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Homogeneous Lyapunov Functions and Necessary Conditions for Stabilization*

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Abstract. We provide necessary conditions for the stabilization of nonlinear control systems with the additional requirement that a time-invariant *homogeneous* Lyapunov function exists for the closed-loop system.

Key words. Continuous feedback stabilization, Necessary conditions, Homogeneity, Lyapunov functions.

1. Introduction

This paper deals with the existence problem of a continuous time-invariant stabilizing feedback for a nonlinear control system. It is well known that the system has to satisfy a number of topological conditions which are generally not precluded by good controllability properties.

Brockett's result [2] in this area emphasizes that these necessary conditions can be interpreted as resulting from the existence of a (time-invariant) Lyapunov function for the closed-loop stabilized system. The general goal of this paper is to illustrate how an additional *homogeneity* assumption on the Lyapunov function leads to additional necessary conditions for the original system.

For *homogeneous* systems, the conditions that we are about to introduce turn out to be necessary for *homogeneous stabilization*, i.e., for the existence of a stabilizing feedback leaving the closed-loop system homogeneous, since in this case the existence of a homogeneous Lyapunov function is guaranteed (see [16]). In general, these conditions are necessary for a "homogeneous Lyapunov design" of the

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stabilizing feedback; for affine systems, they are necessary for the existence of a homogeneous control Lyapunov function (with a small control property). Control Lyapunov functions play an increasing role in the stabilization literature and natural candidates usually exhibit homogeneity properties.

The major contribution in this paper, contained in Section 3, is to show that particular *subsets* of the level sets of a homogeneous Lyapunov function can be used to obtain necessary conditions for asymptotic stability. In particular, for an n-dimensional system of differential equations admitting a homogeneous Lyapunov function, we characterize an index condition based on an $(n-2)$-dimensional subset of the level sets. This index condition is independent of the particular dilation with respect to which the Lyapunov function is assumed to be homogeneous.

In Section 3 we derive natural applications in a stabilization context. In Section 4, assuming feedback stabilization of a given control system, we derive necessary conditions for the existence of a homogeneous Lyapunov function for the stabilized system. Two conditions are obtained for the existence of a homogeneous control Lyapunov function. In particular, we recover a necessary condition for homogeneous stabilization, previously obtained by Dayawansa [8]. Finally, a homology condition is derived from our index condition. Unlike the previous ones, this condition is formulated independently of a particular dilation.

In Section 5 we discuss the interest of adding a dimension for stabilization. Under the extra assumption of the existence of a homogeneous Lyapunov function for the (extended) stabilized system, we illustrate by simple examples the benefit and some limitations of dynamic feedback in the stabilization problem. In particular, we provide simple examples of systems which are homogeneous, satisfy the necessary conditions for homogeneous stabilization existing in the literature, and are not stabilizable by homogeneous feedback.

2. Preliminaries

2.1. Necessary Conditions for the Stabilization Problem

Throughout this paper we adopt the following formulation for the stabilization problem: a **control system** is defined in an open set Ω of $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\dot{x} = f(x, u), \tag{S}$$

where $u \in \mathbb{R}^m$ is the control, $x \in \mathbb{R}^n$ is the state, and $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$; we assume the equilibrium condition

$$f(0, 0) = 0.$$

Then we investigate the existence of a **stabilizing feedback** for (S), i.e., a mapping u from $\mathbb{R}^n$ into $\mathbb{R}^m$ such that:

- $u \in C^0(\mathbb{R}^n; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^m)$.
- $u(0) = 0$.

- The null solution $x = 0$ of the closed-loop system

$$\dot{x} = F(x) := f(x, u(x)) \qquad (S_{cl})$$

is locally asymptotically stable.

The above class of controls is motivated as follows: on the one hand, smoothness of the control outside the origin is not a restriction compared with continuity (see [4]) and guarantees existence and uniqueness of solutions over the time interval $[0, +\infty)$ for every initial condition $x_0 \neq 0$; on the other hand, by imposing only continuity *at* the origin, we can consider situations in which the linearized system possesses unstable uncontrollable modes. Notice that by assumption $F \in C^1(\mathbb{R}^n \backslash \{0\}; \mathbb{R}^n) \cap C^0(\mathbb{R}^n; \mathbb{R}^n)$ and possesses no linearization at the origin in general.

It is known that in addition to some controllability requirements (in particular the system must be asycontrollable, see, for instance, [20]), the above stabilization problem is subject to topological necessary conditions. In particular, a standard result on asymptotic stability (see Theorem 52.1 of [15]) asserts that the closed-loop vector field F satisfies the following property: the index[1] of $-F$ at the origin is equal to one. A necessary condition for the stabilization of (S) is therefore the existence of a feedback $u \in C^0(\mathbb{R}^n; \mathbb{R}^m)$ such that $u(0) = 0$ and such that the index of the mapping $-F(\cdot) = -f(\cdot, u(\cdot))$ at the origin is equal to one. In the following we refer to this necessary condition as the *index condition* for stabilization.

The popular *Brockett's necessary condition* for stabilization [2], requiring that the image of the mapping f contains a neighborhood of the origin, can be derived from the above index condition. More recently, a stronger necessary condition, also derived from the index condition, has been obtained by Coron [3]. Brockett's and Coron's necessary conditions are known to be insufficient for stabilization, even when the system is locally controllable. They remain necessary for *dynamic* stabilization, i.e., stabilization of the extended system

$$\begin{aligned} \dot{x} &= f(x, u), \\ \dot{y} &= v, \end{aligned} \qquad (S_{dyn})$$

with $y \in \mathbb{R}^k$ an additional state vector and $v \in \mathbb{R}^k$ an additional control vector.

2.2. Homogeneity and Stabilization

When the differential equation (S_{cl}) admits a linearization at the origin, the first theorem of Lyapunov asserts that asymptotic stability of the *linear* approximation is sufficient for (local) asymptotic stability of the original system. Natural extensions have been obtained in the literature (see, for instance, [9] and [16]), showing that asymptotic stability of any *homogeneous* approximation is sufficient for (local)) asymptotic stability of the original system. This result applies to a general notion

[1] Let $F \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ have an isolated zero at the origin, and let U be a bounded domain enclosing the origin, sufficiently small such that F has no zero in $U \backslash \{0\}$. Then the *index* of F at the origin is equal to the topological degree of F in ∂U, noted $\deg(F, \partial U)$ (see, for instance, p. 124 of [10]).

of homogeneity, defined as follows: for each n-tuple $(r_1, \ldots, r_n)$ with $0 < r_i < \infty$ for each $i \in \{1, \ldots, n\}$, the mapping $h \in C((0, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$: $(s, x) \to h(s, x) = (s^{r_1}x_1, \ldots, s^{r_n}x_n)^T$ defines a **dilation** on $\mathbb{R}^n$. A vector field $F(x)$ is homogeneous of degree τ with respect to the dilation $h(s, x)$ if, for each i,

$$f_i(h(s, x)) = s^{r_i+\tau} f_i(x).$$

The **Euler vector field** associated to a dilation $h(s, x) = (s^{r_1}x_1, \ldots, s^{r_n}x_n)^T$ is defined by $v(x) = (r_1 x_1, \ldots, r_n x_n)^T$. The **homogeneous rays** are the solution curves of the differential equation $\dot{x} = v(x)$.

Homogeneous *control* systems play a similar role as local approximations of general nonlinear control systems. Their role was first emphasized in the framework of *local controllability*. Most of the existing necessary or sufficient conditions for local controllability have been derived by constructing particular approximations which retain the controllability property (see [14] for a survey). By construction, these approximations are homogeneous in the following (extended) sense:

Definition 1. The system (S) is **homogeneous** (of order τ) if there exists a dilation $h(s, x) = (s^{r_1}x_1, \ldots, s^{r_n}x_n)^T$ on $\mathbb{R}^n$ and a dilation $\tilde{h}(s, u) = (s^{q_1}u_1, \ldots, s^{q_m}u_m)$ on $\mathbb{R}^m$ such that,

$$\forall s > 0, \qquad f_i(h(s, x), \tilde{h}(s, u)) = s^{\tau + r_i} f_i(x, u), \qquad i = 1, \ldots, n.$$

Definition 2. The mapping $f_h(x, u)$ (or equivalently the control system (S_h)) is a (local) **homogeneous approximation** for the mapping $f(x, u)$ (the control system (S), respectively) if there exists a mapping $g \in C^0(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ such that

$$f(x, u) = f_h(x, u) + g(x, u)$$

and such that, for each $i \in \{1, \ldots, n\}$,

$$\frac{g_i(h(s, x), \tilde{h}(s, u))}{s^{\tau + r_i}} \to 0 \qquad \text{as} \quad s \to 0 \tag{1}$$

uniformly on the sphere S^{n+m-1}.

Homogeneous stabilization for homogeneous systems consists in restricting the class of admissible stabilizing feedbacks to those which lead to a homogeneous closed-loop system. This imposes the constraint

$$u(h(s, x)) = \tilde{h}(s, u(x)), \tag{2}$$

which relates the homogeneity of the feedback to the control dilation $\tilde{h}(s, u)$; namely, the jth component of the feedback law must be homogeneous of degree q_j with respect to the (state) dilation $h(s, x)$:

$$u_j(h(s, x)) = s^{q_j} u_j(x), \qquad j = 1, \ldots, m.$$

Definition 3. A homogeneous system is *stabilizable by homogeneous feedback* if there exists a stabilizing feedback law $u \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R}^m) \cap C^0(\mathbb{R}^n; \mathbb{R}^m)$ such that the closed-loop differential equation

$$F_H(x) := f(x, u_H(x)) \qquad (S_{\text{hom}})$$

is homogeneous, i.e., such that (2) holds.

By definition, homogeneous stabilization of a homogeneous approximation is sufficient for (local) stabilization of the original system because, with a homogeneous feedback, the higher-order terms neglected in the open-loop approximation become higher-order terms in the closed-loop system. In this sense, homogeneous stabilization is instrumental in the general stabilization problem.

2.3. *Homogeneous Lyapunov Functions*

Definition 4. The differential equation (S_{cl}) is said to admit a **homogeneous Lyapunov function** if, for some dilation $h(s, x)$, some positive integer p, and some real number k larger than $p \cdot \max_{1 \le i \le n} r_i$, there exists a function $V \in C^p(\mathbb{R}^n; [0, +\infty)) \cap C^\infty(\mathbb{R}^n \setminus \{0\}; [0, +\infty))$ such that

(i) $V(x) = 0 \Leftrightarrow x = 0$.
(ii) $\exists \varepsilon > 0, \forall x \in B(0, \varepsilon) \setminus \{0\}, \nabla V(x) \cdot F(x) < 0$.
(iii) V is homogeneous of order k:

$$\forall s > 0, \qquad V(h(s, x)) = s^k V(x). \qquad (3)$$

Specializing in a natural way the concept of control Lyapunov function (see, for instance, [1] and [21]), we say that the function V above is a **homogeneous control Lyapunov function** for the system (S) if condition (ii) is replaced by

(ii′) $\exists \varepsilon > 0, \forall x \in B(0, \varepsilon) \setminus \{0\}, \exists u \in \mathbb{R}^m, \nabla V(x) \cdot f(x, u) < 0$.

Using standard terminology, the homogeneous Lyapunov function is said to satisfy the "small control property" if (ii′) is satisfied with the additional constraint that $\|u\| \to 0$ as $\|x\| \to 0$. A recent converse Lyapunov theorem due to Rosier [16] asserts the existence of a **homogeneous** Lyapunov function for a homogeneous asymptotically stable differential equation. More precisely, if F is **homogeneous** with respect to some dilation $h(s, x)$ and if the null solution $x = 0$ of (S_{cl}) is asymptotically stable, then the system admits a Lyapunov function which satisfies condition (3) above. Moreover, the result is global, i.e., (ii) holds for $x \in \mathbb{R}^n \setminus \{0\}$.

Homogeneous Lyapunov functions have special properties which impose particular conditions on the vector field F. In the following proposition elementary properties are recalled which naturally follow from the additional property (3):

Proposition 1. *Let $V(x)$ be a positive definite function, homogeneous of order k. Then, for each $s > 0$, the following properties are satisfied*:

(a) *The level set* $V^s := \{x \mid V(x) = s\}$ *of* V *is homogeneous, i.e.,* $V^s = h(s^{1/k}, V^1)$.
(b) *The function* $\nabla V(x) \cdot \nu(x)$ *is constant on* V^s. *More generally,*

$$\forall x \neq 0, \qquad \nabla V(x) \cdot \nu(x) = kV(x). \tag{4}$$

(c) V^s *is homeomorphic to* S^{n-1}.
(d) *For each* $i \in \{1, \ldots, n\}$, $\partial V/\partial x_i$ *is homogeneous of order* $(k - r_i)$:

$$\frac{\partial V}{\partial x_i}(h(s, x)) = s^{k-r_i}\frac{\partial V}{\partial x_i}(x). \tag{5}$$

Proof. (a) is a direct consequence of (3).
(b) We evaluate $V(x)$ along the homogeneous rays:

$$V(h(e^s, x)) = \int_{-\infty}^{s} \nabla V(h(e^\tau, x)) \cdot \nu(h(e^\tau, x))\, d\tau. \tag{6}$$

By (3), the left-hand side of (6) is equal to $e^{ks}V(x)$. Differentiation of both sides of (6) with respect to s leads to

$$ke^{ks}V(x) = \nabla V(h(e^s, x)) \cdot \nu(h(e^s, x)). \tag{7}$$

The above equality evaluated in $s = 0$ gives (4).
(c) Consider the projection of V^s onto S^{n-1} along the homogeneous rays. For each $y \in S^{n-1}$, the homogeneous ray $h(\cdot, y)$ crosses V^s exactly once (if $V(y) = a$, then $h(e^t, y) \in V^s$ (only) for $t = \ln(s/a)^{1/k}$); the projection is therefore a continuous bijection between compact spaces, and therefore a homeomorphism. Finally, (d) is obtained by differentiating both members of (3) with respect to x_i; this gives

$$s^{r_i}\frac{\partial V}{\partial x_i}(h(t, x)) = s^k\frac{\partial V}{\partial x_i}(x). \quad \blacksquare \tag{8}$$

We stress that property (c) of the above proposition is a consequence of the homogeneity. In general, the level sets of a Lyapunov function are merely homotopy spheres [22], hence (c) cannot be proven because of the lack of a proof of the Poincaré conjecture. Property (b) of Proposition 1, usually called the Euler property, emphasizes that the vector field F *nowhere points outward "radially,"* i.e., that, for $x \neq 0$, the equality

$$F(x) = \lambda \nu(x) \tag{9}$$

is satisfied for no nonnegative λ. As an immediate consequence, we obtain the following corollary:

Corollary 1. *If* V *is a homogeneous Lyapunov function for* (S_{cl}), *then, for any scalar function* $\lambda \in C(\mathbb{R}^n; [0, +\infty))$, V *is also a Lyapunov function for the new system*

$$\dot{x} = F(x) - \lambda(x)\nu(x).$$

Proof. V is a Lyapunov function for the new system since, from (4),

$$\dot{V} = \nabla V(x) \cdot F(x) - k\lambda(x)V(x) < \nabla V(x) \cdot F(x) < 0, \qquad \forall x \neq 0. \quad \blacksquare$$

3. More Properties for Homogeneous Lyapunov Functions

This section contains the main contribution of the paper. Assuming that V is a homogeneous Lyapunov function for (S_{cl}), we derive an additional property for V which in turn leads to additional constraints on the vector field F for asymptotic stability. For simplicity, the results of Sections 3.2 and 3.3 assume homogeneity of the vector field F. We show in Section 3.4 how the results can be extended to general differential equations.

3.1. Motivation

Considering the differential equation (S_{cl}), we isolate the p first components of the vector field F, $1 \leq p \leq n$, and define the mapping

$$(F)_p: \mathbb{R}^n \to \mathbb{R}^p: x \to (F_1(x), \ldots, F_p(x))^T.$$

In the following we denote by E_p the canonical subspace

$$E_p = \{x \in \mathbb{R}^n | x_{p+1} = \cdots = x_n = 0\}$$

and by $(E_p)^\perp$ the orthogonal complement of E_p in $\mathbb{R}^n$. The projection of $x \in \mathbb{R}^n$ onto E^p (parallel to $(E_p)^\perp$) is denoted by $\pi_p(x)$. With this notation, $(F)_p = \pi_p \circ F$.

Then we investigate the possibility of characterizing particular properties of the mapping $(F)_p$ from particular properties of the Lyapunov function V.

Our motivation is twofold:

- First, we would like to "refine" the index condition recalled in Section 2.1. The index condition can be considered as resulting from a *global* property of the level sets in the following sense: evaluate the index of $-F$ by means of $\deg(-F, V^s)$ for $s > 0$ sufficiently small; by definition (see Definition 4 (ii)), the continuous mapping

$$\Phi \in C^0([0, 1] \times \mathbb{R}^n; \mathbb{R}^n): \Phi(t, x) = (1 - t)\nabla V + tF(x)$$

  defines a homotopy between $-F$ and ∇V; since the degree is unchanged by homotopy, the identity $\deg(\nabla V, V^s) = 1$ gives the index condition. By analogy, it is tempting to derive an index condition on $(F)_p$ by evaluating a degree quantity on a well-chosen subset of the level set V^s. Typically, the index condition on F does not distinguish the asymptotically stable vector field $\Psi(x) = (-x_1, -x_2, -x_3)$ from the unstable vector field $\Phi(x) = (-x_1, x_2, x_3)$. On the contrary, the index of the planar vector field $(\Psi)_2(x) = (-x_1, -x_2)$ is different from the index of the vector field $(\Phi)_2(x) = (-x_1, x_2)$. We would like to capture this distinction in more general situations.
- There exists a second interest in characterizing necessary conditions for the asymptotic stability of (S_{cl}) from the mapping $(F)_p$. The information about the mapping $(F)_p$ might be better than the information about the mapping F. This is particularly true in the framework of *dynamic* feedback stabilization: dynamic feedback stabilization results in an asymptotically stable $(n + p)$-dimensional differential equation (the n original equations plus the k added

equations defining the dynamic extension); necessary conditions for dynamic stabilization require the properties of a subsystem (that is, the n original equations) of the stabilized system to be characterized, independently of the added dynamics.

3.2. Contracting Subsets

Definition 5. Let $s > 0$ be sufficiently small. V_p^s is called **a contracting p-subset** for (S_{cl}) if there exists a Lyapunov function V for (S_{cl}) such that:

(i) V_p^s is a compact p-submanifold of the level set V^s of V; it is the disjoint union of two points if $p = 1$ and connected if $p > 1$.
(ii) $\forall x \in V_p^s, \forall k \in \{p+1, \dots, n\}, (\partial V/\partial x_k)(x) = 0$.
(iii) $\pi_p(V_p^s)$ separates[2] the origin from ∞ in E_p.

Following the above definition, it is not difficult to prove that if V is a Lyapunov function for (S_{cl}), then V^s is an n-contracting subset for V^s (see Lemma 3.1 in [17]). We can therefore interpret a p-contracting subset as a subset of V^s which preserves the topological properties (i) and (iii) of a level set, and which has the additional property (ii).

A natural question is whether contracting subsets exist in general. One easily proves (without any homogeneity assumption on the system or on the Lyapunov function) the existence of a contracting 1-subset $V_1^s = \{x^-, x^+\}$ (see Proposition 3.2 of [17]). On the contrary, establishing the existence of contracting p-subsets for $1 < p < n$ is nontrivial in general. The following theorem provides an affirmative answer for $p = n - 1$ under the assumption that the system is homogeneous.

Theorem 1. *Let F be homogeneous in (S_{cl}) and $n > 2$. If the null solution $x = 0$ is asymptotically stable, then there exists a homogeneous Lyapunov function V for (S_{cl}) such that, for each $s > 0$, V^s contains a contracting $(n-1)$-subset V_{n-1}^s.*

Proof. The construction of V_{n-1}^s is achieved in four steps, each of which is proven in Section 6.1.

Step 1. Under the assumptions of the theorem, we prove the existence of a homogeneous Lyapunov function V for (S_{cl}) such that the set defined by

$$M := \left\{x \in S^{n-1}: \frac{\partial V}{\partial x_n}(x) = 0\right\} \tag{10}$$

is a compact manifold of codimension 1 in S^{n-1}.

Step 2. Denote by $x^+ := (0, \dots, 0, 1)^T$ (resp. $x^- := (0, \dots, 0, -1)^T$) the north (resp. south) pole of S^{n-1}. Then M separates x^+ and x^- in S^{n-1}.

[2] The separation property is to be understood in the usual topological sense: $E_p \backslash \pi_p(V_p^s)$ is the union of two disjoint open sets A and B, with $0 \in A$ and B unbounded.

Step 3. Let $n > 2$. Then there exists a connected component of M, say M^*, that separates x^+ and x^- on S^{n-1}.

Step 4. For each $s > 0$, the set defined by the projection of M^* onto V^s along the homogeneous rays is a contracting $(n-1)$-subset. ∎

Corollary 2. *Let F be homogeneous in (S_{cl}) and $n = 3$. Then the system admits a homogeneous Lyapunov function V such that, for each $s > 0$, V^s contains a sequence of nested (homogeneous) contracting subsets*

$$(V^c)_1 \subset (V^c)_2 \subset (V^c)_3 = V^c.$$

Proof. See Section 6.2.

A natural question is whether the conclusions of Corollary 2 extend to higher-dimensional systems. The definition of contracting subsets naturally suggests a recursive construction from $p = n$ until $p = 1$: at each step of the construction an addition component of $\nabla V(x)$ is set to zero. The construction we have adopted for the proof of Theorem 1 uses special connectedness properties of the sphere. In general there is no reason to expect that $M^* \approx S^{n-2}$ and more generally that $(V^s)_i \approx S^{i-1}$ for $i \in \{3, \ldots, n-1\}$. As a consequence an alternative argument to Step 3 must be found for a generalization of Theorem 1, i.e., for the construction of $(V^s)_{i-1}$ from $(V^s)_i$. More generally the following question can be asked. Let F be homogeneous in (S_{cl}). Does there exist a homogeneous Lyapunov function V for (S_{cl}) such that, for each $s > 0$, V^s contains a sequence of nested homogeneous contracting subsets

$$(V^s)_1 \subset \cdots \subset (V^s)_{n-1} \subset V^s?$$

3.3. *An Index Condition on $(F)_{n-1}$*

Recalling our interpretation of the index condition for F resulting from a *global* property of the level sets V^s, it is very natural to expect an index-like condition for the mapping $(F)_p$ resulting from the existence of p-contracting subsets V_p^s. With this in mind we introduce the following definition. We denote by K_p the cone

$$K_p := \bigcup_{s>0} h(s, V_p^1). \tag{11}$$

Definition 6. Let G a continuous mapping from K_p into $\mathbb{R}^p$. Suppose that, for some $\varepsilon > 0$, G is nonsingular (i.e., $G(x) \neq 0$) in $K_p \cap B(0, \varepsilon)$. Then for s sufficiently small, the degree of G on V_p^s, i.e., $\deg(G, V_p^s)$, is well defined and independent of s. We call this value the **(generalized) index** of G at 0.

Remark 1. Notice that if G is a vector field in $\mathbb{R}^n$ and if the origin is an isolated zero of G, then the (generalized) index of G at the origin in the cone $\mathbb{R}^n \setminus \{0\}$ is the (classical) index of G at the origin in $\mathbb{R}^n$.

Although extensions exist (see [10]), the definition of topological degree is usually considered for continuous mappings between *orientable* compact connected manifolds without boundaries. By definition, a contracting subset is a compact connected manifold without boundary. However, a contracting subset need not be orientable; in such a situation, the degree is an integer mod 2. For $p = n - 1$, orientability of V_{n-1}^s can be easily established (see Lemma 3.3 of [17]). So throughout this paper the (generalized) index of Definition 6 is an integer.

Theorem 2. *Suppose that there exists a homogeneous Lyapunov function for* (S_{cl}). *For each* $p \in \{1, n-1, n\}$, *consider the cone* K_p *defined by* (11). *Then the (generalized) index of* $(-F)_p$ *at zero in* K_p *is equal to one.*

Proof. See Section 6.3.

Example 1. Let $n = 3$ and $F(x) = (x_1, -x_2, x_3)^T$. We want to "detect" the instability of (S_{cl}) by means of topological necessary conditions. The classical index condition is satisfied here but not *all* the conditions of Theorem 2: the index of $-(F)_2$ is equal to -1 regardless of the location of the manifold M^* on S^2. Also the index of $-(F)_1$ is necessarily equal to -1.

Example 2. Let $n = 3$ and $F(x) = (x_3 x_1, -x_3 x_2, \alpha(x))$ with α any function such that F is homogeneous. Since $(F)_2$ may not vanish in M^*, the sign of x_3 is necessarily invariant in M^*. As a consequence, $(F)_2$ is homotopic to $(x_1, -x_2)$ on M^* and $\deg(-(F)_2, M^*) = -1$. We conclude that the null solution of (S_{cl}) is not asymptotically stable. Notice that the index condition on $(F)_1$ can be satisfied in this example. This illustrates the important role played by the topological properties of M^* (separation and connectedness).

3.4. Extensions to Nonhomogeneous Systems

At this point is must be emphasized that the homogeneity assumption on the vector field F plays a minor role in the developments of the previous sections. The proof of Theorem 1 requires a slight modification of the original Lyapunov function on the unit sphere $S(0, 1)$. By using the homogeneity properties of F, the local modification can be extended globally, leading to a new Lyapunov function which is still homogeneous (see (41)).

If the vector field F is nonhomogeneous, the modification of the original homogeneous Lyapunov function may result in a positive definite function which maintains the properties of the original Lyapunov function only locally around $S(0, 1)$. Nevertheless, this is sufficient to construct the manifold M^* and to establish the degree property of $(F)_{n-1}$ in M^*. As shown in the next sections, this is sufficient for the applications. Repeating the construction on each sphere $S(0, \varepsilon)$, $\varepsilon > 0$, we obtain the following theorem:

Theorem 3. *Suppose that system* (S_{cl}) *admits a homogeneous Lyapunov function. Then, for each* $\varepsilon > 0$, *there exists a positive definite function* V, *possibly nonhomo-*

geneous, and a compact connected orientable manifold $M^* \subset S(0, \varepsilon)$ *such that:*

(a) M^* *separates the north pole from the south pole on* $S(0, \varepsilon)$.
(b) $\forall x \in M^*, (\partial V/\partial x_n)(x) = 0$.
(c) $\forall x \in S(0, \varepsilon), \nabla V(x) \cdot F(x) < 0$ *and* $\nabla V(x) \cdot \nu(x) > 0$.
(d) $\deg(-(F)_{n-1}, M^*) = 1$.

The proof of the above theorem identically follows the construction of Theorem 1.

Remark 2. It must be emphasized that conditions (a), (b), and (d) of Theorem 3 are formulated independently of the particular dilation with respect to which V is assumed to be homogeneous. These conditions are reasonably expected to hold under weaker assumptions than homogeneity of V.

4. Necessary Conditions for Homogeneous Lyapunov Design

In this section we provide necessary conditions for the following problem: Does there exist a stabilizing feedback for the control system (*S*) such that there exists a **homogeneous** Lyapunov function for the closed-loop system?

For *homogeneous* systems, necessary conditions for the problem above are necessary conditions for homogeneous stabilization. This is a consequence of the particular converse Lyapunov theorem recalled in Section 2.3. For *general* systems, necessary conditions for the above problem are necessary conditions for a homogeneous Lyapunov design of the stabilizing feedback. In particular, we obtain necessary conditions for the existence of a *homogeneous* control Lyapunov function.

4.1. An Hautus-Like Condition

The first condition is a direct consequence of Corollary 1. It has been previously proved in [5].

Proposition 2. *Let* V *be a homogeneous control Lyapunov function for* (*S*). *Then, for each function* $\lambda \in C^0(\mathbb{R}^n; [0, +\infty))$, V *is also a homogeneous control Lyapunov function for the system*

$$\dot{x} = f(x, u) - \lambda(x)\nu(x). \tag{12}$$

Proof. Let V a homogeneous control Lyapunov function for (*S*). By definition, the following holds: given $x \neq 0$, there exists a $u \in \mathbb{R}^m$ such that

$$\nabla V(x) \cdot f(x, u) < 0.$$

By property (b) of Proposition 1, this implies

$$\nabla V(x) \cdot (f(x, u) - \lambda(x)\nu(x)) < 0,$$

which shows that V is a homogeneous control Lyapunov function for (12). ■

Corollary 3. *Let* (*S*) *be homogeneous of order* τ *and stabilizable* (*resp. dynamically stabilizable*) *by homogeneous feedback. Then, for each function* $\lambda \in C^0(\mathbb{R}^n; [0, +\infty))$

homogeneous of order τ, the homogeneous system

$$\dot{x} = f(x, u) - \lambda(x)v(x) \tag{13}$$

is stabilizable (resp. dynamically stabilizable) by homogeneous feedback.

Example 3. Consider the single input planar system

$$\begin{aligned} \dot{x}_1 &= x_1 + u, \\ \dot{x}_2 &= 3x_2 + x_1 u^2. \end{aligned} \tag{14}$$

This system is homogeneous with respect to the dilation $h(s, x_1, x_2, u) = (sx_1, s^3x_2, su)$. By Corollary 3, a Lyapunov function for (14) cannot be homogeneous with respect to the dilation $h(s, x_1, x_2) = (sx_1, s^3x_3)$ since this would imply stabilization of the system

$$\begin{aligned} \dot{x}_1 &= u, \\ \dot{x}_2 &= x_1 u^2, \end{aligned} \tag{15}$$

which does not satisfy Brockett's necessary condition for (dynamic) stabilization. We conclude in particular that system (14) is not (dynamically) stabilizable by homogeneous feedback.

Remark 3. In general, the necessary condition of Corollary 3 is *not* necessary for (nonhomogeneous) stabilization. We have shown in [18] that system (14) is stabilizable by nonhomogeneous feedback.

Remark 4. The condition of Corollary 3 can be compared with the Popov–Bellevitch–Hautus criterion for stabilization of *linear* systems (see [13]). For linear systems, the condition also becomes sufficient when considering *complex* values for $\lambda = \lambda_1 + i\lambda_2$ with nonnegative real part ($\lambda_1 \geq 0$).

4.2. The Hautus-Like Condition and Contracting Subsets

The second condition is a direct consequence of the existence of contracting $(n-1)$ subsets for homogeneous Lyapunov functions.

Proposition 3. *Assume that (S) is stabilizable and that V is a homogeneous Lyapunov function for the closed-loop system. Then there exists a mapping $u \in C^1(S^{n-1}; \mathbb{R}^m)$ such that there exists no path from x^+ to x^- entirely contained in*

$$A_{+0} := \{x \in S^{n-1}: (F)_{n-1}(x) = \lambda(v)_{n-1}(x), \lambda \geq 0\},$$

where $F(\cdot) = f(\cdot, u(\cdot))$.

Proof. Let u be a stabilizing feedback for (S) such that V is a homogeneous Lyapunov function for the closed-loop system and denote by F the (asymptotically stable) closed-loop vector field. By Theorem 1, there exists a compact connected manifold M^* that separates x^+ and x^- on S^{n-1}, and such that $\partial V/\partial x_n$ identically vanishes on M^* (modulo a possible redefinition of V). Suppose that our necessary condition is violated, i.e., that there exists a path from x^+ to x^- entirely contained in A_{+0}. Then by the separation property, M^* necessarily intersects A_{+0} at some point $\bar{x}$. By definition,

$$\nabla V(\bar{x})\cdot F(\bar{x}) = (\nabla V)_{n-1}(\bar{x})\cdot(F)_{n-1}(\bar{x}) = \lambda\nabla V(\bar{x})\cdot v(\bar{x}) \geq 0,$$

which contradicts the assumption that V is a Lyapunov function for the closed-loop system. ■

The following corollary immediately follows as a particular case.

Corollary 4. *Let (S) be of the particular form*

$$\begin{cases} \dot{x}_1 &= F_1(x_1, \ldots, x_n), \\ \vdots & \vdots \\ \dot{x}_{n-1} &= F_{n-1}(x_1, \ldots, x_n), \\ \dot{x}_n &= u. \end{cases} \tag{16}$$

Suppose that V is a homogeneous control Lyapunov function for (S) and that the small control property is satisfied. Then there exists no path from x^+ to x^- entirely contained in A_{+0}.

Remark 5. If system (S) is homogeneous, then the necessary condition of Corollary 4 is necessary for homogeneous stabilization. This was independently proven by Dayawansa in [7]. Our approach emphasizes that the result holds from the sole assumption that the closed-loop system (not necessarily homogeneous) admits a homogeneous Lyapunov function.

Remark 6. In general, the condition of Corollary 4 is not necessary for stabilization, even if the control system is homogeneous, affine, and controllable: system (14) with an integrator is of the form (16), homogeneous, and controllable. However, it is stabilizable by nonhomogeneous feedback (see [18] for details).

Remark 7. The degree condition $\deg(-(F)_{n-1}, M^*) = 1$ derived in the previous section is implied by the condition of Proposition 3: indeed, if this last condition is satisfied, then it is always possible to separate S^{n-1} by means of a compact connected manifold M^* of codimension one which avoids the set A_{+0}. Then the mapping

$$\Phi \in C([0, 1] \times \mathbb{R}^n; \mathbb{R}^{n-1})\colon \Phi(t, x) = (t-1)(v)_{n-1}(x) + t(F)_{n-1}(x)$$

does not vanish on $[0, 1] \times M^*$ since this would imply, for some $(\bar{t}, \bar{x})$,

$$(F)_{n-1}(\bar{x}) = \frac{\bar{t}}{1-\bar{t}}(v)_{n-1}(\bar{x}),$$

which contradicts the assumption $M^* \cap A_{+0} = \varnothing$. As a consequence, the mapping

Φ defines a homotopy on M^* between $\Phi(0, \cdot) = -(v)_{n-1}$ and $\Phi(1, \cdot) = (F)_{n-1}$ and the index condition on $(F)_{n-1}$ is satisfied.

4.3. Necessary Conditions Independent of a Particular Dilation

The conditions of Propositions 2 and 3 are explicitly dependent (through the Euler vector field) on the particular dilation with respect to which the Lyapunov function is homogeneous. In this sense, they are very sensitive to the particular choice of the homogeneous Lyapunov function. As an illustration, consider system (14); a simple argument allows us to show that a Lyapunov function for the closed-loop system cannot be homogeneous with respect to the dilation $h(sx_1, sx_3) = (sx_1, s^3x_3)$ (see Example 3). However, the argument fails if any other dilation is considered for the Lyapunov function. In contrast, the degree condition $\deg(-(F)_{n-1}, M^*) = 1$ of Theorem 3 is independent of a particular dilation.

The following example, originally considered in [3], illustrates a situation where Proposition 2 gives conclusions only for one particular dilation while a degree argument provides conclusions for any dilation.

Example 4 [3]. Consider the single-control planar system

$$\begin{aligned} \dot{x}_1 &= u^2(x_1 - u), \\ \dot{x}_2 &= u^2(x_2 - x_1). \end{aligned} \tag{17}$$

System (17) is homogeneous with $\tau = 2$ and $r_1 = r_2 = r_3 = 1$. It has been shown in [3] that it is locally controllable and satisfies the index condition but is not stabilizable. The same conclusion holds when adding a pure integrator. Now consider the third-order dynamical extension

$$\begin{aligned} \dot{x}_1 &= u^2(x_1 - u), \\ \dot{x}_2 &= u^2(x_2 - x_1), \\ \dot{y} &= v. \end{aligned} \tag{18}$$

Let $h(s, (x, y))$ be an arbitrary dilation. Assume that, for some feedback $(u(x, y), v(x, y))$, V is a Lyapunov function for the closed-loop system (18). We show that V cannot be homogeneous with respect to $h(s, (x, y))$.

First suppose that $(r_1, r_2) = (1, 1)$. Then the conclusion follows from Corollary 2: choosing $\lambda(x) = u^2(x)$, we conclude that the dynamical homogeneous stabilization of (17) would imply the dynamical homogeneous stabilization of

$$\begin{aligned} \dot{x}_1 &= -u^3, \\ \dot{x}_2 &= -x_1 u^2 \end{aligned}$$

which does not satisfy Brockett's necessary condition for dynamic stabilization. This simple argument fails when considering a different dilation.

Next assume $(r_1, r_2) \neq (1, 1)$. Choose $\varepsilon > 0$ arbitrary small. By Theorem 3, there exists a simple closed curve M^* on $S(0, \varepsilon)$, separating the north pole from the south pole, such that conditions (b) and (c) of Theorem 3 are satisfied. This imposes in particular that $u(x)$ does not vanish on M^*. Assume that $u(x) > 0$ in M^* (a similar argument applies if $u(x) < 0$ in M^*). Using the proof of Corollary 2, it is easily shown that M^* contains a pair $\{x^-, x^+\}$, with $x_1^- < 0$ and $x_1^+ > 0$ such that $F_1(x^-) > 0$ and $F_1(x^+) < 0$. However, this is a contradiction because by assumption $x_1^- - u(x_1^-) < 0$. We conclude that V is not a Lyapunov function for system (18).

We do not know if system (18) is stabilizable, and, in particular, if the above argument applies when V is not homogeneous.

As recalled in the preliminaries, a homology condition, necessary for stabilization, was derived by Coron in [3] from the (classical) index condition. Following the same lines, we can derive a homology condition from the degree condition $\deg(-(F)_{n-1}, M^*) = 1$ of Theorem 3. This homology condition is necessary for the existence of a homogeneous Lyapunov function for the stabilized system and is formulated independently of a particular dilation. We do not know if the homogeneity assumption on V might be removed. Following the notations used in [3], for an integer $k < n$, we denote by $H_k(X)$ the kth singular homology group of a topological space X with integer coefficients. If f is a continuous mapping from X to Y, it induces a homomorphism f_* from $H_k(X)$ into $H_k(Y)$.

Theorem 4. *Let, for* $\varepsilon \in (0, \infty]$,

$$\Omega_\varepsilon^{n-1} = \{(x, u) \in \Omega: \pi_{n-1} \circ f(x, u) \neq 0, \|x\| + \|u\| < \varepsilon\}. \tag{19}$$

Assume that (S) is stabilizable. Then a necessary condition for the existence of a homogeneous Lyapunov function for the closed-loop system (S_{cl}) *is that, for all* $\varepsilon \in (0, \infty]$,

$$(\pi_{n-1} \circ f)_*(H_{n-2}(\Omega_\varepsilon^{n-1})) = H_{n-2}(\mathbb{R}^{n-1}\backslash\{0\}). \tag{20}$$

Remark 8. For an affine single control system of the form (16), condition (20) does not add a new condition with respect to Coron's condition. More generally, it is shown in [3] that, for affine systems of the form

$$\dot{x} = f(x, y),$$

$$\dot{y} = u,$$

where $(x, y) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ and $u \in \mathbb{R}^m$, a necessary condition for stabilization is given by

$$(f)_*(H_{n-m-1}(\Omega_\varepsilon^n)) = H_{n-m-1}(\mathbb{R}^{n-m}\backslash\{0\}))$$

with, for each $\varepsilon \in (0, \infty]$,

$$\Omega_\varepsilon^n = \{(x, y) \in \mathbb{R}^{n-m} \times \mathbb{R}^m: f(x, y) \neq 0 \text{ and } \|x\| + \|y\| < \varepsilon\}.$$

5. Adding a Dimension for Stabilization

As an application of the results of Section 3, we discuss in this section the interest and some limitations of adding an integrator for the stabilization problem, once again assuming the existence of a *homogeneous* Lyapunov function for the (extended) closed-loop system.

5.1. The Index Condition: Explicit Versus Implicit

The first question we address is the role of the index condition when adding dimensions: although Brockett's condition and Coron's condition are necessary for dynamic feedback stabilization, it was shown by Coron and Praly [6] that the index condition is not necessary for dynamic feedback stabilization. We believe that the results of Section 3 provide a simple explanation of why the addition of dimensions may or may not help when the index condition is not satisfied on the original system.

For the purpose of illustration, we restrict ourselves to single-input ($m = 1$) systems (S) and consider one-dynamical ($k = 1$) extensions (S_{ext}). The subsequent degree calculations on the unit sphere can be reproduced on arbitrary small spheres centered at the origin (as a consequence, this is not a loss of generality with respect to the *local* stabilization problem).

The index condition is satisfied on the original system (S) if, for some mapping $\tilde{u} \in C(\mathbb{R}^n; \mathbb{R})$ satisfying $\tilde{u}(0) = 0$, the following holds:

$$\deg(-f(\cdot, \tilde{u}(\cdot)), S^{n-1}) = 1. \tag{21}$$

On the other hand, the "refined" degree condition of Theorem 2 is satisfied for the extended system (S_{ext}) if, for some mapping $\bar{u} \in C(\mathbb{R}^{n+1}; \mathbb{R})$ satisfying $\bar{u}(0) = 0$, the following holds:

$$\deg(-f(\cdot, \bar{u}(\cdot)), M^*) = 1, \tag{22}$$

where M^* is a suitable manifold of codimension one in S^n.

Conditions (21) and (22) can be compared as follows: if (21) is satisfied, then (22) holds with $\bar{u}(x, y) = \tilde{u}(x)$ and with M^* defined as the (radial) projection on S^n of the manifold

$$M' := \{(x, y) \in S^{n-1} \times \mathbb{R}: y = \tilde{u}(x)\}.$$

The necessary condition of Theorem 2 is therefore satisfied for the extended system if the index condition is satisfied for the original system. Moreover, the manifold M^* can be chosen to be the graph of an *explicit* function of the original coordinates. On the contrary, condition (22) does not imply condition (21). In particular, the manifold M^* might be the graph of an *implicit* function $h(x, y) = 0$. We illustrate the above considerations by means of simple examples.

The first example, originally considered in [6], provides a homogeneous system which does not satisfy the index condition and therefore is not stabilizable. Adding a dimension, a system is obtained which satisfies the index condition and also the "refined" index condition of Theorem 2. The extended system is indeed shown to be stabilizable by homogeneous feedback in [6].

Example 5 [6]. Consider the n-dimensional system ($n \geq 2$)

$$\dot{x} = -\sigma_1(x, u)x \tag{23}$$

with

$$\sigma_1(x, u) = \|x\|^6 - C^2\left(u^3 - \left(\sum_{i=1}^{n-1} x_i^2\right)u + x_n^3\right)^2. \tag{24}$$

The system is homogeneous with $\tau = 6$, and $r_1 = \cdots = r_n = 1$. The index condition is not satisfied for n odd and C sufficiently large: Indeed, the index of the closed-loop system is defined only if σ_1 does not vanish in a neighborhood of the origin. If σ_1 is positive outside the origin, the vector field $-f(\cdot, u(\cdot))$ is homotopic to x and therefore has an index $+1$. If σ_1 is negative outside the origin, the vector field $-f(\cdot, u(\cdot))$ is homotopic to $-x$ and the index is therefore -1 if n is odd.

For C large enough, it is shown in [6] that each continuous feedback $u(x)$ such that 0 is an isolated singularity of the closed-loop system implies $\sigma_1 < 0$. The index condition is therefore not satisfied.

In contrast, the set $M^* := \{(x, y) \in S^{n-1}: y^3 - (\sum_{i=1}^{n-1} x_i^2)y + x_n = 0\}$ may serve as an $(n-1)$-contracting subset. For each $(x, y) \in M^*$, we have $\sigma_1(x, y) = \|x\|^6 > 0$ and therefore $\deg(-f(\cdot, u(\cdot)), M^*) = 1$ if $u(x, y) := y$. It follows that the refined index condition of Theorem 2 is satisfied.

It is shown in [6] that a homogeneous stabilizing feedback indeed exists for the extended system

$$\begin{aligned} \dot{x} &= -\sigma_1(x, y)x, \\ \dot{y} &= u. \end{aligned} \tag{25}$$

The second example also starts from a homogeneous system which does not satisfy the index condition. Adding one dimension, a system is obtained which is asycontrollable and satisfies the index condition. However, the refined index condition of Theorem 2 is not satisfied, showing that no homogeneous Lyapunov function exists for the closed-loop extended system. In particular, the extended system is not stabilizable by homogeneous feedback, and, when adding a pure integrator, the resulting affine system does not admit a homogeneous control Lyapunov function.

Example 6. Consider system (23) with σ_1 replaced by

$$\sigma_2(x, u) = x_n u^3 - \left(\sum_{i=1}^{n-1} x_i^2\right)^{3/2} (u + x_n). \tag{26}$$

The system $\dot{x} = -\sigma_2(x, u)x$ is homogeneous of order four with respect to the standard dilation. Consider on S^{n-1} the local coordinate $\theta = x_n/\sum_{i=1}^{n-1} x_i^2 \in (-\infty, +\infty)$. Figure 1 shows the graph of the function $\sigma_2(\bar{x}_1, \ldots, \bar{x}_{n-1}, x_n, u) = \sigma_2(\theta, u) = 0$ in the plane (θ, u).

Similarly to the previous example, each continuous feedback $u(x)$ such that 0 is an isolated singularity of the closed-loop system implies $\sigma_2 < 0$. The index condition is therefore not satisfied when n is odd.

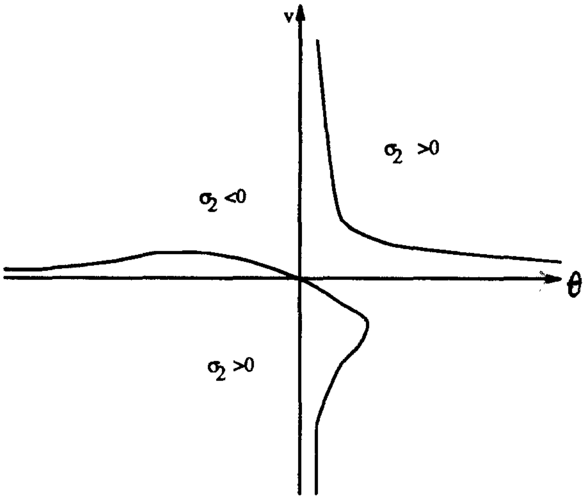

Fig. 1. Graph of $\sigma_2(\theta, u) = 0$.

Adding a dimension, the $(n + 1)$-dimensional system

$$\dot{x} = -\sigma_2(x, u)x, \qquad \dot{y} = v \tag{27}$$

is obtained. The index condition is now satisfied by choosing the feedback $u = u(x)$, $v = y$ with u a (homogeneous) function such that $\sigma_2 < 0$ for $x \neq 0$. The closed-loop vector field is homotopic to $(x, y)^T$ and the index is $+1$ since $n + 1$ is even. System (27) is also asycontrollable by inspection of the control Lyapunov function $V(x, y) = x^T x + y^2$. For each $x \neq 0$, there exists a u_x such that $\sigma_2(x, u_x) > 0$; therefore, for each $(x, y) \neq (0, 0)$, there exists a couple $(u, v) := (u_x, -y))$ such that $\dot{V}(x, y, v, u) < 0$. This implies asycontrollability (see [20]).

In contrast to the first example, the refined index condition of Theorem 2 is not satisfied. In other words, it does not help in the present example to consider another manifold in order to satisfy condition (22): it is easily verified on Fig. 1 that not only every continuous *function* $v(\theta)$ defined in $(-\infty, +\infty)$ but also every continuous *path* defined in the plane (θ, v) for $\theta \in (-\infty, +\infty)$ either lies in the region $\sigma_2(x, u) > 0$ or intersects the region $\sigma_2(x, u) = 0$. We conclude that no homogeneous Lyapunov function exists for the extended closed-loop system.

Remark 9. Notice that in both examples above the usual necessary conditions for stabilization are satisfied on the extended system. *Asycontrollability* is established by means of the control Lyapunov function $V(x, y) = x^T x + y^2$: for each (x, y), there exists a (u, v) such that $\dot{V}(x, y, u, v) < 0$. The *index condition* is satisfied since, in each case, a (homogeneous) feedback exists such that,

$$\forall x \neq 0, \qquad \sigma_i(x, u(x)) < 0, \qquad i = 1, 2.$$

Choosing $v = -y$, the index condition is satisfied for the extended system.

5.2. Beyond Topological Conditions

The question addressed in the previous section can be summarized as follows: *Suppose that (S) does not satisfy the index condition. When does it prevent the existence of a homogeneous Lyapunov function for the extended system* (S_{ext})?

Going beyond the index condition, we first remark that the discussion of the above section can be slightly extended as follows: suppose that, for every feedback $u(x)$, the closed-loop system $f(x, u(x))$ satisfies, for some nonnegative λ and some $x \neq 0$, the relation $f(x, u(x)) = \lambda v(x)$. According to the previous discussion, this may or may not prevent the existence of a homogeneous Lyapunov function for the extended system.

More generally, the natural question is: *Suppose that (S) does not admit a homogeneous Lyapunov function. When does it prevent the existence of a homogeneous Lyapunov function for the extended system* (S_{ext})? For the purpose of illustration, consider the following planar single-input homogeneous system:

$$\begin{aligned} \dot{x}_1 &= -\sigma_2(x, u)(x_1 + x_2), \\ \dot{x}_2 &= -\sigma_2(x, u)(x_2 - x_1), \end{aligned} \tag{28}$$

where, as previously,

$$\sigma_2(x, u) = x_2 u^3 - x_1^3(u + x_2). \tag{29}$$

In compact notation, (28) is the scalar complex system $\dot{z} = -\sigma_2(z, u)(z + iz)$ with $z = x_1 + ix_2$. This system is clearly not stabilizable since each continuous feedback $u(z)$ such that $z = 0$ is an isolated singularity for the closed-loop system imposes $\sigma_2(z, u(z)) < 0$ in a neighborhood of the origin. It is natural to wonder if the addition of an integrator may help for stabilization. Noting that z is never parallel to iz, it is easily noticed that the set A_{+0} of Proposition 4 reduces in this case to the set $\sigma_2^{-1}(0)$ which does not "link" the north pole and the south pole of S^2. As a consequence, system (28) passes all the above tests for (homogeneous) stabilization. However, it is obvious that, whatever the choice of the manifold M^*, the condition $M^* \cap A_{+0} = \varnothing$ implies,

$$\forall (x, y) \in M^*, \qquad \sigma_2(x, y) < 0.$$

On the other hand, the homogeneous function $H(x) := x_1^2 + x_2^2$ is a homogeneous positive definite function which satisfies,

$$\forall (x, y) \in \mathbb{R}^3, \qquad \sigma_2(x, y) < 0 \quad \Rightarrow \quad \nabla H(x) \cdot f(x, y) > 0. \tag{30}$$

The last relation can be shown to preclude the existence of a homogeneous Lyapunov function for the system, leading to the following result [19]:

Proposition 4. *System* (28) *augmented by an integrator, i.e.,*

$$\begin{aligned} \dot{x}_1 &= -\sigma_2(x, y)(x_1 + x_2), \\ \dot{x}_2 &= -\sigma_2(x, y)(x_2 - x_1), \\ \dot{y} &= u, \end{aligned} \tag{31}$$

does not admit a homogeneous control Lyapunov function with respect to the standard dilation. In particular, it is not stabilizable by homogeneous feedback.

Remark 10. We proved the incompatibility between the "instability" homogeneous function H and a homogeneous Lyapunov function for the particular system (31). It is not clear if the existence of a contracting $(n-1)$-contracting subset is in general not compatible with the existence of a homogeneous positive definite function $H(x)$ satisfying,

$$\forall (x, y) \in M^*, \qquad \nabla H(x) \cdot f(x, y) > 0.$$

6. Proofs of the Main Results

6.1. Proof of Theorem 1

Step 1. For notational convenience, we use throughout the proof the notation $\partial_n V$ for the function $(\partial V/\partial x_n)|_{S^{n-1}}$. We denote by $x^+ := (0, \ldots, 0, 1)^T$ (resp. $x^- := (0, \ldots, 0, -1)^T$) the north (resp. south) pole of S^{n-1}.

Let V a homogeneous Lyapunov function for (S_{cl}). Rosier's result asserts the existence of such a function. By (4), notice that

$$\nabla V(x^+) \cdot v(x^+) = r_n\, \partial_n V(x^+) > 0 \tag{32}$$

and

$$\nabla V(x^-) \cdot v(x^-) = -r_n\, \partial_n V(x^-) > 0. \tag{33}$$

Since $r_n > 0$, it follows from (32), (33), and the continuity of V that $M = (\partial_n V)^{-1}(0)$ is nonempty; it is closed as the inverse image of a closed set and thus compact since it is included in S^{n-1}.

In order to prove that M is a manifold of codimension one in S^{n-1}, we show that 0 is a regular value for the function $(\partial_n V)$ or, equivalently, that $(0, 1)$ is a regular value for the mapping $(\partial V/\partial x_n, x^T x)$. Supoose that this is not true. Then there exists a point $x^* \in S^{n-1}$ such that

$$\begin{aligned} &\partial_n V(x^*) = 0, \\ \exists \lambda \in \mathbb{R}, \qquad &\nabla(\partial_n V)(x^*) = \lambda x^*. \end{aligned} \tag{34}$$

We show in the rest of the proof that, for a generic homogeneous Lyapunov function, this set of conditions is satisfied for no point of S^{n-1}. This means that, up to a slight modification of V, the set M is a manifold, which ends the proof. ∎

For a real constant $\varepsilon > 0$, consider the set $\mathcal{U}$ of functions $W \in C^\infty(S^{n-1}, \mathbb{R})$ such that, for each $x \in S^{n-1}$,

$$|V(x) - W(x)| < \varepsilon, \tag{35}$$

$$\forall i \in \{1, \ldots, n\}\}, \qquad \left|\frac{\partial W}{\partial x_i} - \frac{\partial V}{\partial x_i}\right| < \varepsilon. \tag{36}$$

The above set $\mathscr{U}$ is an open neighborhood of V in the topology of Whitney. Define

$$\delta_1 := \min\{-\nabla V(x)\cdot F(x): x \in S^{n-1}\} \tag{37}$$

and

$$\delta_2 := \max\{\|F(x)\|: x \in S^{n-1}\}. \tag{38}$$

By compactness of S^{n-1} and by property (ii) of Definition 4, δ_1 and δ_2 exist and are strictly positive. Choose $\varepsilon > 0$ small enough such that,

$$\forall W \in \mathscr{U}, \quad \forall x \in S^{n-1}, \qquad \|\nabla V(x) - \nabla W(x)\| < \frac{\delta_1}{2\delta_2}. \tag{39}$$

It follows from (37) and (38) that,

$$\forall W \in \mathscr{U}, \quad \forall x \in S^{n-1}, \qquad \nabla W(x)\cdot F(x) \leq -\frac{\delta_1}{2} < 0. \tag{40}$$

Extending each $W \in \mathscr{U}$ in $\mathbb{R}^n\backslash\{0\}$ by homogeneity and defining $W(0) = 0$, we conclude that each $W \in \mathscr{U}$ is a homogeneous Lyapunov function for (S_{cl}). Indeed, it follows from property (d) of Theorem 1 that, for each $s > 0$ and for each $x \in S^{n-1}$,

$$\nabla W(h(s, x))\cdot F(h(s, x)) = t^{\tau+k}\nabla W(x)\cdot F(x) < 0. \tag{41}$$

Since h is onto, we obtain $\nabla W(x)\cdot F(x) < 0$ for all $x \neq 0$.

For each $W \in \mathscr{U}$, define the set $\Omega(W) \subset S^{n-1}$ of points which satisfy the n independent constraints (34). The transversality theorem of Thom (see, for instance, [12]) asserts that $\Omega(W)$ is empty for almost every $W \in \mathscr{U}$. Thus there exists a W in $\mathscr{U}$, arbitrarily close to the original Lyapunov function, such that the set of conditions (34) is satisfied for no point in $\mathbb{R}^n\backslash\{0\}$. For this homogeneous Lyapunov function W, M is a compact manifold of codimension one in S^{n-1}. ■

Step 2. The continuous function $\partial V/\partial x_n$ maps S^{n-1} onto a closed interval I of the real line. By (32) and (33), $0 \in \operatorname{int} I$. The origin is therefore a cut point of I and its inverse image separates S^{n-1}. ■

Step 3. Because M is a compact submanifold of the compact manifold S^{n-1}, it has only a finite number of connected components. If the union of these components separates x^+ and x^- on S^{n-1}, then necessarily one of the components also achieves the separation. This results from the following topological property of the n-sphere (see Theorem 8-29 in [11]): if A and B are disjoint closed subsets of S^n, $n > 1$, and if neither A nor B separates the point x from the point y in S^n, then $A \cup B$ does not separate x from y in S^n. ■

Step 4. By construction, M^* possesses all the properties of a contracting $(n-1)$-subset except that $M^* \not\subset V^s$ for some $s > 0$. The last step of the proof follows from property (d) of Proposition 1 which implies in particular,

$$\forall t > 0, \qquad \frac{\partial V}{\partial x_n}(x) = 0 \;\Rightarrow\; \frac{\partial V}{\partial x_n}(h(t, x)) = 0. \tag{42}$$

Let $s > 0$ and consider the level set V^s of V. Define V^s_{n-1} as the projection of M^* onto V^s along the homogeneous rays. We show that V^s_{n-1} satisfies properties (i)–(iii) of Definition 5. The argument used for the proof of (c) in Proposition 1 shows that this projection is a homeomorphism. It follows that V^s_{n-1} is a connected compact manifold of codimension one in V^s. This establishes (i). By (42), (ii) is also satisfied in V^s_{n-1}. For (iii), notice that π_{n-1} and $h(t, \cdot)$ commute, i.e., for each $x \neq 0$, and for each $t > 0$,

$$\pi_{n-1}(h(t, x)) = h(t, \pi_{n-1}(x)). \tag{43}$$

It follows that $\pi_{n-1}(V^s_{n-1})$ is the projection of $\pi_{n-1}(M^*)$ onto V^s along the homogeneous rays. Since $\pi_{n-1}(M^*)$ separates the origin from infinity in E_{n-1}, (iii) is satisfied. ∎

6.2. *Proof of Corollary 2*

By Theorem 1, there exists a Lyapunov function $V(x)$ for (S_{cl}) such that, for each $s > 0$, V^s contains a contracting 2-subset V^s_2. A Lyapunov 1-subset can be constructed in V^s_2 as follows: Let $M^* \subset S^2$ be the connected manifold constructed in Theorem 1 in order to generate V^s_2. Consider a continuous path γ: $[0, 1] \to M^*$ such that,

$$\forall t \in [0, 1], \qquad x_1(\gamma(t)) \geq 0,$$
$$\gamma(0) = x^N = (0, x^N_2, x^N_3)^T \qquad \text{with} \quad x^N_2 > 0,$$
$$\gamma(1) = x^S = (0, x^S_2, x^S_3)^T \qquad \text{with} \quad x^S_2 > 0.$$

By property (iii) of the contracting subsets, γ exists. By property (4), notice that

$$\nabla V(x^N) \cdot v(x^N) > 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x_2}(x^N) > 0$$

and similarily

$$\nabla V(x^S) \cdot v(x^S) > 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x_2}(x^S) < 0.$$

By continuity of the partial derivatives of V, we conclude that there exists $x^+ \in M^*$ such that

$$x^+_1 > 0, \qquad \frac{\partial V}{\partial x_2}(x^+) = \frac{\partial V}{\partial x_3}(x^+) = 0 \quad \text{and} \quad \frac{\partial V}{\partial x_1}(x^+) > 0.$$

Using the same procedure in the region $x_1 \leq 0$, we construct $x^- \in M^*$ such that

$$x^-_1 > 0, \qquad \frac{\partial V}{\partial x_2}(x^-) = \frac{\partial V}{\partial x_3}(x^-) = 0 \quad \text{and} \quad \frac{\partial V}{\partial x_1}(x^-) > 0.$$

Up to a projection onto V^s along the homogeneous rays, $\{x^-, x^+\}$ is a contracting 1-subset in V^s_2. This ends the proof. ∎

6.3. *Proof of Theorem 2*

The proof is direct for $p = 1$. For $p = n$, this is just a reformulation of the classical index condition. Hereafter we consider the case $p = n - 1$.

Consider the manifold $M^* = S^{n-1} \cap K_{n-1}$. We compute the topological degree of the mapping

$$\Phi\colon M^* \to S^{n-1}\colon \theta \to \frac{(-F)_{n-1}(x)}{\|(-F)_{n-1}(x)\|}.$$

Define

$$\Psi := \frac{(v)_{n-1}(x)}{\|(v)_{n-1}(x)\|}$$

and consider the mapping

$$\varphi \in C^0([0, 1] \times M^*, S^{n-2})\colon \varphi(t, x) = \frac{t(v)_{n-1}(x) - (1-t)(F)_{n-1}(x)}{\|t(v)_{n-1}(x) - (1-t)(F)_{n-1}(x)\|}.$$

Notice that $\varphi(0, x) = \Phi(x)$ and that $\varphi(1, x) = \Psi(x)$. On the other hand, noting that $(\partial V/\partial x_n)(x) = 0$ on M^*, we have,

$$\forall (t, x) \in [0, 1] \times M^*, \qquad \varphi(t, x)\cdot(\nabla V)_{n-1}(x) > 0$$

and then also,

$$\forall (t, x) \in [0, 1] \times M^*, \qquad \varphi(t, x) \neq 0.$$

As a consequence, φ defines a homotopy between Φ and Ψ. Since Ψ is in turn homotopic to π_{n-1} on M^*, it remains to prove that the degree of the mapping

$$\pi_{n-1}|_{M^*}\colon M^* \to \mathbb{R}^{n-1}\colon x \to (x_1, \ldots, x_{n-1})$$

is equal to one.

Let $(S^{n-1})_+ := \{x \in S^{n-1} | x_n > 0\}$. If $M^* \subset (S^{n-1})_+$, then M^* is homeomorphic to its parallel projection in the hyperplane $x_n = 0$. The degree of a homeomorphism is $+1$ or -1 depending on the orientation of M^*. We conclude that the mapping $\pi_{n-1}|_\Gamma$ has degree one (with a suitable orientation of M^*).

If M^* is not included in $(S^{n-1})_+$, it can be "pushed" into $(S^{n-1})_+$ by means of a mapping which preserves the degree: let D_0 be a small disk on S^{n-1}, centered at the south pole and separated from M^*. Consider a continuous mapping

$$G\colon S^{n-1}\backslash D_0 \times [0, 1] \to S^{n-1}\backslash D_0,$$

such that $G(\cdot, t)$ is injective, and for all $x \in S^{n-1}\backslash D_0$ we have

$$G(\cdot, 0) = x; \qquad G(x, 1) \in (S^{n-1})_+.$$

Finally we impose that the north pole of S^{n-1} is a fixed point of $G(\cdot, t)$ for all $t \in [0, 1]$.

We consider the continuous mapping

$$g\colon M^* \times [0, 1] \to S^{n-1}\colon (\theta, t) \to G(\theta, t).$$

Since $G(\cdot, t)$ is injective and maps the north pole onto itself for each $t \in [0, 1]$, $\pi_2 \circ g$ does not vanish on $M^* \times [0, 1]$. As a consequence, the degree of $\pi_{n-1} \circ g(\cdot, t)$ is independent of t. By definition, $\pi_{n-1} \circ g(\cdot, 0) = \pi_{n-1}|_{M^*}$; on the other hand, $g(S^1, 1) \subset (S^{n-1})_+$ which implies that $\pi_{n-1} \circ g(\cdot, 1)$ is a homeomorphism. We conclude that the degree of $\pi_{n-1}|_\Gamma$ is equal to one (with a suitable orientation of M^*), which ends the proof. ■

6.4. *Proof of Theorem 4*

The proof is an adaptation of the proof of Theorem 2 in [3] and a consequence of Theorem 3. Let $u(x)$ be a stabilizing feedback, let $V(x)$ be a homogeneous Lyapunov function, and, for each $\delta > 0$ sufficiently small, denote by M_δ^* the manifold M^* defined in Theorem 3. Consider as in [3] the commutative diagram

$$\begin{array}{ccc} \overline{M}_\delta^* & \xrightarrow{\;h\;} & \mathbb{R}^{n-1}\setminus\{0\} \\ {\scriptstyle v}\big\downarrow & \nearrow_{\pi_{n-1}\circ f} & \\ \Omega_\varepsilon^{n-1} & & \end{array} \tag{44}$$

where $h(x) := \pi_{n-1} \circ f(x, u(x))$, $v(x) := (x, u(x))$, and δ is small enough such that,

$$\forall x \in \overline{M}_\delta^*, \qquad v(x) \in \Omega_\varepsilon^{n-1}. \tag{45}$$

By Theorem 2,

$$\deg(\pi_{n-1} \circ -f(\cdot, u(\cdot)), M_\delta^*) = 1.$$

This implies

$$h_*(H_{n-2}(M_\delta^*)) = H_{n-2}(\mathbb{R}^{n-1}\setminus\{0\}). \tag{46}$$

Theorem 4 follows from (46) and the diagram (44). ■

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