Commun. Math. Phys. 104, 243–250 (1986)

Communications in Mathematical Physics


# The Spectrum of a Schrödinger Operator in $L_p(\mathbb{R}^\nu)$ is $p$-Independent

Rainer Hempel and Jürgen Voigt

Mathematisches Institut der Universität München, Theresienstr. 39, D-8000 München 2, Federal Republic of Germany

**Abstract.** Let $H_p = -\frac{1}{2}\Delta + V$ denote a Schrödinger operator, acting in $L_p(\mathbb{R}^\nu)$, $1 \leqq p \leqq \infty$. We show that $\sigma(H_p) = \sigma(H_2)$ for all $p \in [1, \infty]$, for rather general potentials $V$.

**Introduction.** In [12, 13], B. Simon conjectured that $\sigma(H_p)$ is $p$-independent, where $H_p = -\frac{1}{2}\Delta + V$ is a general Schrödinger operator in $L_p(\mathbb{R}^\nu)$. Partial results on this problem are contained in Simon [12], Sigal [10], Hempel, Voigt [5].

In the notations of Sect. 1, our main result reads as follows.

**Theorem.** *Let $V = V_+ - V_-$, $V_\pm \geqq 0$, where $V_+$ is admissible, and $V_- \in \hat{K}_\nu$ with $c_\nu(V_-) < 1$. Then $\sigma(H_p) = \sigma(H_2)$ for $1 \leqq p \leqq \infty$.*

*In addition, if $\lambda$ is an isolated eigenvalue of finite algebraic multiplicity $k$ of $H_p$, for some $p \in [1, \infty]$, then the same is true for all $p \in [1, \infty]$.*

The proof of this result is contained in Propositions 2.1, 3.1, and 2.2.

In Sect. 2 we prove the inclusion $\sigma(H_2) \subset \sigma(H_p)$, following ideas of Simon and Davies.

In Sect. 3 we show that the integral kernel of $(H_2 - z)^{-n}$, for $n \in \mathbb{N}$, $n > \nu/2$, defines an analytic $\mathscr{B}(L_p(\mathbb{R}^\nu))$-valued function on $\rho(H_2)$, which coincides with $(H_p - z)^{-n}$ for $z$ real and sufficiently negative. This implies $\sigma(H_p) \subset \sigma(H_2)$, by unique continuation.

A different situation, where an integral kernel determines operators with $p$-*dependent* spectrum, can be found in Jörgens [6; IV, Aufg. 12.11 (b)]; note that the kernel in Jörgens' example is the resolvent kernel of the differential operator

$$-\frac{d}{dx}x^2\frac{d}{dx} \quad \text{on} \quad (0, \infty), \quad \text{at } z = -2.$$

## 1. Schrödinger Operators in $L_p(\mathbb{R}^\nu)$

First we recall briefly several facts concerning the semigroup associated with the heat equation. For brevity, we shall write $L_p$ instead of $L_p(\mathbb{R}^\nu)$, in the sequel (analogously, $C_c^\infty := C_c^\infty(\mathbb{R}^\nu)$, etc.). For $t \in \mathbb{C}$, Re $t > 0$, we define $k_t \in L_1$ by

$$k_t(x) := (2\pi t)^{-\nu/2} \exp(-|x|^2/2t).$$

For $1 \leqq p \leqq \infty$ we define $U_{0,p}(t) \in \mathscr{B}(L_p)$ ($t \in \mathbb{C}$, Re $t > 0$) by

$$U_{0,p}(t)f := k_t * f \quad (f \in L_p),$$

and further $U_{0,p}(0) = I$. For $1 \leqq p < \infty$, $U_{0,p}(\cdot)$ is a holomorphic semigroup of angle $\pi/2$; let $-H_{0,p}$ denote its generator. Further denote $H_{0,\infty} := H_{0,1}{}^*$.

Next we introduce the class of potentials $V$ to be considered in this paper. Following Voigt [14], we define classes of potentials by

$$\hat{K}_\nu := \{V \in L_{1,\mathrm{loc}};\ \operatorname*{ess\,sup}_{x \in \mathbb{R}^\nu} \int_{|x-y| \leqq 1} |g_\nu(x-y)|\,|V(y)|\,dy < \infty\},$$

where $g_\nu$ is the usual fundamental solution of $\frac{1}{2}\Delta$. Note that this class is slightly larger than the class $K_\nu$ in Aizenman, Simon [1], Simon [13]. For $V \in \hat{K}_\nu$ we define

$$c_\nu(V) := \lim_{\alpha \downarrow 0} \Big(\operatorname*{ess\,sup}_{x \in \mathbb{R}^\nu} \int_{|x-y| \leqq \alpha} |g_\nu(x-y)|\,|V(y)|\,dy\Big).$$

Obviously $\hat{K}_\nu \subset L_{1,\mathrm{loc,unif}}$ for all $\nu \in \mathbb{N}$, $\hat{K}_1 = L_{1,\mathrm{loc,unif}}$, and $c_1(V) = 0$ for all $V \in \hat{K}_1$.

A potential $V \geqq 0$ will be called *admissible* if $Q(H_{0,2}) \cap Q(V)$ is dense in $L_2$; cf. Voigt [14]. In particular, $V \geqq 0$ is admissible if $V \in L_{1,\mathrm{loc}}(G)$, where $G = \mathring{G} \subset \mathbb{R}^\nu$ is such that $\mathbb{R}^\nu \backslash G$ is a (closed) set of Lebesgue measure zero.

Throughout this paper we shall assume

$$\begin{aligned} &V = V_+ - V_-, \quad V_\pm \geqq 0, \\ &V_- \in \hat{K}_\nu \quad \text{with } c_\nu(V_-) < 1, \quad V_+ \text{ admissible.} \end{aligned} \tag{1.1}$$

In the following proposition we denote the truncation of $V$ by

$$V^{(n)} := (\operatorname{sgn} V)(|V| \wedge n) \quad (n \in \mathbb{N}).$$

**1.1. Proposition.** *Let $V$ satisfy* (1.1), *and let* $1 \leqq p < \infty$. *Then, for $t \geqq 0$, the limit*

$$U_p(t) := s - \lim_{n \to \infty} \exp(-t(H_{0,p} + V^{(n)}))$$

*exists, and $(U_p(t); t \geqq 0)$ is a $C_0$-semigroup on $L_p$. The Feynman-Kac formula*

$$U_p(t)f(x) = E_x\left\{\exp\left(-\int_0^t V(b(s))ds\right) f(b(t))\right\}$$

*holds for all $f \in L_p$.*

Here, $E_x$ and $b(\cdot)$ are as in Simon [13]; cf. Reed, Simon [9], Simon [11]. The proof of this proposition can be found in Voigt [14; Proposition 5.8(a), Proposition 2.8, Remark 5.2(b), Proposition 3.2, Proposition 6.1(c)].

We denote the generator of $(U_p(t); t \geqq 0)$ by $-H_p$, for $1 \leqq p < \infty$, and we shall henceforth write $U_p(t) = \exp(-tH_p)$. Also, $H_\infty = H_1^*$. More detailed information about the operators $H_p$, in particular for $p = 1$, $p = 2$ can be found in Voigt [14].

Note that $H_2$ is the form sum of $-\frac{1}{2}\Delta$ and $V$; cf. Voigt [14; Remark 6.2(c)]. (It follows from Devinatz [3; Lemma 4] that $V_-$ is $H_{0,2}$-form small.)

## 2. $\sigma(H_2) \subset \sigma(H_p)$

In this section we show that interpolation, duality, and $p$–$q$-smoothing lead to the following result.

**2.1. Proposition.** *Let $V$ satisfy* (1.1). *Then $\rho(H_p) \subset \rho(H_2)$ for all $p \in [1, \infty]$, and*

$$(H_p - z)^{-1} | L_p \cap L_2 = (H_2 - z)^{-1} | L_p \cap L_2 \quad (z \in \rho(H_p)).$$

This result was stated in Simon [12, 13]. The argument given there was based on interpolation between the resolvents $(H_p - z)^{-1}$ and $(H_{p'} - z)^{-1}$, for $z \in \rho(H_p) = \rho(H_{p'})$. It is not immediate, however, that these resolvents coincide on $L_p \cap L_{p'}$, as can be seen from Jörgens' example mentioned in the introduction. This gap in Simon's argument was closed by E. B. Davies (private communication). Compare also Hempel, Voigt [5; Proposition 3.1].

*Proof of Proposition 2.1.* (i) (due to E. B. Davies) Let $1 \leqq p < q \leqq \infty$, $t > 0$. Then $e^{-tH_p} \in \mathscr{B}(L_p, L_q)$; cf. Voigt [14; Proposition 6.3]. This implies

$$e^{-tH_p} H_p \subset H_q e^{-tH_p}. \tag{2.1}$$

Assume additionally $\lambda \in \rho(H_p) \cap \rho(H_q)$. Then (2.1) implies

$$(H_q - \lambda)^{-1} e^{-tH_p} = e^{-tH_p} (H_p - \lambda)^{-1}.$$

For $t \to 0$ we obtain

$$(H_p - \lambda)^{-1} | L_p \cap L_q = (H_q - \lambda)^{-1} | L_p \cap L_q. \tag{2.2}$$

(This holds also for $q = \infty$ because $e^{-tH_p} f$ is $\sigma(L_\infty, L_1)$-continuous for $f \in L_p \cap L_\infty$.)

(ii) Let $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$, and let $\lambda \in \rho(H_p)$ $(= \rho(H_{p'}))$. Then $(H_{p'} - \lambda)^{-1} | L_p \cap L_{p'} = (H_p - \lambda)^{-1} | L_p \cap L_{p'}$, by (2.2). The Riesz–Thorin convexity theorem implies that $(H_p - \lambda)^{-1}$ is continuous as an operator $R_\lambda$ on $L_2$.

For $f \in L_2 \cap L_p$, (2.1) implies

$$(H_2 - \lambda) e^{-tH_p} (H_p - \lambda)^{-1} f = e^{-tH_p} f.$$

For $t \to 0$ we obtain $(H_2 - \lambda)$ $(H_p - \lambda)^{-1} f = f$. This implies $(H_2 - \lambda) R_\lambda = I$, and hence $\lambda \in \rho(H_2)$. ■

**2.2. Proposition.** *Let $V$ satisfy (1.1), and let $1 \leqq p \leqq \infty$. Assume that $\lambda$ is an isolated point of $\sigma(H_p)$. Then $\lambda$ is an eigenvalue of $H_p$ with finite algebraic multiplicity if and only if the same is true for $H_2$. In this case, $\lambda$ is real and a pole of first order of the resolvents of $H_p$ and $H_2$, and the multiplicities of $\lambda$ as an eigenvalue of $H_p$ and $H_2$ coincide.*

*Proof.* Without restriction $p < \infty$. (Duality for $p = \infty$.) Note first that the selfadjoint operator $H_2$ can only have real eigenvalues which are poles of first order of the resolvent of $H_2$. Now the assertions follow from Proposition 2.1 and Auterhoff [2; Theorem 1.5]; see also Hempel, Voigt [5; Theorem 1.3]. ■

## 3. $\sigma(H_p) \subset \sigma(H_2)$

In this section we shall derive properties of the integral kernel of $(H_2 - z)^{-n}$, for $n \in \mathbb{N}$, $n > \nu/2$, in order to show the following result.

**3.1. Proposition.** *Let $V$ satisfy (1.1). Then $\rho(H_2) \subset \rho(H_p)$, for all $p \in [1, \infty]$.*

The proof relies on the following two auxiliary results which will be proved below.

**3.2. Lemma.** *Let $X$ be a Banach space, $T$ a closed operator in $X$, $\rho(T) \neq \emptyset$. Then $\rho(T)$ is the domain of holomorphy of $(T - z)^{-n}$, for $n = 1, 2, \ldots$.*

**3.3. Proposition.** *Let $V$ satisfy (1.1), and let $n \in \mathbb{N}$, $n > \nu/2$.*

(a) *Then $(H_2 - z)^{-n}$ is an integral operator, for $z \in \rho(H_2)$.*

(b) *Let $G^{(n)}(x, y; z)$ denote the integral kernel of $(H_2 - z)^{-n}$. Then, for any $K \subset\subset \rho(H_2)$*[1] *there exist constants $C, \eta > 0$ such that*

$$|G^{(n)}(x, y; z)| \leqq Ce^{-\eta|x-y|} \quad (z \in K, x, y \in \mathbb{R}^\nu).$$

*Proof of Proposition 3.1.* By duality, it is sufficient to consider the case $1 \leqq p \leqq 2$. Fix $n \in \mathbb{N}$, $n > \nu/2$, and let $G^{(n)}(x, y; z)$ be as in Proposition 3.3.

First we show that $G^{(n)}(\cdot, \cdot; z)$ defines an analytic $\mathscr{B}(L_p)$-valued function $G_p^{(n)}(z)$ on $\rho(H_2)$. To prove this, we remark that for any $\phi, \psi \in C_c^\infty$, the mapping

$$\rho(H_2) \ni z \mapsto \int\int G^{(n)}(x, y; z)\phi(y)\bar{\psi}(x)dxdy$$

is holomorphic. Furthermore, for any $K \subset\subset \rho(H_2)$, there exists a constant $C'$ such that

$$\| G_p^{(n)}(z) \|_{\mathscr{B}(L_p)} \leqq C' \quad (z \in K),$$

by the estimates in Proposition 3.3(b) and Young's inequality (cf. Reed, Simon [9; p. 32]).

Next, the fact that $e^{-tH_p}$ coincides with $e^{-tH_2}$ on $L_p \cap L_2$ implies that $G_p^{(n)}(z)$ coincides with $(H_p - z)^{-n}$ for $z$ real and sufficiently negative.

It follows by unique continuation that the domain of holomorphy of $(H_p - z)^{-n}$ contains $\rho(H_2)$. Hence, $\rho(H_p) \supset \rho(H_2)$, by Lemma 3.2 above. ■

Let us now prove the auxiliary results.

*Proof of Lemma 3.2.* Clearly, $(T - z)^{-n}$ is holomorphic on $\rho(T)$. Let spr$(A)$ denote the spectral radius of an operator $A \in \mathscr{B}(X)$. From the well-known facts (cf. Kato [7; p. 27, p. 37])

$$\operatorname{spr}((T - \zeta)^{-1}) = \inf_{n \in \mathbb{N}} \|(T - \zeta)^{-n}\|^{1/n},$$
$$\operatorname{spr}((T - \zeta)^{-1}) \geqq \operatorname{dist}(\zeta, \sigma(T))^{-1} \qquad (\zeta \in \rho(T)),$$

it is clear that $\|(T - \zeta)^{-n}\| \geqq \operatorname{dist}(\zeta, \sigma(T))^{-n}$ $(\zeta \in \rho(T))$. ■

For several reasons, we include a proof of Proposition 3.3 (instead of simply referring to Simon [13; Theorem B.7.1 (c')]): The estimate given in [13; loc. cit.] is

[1] $K \subset\subset \rho(H_2)$ means: $\bar{K}$ compact and $\bar{K} < \rho(H_2)$

not uniform for $z \in K \subset\subset \rho(H_2)$ (although one might be willing to believe that it must be true). Also, the proof of the (essential) Lemma B.7.11 in [13] is very sketchy, and it is our aim to give a complete proof of reasonable length. Finally, our proof will show that it is advantageous to consider $(H_p - z)^{-n}$, $n > \nu/2$, $n \in \mathbb{N}$, instead of arguing with $(H_p - z)^{-1}$ directly (which would be possible, but involve more estimates, like [13; Theorem B.7.2 (1), (2), (4)]).

Since we shall have to consider $e^{-tH_p}$ as an operator from $L_p$ to $L_q$, $q \geqq p$, we shall frequently drop the subscript $p$ and simply write $H = -\frac{1}{2}\Delta + V$, in the sequel. The proof will involve several steps, following rather closely the outline given in [13; proof of Lemma B.7.11]. For the remainder of this section, the assumptions of Proposition 3.3 are always assumed to hold.

**3.4. Lemma.** *Let $1 \leqq p \leqq q \leqq \infty$, $\varepsilon_0 > 0$. Then there exist constants $C = C(p, q, \varepsilon_0)$, $A = A(p, q, \varepsilon_0)$, such that for $\varepsilon \in \mathbb{R}^\nu$, $|\varepsilon| \leqq \varepsilon_0$, $t > 0$, we have*

$$\| e^{\varepsilon \cdot x} e^{t\Delta} e^{-\varepsilon \cdot x} \|_{p,q} \leqq C t^{-\gamma} e^{At},$$

*where $\gamma := (\nu/2)(p^{-1} - q^{-1})$.*

*Proof* (compare Simon [13; Lemma B.6.1]). Let $\varepsilon \in \mathbb{R}^\nu$, $|\varepsilon| \leqq \varepsilon_0$. Clearly,

$$K_\varepsilon(x, y; t) := (2\pi t)^{-\nu/2} e^{\varepsilon \cdot (x-y)} \exp\left( -\frac{|x-y|^2}{2t} \right),$$

is the kernel of $e^{\varepsilon \cdot x} e^{(t/2)\Delta} e^{-\varepsilon \cdot x}$. By Young's inequality (cf. Reed, Simon [9; p. 32]), it is enough to estimate $\| K_\varepsilon(0, \cdot; t) \|_s$, for $s := (1 + q^{-1} - p^{-1})^{-1}$. Now,

$$\| K_\varepsilon(0, \cdot; t) \|_s \leqq c t^{-(\nu/2)(1-s^{-1})} \Big[ \int_{\mathbb{R}^\nu} e^{s\varepsilon_0 \sqrt{t}|\eta| - (s/2)|\eta|^2} d\eta \Big]^{1/s},$$

and the term in square brackets can be estimated by

$$\int_{|\eta| \leqq 4\varepsilon_0\sqrt{t}} e^{s\varepsilon_0\sqrt{t}|\eta|} d\eta + \int_{|\eta| > 4\varepsilon_0\sqrt{t}} e^{-(s/4)|\eta|^2} d\eta \leqq c' t^{\nu/2} e^{4s\varepsilon_0^2 t} + c''. \qquad \blacksquare$$

**3.5. Proposition** *(compare* [13; Eq. (B 11)]). *For all $1 \leqq p \leqq q \leqq \infty$ there exist constants $C = C(p, q)$, $A = A(p, q)$ such that for all $t > 0$ we have*

$$\| e^{-tH} \|_{p,q} \leqq C t^{-\gamma} e^{At},$$

*where $\gamma = (\nu/2)(p^{-1} - q^{-1})$.*

*Proof.* This follows from Devinatz [3; Lemma 2] combined with duality and interpolation as described in Voigt [14; proof of Proposition 6.3]. Under the slightly stronger assumption $c_\nu(V) = 0$ a simpler proof can be found in Simon [13; loc. cit.]. $\blacksquare$

**3.6. Lemma** *(compare* [13; *Lemma* B.6.2(b)]). *Let $1 < c < c_\nu(V)^{-1}$, $1/c + 1/c' = 1$. Then, for any $\varepsilon \in \mathbb{R}^\nu$,*

$$\| e^{\varepsilon \cdot x} e^{-tH} e^{-\varepsilon \cdot x} \|_{p,q} \leqq \| e^{-t(-(1/2)\Delta + cV)} \|_{p,q}^{1/c} \| e^{c'\varepsilon \cdot x} e^{(t/2)\Delta} e^{-c'\varepsilon \cdot x} \|_{p,q}^{1/c'}.$$

*Proof.* Let $\varepsilon \in \mathbb{R}^\nu$ and write $w(x) = e^{\varepsilon \cdot x}$. Also, let $h \in C_c^\infty$, $g := w^{-1} h$. Factorizing

$|g| = |h|^{1/c} \cdot |w^{-c'}h|^{1/c'}$, it follows by Hölder's inequality in function space that

$$|(e^{-tH}g)(x)| \leqq [(e^{-t(-(1/2)\Delta + cV)}|h|)(x)]^{1/c} \cdot [(e^{(t/2)\Delta}|w^{-c'}h|)(x)]^{1/c'}.$$

Now, multiplying by $|w(x)|$, taking $q^{th}$ powers and integrating, we obtain

$$\begin{aligned}\int |we^{-tH}w^{-1}h|^q dx &\leqq \int [e^{-t(-(1/2)\Delta + cV)}|h|]^{q/c}[w^{c'}e^{(t/2)\Delta}w^{-c'}|h|]^{q/c'} dx \\ &\leqq \{\int (e^{-t(-(1/2)\Delta + cV)}|h|)^q dx\}^{1/c} \cdot \{\int (w^{c'}e^{(t/2)\Delta}w^{-c'}|h|)^q dx\}^{1/c'},\end{aligned}$$

which implies

$$\| we^{-tH}w^{-1}h \|_q \leqq \| e^{-t(-(1/2)\Delta + cV)} \|_{p,q}^{1/c} \| h \|_p^{1/c} \| w^{c'}e^{(t/2)\Delta}w^{-c'} \|_{p,q}^{1/c'} \| h \|_p^{1/c'}. \qquad \blacksquare$$

**3.7. Proposition** *(compare* [13; *Theorem* B.6.3]). *Let* $1 \leqq p \leqq q \leqq \infty$, $\alpha > \gamma = (\nu/2)(p^{-1} - q^{-1})$, *and* $\varepsilon_0 > 0$. *Then, for z real and sufficiently negative, there exists a constant C such that*

$$\| e^{\varepsilon\cdot x}(H - z)^{-\alpha}e^{-\varepsilon\cdot x} \|_{p,q} \leqq C \quad (\varepsilon \in \mathbb{R}^\nu, |\varepsilon| \leqq \varepsilon_0).$$

*Proof.* For $\phi \in C_c^\infty$, we have (with $w := e^{\varepsilon\cdot x}$)

$$(H - z)^{-\alpha}(w^{-1}\phi) = c_\alpha \int_0^\infty e^{tz}t^{\alpha-1}e^{-tH}(w^{-1}\phi)dt,$$

and hence

$$\begin{aligned}\| w(H - z)^{-\alpha}w^{-1}\phi \|_q &\leqq c_\alpha \int_0^\infty \| we^{-tH}w^{-1} \|_{p,q} e^{tz}t^{\alpha-1} dt \cdot \| \phi \|_p \\ &\leqq c_\alpha \int_0^\infty \| e^{-t(-(1/2)\Delta + cV)} \|_{p,q}^{1/c} \| w^{c'}e^{(t/2)\Delta}w^{-c'} \|_{p,q}^{1/c'} e^{tz}t^{\alpha-1} dt \cdot \| \phi \|_p\end{aligned}$$

(by Lemma 3.6)

$$\leqq c_\alpha \int_0^\infty [C_1 t^{-\gamma}e^{A_1 t}]^{1/c}[C_2 t^{-\gamma}e^{A_2 t}]^{1/c'} e^{tz}t^{\alpha-1} dt \cdot \| \phi \|_p$$

(by Proposition 3.5 and Lemma 3.4)

$$\leqq C_3 \int_0^\infty t^{-\gamma+\alpha-1}e^{At+tz} dt \cdot \| \phi \|_p \leqq C_4 \cdot \| \phi \|_p,$$

provided $A + z < 0$. $\blacksquare$

**3.8. Proposition.** *For any* $K \subset\subset \rho(H_2)$, *there exist* $\varepsilon_0 = \varepsilon_0(K) > 0$ *and a constant* $C = C(K, \varepsilon_0)$ *such that* $K \subset \rho(e^{\varepsilon\cdot x}H_2e^{-\varepsilon\cdot x})$ *for* $|\varepsilon| \leqq \varepsilon_0$, *and*

$$\| e^{\varepsilon\cdot x}(H_2 - z)^{-1}e^{-\varepsilon\cdot x} \| = \| (e^{\varepsilon\cdot x}H_2e^{-\varepsilon\cdot x} - z)^{-1} \| \leqq C \quad (|\varepsilon| \leqq \varepsilon_0, z \in K).$$

*Proof.* As $W_2^1$ contains the form domain of $H_2$, the operators $\partial_j$ are $|H_2|^{1/2}$-bounded and hence $H_2$-bounded with relative bound zero ($j = 1, \ldots, \nu$). This implies

$$e^{\varepsilon\cdot x}H_2e^{-\varepsilon\cdot x} = H_2 + \varepsilon\cdot\nabla - \tfrac{1}{2}\varepsilon^2,$$

for all $\varepsilon \in \mathbb{R}^\nu$. Now the identity

$$(H_2 + \varepsilon\cdot\nabla - \tfrac{1}{2}\varepsilon^2 - z) = (I + (\varepsilon\cdot\nabla - \tfrac{1}{2}\varepsilon^2)(H_2 - z)^{-1})(H_2 - z)$$

implies the desired conclusion. ■

We can now finally proceed to the proof of Proposition 3.3.

*Proof of Proposition 3.3.* Fix $n\in\mathbb{N}$, $n > v/2$, and choose $w$ real and so negative that, by Proposition 3.7,

$$\| e^{\varepsilon\cdot x}(H-w)^{-n/2}e^{-\varepsilon\cdot x}\|_{1,2}+\| e^{\varepsilon\cdot x}(H-w)^{-n/2}e^{-\varepsilon\cdot x}\|_{2,\infty}\leqq C \tag{3.1}$$

for all $|\varepsilon|\leqq 1$, with some constant $C$.

Now let $K\subset\subset\rho(H_2)$ and $z\in K$. Taking $n^{th}$ powers of the resolvent equation

$$(H_2-z)^{-1}=(H_2-w)^{-1}+(z-w)(H_2-w)^{-1}(H_2-z)^{-1},$$

we obtain

$$(H_2-z)^{-n}=(H_2-w)^{-n}\sum_{j=0}^{n}\binom{n}{j}(z-w)^j(H_2-z)^{-j}. \tag{3.2}$$

To prove Proposition 3.3, it is clearly enough to show that, for any $0\leqq j\leqq n$, the operator

$$(H_2-w)^{-n}(H_2-z)^{-j}=(H_2-w)^{-n/2}(H_2-z)^{-j}(H_2-w)^{-n/2} \tag{3.3}$$

is an integral operator with kernel $G_{nj}(x,y;z)$ satisfying

$$|G_{nj}(x,y;z)|\leqq C_{nj}e^{-\alpha_{nj}|x-y|}\quad (z\in K, x,y\in\mathbb{R}^v), \tag{3.4}$$

with some positive constants $C_{nj}$, $\alpha_{nj}$.

So let $0\leqq j\leqq n$. By Proposition 3.8, there exists $\varepsilon_0>0$ such that

$$\| e^{\varepsilon\cdot x}(H_2-z)^{-j}e^{-\varepsilon\cdot x}\|\leqq C'\quad (|\varepsilon|\leqq\varepsilon_0, z\in K). \tag{3.5}$$

By (3.3) we have

$$e^{\varepsilon\cdot x}(H_2-w)^{-n}(H_2-z)^{-j}e^{-\varepsilon\cdot x}=(e^{\varepsilon\cdot x}(H_2-w)^{-n/2}e^{-\varepsilon\cdot x})(e^{\varepsilon\cdot x}(H_2-z)^{-j}e^{-\varepsilon\cdot x})\cdot(e^{\varepsilon\cdot x}(H_2-w)^{-n/2}e^{-\varepsilon\cdot x}),$$

and hence it follows from (3.1), (3.5), that

$$\| e^{\varepsilon\cdot x}(H_2-w)^{-n}(H_2-z)^{-j}e^{-\varepsilon\cdot x}\|_{1,\infty}\leqq C''\quad (|\varepsilon|\leqq\varepsilon_0, z\in K).$$

Now it follows from a classical theorem of Dunford and Pettis ([4; Theorem 2.2.5, p. 348]; see also Simon [13; Cor. A.1.2]), that the operator $e^{\varepsilon\cdot x}(H_2-w)^{-n}(H_2-z)^{-j}e^{-\varepsilon\cdot x}$ is an integral operator, and its kernel $G_{nj,\varepsilon}(x,y;z)$ satisfies the estimate

$$\| G_{nj,\varepsilon}(\cdot,\cdot;z)\|_\infty\leqq C'''\quad (|\varepsilon|\leqq\varepsilon_0, z\in K). \tag{3.6}$$

In particular, the above statements apply to $\varepsilon=0$, and we see that $(H_2-w)^{-n}\times(H_2-z)^{-j}$ is an integral operator with $L_\infty$-kernel $G_{nj}(\cdot,\cdot;z)$; clearly,

$$e^{\varepsilon\cdot(x-y)}G_{nj}(x,y;z)=G_{nj,\varepsilon}(x,y;z).$$

Therefore (3.6) implies

$$e^{\varepsilon_0|x-y|}|G_{nj}(x,y;z)|\leqq C'''\quad (z\in K). \qquad ■$$

*Acknowledgement.* One of the authors (R. H.) would like to thank P. Deift (Courant Institute, New York) for several useful discussions.

## References


1. Aizenman, M., Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators. Commun. Pure Appl. Math. **35**, 209–273 (1982)
2. Auterhoff, J.: Interpolationseigenschaften des Spektrums linearer Operatoren auf $L^p$-Räumen. Math. Z. **184**, 397–406 (1983)
3. Devinatz, A.: Schrödinger operators with singular potentials. J. Oper. Theory **4**, 25–35 (1980)
4. Dunford, N., Pettis, B. J.: Linear operations on summable functions. Trans. Am. Math. Soc. **47**, 323–392 (1940)
5. Hempel, R., Voigt, J.: On the $L_p$-spectrum of Schrödinger operators. J. Math. Anal. Appl. (to appear)
6. Jörgens, K.: Lineare Integraloperatoren. Stuttgart: B. G. Teubner, 1970
7. Kato, T.: Perturbation theory for linear operators. Second edition. Berlin Heidelberg, New York: Springer 1976
8. Reed, M., Simon, B.: Methods of modern mathematical physics I: Functional analysis. Revised and enlarged edition. New York: Academic Press 1980
9. Reed, M., Simon, B.: Methods of modern mathematical physics II: Fourier analysis, self-adjointness. New York: Academic Press 1975
10. Sigal, I. M.: A generalized Weyl theorem and $L^p$-spectra of Schrödinger operators. Preprint
11. Simon, B.: Functional integration and quantum physics. New York: Academic Press 1979
12. Simon, B.: Brownian motion, $L^p$ properties of Schrödinger operators and the localization of binding. J. Funct. Anal. **35**, 215–225 (1980)
13. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. New Ser. **7**, 447–526 (1982)
14. Voigt, J.: Absorption semigroups, their generators, and Schrödinger semigroups, J. Funct. Anal. (to appear)


Communicated by B. Simon

Received July 22, 1985; in revised form September 26, 1985