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Localization in ν-Dimensional Incommensurate Structures

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Abstract. We exhibit a class of quasi-periodic unbounded potential in the ν-dimensional discrete Schrödinger equation, for which the spectrum is only pure point, with exponentially localized states and a dense set of eigenvalues in $\mathbb{R}$.

Introduction

In a recent paper Fishman et al. [7] gave a solution of the Schrödinger equation

$$\Psi(n+1)+\Psi(n-1)+\lambda\tan\pi(x-n\omega)\Psi(n)=E\Psi(n), \tag{1}$$

where ω is an irrational number, $x\in\mathbb{R}$ and $\lambda>0$. Actually, provided ω satisfies a diophantine condition, they gave an explicit expression of the eigenfunctions which turns out to decrease exponentially, and an implicit expression for the corresponding eigenvalues, leading to a dense set in the spectrum.

This result was interesting both because the solution was complete and also because there is no continuous part in the spectrum. Examples of discrete non-self-adjoint operators with only pure point spectrum were already known by Sarnak [11]. However in the self-adjoint case, with a bounded quasi-periodic potential we expect in general that aside from the pure point part there is a singular continuous spectrum. This seems to be the case for the almost Mathieu equation:

$$\Psi(n+1)+\Psi(n-1)+2\lambda\cos2\pi(x-n\omega)\Psi(n)=E\Psi(n), \tag{2}$$

where it has been proved by Bellissard et al. [2] that if λ is big enough, and ω satisfies a diophantine condition

$$\forall n,m\in\mathbb{Z},\qquad n\neq0,\qquad |n\omega+m|\geqq\frac{\gamma}{|n|^{\sigma}},\qquad \sigma>1, \tag{3}$$

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most of the spectrum is pure point. However if x is chosen very close to the set $\{n\omega+m; n, m\in\mathbb{Z}\}$, namely if there is an infinite sequence q_n, p_n of integers such that

$$|x-q_n\omega-p_n|<\frac{1}{n^{q_n}}, \qquad n=1,2,\ldots, \tag{4}$$

a tunneling effect may occur which may give extended states.

The same phenomena is observed if ω is a Liouville number such that an infinite sequence exists for which

$$|q_n\omega-p_n|<\frac{1}{|n|^{q_n}}, \qquad n=1,2,\ldots, \tag{5}$$

In this situation Gordon [8] and Avron and Simon [1] proved that there is no eigenvalue, all the states are extended, and the spectrum is singular continuous.

It is therefore interesting to find a class of potentials for which only pure point spectrum arises.

The existence of such potentials was recently proved via the inverse spectral method and solving the small divisor problem by Craig [5].

In a sense the inverse method allows an easier control of resonances at large distance, but the price to pay is that the potential itself is not explicitly known, even though the method is constructive. In the same way Pöschel [9] using the Craig method is able to construct a large class of limit periodic potentials with pure point spectrum, and dense family of eigenvalues in a set which can be essentially a Cantor set of zero Lebesgue measure, or a full interval.

In our work we consider the direct problem. We construct a class $\mathscr{P}$ of quasi-periodic functions having singularities, containing the Fishman et al. and the Sarnak examples for sufficiently large coupling, for which the Schrödinger operator:

$$\varepsilon\sum_{|e|=1}\Psi(n-e)+V(x-\omega n)\Psi(n)=E\Psi(n), \qquad n\in\mathbb{Z}^\nu \tag{6}$$

has only pure point spectrum at small ε with exponentially localized states, and a dense set of eigenvalues in the real line. In Eq. (6), V is a function in the class $\mathscr{P}$ and is periodic of period 1, x is a real number, and if $n=(n_1,\ldots,n_\nu)\in\mathbb{Z}^\nu$, then

$$\begin{aligned}|n|&=\sum_{\mu=1}^{\nu}|n_\mu|,\\ \omega n&=\sum_{\mu=1}^{\nu}\omega_\mu\cdot n_\mu.\end{aligned} \tag{7}$$

Here ω is a ν-dimensional vector obeying a diophantine condition:

$$\exists\gamma>0, \qquad \sigma>\nu \quad |\omega n+m|\geqq\frac{\gamma}{|n|^\sigma} \quad \forall m\in\mathbb{Z}, \qquad n\in\mathbb{Z}^\nu. \tag{8}$$

Of course the diophantine condition is essential. Otherwise we expect a singular spectrum to occur [1, 7].

As in [2] we use essentially techniques based on the Kolmogoroff, Arnold Moser method, see also [6, 10].

It has also been used by Craig; however we can perform estimates for the direct problem because of the properties of the potentials in the class $\mathscr{P}$.

Essentially $V\in\mathscr{P}$ has the property that

$$|V(x-\omega n)-V(x-\omega m)|^{-1}\leqq K|n-m|^{\sigma}. \tag{9}$$

This diophantine estimate is exactly the condition needed in a perturbation theory to avoid a tunneling effect at large distance.

The paper is organized as follows. In the first section we describe the class $\mathscr{P}$ and its properties. In the second one, we introduce the technical machinery, namely an algebra of holomorphic kernels. A wider algebra has already been used by Craig [5], but these algebras occur naturally in almost periodic Schrödinger operators as pointed out by Bellissard and Testard [3]. The third section is devoted to the precise exposition of the results. The last section concerns the proof of the main theorem.

I The Class $\mathscr{P}$

If $R>0$, $\mathscr{H}_R$ denotes the set of period 1 holomorphic bounded functions on

$$\mathscr{D}_R=\{z\in\mathbb{C}, |\mathscr{I}m\, z|<R\}, \tag{I.1}$$

equiped with the sup-norm

$$\|f\|_R=\sup_{z\in\mathscr{D}_R}|f(z)|. \tag{I.2}$$

Then $\mathscr{P}_R$ is the set of period-one meromorphic functions d on $\mathscr{D}_R$ such that there is a constant $C>0$ with

$$|d(z)-d(z-a)|\geqq C\|a\|, \quad \forall a\in\mathbb{R}, \quad \forall z\in\mathscr{D}_R \tag{I.3}$$

with the notation:

$$a\in\mathbb{R}, \qquad \|a\|=\inf_{m\in\mathbb{Z}}|a+m|. \tag{I.4}$$

Then $|d|_R$ is defined as the biggest possible value of C in (I.3). $\mathscr{P}$ is then $\bigcup_{R>0}\mathscr{P}_R$.

Examples of elements of $\mathscr{P}$ are the following:

Example 1 (Sarnak [11]).

$$d_1(z)=\exp 2i\pi z.$$

Then

$$|d_1|_R\geqq 4\exp(-2\pi R). \tag{I.5}$$

Example 2 (Fishman et al. [7]).

$$d_2(z)=\tan\pi z,$$

with

$$|d_2|_R\geqq 2(\cosh 2\pi R)^{-1}. \tag{I.6}$$

The set $\mathscr{P}_R$ is not a linear space, however, we get immediately:

Lemma I.1. *If* $d \in \mathscr{P}_R$ *then* $\lambda d \in \mathscr{P}_R$, $\forall \lambda \in \mathbb{C}$, $\lambda \neq 0$ *and*

$$|\lambda d|_R = |\lambda| |d|_R . \tag{I.7}$$

On the other hand $\mathscr{P}$ is stable under some perturbations:

Lemma I.2. *Let* d *be in* $\mathscr{P}_R$ *and* g *be in* $\mathscr{H}_R$. *If* $R > \varrho > 0$ *is such that*

$$\|g\|_R < \varrho |d|_R , \tag{I.8}$$

then $d + g \in \mathscr{P}_{R-\varrho}$ *and*

$$\big| |d+g|_{R-\varrho} - |d|_{R-\varrho} \big| \leqq \varrho^{-1} \|g\|_R . \tag{I.9}$$

Proof. By (I.8), the function

$$\varphi(z) = d(z) - d(z-a) + g(z) - g(z-a) \tag{I.10}$$

never vanishes. For

$$\frac{|g(z) - g(z-a)|}{\|a\|} \leqq \sup_{z \in \mathscr{D}_R} \left| \frac{dg}{dz}(z) \right| , \tag{I.11}$$

using the Cauchy formula, if $z \in \mathscr{D}_{R-\varrho}$ we get

$$\left| \frac{dg}{dz}(z) \right| \leqq \left| \oint_\gamma \frac{1}{2i\pi} \frac{g(z')}{(z'-z)^2} dz' \right| , \tag{I.12}$$

where γ is any path contained in $\mathscr{D}_R$ and enclosing z. Thus:

$$\left| \frac{dg}{dz}(z) \right| \leqq \frac{|\gamma|}{2\pi \operatorname{dist}(z, \gamma)^2} \|g\|_R . \tag{I.13}$$

Since $z \in \mathscr{D}_{R-\varrho}$, we get $\operatorname{dist}(z, \gamma) \geqq \varrho$ and choosing γ as the circle of radius ϱ around z we get:

$$\left\| \frac{dg}{dz} \right\|_{R-\varrho} \leqq \frac{1}{\varrho} \|g\|_R . \tag{I.14}$$

Thus, if $\|g\|_R < |d|_R \varrho$, $\varphi(z)$ never vanishes. On the other hand

$$\left| \inf_{z \in \mathscr{D}_{R-\varrho}} \frac{|\varphi(z)|}{\|a\|} - \inf_{z \in \mathscr{D}_{R-\varrho}} \frac{|d(z) - d(z-a)|}{\|a\|} \right| \leqq \left\| \frac{dg}{dz} \right\|_{R-\varrho} , \tag{I.15}$$

which gives (I.9)

If f is holomorphic in some domain of $\mathbb{C}$ we note:

$$f^*(z) = \bar{f}(\bar{z}) . \tag{I.16}$$

Lemma I.3. *Let* f *be holomorphic on* $\mathscr{D}_\varrho$. *We assume* $f = f^*$ *and*

$$|f(z) - f(z')| \geqq C |z - z'| , \quad \forall z, z' \in \mathscr{D}_\varrho , \tag{I.17}$$

where $C > 0$. *Then if* $d \in \mathscr{P}_R$, $d = d^*$, *there is* $R_\varrho > 0$ *such that* $f \circ d \in \mathscr{P}_{R_\varrho}$ *and*

$$|f \circ d|_{R_\varrho} \geqq C |d|_{R_\varrho} . \tag{I.18}$$

Proof. Since $\operatorname{Im} d(z) \to 0$ as $\operatorname{Im} z \to 0$, there is $R_\varrho > 0$ such that if $|\operatorname{Im} z| < R_\varrho$ then $|\operatorname{Im} d(z)| < \varrho$. Therefore if $z \in \mathscr{D}_{R\varrho}$, $f \circ d(z)$ is well defined and meromorphic.

Now:

$$|f \circ d(z) - f \circ d(z-a)| \geqq C|d(z) - d(z-a)| \geqq C|d|_{R\varrho} \|a\| . \tag{I.19}$$

Example 3. Taking $f(z) = z + \frac{1}{3}z^3$ and d_2 as in Example 2, we get:

$$f \circ d(z) = \tan \pi z + \tfrac{1}{3} \tan^3 \pi z$$

and

$$|f \circ d|_R > 0 \qquad \text{for } R \text{ small enough}.$$

Lemma I.4. *If $d \in \mathscr{P}_R$, $d = d^*$, there is a unique $x \in [0, 1[$ such that the real poles of d are: $\{x + n, n \in \mathbb{Z}\}$. Moreover d is strictly monotonic in each interval $[x+n, x+n+1[$ and the set of values of $d(z)$, $z \in \mathbb{R}$ is the real line $\mathbb{R}$.*

Proof. If d had no pole on $\mathbb{R}$, then its restriction on $\mathbb{R}$ would be $\mathscr{C}^1$. Since $|d(x) - d(x-a)| \geqq |d|_R \|a\|$, $\forall a \in \mathbb{R}$, the first derivative $\frac{d}{dz} d$ on $\mathbb{R}$ would be strictly monotonic. But this is impossible because d has period 1. Thus d has real poles.

Since d is meromorphic, there is only a finite number of them in each interval of length 1. Let $0 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_N < 1$ be the poles in $[0, 1[$. We claim that $N = 1$.

For d is certainly strictly monotonic in each interval $]x_i, x_{i+1}[$ (with the convention $x_{N+1} = x_1 + 1$). Then, for a given $s \in \mathbb{R}$, there are $y_1, y_2, \ldots, y_N$; $y_i \in]x_i, x_{i+1}[$ such that $d(y_i) = s$ and clearly if $N > 1$, $\|y_i - y_{i+1}\| > 0$ for some i. This is contradictory with (I.3).

II. The Algebra of Holomorphic Kernels

For $r > 0$, $R > 0$ and $\nu \in \mathbb{N}$, we denote by $\mathfrak{A}_{R,r,\nu}$ (or $\mathfrak{A}_{R,r}$ if no confusion arises) the set of kernels $\mathbf{m} = (\mathbf{m}(z, n)) n \in \mathbb{Z}^\nu$, $z \in \mathscr{D}_R$, where for each $n \in \mathbb{Z}^\nu$, the map $z \mapsto \mathbf{m}(z, n)$ belongs to $\mathscr{H}_R$, and

$$\|\mathbf{m}\|_{R,r} = \sup_{z \in \mathscr{D}_R} \sum_{k \in \mathbb{Z}^\nu} |\mathbf{m}(z, k)| e^{r|k|} \tag{II.1}$$

is finite. Then $\mathfrak{A}_{R,r}$ is a Banach space.

Let now ω be a ν-dimensional vector

$$\omega = (\omega_1, \ldots, \omega_\nu) \in \mathbb{R}^\nu . \tag{II.2}$$

For $n = (n_1, \ldots, n_\nu) \in \mathbb{Z}^\nu$, we denote by $\omega \cdot n$ the inner product $\sum_{\mu=1}^{\nu} \omega_\mu n_\mu$, and $|n| = \sum_{\mu=1}^{\nu} |n_\mu|$.

The vector ω is rational if there is $n \neq 0$ in $\mathbb{Z}^\nu$ such that $\omega \cdot n \in \mathbb{Z}$. Otherwise it is irrational. In the later case the set of points $\{\omega \cdot n + m;\ n \in \mathbb{Z}^\nu, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^\nu$.

There is a set of full-Lebesgue measure of vectors in $\mathbb{R}^\nu$ for which there are $\gamma > 0$, $\sigma > \nu$

$$\|\omega \cdot n\| \geqq \frac{\gamma}{|n|^\sigma} . \tag{II.3}$$

In this case ω is called diophantine. On the other hand, ω is called a Liouville vector, if it is irrational and if there is an infinite set $(n_k)_{k\geqq 0}$ in $\mathbb{Z}^\nu$ for which

$$\|\omega\cdot n\| \leqq \frac{1}{|n_k|^k}. \tag{II.4}$$

Giving $\omega\in\mathbb{R}^\nu$, we define in $\mathfrak{A}_{R,r}$ an algebraic stucture by:

$$(\mathbf{m}_1\cdot\mathbf{m}_2)(z,n) = \sum_{l\in\mathbb{Z}^\nu} \mathbf{m}_1(z,l)\mathbf{m}_2(z-\omega l, n-l). \tag{II.5}$$

A simple calculation shows that

$$\|\mathbf{m}_1\cdot\mathbf{m}_z\|^{\sim}_{R,r} \leqq \|\mathbf{m}_1\|^{\sim}_{R,r}\cdot\|\mathbf{m}_2\|^{\sim}_{R,r}. \tag{II.6}$$

An involution is given by

$$\mathbf{m}^*(z,n) = \overline{\mathbf{m}(\bar{z}-\omega n, -n)}. \tag{II.7}$$

If we define a new norm by

$$\|\mathbf{m}\|_{R,r} = \mathrm{Max}(\|\mathbf{m}\|^{\sim}_{R,r}, \|\mathbf{m}^*\|^{\sim}_{R,r}), \tag{II.8}$$

then $\mathfrak{A}_{R,r}$ becomes a Banach $*$-algebra denoted by $\mathfrak{A}^{\omega}_{R,r}$.

Of course $\mathfrak{A}^{\omega}_{R,r}$ increases when R and r decreases.

Examples of elements of $\mathfrak{A}^{\omega}_{R,r}$ are given as follows:

(i) if $g\in\mathscr{H}_R$, then g can be considered as an element of $\mathfrak{A}^{\omega}_{R,r}$ by putting:

$$\mathbf{g}(z,n) \equiv g(z)\delta_{n,0}. \tag{II.9}$$

Such a kernel is called *diagonal.*

(ii) If $e\in\mathbb{Z}^\nu$, $\mathbf{u}_e$ is the kernel

$$\mathbf{u}_e(z,n) = \delta_{n,e}. \tag{II.10}$$

One can easily see that

$$\begin{aligned}
&\text{(a)} && \mathbf{u}_0 = \mathbb{1} \quad \text{is an identify},\\
&\text{(b)} && \mathbf{u}_e^*\mathbf{u}_e = \mathbf{u}_e\mathbf{u}_e^* = \mathbb{1}, \quad \forall e\in\mathbb{Z}^\nu,\\
&\text{(c)} && \mathbf{u}_{e_1}\mathbf{u}_{e_2} = \mathbf{u}_{e_1+e_2}, \quad \forall e_1, e_2\in\mathbb{Z}^\nu,
\end{aligned} \tag{II.11}$$

(iii) the Laplace kernel is then given by

$$\Delta = \sum_{e;\,|e|=1} \mathbf{u}_e. \tag{II.12}$$

A canonical set of representations of $\mathfrak{A}^{\omega}_{R,r}$ in $\mathbf{l}^2(\mathbb{Z}^\nu)$ is given by

$$\Pi_z(\mathbf{m})\Psi(n) = \sum_{l\in\mathbb{Z}^\nu} \mathbf{m}(z-\omega n, l-n)\Psi(l), \tag{II.13}$$

where $\Psi\in\mathbf{l}^2(\mathbb{Z}^\nu)$, $z\in\mathscr{D}_R$ and $\mathbf{m}\in\mathfrak{A}^{\omega}_{R,r}$.

We can check immediately:

$$\begin{aligned}
&\text{(i)} && \|\Pi_z(\mathbf{m})\| \leqq \|\mathbf{m}\|_{R,r}, \quad \forall z\in\mathscr{D}_R,\\
&\text{(ii)} && \Pi_z(\mathbf{m}_1\cdot\mathbf{m}_z) = \Pi_z(\mathbf{m}_1)\Pi_z(\mathbf{m}_2),\\
&\text{(iii)} && \Pi_z(\mathbf{m}^*) = \Pi_{\bar{z}}(\mathbf{m})^*.
\end{aligned} \tag{II.14}$$

In particular, any diagonal element of $\mathfrak{A}^{\omega}_{R,r}$ gives rise to a multiplication operator, i.e. a diagonal matrix on $\mathbb{Z}^{\nu}$.

In view of our problem one will extend Π_z to $\mathscr{P}_R$ by means of unbounded operators. Namely, if $d \in \mathscr{P}_R$, we define $\mathscr{D}_z(\mathbf{d})$ to be the set of vectors $\Psi \in \mathbf{l}^2(\mathbb{Z}^{\nu})$ such that $\Psi(n)=0$ whenever $z-n\omega$ is a pole of $\mathbf{d}$, and $\sum_n |\mathbf{d}(z-n\omega)\Psi(n)|^2 < +\infty$. Then if $\Psi \in \mathscr{D}_z(\mathbf{d})$

$$\Pi_z(\mathbf{d})\,\Psi(n) = \mathbf{d}(z-n\omega)\,\Psi(n) \tag{II.15a}$$

if $\mathbf{d}(z-\omega n)$ is finite and

$$\Pi_z(\mathbf{d})\,\Psi(n) = 0 \qquad \text{otherwise}. \tag{II.15b}$$

If $\mathbf{d}=\mathbf{d}^*$, then $\Pi_z(\mathbf{d})$ is self adjoint when $z \in \mathbb{R}$.

We remark that for all z except on a countable set, $\mathscr{D}_z(\mathbf{d})$ is a dense subspace of $l^2(\mathbb{Z})$. Whereas if z is such that for some n_0, (which is unique by Lemma I.4), $z-\omega n_0$ is a pole of $\mathbf{d}$, $\mathscr{D}_z(\mathbf{d})$ is dense in the hyperplane $\Psi(n_0)=0$. In this latter case, it is not difficult to see that if P_{n_0} denotes the projection on this hyperplane, it commutes with any operator of the form $\Pi_z(\mathbf{d}+\mathbf{m})$ $\mathbf{m} \in \mathfrak{A}^{\omega}_{R,r}$, and the results of the next section have an obvious extension to this case.

We shall consider the set $\hat{\mathfrak{A}}^{\omega}_{R,r}$ obtained from $\mathfrak{A}^{\omega}_{R,r}$ adding the elements of $\mathscr{P}_R$.

In this set-up, if $V \in \mathscr{P}_R$ the Schrödinger operator given by Eq. (6) of the Introduction can be seen as the operator $\Pi_x(\Delta + V)$.

III. The Main Results

Theorem 1. *Let given $R>0$, $r>0$, $\omega \in \mathbb{R}^{\nu}$ satisfying the diophantine condition:*

$$\|\omega \cdot n\| \geqq \frac{\gamma}{|n|^{\sigma}} \quad \textit{with} \quad \gamma>0, \sigma>\nu \quad \forall n \in \mathbb{Z}^{\nu}. \tag{III.1}$$

If $V \in \mathscr{P}_R$, there is a positive constant ε_c, depending on R, r, γ, σ and $|V|_R$ only, such that if $\mathbf{m} \in \mathfrak{A}^{\omega}_{R,r}$, $\|\mathbf{m}\|_{R,r} < \varepsilon_c$ there exists an invertible element $\mathbf{u} \in \mathfrak{A}^{\omega}_{R,r}$ and $\hat{V} \in \mathscr{P}_{R/2}$ with

$$\begin{aligned}
&(1) && \mathbf{u}(V+\mathbf{m})\mathbf{u}^{-1} = \hat{V}, \\
&(2) && \mathrm{Max}(\|\mathbf{u}-\mathbb{1}\|_{R/2,r/2}, \|\mathbf{u}^{-1}-\mathbb{1}\|_{R/2,r/2}) \leqq K_1 \|\mathbf{m}\|_{R,r}, \\
&(3) && V-\hat{V} \in \mathscr{H}_{R/2} \quad \textit{and} \quad \|V-\hat{V}\|_{R/2} \leqq K_2 \|\mathbf{m}\|_{R,r}, \\
&&& |\hat{V}|_{R/2} \geqq \tfrac{1}{2}|V|_R.
\end{aligned} \tag{III.2}$$

If in addition $\mathbf{m}+V$ is self-adjoint, then u is unitary and $\hat{V}=\hat{V}^*$.

As a first consequence we note:

Corollary 1. *Let m and V as in the previous theorem. Then the operator $H_z = \Pi_z(\mathbf{m}+V)$ has a complete set of eigenvectors which are exponentially localized. The corresponding eigenvalues are the set*

$$\{\hat{V}(z-\omega n);\, n \in \mathbb{Z}^{\nu};\, z-\omega n \quad \textit{is not pole of } V\}. \tag{III.3}$$

Remark. The completeness of the set of eigenvectors may be understood in the hyperplane defined by $\mathscr{D}_z(\mathbf{d})$. The domain does not depend on $\mathbf{m}$.

Corollary 2. *Let* $\mathbf{m}$ *and* V *as in Theorem 1 with in addition* $\mathbf{m}=\mathbf{m}^*$, $V=V^*$. *Then* $\forall x\in\mathbb{R}$, $H_x=\Pi_x(\mathbf{m}+V)$ *is self-adjoint; its spectrum is* $\mathbb{R}$, *and it is only pure point. All its eigenvectors are exponentially localized.*

Some Applications

1) If one considers one of the examples treated by Sarnak where $V(z)=\exp 2i\pi z$, the operator

$$H_z\Psi(n)=\varepsilon\Delta\Psi+e^{2i\pi(z-\omega n)}\Psi(n) \tag{III.4}$$

has only point spectrum for $\varepsilon\|\Delta\|_{R,r}<\varepsilon_c$, with exponentially localized eigenstates.

The special case $\gamma=1$ and

$$H_x\Psi(n)=\varepsilon\Psi(n-1)+e^{2i\pi(x-n\alpha)}\Psi(n), \tag{III.5}$$

where α is a diophantine number in $\mathbb{R}$, is interesting because, Sarnak, [10], proved that if $\varepsilon<1$ the spectrum is pure point with exponentially localized eigenstates, whereas for $\varepsilon>1$ we have only an essential spectrum, with eigenfunctions being of Block wave types. Moreover, if $\varepsilon<1$ the spectrum is the unit circle S^1, whereas if $\varepsilon>1$ it is equal to $|\varepsilon|S^1$.

One could also prove (Bellissard [4]) that if $\varepsilon=1$ the spectrum is the unit disk. Sarnak's result is now a consequence of Theorem 1 (Corollary 1).

2) The Schrödinger operator of Fishman et al. belongs to the class described in Corollary 2 provided $\varepsilon\|\Delta\|_{R,r}<\varepsilon_c$.

Proof of Corollaries 1 and 2. Corollary 2 is an immediate consequence of Corollary 1 and Lemma I.4. Let us now prove Corollary 1.

By Theorem 1, there is an operator $U_z=\Pi_z(u)$ such that (at least formally):

$$U_zH_zU_z^{-1}=\hat{V}_z(=\Pi_z(\hat{V}))\,. \tag{III.6}$$

Clearly $\hat{V}_z$ is a multiplication operator, and therefore the eigenvalues are

$$\{\hat{V}(z-\omega n), n\in\mathbb{Z}^\nu, z-\omega n \text{ is not a pole of } \hat{V}\}\,. \tag{III.7}$$

We remark that since $V-\hat{V}\in\mathscr{H}_{R/2}$, the poles of $\hat{V}$ coincide with the poles of V in $\mathscr{D}_{R/2}$.

Since $\hat{V}\in\mathscr{P}_{R/2}$ one gets

$$|\hat{V}(z-\omega n)-\hat{V}(z-\omega l)|\geqq|\hat{V}|_{R/2}\,\|\omega\cdot(l-n)\|\,, \tag{III.8}$$

and therefore each eigenvalue has multiplicity one. The corresponding eigenstate is given by

$$\Psi_n(l)=U_z^{-1}\delta_n(l)=\mathbf{u}^{-1}(z-l\omega, n-l) \tag{III.9}$$

[here $\delta_n(l)=\delta_{nl}$]. Since $\mathbf{u}^{-1}\in\mathfrak{A}^{\omega}_{R,r}$ we get an exponential decreasing

$$|\Psi_n(l)|\leqq\|\mathbf{u}^{-1}\|_{R/2,r/2}e^{-\frac{r}{2}|n-l|}\,. \tag{III.10}$$

However, (III.6) is not well defined. Indeed the left hand side is defined only on the domain $U_z^{-1}\mathscr{D}_z(V)$. Clearly $\mathscr{D}_z(V)$ and $\mathscr{D}_z(\hat{V})$ coincide thanks to Theorem 1, (3), and from (III.6) one gets:

$$\Psi \in U_z^{-1}\mathscr{D}_z(V) \Rightarrow \|\hat{V}_z\Psi\| < +\infty \Leftrightarrow \Psi \in \mathscr{D}_z(\hat{V}), \tag{III.11}$$

because U_z is bounded. Thus:

$$U_z^{-1}\mathscr{D}_z(V) \subset \mathscr{D}_z(V). \tag{III.12}$$

In much the same way one gets $H_z = U_z^{-1}\hat{V}_z U_z$ on $U_z\mathscr{D}_z(\hat{V})$ and one finds:

$$U_z\mathscr{D}_z(\hat{V}) = U_z\mathscr{D}_z(V) \subset \mathscr{D}_z(V). \tag{III.13}$$

Thus $\mathscr{D}_z(V)$ is invariant by U_z. Therefore the eigenstate given by (III.9) belongs to $\mathscr{D}_z(V)$ and since U_z has a bounded inverse, they are complete [dense in the closure of $\mathscr{D}_z(V)$].

IV. The Proof of Theorem 1

Let $V, \mathbf{m}$ be as in Theorem 1. Without loss of generality we can assume

$$\mathbf{m}(z,0) = 0. \tag{IV.1}$$

For otherwise the function $\mathbf{m}(z,0)$ can be absorbed in V.

We define a kernel $\mathbf{w}$ as follows:

$$\begin{aligned} &\text{if} \quad n=0, \quad \mathbf{w}(z,0)=0, \\ &\text{if} \quad n\neq 0, \quad \mathbf{w}(z,n) = \frac{\mathbf{m}(z,n)}{V(z)-V(z-\omega-n)}. \end{aligned} \tag{IV.2}$$

Lemma IV.1. *For any* $\delta > 0$, $\mathbf{w} \in \mathfrak{A}^{\omega}_{R,r-\delta}$ *and*

$$\|\mathbf{w}\|_{R,r-\delta} \leqq \frac{C(\sigma)}{\gamma\delta^{\sigma}|V|_R}\|\mathbf{m}\|_{R,r}. \tag{IV.3}$$

Moreover, if $\mathbf{m} = \mathbf{m}^*$, $V = V^*$, *then* $\mathbf{w} = -\mathbf{w}^*$.

Proof. It is enough to estimate $\|\mathbf{w}\|^{\sim}_{R,r-\delta}$. We get [thanks to $V \in \mathscr{P}_R$ and Eq. (III.1)]

$$\sum_{n\in\mathbb{Z}^\nu} |\mathbf{w}(z,n)|e^{(r-\delta)|n|} \leqq \frac{1}{|V|_R\gamma}\sum_{n\in\mathbb{Z}^\nu} |n|^{\sigma}e^{-\delta|n|}|\mathbf{m}(z,n)|e^{r|n|}. \tag{IV.4}$$

Since

$$|n|^{\sigma}e^{-\delta|n|} \leqq \frac{C(\sigma)}{\delta^{\sigma}} = e^{-\sigma}\frac{\sigma^{\sigma}}{\delta^{\sigma}}, \tag{IV.5}$$

one gets immediately (IV.3). The end of the lemma is easy to check.

Lemma IV.2. *One has*

$$e^{\mathbf{w}}(V+\mathbf{m})e^{-\mathbf{w}} = V + \bar{\mathbf{m}}, \tag{IV.6}$$

with $\bar{\mathbf{m}} \in \mathfrak{A}^{\omega}_{R,r-\delta}$, $\forall \delta > 0$ *and*

$$\|\bar{\mathbf{m}}\|_{R,r-\delta} \leqq \|\mathbf{m}\|_{R,r}\|\mathbf{w}\|_{R,r-\delta}e^{2\|\mathbf{w}\|_{R,r-\delta}}. \tag{IV.7}$$

Proof. Using Definition (IV.2) of $\mathbf{w}$ one sees that

$$A_{\mathbf{w}}(V) \equiv \mathbf{w}V - V\mathbf{w} = \mathbf{m}. \tag{IV.8}$$

Thus,

$$\begin{aligned} e^{\mathbf{w}}(V+\mathbf{m})e^{-\mathbf{w}} &= V + \sum_{k=1}^{\infty} A_{\mathbf{w}}^{k}(V)/k! + \sum_{k=1}^{\infty} A_{\mathbf{w}}^{k-1}(\mathbf{m})/(k-1)! \\ &= V + \sum_{k=1}^{\infty} A_{\mathbf{w}}^{k}(\mathbf{m})/(k-1)!(k+1) \equiv V + \bar{\mathbf{m}}. \end{aligned} \tag{IV.9}$$

Since $\mathbf{w} \in \mathfrak{A}_{R,r-\delta}^{\omega}$ and $\mathbf{m} \in \mathfrak{A}_{R,r}^{\omega} \subset \mathfrak{A}_{R,r-\delta}^{\omega}$, $\bar{\mathbf{m}}$ belongs to $\mathfrak{A}_{R,r-\delta}^{\omega}$ once it is proved that the series converges. But this is easy to check since:

$$\|A_{\mathbf{w}}(\mathbf{m})\|_{R,r-\delta} \leqq 2\|\mathbf{w}\|_{R,r-\delta}\|\mathbf{m}\|_{R,r-\delta}, \tag{IV.10}$$

and therefore

$$\|\bar{\mathbf{m}}\| = \left\| \sum_{k=1}^{\infty} A_{\mathbf{w}}^{k}(\mathbf{m})/(k-1)!\,(k+1) \right\| \leqq \|\mathbf{w}\|_{R,r-\delta}\|\mathbf{m}\|_{R,r-\delta} e^{2\|\mathbf{w}\|_{R,r-\delta}}. \tag{IV.11}$$

We now define $\hat{V}$ and $\hat{\mathbf{m}}$ as follows:

$$\begin{aligned} \hat{V}(z) &= V(z) + \bar{\mathbf{m}}(z,0), \\ \hat{\mathbf{m}}(z,n) &= \bar{\mathbf{m}}(z,n) \qquad \text{if } n \neq 0, \\ &= 0 \qquad \text{otherwise}. \end{aligned} \tag{IV.12}$$

Then, thanks to Lemma I.2 we get:

Lemma IV.3. *If $R > \varrho > 0$ is such that*

$$\|\bar{\mathbf{m}}\|_{R-r-\delta} < \varrho |V|_R, \tag{IV.13}$$

then $\hat{V} \in \mathscr{P}_{R-\varrho}$ and

$$|\hat{V}|_{R-\varrho} \geqq |V|_R - \frac{\|\bar{\mathbf{m}}\|_{R,r-\delta}}{\varrho} > 0. \tag{IV.14}$$

The strategy now is very simple. Starting from $V \in \mathscr{P}_R$ and $\mathbf{m} \in \mathfrak{A}_{R,r}^{\omega}$ satisfying (IV.1), we get $\hat{V} \in \mathscr{P}_{R-\varrho}$, $\hat{\mathbf{m}} \in \mathfrak{A}_{R-\varrho,r-\delta}^{\omega}$ satisfying again (IV.1), provided the estimate (IV.13) is true. We thus proceed recursively by defining a sequence $(V_l, \mathbf{m}_l)_{l=0,1}$ with:

$$V_0 = V, \mathbf{m}_0 = \mathbf{m}, \quad \text{and} \quad V_{l+1} = \hat{V}_l, \mathbf{m}_{l+1} = \hat{\mathbf{m}}_l. \tag{IV.15}$$

Since at each step estimates must be checked we introduce the following parameters:

$$\delta_l = \frac{r}{2^{l+2}}, \qquad \varrho_l = \frac{R}{2^{l+2}}. \tag{IV.16}$$

This choice is given to get $\sum_0 \delta_l = \frac{r}{2}$, $\sum_0 \varrho_l = \frac{R}{2}$.

Now we define r_l, R_l by:

$$\begin{aligned} r_0 &= r, \qquad R_0 = R, \\ r_{l+1} &= r_l - \delta_l = \frac{r}{2} + \frac{r}{2^{l+1}}, \qquad R_{l+1} = R_l - \varrho_l = \frac{R}{2} + \frac{R}{2^{l+1}}. \end{aligned} \tag{IV.17}$$

We assume that if $l \leqq L$ we have been able to construct $V_l, \mathbf{m}_l$ using (IV.15) with

$$\|\mathbf{m}_l\|_{R_l, r_l} = \varepsilon_l, \quad V_l \in \mathscr{P}_{R_l}, \quad |V_l|_{R_l} = d_l > \tfrac{1}{2} d_0 = |V|_R. \tag{IV.18}$$

We want to construct V_{L+1}, m_{L+1} using (IV.15). First of all, $\mathbf{w}_{L+1}$ is constructed via Eq. (IV.2) and we get (Lemma IV.1):

$$\|\mathbf{w}_{L+1}\|_{R_L, r_{L+1}} \leqq \frac{C(\sigma) 2^{\sigma(L+2)}}{\gamma r^\sigma d_L} \varepsilon_L. \tag{IV.19}$$

From Lemma IV.2:

$$\varepsilon_{L+1} = \|\bar{\mathbf{m}}_L\|_{R_{L+1}, r_{L+1}} \leqq \frac{C(\sigma) 2^{\sigma(L+2)}}{\gamma r^\sigma d_L} \varepsilon_L^2 \exp\left[\left(\frac{2C(\sigma)}{\gamma r^\sigma d_L} s^{\sigma(L+2)} \varepsilon_L\right)\right]. \tag{IV.20}$$

In view of the Lemma IV.3 we need the constraints

$$\varepsilon_{L+1} < \frac{R d_L}{2^{L+2}}, \tag{IV.21}$$

and

$$d_{L+1} \geqq d_L - \varepsilon_{L+1} \frac{2^{L+2}}{R}. \tag{IV.22}$$

In order to solve this set of recursive estimates let us introduce the sequence

$$\eta_0 > 0, \quad A > 0, \quad \eta_l = \frac{1}{A 2^{\sigma(l+3)}} (\eta_0 8^\sigma A)^{2^l}. \tag{IV.23}$$

We check immediately (the proof is left to the reader).

Lemma IV.4. *η_l satisfies the recursion relation:*

$$\eta_{l+1} = A 2^{\sigma(l+2)} \eta_l^2. \tag{IV.24}$$

Lemma IV.5. (i) *If $\eta_0 \leqq (8^\sigma A)^{-1}$, then* $\lim_{l \to \infty} \eta_l = 0$.

(ii) *If in addition $\eta \leqq 4^{-\sigma}$, then*

$$2^{\sigma(l+2)} \eta_l \leqq 1, \quad \forall l \geqq 0. \tag{IV.25}$$

Let us define ε_c as the maximum value of η_0 such that:

$$\text{(i)} \quad \eta_0 \leqq (8^\sigma A)^{-1}, \quad \eta_0 \leqq 4^{-\sigma},$$
$$\text{(ii)} \quad \sum_{i=0}^{\infty} 2^{i+1} \eta_i \leqq \frac{R}{2} d_0. \tag{IV.26}$$

We shall choose A by:

$$A = \frac{C(\sigma)}{\gamma r^\sigma} \frac{2}{d_0} \exp\left(\frac{4C(\sigma)}{\gamma r^\sigma d_0}\right). \tag{IV.27}$$

Lemma IV.6. *Assume that $\eta_0 = \varepsilon_0 < \varepsilon_c$, and that if $1 \leqq l \leqq L$, one has:*

$$\text{(i)} \quad \varepsilon_l \leqq \eta_l,$$
$$\text{(ii)} \quad d_l \geqq d_0 - \frac{1}{R} \sum_{i=0}^{l} 2^{l+1} \eta_i. \tag{IV.28}$$

Then, ε_{L+1} *and* d_{L+1} *satisfy again* (IV.28).

Proof. Since $\varepsilon_L \leqq \eta_L$, using Lemma IV.5 one gets:

$$2^{\sigma(L+2)}\varepsilon_L \leqq 1. \tag{IV.29}$$

Moreover, by (IV.28) and (IV.26) we have:

$$d_L > \frac{d_0}{2}. \tag{IV.30}$$

Thus from (IV.20), (IV.24), (IV.27) we obtain:

$$\varepsilon_{L+1} \leqq A 2^{(L+2)\sigma}\varepsilon_L^2 \leqq A 2^{(L+2)}\eta_L^2 = \eta_{L+1}. \tag{IV.31}$$

From (IV.22) we get:

$$d_{L+1} \geqq d_L - \frac{2^{L+2}}{R}\eta_{L+1} \geqq d_0 - \sum_{i=0}^{L+1} 2^{i+1}\frac{\eta_i}{R}. \tag{IV.32}$$

As a conclusion if $\varepsilon < \varepsilon_c$ the sequence $(V_l, \mathbf{m}_l)$ exists and since

$$R_l \geqq \frac{R}{2}, \quad r_l \geqq \frac{r}{2}, \quad \forall l, \tag{IV.33}$$

$$\lim_{l\to\infty} \|\mathbf{m}_l\|_{R/2, r/2} = 0, \tag{IV.34}$$

$$V_l(z) - V(z) = \sum_{i=0}^{l-1} \bar{\mathbf{m}}_i(z, 0) = g_l \in \mathscr{H}_{R/2}, \quad \forall l, \tag{IV.35}$$

and $\hat{V} - V = \lim g_l$ exists in $\mathscr{H}_{R/2}$. For each $l \in \mathbb{N}$

$$V_l + \mathbf{m}_l = u_l(V + \mathbf{m})u_l^{-1} \tag{IV.36}$$

with

$$u_l = e^{\mathbf{w}_l} e^{\mathbf{w}_{l-1}} \dots e^{\mathbf{w}_1}. \tag{IV.37}$$

If $V^* = V$, $\mathbf{m}^* = \mathbf{m}$, u_l is unitary.

Thanks to (IV.19) this product converges as $l \to \infty$ to $u \in \mathfrak{A}^{\omega}_{R/2, r/2}$ and

$$\|u^{\pm 1} - \mathbb{1}\|_{R/2, r/2} \leqq \left(\sum_{l=1}^{\infty} \|\mathbf{w}_l\|_{R_l, r_l}\right) \exp\left(\sum_{l=1}^{\infty} \|\mathbf{w}_l\|_{R_l, r_l}\right) \leqq K \cdot \varepsilon_0. \tag{IV.38}$$

Finally, the estimate (IV.26)–(IV.28) gives:

$$|\hat{V}|_{R/2} \geqq \tfrac{1}{2}|V|_R. \tag{IV.39}$$

This achieves the proof of Theorem 1.

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