Econ. Theory 7, 125–138 (1996)

Economic Theory

On fair allocations and monetary compensations*

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Received: June 1, 1994; revised version December 19, 1994

Summary. In this paper we study fair division problems with the special feature that there exists only one transferable good that everyone likes. This good will be used to compensate some individuals for their differences in other non-transferable resources (like talents or handicaps). In this context we test the traditional no-envy solution and we verify that: 1) its ethical content can be a matter of discussion, and 2) frequently it does not select a non-empty set of allocations. We propose an extension of this criterion that partially solves the existence problem while also retaining the main ethical properties of the preceding solution.

1. Introduction

A fair division problem can be stated as follows: how to divide a bundle of goods among a group of individuals or institutions if every one of them has an identical right to the resources? The work of Roemer [8, 9] emerges from the literature on these problems. His innovative assumption (as in Dworkin [3]) is that individuals are unequally endowed with *personalized goods*, which are goods that can only be profitable for some particular agents. Those goods may be thought as representing talents, or skills, or capabilities or, more widely, and kind of goods that do not enter the distribution problem under consideration, because they were distributed previously. The aim of Roemer's model was to characterize several well known solutions to bargaining games in this framework. A crucial assumption is that individual preferences can be represented by utility functions that allow for cardinal measure and interpersonal comparisons. The "Welfare Egalitarian Solution" can be considered as Roemer's own proposal to implement the egalitarian principle in such a class of distribution problems.

* Comments and criticisms on earlier drafts by Ahmet Alkan, Marc Fleurbaey, Louis Gevers, Leonid Hurwicz, Mathew Jackson and an anonymous referee are gratefully acknowledged. Financial support has been provided by Spanish DGICYT, PT 90-0654 and Protocolo Aquitania-Navarra-País Vasco.
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On the other hand there are many works on distribution problems that rely only on *ordinal* preferences. The most usual strategy consists in proposing a notion of fairness to solve the problem using that selected notion. The idea of fairness that has played the most important role in economic analysis is the concept of no-envy (Foley [5]). An allocation is said to be *envy-free* if no agent would prefer someone else's consumption bundle to his. Besides, it is very usual – although probably misleading – to consider that a *fair* allocation is an allocation that is both Pareto efficient and envy-free.

But in many relevant economic environments it is very easy to see that the requirements of Pareto efficiency and absence of envy are incompatible. To get around this obstacle several authors proposed different concepts of fair allocations. Two interesting surveys on these issues can be found in Thomson and Varian [11] and Arnsperger [1].

Fleurbaey [4] considers a model in which there is a fixed quantity of money to be divided among a set of individuals and there are also some other resources on which individuals have different initial claims (personalized goods in the sense defined by Roemer). Now, when we propose a distribution of the transferable good, we must have in mind the very likely unequal distribution of personalized goods. That is, the distribution of the transferable goods must be addressed to *compensate the individuals for their differences in capabilities.*

Fleurbaey proposes to apply the no-envy test and to extend it to the personalized goods that cannot be transferred. That is, a distribution of the transferable goods can be considered envy-free if every individual prefers his allocation – made up of *both transferable and personalized goods* – than the allocation of someone else. Although the distribution of personal resources is irreversible, nothing prevents the agents to imagine how they would feel if they would enjoy other personal resources distinct from the ones they posses. Quoting Arrow [2, p. 115]: "... *the characteristics that define an individual are included in the comparison. In effect, these characteristics are put on a par with the items usually regarded as constituting an individual's wealth. The possession of tools would ordinary be regarded as part of the social state; why not the possession of the skills to use the tools and the intelligence which lies behind those skills? Individuals, in appraising each other's states of well-being, consider not only material possessions but also find themselves desiring this man's scope and that man's art.*"

Fleurbaey also finds that a fair allocation satisfies certain additional compensation properties that might be more or less acceptable from an equity point of veiw. Finally, he gives conditions for the existence of fair allocations. We want to remark that all along this approach, utility functions are of the ordinal non-comparable type, though each agent can assess how he would feel if he was endowed with the skill of any other agent. Yet the conditions needed to guarantee the existence of an ordinal fair solution are strong. Moreover, the notion of fairness as no-envy becomes too demanding when these personalized goods enter the problem. Consequently, it is possible to present several distribution problems in a resonably large domain that are not resolved because no allocation passes the no-envy test. Thus, as Fleurbaey [4] points out, there is a trade off between the ethical value of the no-envy approach and its practical relevence in the sense of selecting allocations for a sufficient large domain of problems.

In this work we propose a new and less demanding notion of the no-envy test. It will be based on the assumption that every individual can compare not only his position with the positions of the other individuals but also that everyone can make comparisons between the positions of any pair of individuals and report which one is the position he prefers. In other words, we say that to make a judgment such as "the position of individual i is better, worse or similar than the position of individual j" it is not necessary that such judgement had to be reported, precisely by one of those two individuals. At first glance, every individual can do such statements. He only needs to imagine himself in the place of another person. There is a close relation of this approach with the notion of *extended sympathy* considered, for example, in Arrow [2] and Sen [10].

Our solution will be an extension of the no-envy solution because it will select the set of envy-free allocations whenever it is non-empty. Now, suppose there are no envy-free allocations. Then, for all allocations, at least one individual, say i, envies another one j. We can require the rest of individuals to evaluate the positions of both i and j. Our interest will be on how many individuals support the opinion "i is worse than j". We think that the feeling of an individual about his relative welfare will be more or less justifiable depending upon *how many individuals* would share it. A given allocation should be rejected when almost all of the individuals agree on the fact that "i is worse than j", but not when there is only one individual who makes such assertion (i, presumably). The solution that we propose takes into account these considerations. For each distribution problem, it selects precisely the set of feasible allocations in which the number of individuals that support someone's demand of new compensations is minimum.

The rest of the paper is organised as follows. Section 2 presents the model and some compensation properties proposed by Fleurbaey. Section 3 presents some results concerning the no-envy solution and the new solution. Sufficient conditions for the existence of both of them are established. Finally, section 4 gathers some final remarks.

2. The model

We consider fair distribution problems in which there are two types of resources. First, a set of *nontransferable* resources (at least one for each individual), and second, one transferable good – call it *money* – that everyone likes. Besides, there is a set $N = \{1, 2, \dots, n\}$ of (at least two) individuals. The number of individuals in N will be n.

Each individual i in N is characterized by two things. First a description y_i of his internal resources taken from a set Y taken from some topological space. This set Y is common for all individuals. A possible interpretation of this set is to consider that $Y \subset \mathbb{R}^l$, where l is the number of relevant characteristics that internal resources are made of. Thus, when a fair distribution problem has to be solved, each individual is endowed with a vector of l characteristics that have been determined by nature, family, social environment, education or sex. Second, a preference relation R_i defined on $\mathbb{R} \times Y$. $\mathscr{R}$ denotes the set of all R_i which are complete, transitive, continuous (upper and lower contour sets for a given point of $\mathbb{R} \times Y$ are closed with respect to R_i for all i) monotonic and satisfy no overwhelming difference. Monotonicity means that:

$$x, x' \in \mathbb{R} \text{ and } x > x', \text{ then } (x, y_j)\, P_i(x', y_j) \text{ for all } i, j. \tag{1}$$

The property of no overwhelming difference is the following:

$$\forall\, i, j, k, \exists \gamma \in \mathbb{R}, \text{ such that if } x \geq \gamma \text{ then } (x, y_i) R_k (0, y_j). \tag{2}$$

In words, there is an amount of money (possibly large) such that everyone considers that any individual would be compensated for not having anybody else's internal resources.

P_i and I_i will be the antisymmetric and reflexive parts of R_i, respectively. Individual preferences are represented by utility functions $u_i(x_j, y_j)$ defined on $\mathbb{R} \times Y$, where x_j is the amount of money that individual j enjoys and y_j is the amount of internal resources of individual j. Notice that though each individual i knows that the distribution of internal resources is irreversible, he may get an impression of how he would feel being in j's position.

To end, let $M \in \mathbb{R}_+$ be a given quantity of money to be distributed among the set of agents N.

A distribution problem is a couple of lists $\{R_i\}_{i\in N}$ where $R_i \in \mathscr{R}$ and $\{y_i\}_{i\in N}$ where $y_i \in Y$ for $i = 1, 2, \ldots, n$ and an amount of money M. We will denote these problems as:

$$\xi = \{n, \{R_i\}_{i\in N}, \{y_i\}_{i\in N}; M\} \text{ or } \xi = \{n, \{\mathrm{u}_i\}_{i\in N}, \{y_i\}_{i\in N}; M\}.$$

We call Σ the set of admissible distribution problems. For a given $\xi \in \Sigma$, an allocation will be a vector of n pairs (x_i, y_i) such that $x_i \geq 0$ for all i. An allocation is feasible if $\sum_{i\in N} x_i \leq M$ and $y_i \in Y$ for all i. The set of feasible allocations for a given problem ξ is denoted by $Z(\xi)$.

A *solution* is a correspondence Φ that associates to each problem $\xi \in \Sigma$, a subset $\Phi(\xi)$ of $Z(\xi)$. An example is the Pareto Solution:

Pareto Solution, P: for all distribution problem $\xi = \{n, \{R_i\}_{i\in N}, \{y_i\}_{i\in N}; M\} \in \Sigma$, $(x, y) \in P(\xi)$ if for no feasible allocation (x', y) it happens that $(x'_i, y_i)\, R_i(x_i, y_i)$ for all i and for some j, $(x'_j, y_j)\, P_j(x_j, y_j)$.

The reader can easily check that in our model it is equivalent to write $P(\xi) = \{(x, y) \in Z(\xi) \mid \Sigma x_i = M\}$. That is, the Pareto solution selects all the allocations where there is no waste of resources. This solution is not very conclusive. From now on, we will focus on solutions selecting a subset of the set $P(\xi)$.

A typical example is the classical envy free solution (Foley [5]) that can be formulated as follows.

The Foley solution, F: for all $\xi \in \Sigma, F(\xi) = \{(x, y) \in Z(\xi) \mid \forall i, j\colon (x_i, y_i)\, R_i(x_j, y_i)\}$.

Each individual compares the bundle of transferable goods that he receives with the bundles that the others receive. Existence of a Foley solution that it is also Pareto optimal is trivially solved with a single good: the equal split of M.

In the present framework this solution loses almost all its ethical content. It seems natural to extend the no-envy test to the whole bundle including non-transferable resources. Fleurbaey [4] proposes as an outcome solution the

allocation in which no agent envies the whole allocation of any other. We call this solution, the envy free or no-envy solution.
The no-envy solution, EF: for all $\xi \in \Sigma$, $EF(\xi) = \{(x, y) \in Z(\xi) | \forall i,j\colon (x_i, y_i)\, R_i(x_i, y_j)\}$.

Before we proceed any further, we want to stress that any proposal of solution for the problems in Σ should be evaluated from two different perspectives. First, it ought to be a "fair" solution, which means that the solution should satisfy certain axioms of fairness that presumably contain some plausible ethical intuitions. Think in the following requirement: "*if an individual owns a given set of internal resources, which are considered worse than another's by all members of the society, she should receive a larger amount of money*" (this is a property that we will call CAH(n) in the sequel). Second, we need a decisive solution, that is, a solution that gives an outcome for a reasonably large domain of problems that we may face.

To check the first requirement, we present the following set of properties originally proposed by Fleurbaey [4]. They are presented in three groups, from less to more demanding.
EWEP (*Equal Welfare for equal preferences*): $(\forall i,j)$ $[R_i = R_j \Rightarrow (x_i, y_i)\, I_i(x_j, y_j)]$.
EWEP*: $[\forall i,j\colon R_i = R_j] \Rightarrow [\forall i,j\colon (x_i, y_i)\, I_i(x_j, y_j)]$.
The first property means that no individual must be hurt because he has a bad endowment of personal characteristics. EWEP* is a weaker version of EWEP. It only requires that whenever *all agents* have the same preferences, then the level of welfare – in ordinal terms – should be the same for everybody.
EREH (*Equal Resources for Equal Handicaps*): $(\forall i,j)$ $[y_i = y_j \Rightarrow x_i = x_j]$. EREH*: $[\forall i,j\colon y_i = y_j] \Rightarrow [\forall i,j\colon x_i = x_j]$.
EREH requires that only differences in personal characteristics, *not in preferences*, should be compensated among individuals. Again, EREH* is weaker than EREH. Both EWEP and EREH derive from the principle of horizontal equity, that is, equal treatment of equals. This principle should take the following shape:

$$(\forall i,j)\ [R_i = R_j \text{ and } y_i = y_j \Rightarrow x_i = x_j]$$

Obviously, this condition is a weaker requirement than EWEP and EREH, and so it is also less useful.

Now we need some indication of how to carry out monetary compensations when individuals differ in their personal characteristics. Before, we need to add some notation. Let $\Pi\,(N)$ be the set of non-empty coalitions of N. For each i we define Ω^i as the collection of all elements of $\Pi\,(N)$ containing individual i, that is:

$$\Omega^i = \{S \in \Pi\,(N) | i \in S\}.$$

S^i will denote an element of Ω^i. Finally, let β be an integer between 1 and n.

CAH(β) (*Compensation for Acknowledged Handicaps*): $(\forall i,j)$ $[\exists S^i \in \Omega^i | \#S^i = \beta | \forall k \in S^i, \forall x\colon (x, y_j)\, P_k(x, y_i)] \Rightarrow [x_j < x_i]$.

This property says that, whenever a coalition of β individuals agrees that the individual characteristics of j are better than ones of i – provided that i is in that coalition –, a monetary compensation must be made in i's favour. This is not a single property, but a range of properties. In the case $\beta = 1$, property CAH(1) might be seriously questioned. It requires that if one individual feels that the bundle of any other is better than his, he should receive more income than the other. But this requirement is quite strong. Suppose a situation in which individual i thinks that j is better than himself and j thinks also that i is better than him (which is a very natural case). Each of them would ask for a monetary compensation with respect to the other, but no allocation could satisfy both demands. This is exactly the kind of situations that we want to avoid with the new solution that we propose. On the other hand, when $\beta = n$, CAH(n) is a very natural requirement. It permits to make compensations when there is an agreement on which direction they must be made. However there are other intermediate values of β that also can be interesting.

It is an easy exercise to check that the Pareto solution (P) fulfils none of the properties listed above. For the Foley solution (F) only EREH and EREH* hold. On the other hand, the no-envy solution satisfies all of them, including the strongest form of CAH(β), namely CAH(1).

But we were also interested in finding sufficient conditions for the existence of solutions. As we said before, both P and F always select a nonempty set of allocations. We study the no-envy solution EF and our new solution in the next section.

3. Results

Our first result in this section presents conditions for existence of the solution EF. This result is very closely related to that of Fleurbaey [9] but with two changes. First, condition (b) is slightly different, requiring only a weak preference. Second, the proof follows an alterantive path.

Proposition 1 Any of the following two conditions guarantees the existence of envy-free allocations.

a) $\forall i,j$: $y_i = y_j$.

b) $\forall i,j$: $R_i = R_j = R$ and $\forall i,j$: $(M/(n-1), y_i) R (0, y_j)$. (3)

Proof. Existence follows directly from (a) since the division x^* defined by $x_1^* = M/n$ is always an envy-free allocation.

To prove that (b) is also a sufficient condition, we use the fact that when all individuals have the same preferences, they rank the individual endowments in the same way. That is, for all i, j, k:

$$(x, y_j)\, R_k (x, y_i) \text{ or } (x, y_i)\, R_k (x, y_j) \text{ or both.} \quad (4)$$

Assume that individual n owns the best endowment of internal resources y_n, according to the unanimous preference relation R. We are going to construct an allocation that always exists and that it is also envy-free.

Let us define x_k^* as the least amount of money to compensate individual k for his handicap with respect to n.

$$\forall k\text{: } (x_k^*, y_k)\, I (0, y_n).$$

Obviously, $x_n^* = 0$. We will show that x_k^* does exist also for all $k \neq n$.

By condition (2), there is a vector of numbers z_k (with $z_k \geq 0$ for all k, by condition [4]) such that:

$$\forall k: (z_k, y_k) R(0, y_n).$$

Two cases are possible:

1) $(z_k, y_k) I(0, y_n)$. Then, $x_k^* = z_k$.
2) $(z_k, y_k) P(0, y_n)$. By [4], $\forall k$: $(0, y_n) R(0, y_k)$. Again, if $(0, y_n) I(0, y_k)$, then $x_k^* = 0$; otherwise $(0, y_n) P(0, y_k)$. Then we have, for all k: $(z_k, y_k) P(0, y_n)$ and $(0, y_n) P(0, y_k)$. Take the closed interval $[0, z_k]$ and let τ_k be in this interval, such that $\forall t \in [0, z_k]$, if $t > \tau_k$, $(t, y_k) P(0, y_n)$ and for $t < \tau_k$, $(0, y_n) P(t, y_k)$. Such a number τ_k there exists; if not we would get that $\forall t \in [0. z_k]$: $(t, y_k) P(0, y_n)$, a contradiction. By the continuity of the preference relation, it must be that $(\tau_k, y_k) I(0, y_n)$ and thus, $x_k^* = \tau_k$.

We claim the division $(x_1^*, x_2^*, \ldots, x_n^*)$ is envy-free. First we need to show that it is feasible.

By condition [3], $(M/(n-1), y_i) R(0, y_n)$ for all i. Now, $x_k^* \leq (M/(n-1)$. Suppose not. Then, given that $x_k^* > M/(n-1)$, the monotonicity property [1] would imply $(x_k^*, y_k) P(M/(n-1) y_k) R(0, y_n)$ for some k, a contradiction with the definition of x_k^*. Trivially, the sum of all x_k^* is a number lesser than or equal to M, and thus $(x^*, y) \in Z(\xi)$.

Finally, it is also envy-free because for all i: $(x_i^*, y_i) I(0, y_n)$ and by the transitivity of preferences, for all i,j: $(x_i^*, y_i) I(x_j^*, y_j)$. *Q.E.D.*

The interpretation of the second part of condition (b) is that either the personal characteristics must be close among individuals so that the amount $M/(n-1)$ suffices to compensate any differences or that in spite of individuals might be very different, the amount of money is large enough to set the differences aside.

Remark 1

Conditions used in the above proposition 1 are not necessary. The following example makes it clear. Let ξ be a distribution problem with utility profile given by $u_i = x_i + y_i$, for $i = 1, 2, 3$ and $y_i = i$, $M = 3$. Though $R_i = R_j$ for all i, j, condition [3] does not hold whenever $j = 3$, $i = 1$, and yet the division given by $x_1 = 2$, $x_2 = 1$, $x_3 = 0$ produces an envy-free allocation.

Now we can ask whether conditions of result 1 also guarantee the efficiency of the resulting allocation. Because if this is not so it would happen that to get an envy-free allocation we might be wasting resources. Fortunately, the next result shows that condition (b) is also sufficient to have efficiency (case (a) is trivial).

Proposition 2

If $\forall i,j$: $R_i = R_j = R$ and $(M/(n-1), y_i) R(0, y_j)$, then the set $\mathrm{EF}(\xi) \cap P(\xi)$ is nonempty. In fact, for each M, this set is a singleton.

Proof. First recall that $P(\xi) = \{(x, y) \in Z(\xi) | \Sigma x_i = M\}$. By proposition 1 we know that, under the above conditions, $\mathrm{EF}(\xi) \neq \varnothing$. Take any $(x^*, y) \in \mathrm{EF}(\xi)$ and suppose $\Sigma x_i^* < M$.

According to the common preference relation R, $(x_1^*, y_1) I (x_2^*, y_2) I \ldots I (x_n^*, y_n)$. That is, all individual bundles must lie on the same indifference curve of the common indifference map-which is "dense" due to the assumptions on R – in the space $X \times Y$. Now, by the continuity and monotonicity of R and the divisibility of M, there will be an allocation (x, y) in an upper indifference curve, $(x_1, y_1) I (x_2, y_2) I \ldots I (x_n, y_n)$, and such that $\Sigma\, x_i = M$. Then $(x, y) \in \mathrm{EF}(\xi) \cap P(\xi)$.

Moreover, the monotonicity of R implies that such allocation is unique. To see this, suppose we have $(x', y) \in \mathrm{EF}(\xi) \cap P(\xi)$ and $(x', y) \neq (x, y)$. Then, for some pair $\{i, j\}$, $x_i' > x_i$ and $x_j' < x_j$ and obviously $(x_i', y_i) P (x_j', y_j)$, a contradiction of $(x', y) \in \mathrm{EF}(\xi)$. *Q.E.D.*

Example 1

Assume again that the amounts of personal resources the represented by real numbers. Let ξ be a problem with a common utility profile given by the quasi-linear separable utility function $u(x, y) = kx + g(y)$ with $k \geq 0$. Let x^* be a division of money that produces an envy-free and Pareto-dominated allocation. It is possible to obtain an envy-free and Pareto-optimal allocation by equally dividing the amount $M - \Sigma\, x_i^* > 0$, among individuals. For, as x^* is an envy-free division of M, $kx_i^* + g(y_i) = kx_j^* + g(y_j)$ for all i, j. Now, it is also true that if we fix $\delta = (M - \Sigma\, x_i^*)/n$, then $k(x_i^* + \delta) + g(y_i) = k(x_j^* + \delta) + g(y_j)$ for all i, j.

If the utility function takes the form $u(x, y) = xy$ the division of the remaining money should be made *proportionally* to division x^*. This is true for all Cobb-Douglas-type utility functions. To see this, take $u(x, y) = Ax^\beta y^\gamma$ with $A, \beta, \gamma \geq 0$. As x^* is envy-free, then $A(x_i^*)^\beta (y_i)^\gamma = A(x_j^*)^\beta (y_j)^\gamma$ for all i, j or, in logarithms, $\beta \ln x_i^* + \gamma \ln y_i = \beta \ln x_j^* + \gamma \ln y_j$. Dividing $M - \Sigma\, x_i^*$ in proportion to what individuals got in x^*, the quantity $\{(M - \Sigma\, x_i^*)/\Sigma\, x_i^*\} x_i^*$ should be given to each individual. As the term in brackets is a constant, we can write this quantity as kx_i^* and check that $A(x_i^* + kx_i^*)^\beta (y_i)^\gamma = A(x_j^* + kx_j^*)^\beta (y_j)^\gamma$ for all i, j since we have $\beta \ln x_i^* + \beta \ln(1 + k) + \gamma \ln y_i = \beta \ln x_j^* + \beta \ln(1 + k) + \gamma \ln y_j$. That is,

$$\beta \ln x_i^* + \gamma \ln y_i = \beta \ln x_j^* + \gamma \ln y_j.$$

It is obvious that the conditions we need to guarantee the existence of envy-free allocations are rather restrictive. Actually, the EF solution will give an empty set as the outcome of many distribution problems in Σ. The extension of the EF solution that the present paper proposes, may be seen as an attempt to overcome this dilemma. It provides solutions for a larger set of problems for which the EF solution is empty and, at the same time, it retains most of the compensation properties of that solution EF. First we define an α–*equity* allocation. Recall that S^i denotes a set that contains individual i.

Definition: We say that a feasible allocation (x, y) is of α–equity ($\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq n$) for a given problem ξ, when there is no pair of individuals $\{i, j\}$ and a coalition $S^i \subset \Omega^i$ ($\#S^i = \alpha$) such that for all $k \in S^i$: $(x_j, y_j) P_k (x_i, y_i)$.

For any given distribution problem and for any α, let $E(\xi, \alpha)$ be the set of allocations that are of α–equity but not of $(\alpha - 1)$–equity. We make this remark because when an allocation (x, y) is of α–equity, it is also of $(\alpha + 1)$–equity, $(\alpha + 2)$–equity, and so on. Then we assign to every feasible allocation the minimum possible α.

We can check immediately that $E(\xi, 1)$ always contains the same allocations that are selected by the no-envy solution EF. Cenrtainly, if $\alpha = 1$ the only individual in the set S^i must be precisely individual i. Then $E(\xi, 1) = \mathrm{EF}(\xi)$. On the other hand, if $\alpha = n$, allocations in $E(\xi, n)$ are those for which there is no unanimous agreement on the fact that some individual is in a better position than another. We want to point out that the solution proposed by Van Parijs [12] is exactly the set $E(\xi, n)$.

Obviously α might take any value between 1 and n. In fact, we can attach to every feasible allocation a number representing that in such allocation we cannot find α individuals unanimously preferring the position of one individual to the position of another one. We can go forward and pick the set of efficient allocations where that number is minimum. This is the solution that we propose under the name of *Extended sympathy minimum envy*[2].

Extended sympathy minimum envy solution ($ESME(\xi)$): for each $\xi \in \Sigma$, $\mathrm{ESME}(\xi) = E(\xi, \alpha^*) \cap P(\xi)$ where $\alpha^* = \mathrm{Min}\{\alpha \mid E(\xi, \alpha) \neq \varnothing\}$.

It is immediate to check that when $\mathrm{EF}(\xi) \neq \varnothing$, $\mathrm{ESME}(\xi) = \mathrm{EF}(\xi) \cap P(\xi)$ and $\alpha^* = 1$. This idea is in the same vein as the *minimax set* developed by Kramer [7] in a classical social choice framework. His proposal was to choose those social alternatives where the maximum vote against them would be the minimum. This set of allocations coincided with the set of Condorcet winners whenever they exist. To illustrate our proposal we present an example.

Example 2

Let ξ be a distribution problem with $Y = \mathbb{R}$ (the real line). The utility profile is

$$u_1 = x_1 + y_1, u_2 = 2x_2 + y_2, u_3 = 3x_3 + y_3.$$

Personal resources take the values $y_1 = y_3 = 1$, $y_2 = 3$ and $M = 10$. Thus, the best endowment of capabilities is the one corresponding to individual 2, and individuals 1 and 3 have the same skills. The values of vector y_i summarise the following information.

1) $\forall i, j$ with $i = 1, 2, 3, j \neq 2$ and $\forall x$: $(x, y_2)\, P_i(x, y_j)$.

2) $\forall i$ and $\forall x$: $(x, y_1)\, I_i(x, y_3)$.

To simplify the graphical representation, we impose to any solution the condition that individuals equally handicapped (skilled) should be treated equally. Indeed, the next proposition proves that our solution always satisfies this property called EREH. Now, according to this principle, any admissible division of M satisfies the equality $x_1 = x_3$, and we can use this fact to represent a three-agent distribution problem in a two-dimensional space, as it appears in figure 1. The sets of Pareto-optimal points – for diffferent values of M – satisfying the above principle are

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[2] We are specially grateful to the referee for suggesting us this name.

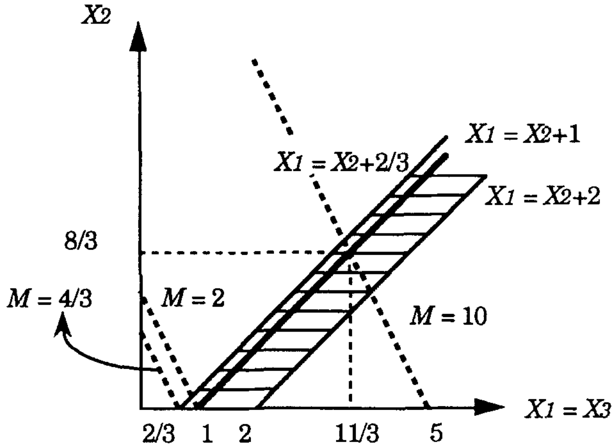

Fig. 1

represented by the dotted lines, where the maximal possible value of x_1 (and x_3) is exactly half the maximal value for x_2. We check that $E(\xi, 3) = \{(x, y) \in Z(\xi) | x_2 + 2/3 \leq x_1 = x_3 \leq x_2 + 2\}$. This set is represented by the shaded area between the lines $x_1 = x_2 + 2/3$ and $x_1 = x_2 + 2$. Its intersection with $P(\xi)$ is the segment between (32/9, 26/9) and (4, 2). Second, we see that $E(\xi, 2) = \{(x, y) \in Z(\xi) | x_2 + 1 = x_1 = x_3\}$. This set is represented by a bold line. Its intersection with $P(\xi)$ is the unique point (11/3, 8/3). Finally, $E(\xi, 1) = \emptyset$: for individual 1 it must be that $x_1 \geq x_2 + 2$ and for individual 2, $x_2 + 1 \geq x_1$. But these two inequalities are incompatible.

Then, the only division of M selected by the solution ESME is (11/3, 8/3, 11/3) and $\alpha^* = 2$.

To conclude, we have in this example that $E(\xi, 1)$ is always empty and that $E(\xi, 2)$ and $E(\xi, 3)$ are not empty whenever $M \geq 2$ and $M \geq 4/3$, respectively. So, for $M \geq 4/3$ ESME(ξ) is also non-empty. By the way, we observe that $E(\xi, 1) \subseteq E(\xi, 2) \subseteq E(\xi, 3)$.

The next result establishes that the solution we propose gives acceptable outcomes when considering the compensation properties of section 2.

Proposition 3

Let $\alpha \in \mathbb{N}$ such that $\alpha > 1$. Any allocation in the set $E(\xi, \alpha)$ satisfies EWEP* and EREH. It also satisfies CAH(β) for all $\beta \geq \alpha$.

Proof. (i) It does not satisfy EWEP.
Consider a problem with three agents. Individuals 1 and 2 share the same preferences represented by $u_i = x_i + y_i$ for $i = 1, 2$ being $u_3 = 2x_3 + y_3$ the utility function for individual 3. Let $y_1 = y_3 = 1$, $y_2 = 3$ and $M = 10$. Allocations belonging to the set $E(\xi, 3)$ produce divisions of money such that $x_2 + 1 \leq x_1 = x_3 \leq x_2 + 2$. One of such divisions is (11/3, 8/3, 11/3). Its associated feasible allocation gives $u_1 = 14/3 \neq 17/3 = u_2$.

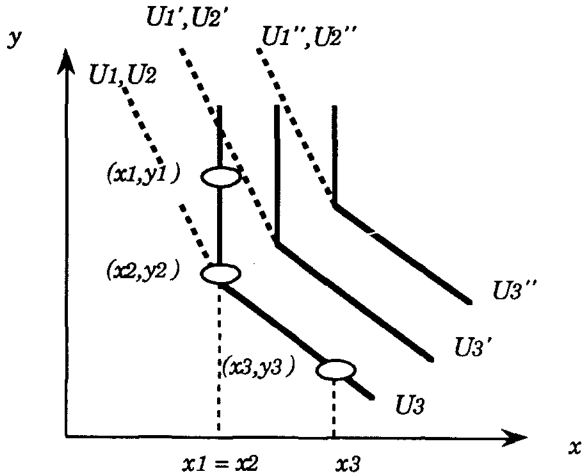

Fig. 2

(ii) It satisfies EWEP*
Suppose not. Then, there are two agents i, j, such that $(x_i, y_i)\, P_i(x_j, y_j)$. This implies that, for all h: $(x_i, y_i)\, P_h(x_j, y_j)$. Thus this allocation does not belong to $E(\xi, \alpha)$ for any α.
(iii) It does also satisfy property EREH
Suppose not. Then there are individuals i, j for which, though $y_i = y_j$, it holds that $x_i > x_j$. By the monotonicity of preferences, $(x_i, y_i)\, P_h(x_j, y_j)$ for all h. This mean that the starting allocation does not belong to $E(\xi, \alpha)$ for any α, which is a contradiction.
(iv) Property CAH(β) is satisfied for all $\beta \geq \alpha$.
First, we use an example to illustrate that some allocations of $E(\xi, \alpha)$ do not satisfy CAH(1) for $\alpha > 1$. It is depicted in figure 2. There are three agents. The proposed allocation belongs to $E(\xi, 3)$. Besides, $(x_1, y_1)\, P_2(x_2, y_2)$. But it holds that $x_1 = x_2$.

Now we prove that CAH(β) is satisfied for all $\beta \geq \alpha$. Let us assume that (x, y) belongs to $E(\xi, \alpha)$ for any $\alpha > 1$ and that the antecedent of property CAH(β) also holds for $\beta = \alpha$. That is, there are α individuals (indexed by k) with the opinion that $\forall x$: $(x, y_j)\, P_k(x, y_i)$. If $x_j \geq x_i$, then we have for those individuals: $(x_j, y_j)\, P_k(x_i, y_i)$, a contradiction. Now, it is immediate to check that whenever CAH(β) is satisfied for $\beta = \alpha$ it is also satisfied for all $\beta > \alpha$. *Q.E.D.*

As a corollary of this proposition we obtain that ESME(ξ) satisfies EWEP*, EREH and CAH(α*). It does not satisfy EWEP, a property that exhibits a non controversial ethical justification. We believe that this is the price to be paid for the existence of an outcome solution.

Example 2 showed that as long as we increase α the sets $E(\xi, \alpha)$ become enlarged. Nevertheless, in one case, these sets remain the same as α increases.

Proposition 4

Let $R_i = R_j = R$ for all i, j and $\alpha_1, \alpha_2 \in \mathbb{N}$ with $1 \leq \alpha_i \leq n$ for $i = 1, 2$. Then $E(\xi, \alpha_1) = E(\xi, \alpha_2)$.

Proof. Suppose, without loss of generality that $\alpha_1 \leq \alpha_2$. Here, we know that $E(\xi, \alpha_1) \subseteq E(\xi, \alpha_2)$. To show $E(\xi, \alpha_2) \subseteq E(\xi, \alpha_1)$. Let $(x, y) \in E(\xi, \alpha_2)$. Assume that $(x, y) \notin E(\xi, \alpha_1)$. Then, there are at least two agents $\{i, j)$ such that $\forall k \in S^i$, where $\# S^i = \alpha_1$, $(x_i, y_i)\, P_k(x_j, y_j)$. As all individuals share the same preference relation, this must be true not only for the members of the set S^i, but for the whole society, and therefore for any particular set of cardinality equal to α_2. Then, $(x, y) \notin E(\xi, \alpha_2)$, a contradiction. Finally, for the case $E(\xi, \alpha_2) = \emptyset$, it holds that $E(\xi, \alpha_1) = \emptyset$ and then $E(\xi, \alpha_1) = E(\xi, \alpha_2)$. *Q.E.D.*

This last exercise, presented as proposition 4, together with the existence theorem for the no-envy solution (proposition 1), shows a possible path to follow in order to generalize proposition 1. We consider as the relevant cases those in which *individuals may differ in their endowment of internal resources and preferences as well.* Next result presents sufficient conditions for the existence of the sets $E(\xi, \alpha)$.

Proposition 5

The following condition guarantees that $E(\xi, \alpha) \neq \emptyset$ for any given α:
For all x such that $(x, y) \in P(\xi)$, there exists i with $x_i > 0$, for whom there are at least $n - \alpha + 1$ individuals (indexed by k) that feel that $\forall j$: $(x_i, y_i)\, R_k(x_j, y_j)$. [5]

Proof. We use for this proof the Knaster-Kuratowski-Mazurkiewicz theorem as presented by Ichiishi [6]:

K-K-M theorem. Let $(z_i)_{i \in F}$ be an affinely independent subset of $\mathbb{R}^n$ (a finite subset $(z_i)_{i \in F}$ of $\mathbb{R}^n$ is called affinely independent if $\sum_{i \in F} r_i z_i = 0$, $r_i \in \mathbb{R}$ and $\sum_{i \in F} r_i = 0$ implies $r_i = 0$ for each $i \in F$). Let $\{C^i\}_{i \in F}$ be a family of closed subsets of the simplex $co(z_i)_{i \in F}$ such that for each $F' \subset F$ it follows that $co(z_i)_{i \in F'} \subset \cup_{i \in F'} C^i$. Then, $\cap_{i \in F} C^i \neq \emptyset$, where $co(z_i)_{i \in F}$ denotes the convex hull of $(z_i)_{i \in F}$.

We can go now to the proof of the result. Let ξ be a fair division problem with a set N of individuals with $\# N = n$. First we define for all i the set E^i in the following way:

$$E^i = \{x \in \mathbb{R}^n_+ \,|\, \Sigma x_i = M \,|\, \forall j \neq i, \exists S \subset N\colon \# S \geq n - \alpha + 1\colon \forall k \in S, (x_i, y_i)\, R_k(x_j, y_j)\}.$$

To see that these sets are closed is straightforward. See also that as α goes from 1 to n, $n - \alpha + 1$ goes from n to 1.

Second, consider the n vectors $(z_i)_{i \in N}$ defined as follows $z_1 = (M, 0, \ldots, 0)$, $z_2 = (0, M, 0, \ldots, 0), \ldots, z_n = (0, \ldots, 0, M)$. We want to prove that for all $(w, y) \in P(\xi)$, w belongs to $\text{co}(z_i)_{i \in N}$ and that for all $v \in \text{co}(z_i)_{i \in N}$, (v, y) also belongs to $P(\xi)$.

Let $(w, y) \in P(\xi)$, then $\sum_{i \in N} w_i = M$. If we fix $\beta_i = w_i / M$, then $\beta_i \geq 0$ and $\sum_{i \in N} \beta_i = 1$. Given that $w = \sum_{i \in N} \beta_i z_i$, then $w \in \text{co}(z_i)_{i \in N}$.
Now let $v \in \text{co}(z_i)_{i \in N}$; then there are n non-negative numbers β_i such that $\sum_{i \in N} \beta_i = 1$ and for which $v = \sum_{i \in N} \beta_i z_i = (\beta_1 M, \beta_2 M, \ldots, \beta_n M)$. Adding up all components of v, we have: $\sum_{i \in N} v_i = M \sum_{i \in N} \beta_i = M$. Then, $(v, y) \in P(\xi)$.

Third, to show that for all $Q \subset N$, it holds true that $\text{co}(z_i)_{i \in Q} \subset \cup_{i \in Q} E^i$. If $Q \neq N$, then $\text{co}(z_i)_{i \in Q} = \{x \in \mathbb{R}^n_+ \,|\, \Sigma x_i = M \,|\, \forall i \notin Q, x_i = 0\}$.

A) In the case $\# Q = 1$, if $x^* \in \text{co}(z_i)_{i \in Q}$, we know that x^* is a vector with M placed in the position corresponding to the unique member of Q and all the rest of the components are zeros. Call this agent h. Then $x_h^* = M$ and $x_j^* = 0$ for all $j \neq h$. Condition [5] guarantees that $x^* \in E^h$ and, therefore, $x^* \in \cup_{i \in Q} E^i$.

B) When $1 < \#Q < n$, we know that if $x^* \in \text{co}(z_i)_{i \in Q}$, for $n - \#Q$ individuals $x_j^* = 0$. For the rest of them, $\#Q, x_i^* > 0$. As x^* is such that $\Sigma\, x_i^* = M$, there must exist an individual h for whom condition [5] guarantees that there are at least $n - \alpha + 1$ individuals (indexed generically by k) whose opinion is that $\forall j$: $(x_i^*, y_i)\, R_k (x_j^*, y_j)$. This means $x^* \in E^h$ and therefore, $x^* \in \cup_{i \in Q} E^i$.

C) If $Q = N$ and $x^* \in \text{co}(z_i)_{i \in N}$, then again condition [5] implies $x^* \in \cup_{i \in N} E^i$. Applying the K-K-M theorem, it holds that the intersection of the n sets E^i is not empty. This implies that there exists a feasible allocation in the set $E(\xi, \alpha)$. *Q.E.D.*

The conditions of proposition 5 are also enough to guarantee that the allocations obtained are efficient. This is obvious since we dealt only with Pareto efficient divisions of M in the proof above.

Besides, we can lean on proposition 5 to establish a very weak condition that it is sufficient to guarantee that ESME(ξ) is not empty. We present it as a corollary.

Corollary

Suppose that for all x such that $(x, y) \in P(\xi)$, there exists i with $x_i > 0$, for whom at least *one* individual k thinks that $\forall j$: $(x_i, y_i)\, \mathbb{R}_k (x_j, y_j)$. Then ESME($\xi$) $\neq \emptyset$.

Proof. By proposition 5, this conidition implies that $E(\xi, n) \cap P(\xi) \neq \emptyset$ and then ESME(ξ) $\neq \emptyset$ with $\alpha^* \leq n$.

The condition of proposition 5 may have the following interpretation. Instead of requiring that preferences of *all* individuals must be the same over *all* allocations (as condition (b) of proposition 1), it demands that the preferences of *a group* of individuals – whose number depends again on α – must agree over *a group* of allocations, namely those Pareto efficient allocations where at least two individuals posses a strictly positive amount of money.

It is easy to see that the greater is α, the weaker is the sufficient condition, since the number of people that must satisfy those conditions decreases.

4. Final remarks

To sum up, we have discussed a new solution concept with the objective of correcting some flaws of the no-envy solution. With this solution every individual had the right to veto all feasible allocations. What we do here is precisely to increase the requirements needed to reject an allocation. That is, only coalitions of size α^* can veto a given allocation. This means that all the allocations that are not selected by the solution are rejected by, at least, α^* individuals.

Our solution will select a set of non-empty allocations for a large part of the domain Σ. What happens when ESME(ξ) $= \emptyset$ and then $E(\xi, n) = \emptyset$? In this case, for any feasible allocation there is at least one couple $\{i,j\}$ such that, $\forall k$: $(x_j, y_j)\, P_k (x_i, y_i)$. That is, everybody thinks that the internal endowments of i and j are so different that it is impossible to compensate individual i with all the existing amount of money M. If we want to propose some allocation, including in this case, we can do $x_j = 0$ for those j's. Then we construct a new problem ξ' with the rest of the individuals provided there are some individuals left. The next step will be to repeat the preceding technique with this reduced problem.

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