) \| = \| u_0 \|, \quad t \in I_T, \tag{4.5}$$

$$\| u_\varepsilon - u \|_{\mathscr{X}(T)} \to 0 \ (\varepsilon \to 0). \tag{4.6}$$

Then, applying Lemma 2.1 and 2.4 to (4.4), we have for any $\delta > 0$,

$$\begin{aligned}\| u_\varepsilon \|_{\mathscr{Y}} &\leq C(\| u_0 \|_{H^2} + (1+T)\| V_\varepsilon u_\varepsilon \|_{\mathscr{Y}^*}) \\ &\leq C(1+T)(\delta \| V_\varepsilon \|_{\tilde{\mathscr{M}}} + T^\kappa \| \dot{V}_\varepsilon \|_{\mathscr{N}}) \| u_\varepsilon \|_{\mathscr{Y}} \\ &\quad + C(\| u_0 \|_{H^2} + C_\delta \| V_\varepsilon \|_{\tilde{\mathscr{M}}} \| u_0 \|),\end{aligned} \tag{4.7}$$

where C and C_δ are independent of $T<1$. Hence choosing $0<\delta<1$ and $0<T<1$ small so that

$$\sup_{\varepsilon>0} C(1+T)(\delta\|V_\varepsilon\|_{\tilde{\mathcal{M}}}+T^\kappa\|\dot{V}_\varepsilon\|_{\mathcal{N}})\leqq C(1+T)(\delta\|V\|_{\tilde{\mathcal{M}}}+T^\kappa\|\dot{V}\|_{\mathcal{N}})<1/2, \tag{4.8}$$

we see that for $0<\varepsilon<1$,

$$\|u_\varepsilon\|_{\mathcal{Y}(T)}\leqq 2C(\|u_0\|_{H^2}+C_\delta\|V\|_{\tilde{\mathcal{M}}}\|u_0\|). \tag{4.9}$$

Estimating similarly as in (4.7) by using Lemma 2.1 and 2.4, we have

$$\begin{aligned}\|u_{\varepsilon_2}-u_{\varepsilon_1}\|_{\mathcal{Y}(T)}\leqq{}& C(1+T)(\delta\|V_{\varepsilon_2}-V_{\varepsilon_1}\|_{\tilde{\mathcal{M}}}+T^\kappa\|\dot{V}_{\varepsilon_2}-\dot{V}_{\varepsilon_1}\|_{\mathcal{N}})\|u_{\varepsilon_2}\|_{\mathcal{Y}(T)}\\ &+C_\delta\|V_{\varepsilon_2}-V_{\varepsilon_1}\|_{\tilde{\mathcal{M}}}\|u_0\|+C_\delta\|V_{\varepsilon_1}\|_{\tilde{\mathcal{M}}}\|u_{\varepsilon_2}-u_{\varepsilon_1}\|_{2,\infty}\\ &+C(1+T)(\delta\|V_{\varepsilon_1}\|_{\tilde{\mathcal{M}}}+T^\kappa\|\dot{V}_{\varepsilon_1}\|_{\mathcal{N}})\|u_{\varepsilon_2}-u_{\varepsilon_1}\|_{\mathcal{Y}(T)}.\end{aligned} \tag{4.10}$$

Thus choosing δ and $T>0$ as in (4.8) and using the relations (4.3), (4.6) and (4.9), we see that $u_\varepsilon\to u$ in $\mathcal{Y}(T)$ as $\varepsilon\to 0$. This proves (4.1). Since $Vu\in\mathcal{Y}^*(T)$, then Lemma 2.5 implies (4.2).

Now we suppose that $V(t,x)$ satisfies the assumption (A.2) and prove (4.1) and (4.2). Except for the way of approximating $V(t,x)$, the proof goes entirely similarly as in that of Lemma 4.1. We extend $V(t,x)$ as

$$\begin{aligned}&V(t,x)=V(-T,x)\quad\text{for}\quad t\leqq -T,\quad x\in\mathbb{R}^n,\\ &V(t,x)=V(T,x)\qquad\text{for}\quad t\geqq T,\quad x\in\mathbb{R}^n,\end{aligned} \tag{4.11}$$

and set for $\varepsilon>0$ small

$$V_\varepsilon(t,x)=\int_{-\infty}^{\infty}V(t+\varepsilon s,x)\chi(s)ds,\quad t\in I_T, \tag{4.12}$$

where $\chi(t)\in C_0^\infty(\mathbb{R}^1)$ is the function of (3.17). It is clear that $V_\varepsilon\in C^1(I_T,L^{\tilde{p}}(\mathbb{R}^n))+C^1(I_T,L^\infty(\mathbb{R}^n))$ and that (4.3) is satisfied. Hence, if $u_\varepsilon(t)$ is the solution of (4.4) with this V_ε, we have $u_\varepsilon\in\mathcal{Y}(T)$ and it satisfies (4.5) $\sim$ (4.6). Then we repeat, word by word, the argument after (4.6) of the proof of Lemma 4.1. This yields (4.1) and (4.2), completing the proof of Theorem 1.3.

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