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Stochastic growth when utility depends on both consumption and the stock level*

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Summary. This paper examines the dynamic behavior of optimal consumption and investment policies in the aggregate stochastic growth model when utility depends on both consumption and the stock level. Such models arise in the study of renewable resources, monetary growth, and growth with public capital. The paper shows that there is a global convergence of optimal policies to a unique stationary distribution if (a) there is sufficient complementarity in the model, or (b) if there is sufficient randomness in production. Two examples illustrate the possibility of multiple stationary distributions. In one, multiple stochastic steady states exist for a generic class of production and utility functions.

1. Introduction and preliminaries

The optimal growth model when utility depends on both consumption and the stock level has proved useful for analyzing a variety of important economic problems related to renewable resources, monetary growth, and growth with public capital. This paper investigates the long run behavior of optimal consumption and investment processes for such problems when production is stochastic. The results extend Nyarko and Olson [1991] and provide conditions on technology and preferences that are sufficient to guarantee the global convergence of optimal consumption and investment to a unique stationary distribution. This question is non-trivial since Kurz [1968] has shown that multiple optimal steady states can exist when utility depends on both consumption and the stock.[1]

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[1] Majumdar [1982] and Majumdar and Mitra [1991] consider a deterministic version of the problem, while Brock and Mirman [1972] and Mirman and Zilcha [1975, 1977] examine the stochastic case where utility depends solely on consumption. Mendelssohn and Sobel [1980] examine the problem of convergence in a model similar to ours, but their results follow from assumptions imposed directly on the Markov transition kernel that governs the evolution of optimal resource stocks. In somewhat related work, Mirman and Spulber [1984] examine a renewable resource model that assumes uniqueness of the

We show that there exists a unique stationary distribution when the utility function exhibits (i) complementarity between consumption, investment, and output, and (ii) a form of balanced growth complementarity. Examples are given to illustrate the existence of multiple stochastic steady states when the sufficient conditions are violated. We also show that a model with multiple optimal stationary distributions can be transformed into a globally convergent model simply by introducing sufficient randomness into production. This suggests that highly variable economies may be less subject to dependence on initial conditions than economies with small productivity shocks. Proofs are relegated to the end of the paper.

The model used in this paper is essentially that developed in Nyarko and Olson [1991]. A brief description is given here, and the reader is referred to that paper for more details. The resource stock, consumption, and investment are denoted by y_t, c_t, and x_t, respectively, where $x_t = y_t - c_t$. Growth in the resource stock is governed by the production function, $y_{t+1} = f(x_t, r_{t+1})$, where r_t is an i.i.d. stochastic process with (common) probability measure γ. The utility of an agent depends on both the resource stock and consumption, and is denoted by $U(c_t, y_t)$. Given an initial stock $y_0 > 0$, the agent maximizes the discounted sum of utility over time, where $\delta \in (0, 1)$ is the discount factor. The production and utility functions are assumed to satisfy the following restrictions throughout the paper.

T.1. For all r, $f(x, r)$ is strictly increasing in x.
T.2. f is concave in x.
T.3. For all r, $f(0, r) = 0$ while $f(x, r) > 0$ if $x > 0$.
T.4. The first and second derivatives of $f(x, r)$ in x exist and are continuous in (x, r).
T.5. There exists a $\bar{y}$ such that $f(x, r) < x$ a.s. for all $x \geq \bar{y}$.
T.6. If f is stochastic (i.e., the distribution of r is nondegenerate) then there exists no $x > 0$ and $y \geq 0$ such that $\gamma(\{r \mid f(x, r) = y\}) = 1$.
T.7. $y_0 \in (0, \bar{y}]$.
U.1. $U(c, y)$ is nondecreasing in y.
U.2. $U(c, y)$ is concave in (c, y).
U.3. $U(c, y)$ is twice continuously differentiable.
U.4. $U_{cc} + U_{cy} < 0$.
U.5. $U_{cy} \geq 0$.
U.6. $U(c, y)$ is strictly increasing in y and $f(x, r)$ is strictly concave in x for each r, or
U.6′. $U(c, y)$ is strictly concave.

Standard dynamic programming arguments imply the existence of stationary optimal consumption and investment policy functions $c_t = C^*(y_t)$ and $X^*(y_t) = y_t - C^*(y_t)$. Furthermore, the following functional equation holds:

$$V(y) = \operatorname*{Max}_{0 \leq c \leq y} U(c, y) + \delta \int V(f(y - c, r))\gamma(dr).$$

limiting distribution. In contrast, the analysis in this paper emphasizes assumptions on primitives of the model. For a further discussion of the related literature the reader is referred to Nyarko and Olson [1991].

The global convergence of optimal policies derived in this paper contrasts with the cyclic or chaotic behavior studied in Benhabib and Nishimura [1985], Majumdar and Mitra [1991], and Nishimura and Yano [1991]. These papers all utilize a deterministic framework in which our condition U.4 is violated.

Given the assumptions above, it is well known that $V(y)$ is nondecreasing, concave,[2] and that the optimal policy functions are uniquely defined and continuous in y. Under U.4 and U.5 the optimal investment and consumption policy functions are strictly increasing and nondecreasing, respectively (Nyarko and Olson [1991, Theorems 2.3 and 2.4]).

Let y_t^*, c_t^* and x_t^* denote the optimal output, consumption, and investment, and let U_c and U_y denote derivatives of U with respect to c and y. It is assumed that the optimal consumption and investment policies are interior so that the following Euler equation holds:[3]

$$U_c(c_t^*, y_t^*) = \delta E\{[U_c(c_{t+1}^*, y_{t+1}^*) + U_y(c_{t+1}^*, y_{t+1}^*)]f'(x_t^*, r_{t+1})\}.$$

It is important to note that utility is not necessarily monotone in c as is typical in renewable resource models where higher consumption levels require more effort. Even so, optimal consumption always occurs in a region of U that is increasing in c, i.e., optimal consumption is less than or equal to that which would be chosen by a myopic decision-maker.

2. Uniqueness of the limiting distribution under complementarity conditions

The convergence of optimal processes to a stationary distribution is characterized in Nyarko and Olson [1991, Theorem 2.5]; however, questions about the number of stationary distributions and whether optimal processes converge locally or globally are not addressed in that paper. In this section we focus on assumptions on the utility function that are sufficient to imply the global convergence of optimal processes to a unique stationary distribution from all inital resource stocks. Assume:

U.7. There exists a $\tilde{y} > 0$ such that for all $y \geq \tilde{y}$, $U_c(c, y) = 0$ implies $U_y > 0$, where $0 < c \leq y$.

U.8. For all $y > 0$, $0 < c < y$ and $\lambda > 1$ such that $U_c(c, y) > 0$ and $U_c(\lambda c, \lambda y) > 0$,

$$\frac{U_y(c, y)}{U_c(c, y)} \geq \frac{U_y(\lambda c, \lambda y)}{U_c(\lambda c, \lambda y)}.$$

T.8. $f(x, r)$ is strictly concave in x.

Assumption U.7 is needed to rule out the possibility of U attaining a maximum at c' and, at the same time, being independent of y at c'. U.8 can be interpreted as a complementarity condition on the decision maker's preferences as consumption and output increase along a balanced growth path between c and y. $-U_y/U_c$ is the slope of indifference curves of U. Hence, U.8 implies that indifference curves for U have decreasing slopes as the stock level and consumption increase along a ray through the origin in (y, c) space. Assumption U.8 is satisfied if utility is stock

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[2] If $U(c, y)$ is strictly concave, then this result can be strengthened to show that $V(y)$ is also strictly concave.

[3] This is true if the usual Inada conditions are imposed on U. It also imposes the implicit requirement that U is not everywhere decreasing in c.

independent so that $U_y = 0$ for all (c, y). In addition, it does not rule out the possibility that $U_c = 0$ for some (c, y).[4] One class of utility functions that satisfies all of our assumptions including U.7 and U.8 is the class $U(c, y) = c^\alpha y^\beta$, where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta < 1$. All our assumptions are also satisfied if utility is independent of the stock.

The following definitions are used in characterizing the convergence of optimal processes. Let $f_m(x) = \min_r f(x, r)$, $f_M(x) = \max_r f(x, r)$, $X_m(x) = X^*(f_m(x))$, $X_M(x) = X^*(f_M(x))$, $z_m = \min\{x > 0 | X_m(x) = x\}$, $z_M = \max\{x > 0 | X_M(x) = x\}$, $x_m = \max\{x > 0 | X_m(x) = x\}$, $x_M = \min\{x > 0 | X_M(x) = x\}$. In addition, let $y_m = f_m(x_m)$ and $y_M = f_M(x_M)$. A unique stationary distribution exists if $x_m \leq x_M$. If $x_m > x_M$, there are at least two stationary distributions and there may exist many stationary distributions in the interval (x_m, x_M) (see Nyarko and Olson [1991] for a further discussion).

To avoid troublesome anomalies we assume:

T.9. There exists a $\theta > 0$ such that $X_m(x) > x$ for all $x \in (0, \theta)$.

T.9 prevents the optimal stock process from converging over time to zero even if the worst state occurs at each date. Sufficient conditions on the primitives of the model for T.9 are:

T.9(a). $\lim_{x \to 0} f_x(x, r) = \infty$ for all r (Inada condition on f).
T.9(b). Either r is drawn from a finite set, or $f(x, r)$ is ordered in r and the minimum shock has positive probability.

Define $r^t = (r_1, \ldots, r_t)$ and for each t and r^t define $X^t(x_0, r^t) = X^*(f(..X^*(f(X^*(f(x_0, r_1)), r_2)), \ldots, r_t))$. Let F_0 be any distribution function for x_0. F_t is defined to be the distribution function for x_t generated by the transition equation $X^t(x_0, r^t)$. We now state the main result of this section.

Theorem 1. *If U.7–U.8 and T.8–T.9 hold in addition to the assumptions of Section 1, then $F_t(x)$ converges uniformly in x to a unique stationary distribution, $F_\infty(x)$, independently of the initial stock y_0. In addition, the support of F_∞ is a subset of $[x_m, x_M]$.*

It is known that multiple steady states may exist if the production function is not concave (see Majumdar et al. [1989]). The two examples below show that multiple optimal steady states may exist if all our assumptions (including concavity of production) are satisfied except U.7 and U.8.

Example 1 – Violation of U.7 leads to multiple optimal stationary distributions

Assume the utility function is independent of y, strictly concave in c, and let $U(c)$ reach a maximum at c' with $U_c > 0$ for $c < c'$ and $U_c < 0$ for $c > c'$. Further, let $g(x)$ be any function that satisfies assumptions T.1–T.5 and T.9(a) such that $\lim_{x \to \infty} g'(x) = 0$

[4] 4.8 holds for significant classes of utility functions including the class of all homothetic utility functions. A larger class is the class of all utility functions homothetic to a point in the region $\Omega = \{(c, y) \in \Re^2 | c \leq y, c \leq 0\}$. This is a subset of the class of quasihomothetic or affine-homothetic reward functions (see Blackorby, Boyce, and Russell [1978]).

and for some $x' > c'$, $g(x') > x' + c'$. Fix any $k > \text{Max}\,\{x'/g(x'-c'), 1\}$. Let $f(x, r)$ be any stochastic production function obeying T.1–T.9(a, b) with $f_m(x) = \min_r f(x, r) = kg(x)$. This defines a large class of production functions. For every production function in this class, Nyarko and Olson [1990] prove that there are multiple optimal stationary distributions, provided the discount rate is sufficiently small.[5]

Example 2 – Violation of U.8 leads to multiple optimal steady states

Define $U(c, y) = 16y - 1/2(c-24)^2$. Then $U_c = 24 - c$, $U_y = 16$, $U_{yy} = U_{cy} = 0$, and $U_{cc} = -1$. U is nondecreasing in y and concave, U.1–U.7 are satisfied, but $U_y(\lambda c, \lambda y)/U_c(\lambda c, \lambda y) = 16/(24 - \lambda c)$ so U.8 is violated. Assume that $f(x) = 10x^{1/2}$. One can check that when $\delta = 0.10$, the Euler equation is satisfied when (y_t, x_t, c_t) are any one of the three stationary triples $(y, x, c) = (10.4695, 1.0961, 9.3734)$, $(29.2739, 8.5696, 20.7043)$, or $(65.2567, 42.5843, 22.6723)$. Maintaining any of these three triples as a steady state is feasible and the transversality condition holds, so each is an optimal steady state.[6]

3. Sufficient variability in production implies the existence of a unique stationary distribution

In this section we show that even if the sufficient conditions of Section 2 fail, a model with a unique stationary distribution can be obtained through a sufficient "stretching out" of the randomness in the production function.[7]

Let $\{f^k(x, r)\}_{k=0}^{\infty}$ be a collection production functions. Assume:

T.10. For each k, $f^k(x, r)$ satisfies assumptions T.1–T.5 and T.9.
T.11. For each $x > 0$, $\lim_{k\to\infty} \text{Max}_r\, f^k(x, r) = \infty$.
T.12. For each $x > 0$, $\lim_{k\to\infty} E\partial f^k(x, r)/\partial x = \infty$.
T.13. For each $x > 0$ and $k \geq 0$, $f_m^k(x) \leq f_m^0(x)$.

Assumption T.11 implies that the production function becomes arbitrarily large in the best state while T.13 ensures that in the worst state the production function is uniformly bounded above. This formalizes the notion of "stretching out" the production function. Assumption T.12 is an additional assumption that requires that the expected marginal product becomes arbitrarily large. To further illustrate the need for T.13 consider production functions defined by $f^k(x, r) = kf(x, r)$ where $f(x, r)$ obeys T.1–T.5. This class of production functions involves a simple change of units so the limiting behavior of optimal policies should not vary with k. Such classes are ruled out by T.13.

Theorem 2. *Let $\{f^k(x, r)\}_{k=0}^{\infty}$ be a collection production functions satisfying T.10–T.13. Suppose that the utility function obeys all assumptions of section 1 (but not necessarily*

[5] This derivation is available from the authors on request.
[6] The transversality condition is $\lim_{t\to\infty} \delta^t U_c(c_t^*, y_t^*)x_t^* = 0$. In Example 2, the transversality condition is satisfied since $U_c(c_t^*, y_t^*)$ and x_t^* are both constant and positive at the three steady states.
[7] This idea is similar to the notion of "very stochastic" employed in Majumdar, Mitra, and Nyarko [1989].

the assumptions of section 2). Then for all k sufficiently large the model with production function $f^k(x,r)$ has a unique non-trivial stationary distribution and the conclusions of Theorem 1 hold.

Theorem 2 implies that the convergence properties of models with monotone transition functions depend substantially on the degree of randomness in the model. It shows that enough variability in production forces the long run behavior of optimal processes to be independent of initial conditions. This suggests that the existence of multiple stationary distributions depends on whether the economy is subject to technology shocks that have large or small variation. The resolution of this question is an issue for further theoretical and empirical investigation.

4. Proofs

The proofs of both theorems are accomplished through a series of subsidiary lemmas. Proofs of these lemmas are given in Nyarko and Olson [1990] and can be obtained from the authors on request.

Proof of Theorem 1. Assume $x_m > x_M$. Define $c_m = C^*(y_m)$ and $c_M = C^*(y_M)$. From the definition of X_m and the fact that x_m is a fixed point of X_m, it follows that $X^*(y_m) = X^*(f_m(x_m)) = X_m(x_m) = x_m$. Hence, x_m and c_m are optimal investment and consumption from y_m. Similarly, x_M and c_M are optimal investment and consumption from y_M.

Lemma 1.1. *If $x_m > x_M$, then for all r, $y_M \geq f(x_M, r)$, $y_m \leq f(x_m, r)$, $c_M \geq C^*(f(x_M, r))$, $c_m \leq C^*(f(x_m, r))$, and $y_m > y_M$ and $c_m \geq c_M$.*

Lemma 1.2. *Under U.7, $U_c(c_m, y_m) > 0$ and $U_c(c_M, y_M) > 0$.*

Lemma 1.3. *If $x_m > x_M$ and U.7 holds, then*

$$\frac{U_y(c_M, y_M)}{U_c(c_M, y_M)} < \frac{U_y(c_m, y_m)}{U_c(c_m, y_m)}.$$

Lemma 1.4. *$x_m > x_M$ implies $c_m/y_m \leq c_M/y_M$.*

Lemma 1.5. *If $x_m > x_M$ and U.8 holds, then*

$$\frac{U_y(c_M, y_M)}{U_c(c_M, y_M)} \geq \frac{U_y(c_m, y_m)}{U_c(c_m, y_m)}.$$

The proof of Theorem 1 follows from the fact that Lemmas 1.3 and 1.5 contradict each other. Thus, it cannot be that $x_m > x_M$. //

Proof of Theorem 2.

Lemma 2.1. *Fix any production function $f(x, r)$ that satisfies T.10. If $Ef'(x_M, r) > 1/\delta$ then $U(c, y)$ attains its global maximum at a unique $\bar{c} > 0$ and $C^*(f_m(x_M)) = C^*(f_M(x_M)) = \bar{c}$.*

Lemma 2.2. *Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a class of production functions obeying T.10–T.13. Suppose further that for all k, the model with the production function $f^k(x,r)$ has more* *than one non-trivial stationary distribution. Then, a)* $E\partial f^k(x_M^k, r)/\partial x > 1/\delta$ *for all* k *sufficiently large, where* x_M^k *is defined in a similar manner as* x_M *for the model with production function* $f^k(x, r)$*; and b)* $\lim_{k\to\infty} x_M^k = 0$.

Now let $\{f^k(x, r)\}_{k=0}^{\infty}$ be a class of production functions satisfying the hypotheses of Lemma 2.2. From Lemmas 2.1, 2.2(a) and T.13 it follows that for k sufficiently large, $0 < \bar{c} = C^*(f_m^k(x_M^k)) \leq f_m^k(x_M^k) \leq f_m^0(x_M^k)$. Taking limits as $k \to \infty$ and using Lemma 2.2(b) then implies that $0 < \bar{c} \leq f_m^0(0) = 0$, which is a contradiction. //

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