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Stability Radii of Infinite-Dimensional Positive Systems*

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Abstract. We show that for infinite-dimensional discrete-time positive systems the complex and real stability radii coincide. Furthermore, we provide a simple formula for the complex stability radius of positive systems by the associated transfer function. We illustrate our results with an example dealing with a simple type of differential-difference equations.

Key words. Infinite-dimensional systems, Discrete-time systems, Robust stability, Stability radii, Banach lattices, Positive systems, Positive operators.

1. Introductory Remarks

This work is motivated by a paper of Hinrichsen and Son on stability radii of finite-dimensional discrete-time positive systems, see [HS]. The stability radius for linear systems, introduced by Hinrichsen and Pritchard, is a measure for the stability robustness of a stable system, see [HP]. It is defined as the smallest (in norm) complex or real perturbation which destabilizes the system. In general, the complex and real stability radii differ. It is therefore natural to investigate for which kinds of systems these two radii coincide. This note is a contribution to this problem and is motivated by the following finite-dimensional result due to Hinrichsen and Son, see p. 13 of [HS]. Consider the discrete-time system $x(t+1) = Ax(t)$, $t \in \mathbb{N}_0$, where $A \in \mathbb{R}_+^{n\times n}$.

Theorem 1.1. *Let $A \in \mathbb{R}_+^{n\times n}$ with spectral radius $r(A) < 1$, $D \in \mathbb{R}_+^{n\times l}$, and $E \in \mathbb{R}_+^{q\times n}$. Furthermore, assume that $\mathbb{K}^l$ and $\mathbb{K}^q$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, are provided with monotonic norms, that is, $|x| \le |y|$ implies $\|x\| \le \|y\|$ for every $x, y \in \mathbb{K}^l$ and $\mathbb{K}^q$, respectively, where $|x| := (|x_i|)$. Moreover, denote by $G(s) := E(sI - A)^{-1}D$ the transfer function associated to (A, D, E), by*

$$r_{\mathbb{K}}(A; D, E) := \inf\{\|\Delta\| \mid \Delta \in \mathbb{K}^{l\times q},\ r(A + D\Delta E) \ge 1\}$$

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the complex and real stability radius for $\mathbb{K} = \mathbb{C}, \mathbb{R}$, *and by*

$$r_+(A; D, E) := \inf\{\|\Delta\| \mid \Delta \in \mathbb{R}_+^{l\times q},\ r(A + D\Delta E) \geq 1\}$$

the positive stability radius, respectively. Then we have

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_+(A; D, E) = \frac{1}{\|G(1)\|}.$$

Our objective is to generalize this theorem to infinite-dimensional discrete-time positive systems. It turns out that some of the ideas behind the proof of Theorem 1.1 can be generalized to the infinite-dimensional case.

The key argument in proving the equality of the complex, real, and positive stability radii in infinite dimensions is that the complex stability radius can be approximated by the norm of *one-dimensional* destabilizing perturbations. Since one-dimensional operators on Banach lattices admit a modulus, the complex stability radius can even be approximated by the norm of one-dimensional *positive* destabilizing perturbations. This observation is essential in proving the equality of the three stability radii.

A formula for the stability radius of positive systems, corresponding to the finite-dimensional counterpart in Theorem 1.1, can be proven using results of the Perron–Frobenius Theory for positive operators. This theory ensures that the spectral radius of a positive operator T is indeed a spectral value, but, unlike the finite-dimensional case, it is not necessarily an eigenvalue of T. Nevertheless, we prove a characterization of the stability radius for infinite-dimensional positive systems which corresponds to the formula in Theorem 1.1.

We conclude this note with an example dealing with a simple type of differential-difference equations.

Throughout this note we use the following notations. Let $E \neq \{0\}$ and $F \neq \{0\}$ be real or complex Banach spaces. Then $\mathscr{L}(E,F) := \{T\colon E \to F \mid T \text{ linear and bounded}\}$, $\mathscr{L}(E) := \mathscr{L}(E,E)$, and E' is the dual space of E. We denote all norms by $\|\cdot\|$ (with one exception in Section 4). Commonly, for $T \in \mathscr{L}(E)$, we denote by $\rho(T) := \{\lambda \in \mathbb{C} \mid R(\lambda, T) := (\lambda I - T)^{-1} \in \mathscr{L}(E)\}$ the *resolvent set* of T, by $\sigma(T) := \mathbb{C}\backslash\rho(T)$ the *spectrum* of T, and by $r(T) := \sup_{\lambda\in\sigma(T)}|\lambda|$ the *spectral radius* of T. Finally, $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$, and $\mathbb{D} := \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

2. Preliminary Results

To make this note more self-contained we give a brief summary of the notions of Banach lattices, positive operators, and moduli of operators as well as some of their basic properties as they are needed. For further reading we refer to [M] and [S].

Banach Lattices

Let $E \neq \{0\}$ be a *real* vector space endowed with an order relation $\leq$, that is, a reflexive antisymmetric and transitive relation such that the following properties are satisfied:

$$x \leq y \quad \Rightarrow \quad x + z \leq y + z, \qquad x, y, z \in E, \tag{2.1}$$

$$x \leq y \quad \Rightarrow \quad \alpha x \leq \alpha y, \qquad x, y \in E, \quad \alpha \in \mathbb{R}_+. \tag{2.2}$$

Then E is called an *ordered* vector space. Denote the *positive* elements of E by $E^+ := \{x \in E \mid x \geq 0\}$, where, of course, $x \geq 0$ means $0 \leq x$. If furthermore the *lattice property* holds, that is, if

$$\sup\{x, y\} \in E, \qquad x, y \in E,$$

then E is called a *vector lattice*. In other words, if for every $x, y \in E$ there exists $z \in E$ such that $x \leq z$, $y \leq z$ and for every $v \in E$ satisfying $x \leq v$, $y \leq v$ it holds that $z \leq v$, then $\sup\{x, y\} = z$. The set E^+ fulfills the following "geometric" properties:

$$E^+ + E^+ \subseteq E^+, \qquad \mathbb{R}_+ E^+ \subseteq E^+, \qquad E^+ \cap (-E^+) = \{0\}. \tag{2.3}$$

In particular, E^+ is a convex cone in E. Conversely, every subset C of E satisfying (2.3) determines an order relation on E by

$$x \leq y \quad :\Leftrightarrow \quad y - x \in C$$

such that (2.1) and (2.2) hold. Moreover, if E is a vector lattice, the set E^+ is *generating*, that is, it satisfies

$$E = E^+ - E^+.$$

The following proposition provides a characterization of the vector lattices among all ordered vector spaces in terms of the "geometric" structure of the cone. A proof of this result can be found on p. 28 of [P1].

Proposition 2.1. *Let E be a real vector space ordered by a cone C. Then the following assertions are equivalent:*

(1) *E is a vector lattice.*
(2) *For every $x, y \in E$ there exists $z \in E$ such that $(x + C) \cap (y + C) = z + C$.*

The following remark indicates to which cones Theorem 1.1 can be generalized in the finite-dimensional setting.

Remark 2.2. If $E = \mathbb{R}^n$ is endowed with an order relation generated by a cone C, Proposition 2.1 states that E is a vector lattice if and only if there exist n linearly independent vectors $v^{(k)} = (v_1^{(k)}, \ldots, v_n{}^{(k)}) \in \mathbb{R}^n$, $k = 1, \ldots, n$, such that

$$C = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i v_i^{(k)} \geq 0,\ k = 1, \ldots, n \right\}.$$

If E is a vector lattice, the *modulus* of $x \in E$ is defined by $|x| := \sup\{x, -x\}$. If $\|\cdot\|$ is a norm on the vector lattice E satisfying the *lattice norm property*, that is, if

$$|x| \leq |y| \quad \Rightarrow \quad \|x\| \leq \|y\|, \qquad x, y \in E, \tag{2.4}$$

then E is called a *normed vector lattice*. The lattice norm property implies $\| |x| \| = \|x\|$ for every $x \in E$. If, in addition, E is norm complete with respect to $\| \cdot \|$, then E is called a *Banach lattice*.

So far the concept of Banach lattices is restricted to real vector spaces. Since it is often necessary to consider complex vector spaces we extend the notion of Banach lattices to the complex case. For this extension all underlying vector lattices E are assumed to be *relatively uniformly complete*, that is, for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} |\lambda_n| \in \mathbb{R}$ and for every $x \in E$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in E it holds that

$$0 \le x_n \le \lambda_n x \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} x_i \in E.$$

Now let E be a relatively uniformly complete vector lattice. The *complexification* of E is defined by $E_{\mathbb{C}} := E \times E$ with the canonical addition and scalar multiplication. It is frequently written as $E_{\mathbb{C}} = E + \imath E$. The *modulus* of $z = x + \imath y \in E_{\mathbb{C}}$ is defined by

$$|z| = \sup_{0\le\varphi\le 2\pi} |(\cos\varphi)x + (\sin\varphi)y| \in E. \tag{2.5}$$

The modulus is homogeneous, that is, $|\alpha z| = |\alpha|\,|z|$ for every $z \in E_{\mathbb{C}}$ and every $\alpha \in \mathbb{C}$, and satisfies the triangle inequality $|z_1 + z_2| \le |z_1| + |z_2|$ for every $z_1, z_2 \in E_{\mathbb{C}}$. A *complex vector lattice* is defined as the complexification of a relatively uniformly complete vector lattice endowed with the modulus (2.5). If E is normed, then

$$\|z\| := \| |z| \|, \qquad z \in E_{\mathbb{C}}, \tag{2.6}$$

defines a norm on $E_{\mathbb{C}}$ satisfying the lattice norm property (2.4). If E is a Banach lattice, then $E_{\mathbb{C}}$, endowed with the modulus (2.5) and the norm (2.6), is called a *complex Banach lattice*.

Standard examples of Banach lattices with respect to their canonical order relations are $\mathscr{C}(K, \mathbb{K}^n)$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, for compact Hausdorff spaces K as well as $L^p(\Omega, \Sigma, \mu)$ for arbitrary measure spaces (Ω, Σ, μ) and every $1 \le p < \infty$.

Positive Operators and Moduli of Operators

Let E and F be real Banach lattices and $T \in \mathscr{L}(E,F)$. If $TE^+ \subseteq F^+$, then T is called *positive* ($T \ge 0$). By $S \le T$ we mean $T - S \ge 0$ for $S \in \mathscr{L}(E,F)$.

Every $\mathbb{R}$-linear map $T \in \mathscr{L}(E,F)$ has a unique $\mathbb{C}$-linear extension $T_{\mathbb{C}} \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})$ given by

$$T_{\mathbb{C}} z := Tx + \imath Ty, \qquad z = x + \imath y \in E_{\mathbb{C}}.$$

An operator $T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})$ is called *real* if $TE \subseteq F$. An operator $T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})$ is called *positive* ($T \ge 0$) if T is real and $TE^+ \subseteq F^+$. We introduce the denotations $\mathscr{L}^{\mathbb{R}}(E_{\mathbb{C}}, F_{\mathbb{C}}) := \{T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}}) | T \text{ real}\}$ and $\mathscr{L}^+(E_{\mathbb{C}}, F_{\mathbb{C}}) := \{T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}}) | T \ge 0\}$. The cone $\mathscr{L}^+(E_{\mathbb{C}}, F_{\mathbb{C}})$ is closed in $\mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})$, however, it is in general not generating. For $T \in \mathscr{L}^+(E_{\mathbb{C}}, F_{\mathbb{C}})$ we emphasize the simple but important fact

$$\|T\| = \sup_{\substack{x \in E^+ \\ \|x\| \le 1}} \|Tx\|. \tag{2.7}$$

An operator $T \in \mathscr{L}(E,F)$ possesses a *modulus* if $|T| := \sup\{T, -T\} \in \mathscr{L}(E,F)$ in the canonical order relation of $\mathscr{L}(E,F)$. It can be shown that, if $\sup_{|z| \le x} |Tz| \in F$ for every $x \in E^+$, then T possesses a modulus $|T|$ and

$$|T|x = \sup_{|z| \le x} |Tz|, \qquad x \in E^+. \tag{2.8}$$

Since E is generating we have that $|T| \in \mathscr{L}^+(E,F)$.

Let $T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})$. If $\sup_{|z| \le x} |Tz| \in F$ for every $x \in E^+$, then it holds by linear extension that $|T| \in \mathscr{L}^+(E_{\mathbb{C}}, F_{\mathbb{C}})$. Denote $\mathscr{L}^{|\cdot|}(E_{\mathbb{C}}, F_{\mathbb{C}}) := \{T \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}}) \mid |T| \in \mathscr{L}(E_{\mathbb{C}}, F_{\mathbb{C}})\}$. We have $|Tx| \le |T|\,|x|$ for every $T \in \mathscr{L}^{|\cdot|}(E_{\mathbb{C}}, F_{\mathbb{C}})$ and every $x \in E_{\mathbb{C}}$ as well as $|Tx| \le T|x|$ for every $T \in \mathscr{L}^+(E_{\mathbb{C}}, F_{\mathbb{C}})$ and every $x \in E_{\mathbb{C}}$. The lattice norm property (2.4) implies that

$$\|T\| \le \|\,|T|\,\|, \qquad T \in \mathscr{L}^{|\cdot|}(E_{\mathbb{C}}, F_{\mathbb{C}}). \tag{2.9}$$

Remark 2.3. In the finite-dimensional setting the above described notions have the following meaning. If $E = \mathbb{R}^m$ and $F = \mathbb{R}^n$, then, of course, $E_{\mathbb{C}} = \mathbb{C}^m$ and $F_{\mathbb{C}} = \mathbb{C}^n$. Moreover, $\mathscr{L}(\mathbb{C}^m, \mathbb{C}^n) = \mathscr{L}^{|\cdot|}(\mathbb{C}^m, \mathbb{C}^n)$ and for $T = (t_{ij}) \in \mathscr{L}(\mathbb{C}^m, \mathbb{C}^n)$ we have $|T| = (|t_{ij}|)$.

3. Discrete-Time Systems

In the rest of this paper let U, X, and Y denote complex Banach lattices as well as $A \in \mathscr{L}(X)$, $D \in \mathscr{L}(U,X)$, and $E \in \mathscr{L}(X,Y)$. We consider the infinite-dimensional discrete-time system

$$x(t+1) = Ax(t), \qquad t \in \mathbb{N}_0. \tag{3.1}$$

Definition 3.1. System (3.1) is called *positive* if $A \in \mathscr{L}^+(X)$.

Stability Radii

Recall that system (3.1) is called *exponentially stable* or *power stable* if there exist $c \ge 1$ and $0 < \beta < 1$ such that

$$\|A^t\| \le c\beta^t, \qquad t \in \mathbb{N}_0.$$

It is well known that the exponential stability of system (3.1) is equivalent to $r(A) < 1$, see p. 516 of [P3].

In this note we consider affine perturbations of the form

$$A \mapsto A + D\Delta E,$$

where the perturbations Δ are of three different types:

$$\Delta \in \mathscr{L}(Y,U), \qquad \Delta \in \mathscr{L}^{\mathbb{R}}(Y,U), \qquad \Delta \in \mathscr{L}^+(Y,U).$$

Now we study a measure of robust stability of linear systems, the stability radius. Let $A \in \mathscr{L}(X)$ be such that $r(A) < 1$. We define the *complex*, *real*, and *positive stability radii* of A by

$$r_{\mathbb{C}}(A; D, E) = \inf\{\|\Delta\| \mid \Delta \in \mathscr{L}(Y, U),\ r(A + D\Delta E) \geq 1\},$$
$$r_{\mathbb{R}}(A; D, E) = \inf\{\|\Delta\| \mid \Delta \in \mathscr{L}^{\mathbb{R}}(Y, U),\ r(A + D\Delta E) \geq 1\},$$

and

$$r_{+}(A; D, E) = \inf\{\|\Delta\| \mid \Delta \in \mathscr{L}^{+}(Y, U),\ r(A + D\Delta E) \geq 1\},$$

respectively, where we set $\inf \emptyset := \infty$. Obviously

$$r_{\mathbb{C}}(A; D, E) \leq r_{\mathbb{R}}(A; D, E) \leq r_{+}(A; D, E). \tag{3.2}$$

Let $G(s) := E(sI - A)^{-1} D \in \mathscr{L}(U, Y)$, $s \in \rho(A)$, denote the transfer function associated to (A, D, E). In proving the main results of this paper we use the following characterization of the complex stability radius. It is proven on p. 267 of [WH].

Proposition 3.2. *Let $A \in \mathscr{L}(X)$ be such that $r(A) < 1$, $D \in \mathscr{L}(U, X)$, and $E \in \mathscr{L}(X, Y)$. Then*

$$r_{\mathbb{C}}(A; D, E) = \frac{1}{\max_{|s|=1} \|G(s)\|}. \tag{3.3}$$

Remark 3.3. In the proof of the previous proposition it is in particular shown that there exists a sequence $(\Delta_n)_{n \in \mathbb{N}}$ in $\mathscr{L}(Y, U)$ such that

(1) rank $\Delta_n = 1$, $n \in \mathbb{N}$,
(2) $1 \leq r(A + D\Delta_n E)$, $n \in \mathbb{N}$, and
(3) $\|\Delta_n\| \downarrow r_{\mathbb{C}}(A; D, E)$, $n \to \infty$.

Equality of the Complex and Real Stability Radii

In the next two lemmata we state the main arguments for proving the equality of the complex and real stability radii for positive systems.

Lemma 3.4. *Let $T \in \mathscr{L}(Y, U)$ be such that* rank $T = 1$. *Then $T \in \mathscr{L}^{|\cdot|}(Y, U)$ and $\|T\| = \|\,|T|\,\|$.*

Proof. Since T is of rank one there exist $a \in Y'$ and $u \in U$ such that

$$T = a \otimes u,$$

that is, $Ty = a(y)u$ for every $y \in Y$. In order to prove that $T \in \mathscr{L}^{|\cdot|}(Y, U)$ it remains to show that $\sup_{|z| \leq y} |Tz| \in U$ for every $y \in Y^{+}$. However, this is immediate since

$$|Tz| = |a(z)u| = |a(z)|\,|u| \leq \|a\|\,\|z\|\,|u| \leq \|a\|\,\|y\|\,|u|,$$

where we have used that $|z| \le y$ implies $\|z\| \le \|y\|$, that is, the lattice norm property (2.4). In particular, we have that $a \in \mathscr{L}^{|\cdot|}(Y, \mathbb{C})$.

It remains to show the identity $\|T\| = \|\,|T|\,\|$. Applying (2.8) we obtain

$$|T|y = \sup_{|z| \le y} |Tz| = \sup_{|z| \le y} |a(z)|\,|u| = (|a|(y))|u|, \qquad y \in Y^+,$$

that is, $|T| = |a| \otimes |u|$. Assume for a moment that $\|a\| = \|\,|a|\,\|$. Then

$$\|T\| = \|a \otimes u\| = \|a\|\,\|u\| = \|\,|a|\,\|\,\|\,|u|\,\| = \|\,|T|\,\|,$$

and the proof is complete.

In order to prove the identity $\|a\| = \|\,|a|\,\|$ it remains to show that $\|\,|a|\,\| \le \|a\|$, see (2.9). Using (2.7) and (2.8) we have

$$\|\,|a|\,\| = \sup_{\substack{y \in Y^+ \\ \|y\| \le 1}} |\,|a|(y)| = \sup_{\substack{y \in Y^+ \\ \|y\| \le 1}} \left| \sup_{|z| \le y} |a(z)| \right| = \sup_{\substack{y \in Y^+ \\ \|y\| \le 1}} \sup_{|z| \le y} |a(z)|.$$

Therefore, there exist sequences $(y_n)_{n \in \mathbb{N}}$ in Y^+ and $(z_n)_{n \in \mathbb{N}}$ in Y satisfying $\|y_n\| \le 1$ and $|z_n| \le y_n$ such that, for an $\varepsilon > 0$,

$$\|\,|a|\,\| - \varepsilon \le |a(z_n)|, \qquad n \text{ large enough.}$$

For $|z_n| \le y_n$ the lattice norm property (2.4) implies $\|z_n\| \le \|y_n\| \le 1$ and we obtain

$$\|\,|a|\,\| - \varepsilon \le |a(z_n)| \le \sup_{\|z\| \le 1} |a(z)| = \|a\|.$$

This completes the proof. ■

Lemma 3.5. *Let $S \in \mathscr{L}^+(X)$ and $T \in \mathscr{L}(X)$ be such that $|Tx| \le S|x|$ for every $x \in X$. Then $r(T) \le r(S)$.*

Proof. Let $n \in \mathbb{N}$. Since $|Tx| \le S|x|$ we have that $|T^n x| \le S^n |x|$ for every $x \in X$. Thus

$$\|T^n x\| \le \|S^n |x|\,\| \le \|S^n\|\,\|\,|x|\,\| = \|S^n\|\,\|x\|, \qquad x \in X,$$

that is, $\|T^n\| \le \|S^n\|$. Hence

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} \le \lim_{n \to \infty} \|S^n\|^{1/n} = r(S). \qquad ■$$

We are now in a position to prove the first main result of this note.

Theorem 3.6. *Let $A \in \mathscr{L}^+(X)$ be such that $r(A) < 1$, $D \in \mathscr{L}^+(U, X)$, and $E \in \mathscr{L}^+(X, Y)$. Then*

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_+(A; D, E).$$

Proof. Suppose that $r_{\mathbb{C}}(A; D, E) < \infty$, as otherwise there is nothing to show. It remains to prove $r_+(A; D, E) \le r_{\mathbb{C}}(A; D, E)$, see (3.2). By Remark 3.3, there exists a sequence $(\Delta_n)_{n \in \mathbb{N}}$ in $\mathscr{L}(Y, U)$ such that rank $\Delta_n = 1$ and $1 \le r(A + D\Delta_n E)$ for every $n \in \mathbb{N}$ and for an $\varepsilon > 0$ it holds that $\|\Delta_n\| \leq r_{\mathbb{C}}(A; D, E) + \varepsilon$ for every n large enough. Furthermore, according to Lemma 3.4, we have that $\Delta_n \in \mathscr{L}^{|\cdot|}(Y, U)$ for every $n \in \mathbb{N}$. Using the positivity of D and E we obtain

$$\begin{aligned} |(A + D\Delta_n E)x| &\leq |Ax| + |D\Delta_n Ex| \leq A|x| + D|\Delta_n Ex| \\ &\leq A|x| + D|\Delta_n|\,|Ex| \leq A|x| + D|\Delta_n|E|x| \\ &= (A + D|\Delta_n|E)|x|, \qquad x \in X, \quad n \in \mathbb{N}. \end{aligned}$$

Now we know by Lemma 3.5 that

$$1 \leq r(A + D\Delta_n E) \leq r(A + D|\Delta_n|E), \qquad n \in \mathbb{N}.$$

Therefore, $|\Delta_n| \in \mathscr{L}^+(Y, U)$ also destabilizes A. Finally, using again Lemma 3.4, it follows, for an $\varepsilon > 0$, that

$$r_+(A; D, E) \leq \|\,|\Delta_n|\,\| = \|\Delta_n\| \leq r_{\mathbb{C}}(A; D, E) + \varepsilon, \qquad n \text{ large enough.}$$

This completes the proof. ■

A Formula for the Complex Stability Radius

Next we derive a simple characterization of the complex stability radius of positive systems using Proposition 3.2. The following proposition concerning the Perron–Frobenius Theory for operators can be found on p. 248 of [M].

Proposition 3.7. *Let $T \in \mathscr{L}^+(X)$. Then:*

(1) $r(T) \in \sigma(T)$.
(2) $R(\lambda, T) \geq 0 \Leftrightarrow \lambda > r(T)$.

Remark 3.8. The proof of part (1) of the previous proposition shows in particular that there exist a sequence $(\lambda_m)_{m\in\mathbb{N}}$ in $\mathbb{R}$ and an $x \in X^+$ such that $\lambda_m \downarrow r(T)$ and $\|(\lambda_m I - T)^{-1}x\| \to \infty$ for $m \to \infty$.

Lemma 3.9. *Let $A \in \mathscr{L}^+(X)$, $D \in \mathscr{L}^+(U, X)$, and $E \in \mathscr{L}^+(X, Y)$. Furthermore, let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $r(A) < \lambda_1 \leq \lambda_2$. Then $\|G(\lambda_2)\| \leq \|G(\lambda_1)\|$.*

Proof. Let $r(A) < \lambda_1 \leq \lambda_2$. By Proposition 3.7(2) we have $R(\lambda_1, A) \geq 0$ and $R(\lambda_2, A) \geq 0$. Using the resolvent equation for operators we obtain

$$R(\lambda_1, A) - R(\lambda_2, A) = (\lambda_2 - \lambda_1)R(\lambda_1, A)R(\lambda_2, A) \geq 0,$$

that is, $0 \leq (\lambda_2 I - A)^{-1} \leq (\lambda_1 I - A)^{-1}$. Multiplying this inequality by E from the left and by D from the right—recall that D and E are positive—we have that $0 \leq G(\lambda_2) \leq G(\lambda_1)$. Since

$$\|G(\lambda_2)u\| \leq \|G(\lambda_2)|u|\,\| \leq \|G(\lambda_1)|u|\,\| \leq \|G(\lambda_1)\|\,\|\,|u|\,\| = \|G(\lambda_1)\|\,\|u\|, \qquad u \in U,$$

the proof is complete. ■

We are now prepared to provide a formula for the complex stability radius of positive systems.

Theorem 3.10. *Let $A \in \mathscr{L}^+(X)$ be such that $r(A) < 1$, $D \in \mathscr{L}^+(U, X)$, and $E \in \mathscr{L}^+(X, Y)$. Then*

$$r_{\mathbb{C}}(A; D, E) = \frac{1}{\|G(1)\|}.$$

Proof. Suppose that $r_{\mathbb{C}}(A; D, E) < \infty$, as otherwise there is nothing to show. By (3.3) it remains to prove that $1/\|G(1)\| \leq r_{\mathbb{C}}(A; D, E)$. Consider the sequence $(\Delta_n)_{n \in \mathbb{N}}$ in $\mathscr{L}(Y, U)$ given in Remark 3.3, that is, rank $\Delta_n = 1$ and $1 \leq r(A + D\Delta_n E)$ for every $n \in \mathbb{N}$ and for an $\varepsilon > 0$ it holds that $\|\Delta_n\| \leq r_{\mathbb{C}}(A; D, E) + \varepsilon$ for every n large enough. Furthermore, according to Lemma 3.4, we have that $\Delta_n \in \mathscr{L}^{|\cdot|}(Y, U)$ for every $n \in \mathbb{N}$. This implies $A_n := A + D|\Delta_n|E \geq 0$ for every $n \in \mathbb{N}$. Now fix $n \in \mathbb{N}$. By Remark 3.8 there exist a sequence $(\lambda_{n,m})_{m \in \mathbb{N}}$ in $\mathbb{R}$ and $x_n \in X^+$ such that $\lambda_{n,m} \downarrow r(A_n)$ and $\|(\lambda_{n,m}I - A_n)^{-1}x_n\| \to \infty$ for $m \to \infty$. Define

$$x_{n,m} := \frac{(\lambda_{n,m}I - A_n)^{-1}x_n}{\|(\lambda_{n,m}I - A_n)^{-1}x_n\|}.$$

Then it follows from

$$\begin{aligned}(r(A_n)I - A_n)x_{n,m} &= (r(A_n)I - \lambda_{n,m}I)x_{n,m} + (\lambda_{n,m}I - A_n)x_{n,m} \\ &= (r(A_n)I - \lambda_{n,m}I)x_{n,m} + \frac{x_n}{\|(\lambda_{n,m}I - A_n)^{-1}x_n\|}\end{aligned}$$

that $\|(r(A_n)I - A_n)x_{n,m}\| \to 0$ for $m \to \infty$. Define $z_{n,m} := (r(A_n)I - A_n)x_{n,m}$. Since $1 \leq r(A + D\Delta_n E)$ the operator $(r(A_n)I - A)^{-1}$ exists and we have

$$(r(A_n)I - A)^{-1}D|\Delta_n|Ex_{n,m} = x_{n,m} - (r(A_n)I - A)^{-1}z_{n,m}. \tag{3.4}$$

Multiplying this equation by E from the left and setting $y_{n,m} := Ex_{n,m}$ we obtain

$$G(r(A_n))|\Delta_n|y_{n,m} = y_{n,m} - E(r(A_n)I - A)^{-1}z_{n,m}.$$

Note that there exists $c > 0$—which may depend on n—such that $c < \|y_{n,m}\|$ for every m large enough. This can be seen as follows. Assume that $y_{n,m} \to 0$ for $m \to \infty$. Then also the left-hand side of (3.4) tends to zero for $m \to \infty$. This implies that

$$\|x_{n,m}\| - \|(r(A_n)I - A)^{-1}z_{n,m}\| \leq \|x_{n,m} - (r(A_n)I - A)^{-1}z_{n,m}\| \to 0.$$

Since $\|x_{n,m}\| = 1$ for every $m \in \mathbb{N}$ and $\|(r(A_n)I - A)^{-1}z_{n,m}\| \to 0$ for $m \to \infty$, this cannot happen.

Therefore, for every m large enough

$$\frac{\|y_{n,m} - E(r(A_n)I - A)^{-1}z_{n,m}\|}{\|y_{n,m}\|} \leq \|G(r(A_n))\| \, \| \, |\Delta_n| \, \|.$$

If $m \to \infty$ we have

$$1 \le \|G(r(A_n))\| \, \| \, |\Delta_n| \, \|,$$

which can be verified as follows. Since

$$1 - \frac{\|E(r(A_n)I - A)^{-1} z_{n,m}\|}{\|y_{n,m}\|} \le \frac{\|y_{n,m} - E(r(A_n)I - A)^{-1} z_{n,m}\|}{\|y_{n,m}\|}$$

$$\le \|G(r(A_n))\| \, \| \, |\Delta_n| \, \|,$$

it is sufficient to show that $\|E(r(A_n)I - A)^{-1} z_{n,m}\| / \|y_{n,m}\| \to 0$ for $m \to \infty$. However, this immediately follows from the definition of $z_{n,m}$ and

$$\frac{\|E(r(A_n)I - A)^{-1} z_{n,m}\|}{\|y_{n,m}\|} \le \frac{\|E(r(A_n)I - A)^{-1}\|}{c} \|z_{n,m}\|.$$

Now Lemma 3.9 implies $\|G(r(A_n))\| \le \|G(1)\|$. Finally, by Lemma 3.4 we obtain that, for an $\varepsilon > 0$,

$$\frac{1}{\|G(1)\|} \le \frac{1}{\|G(r(A_n))\|} \le \| \, |\Delta_n| \, \| = \|\Delta_n\| \le r_{\mathbb{C}}(A; D, E) + \varepsilon, \qquad n \text{ large enough.}$$

This completes the proof. ■

Remark 3.11. If $A \in \mathscr{L}^+(X)$ is assumed to be compact, then the proof of Theorem 3.10 becomes much easier, since in this case $r(A_n)$ is an eigenvalue of A_n, see p. 250 of [M]. The positivity of D and E is necessary in proving Theorems 3.6 and 3.10 since for nonpositive D and E these results are already wrong in the finite-dimensional setting, see p. 14 of [HS].

4. An Example

Let $A_1, A_2 \in \mathbb{R}^{n \times n}$, $r > 0$, and $\varphi \in \mathscr{C}([-r, 0], \mathbb{R}^n)$. Furthermore, let $X := \mathscr{C}([0, r], \mathbb{R}^n)$ be endowed with the supremum norm $\| \cdot \|_\infty$ and the canonical order relation. We consider the infinite-dimensional discrete-time system

$$x_{k+1} = A x_k, \qquad k \in \mathbb{N}_0, \tag{4.1}$$

with initial state $x_0(t) = \varphi(t - r)$, $t \in [0, r]$, and $A \in \mathscr{L}(X)$ defined by

$$(Ax)(t) = e^{A_1 t} x(r) + \int_0^t e^{A_1(t-\tau)} A_2 x(\tau) \, d\tau, \qquad t \in [0, r].$$

System (4.1) can be considered as a model of the differential-difference equation

$$\dot{z}(t) = A_1 z(t) + A_2 z(t - r), \qquad t \in \mathbb{R}_+, \tag{4.2}$$

where $z(t) = \varphi(t)$, $t \in [-r, 0]$, using the transformation

$$x_k(\cdot) := z(kr - r + \cdot)|_{[0,r]} \in X, \qquad k \in \mathbb{N}_0.$$

The unique solution of system (4.2) is given by $z(t) = \varphi(t)$ for $t \in [-r, 0]$ and

$$z(t) = e^{A_1 t}\varphi(0) + \int_0^t e^{A_1(t-\tau)} A_2 z(\tau - r)\, d\tau, \qquad t \in \mathbb{R}_+, \tag{4.3}$$

see p. 6 of [P2].

In the following subsections we show that the notions of positivity and stability are consistent for systems (4.1) and (4.2).

Positivity

We define when a system of type (4.2) is called positive.

Definition 4.1. System (4.2) is called *positive* if, for every positive initial condition $\varphi \in \mathscr{C}([-r, 0], \mathbb{R}^n_+)$, the corresponding solution z satisfies $z(t) \in \mathbb{R}^n_+$ for every $t \in \mathbb{R}_+$.

Recall that a matrix $M = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if $a_{ij} \geq 0$ for $i \neq j$. A matrix $M \in \mathbb{R}^{n\times n}$ is called *positive* if $M \in \mathbb{R}^{n\times n}_+$.

Proposition 4.2. *System* (4.2) *is positive if and only if* A_1 *is a Metzler matrix and* $A_2 \in \mathbb{R}^{n\times n}_+$.

Proof. Assume that system (4.2) is positive, that is, for every $\varphi \in \mathscr{C}([-r, 0], \mathbb{R}^n_+)$, the corresponding solution z satisfies $z(t) \in \mathbb{R}^n_+$ for every $t \in \mathbb{R}_+$. Choose φ such that $\varphi|_{[-r,-r_0]} \equiv 0$ for some $0 < r_0 < r$. Thus the second term of the right-hand side of (4.3) vanishes for every $t \in [0, r - r_0]$, and therefore $e^{A_1 t}\varphi(0) \in \mathbb{R}^n_+$ for every $t \in [0, r - r_0]$. Since this holds for every $\varphi(0) \in \mathbb{R}^n_+$ we have that $e^{A_1 t} \in \mathbb{R}^{n\times n}_+$ for every $t \in [0, r - r_0]$. Now assume that $a_{ij} < 0$ for some $i \neq j$, that is, $A_1 = (a_{ij})$ fails to be a Metzler matrix. Since

$$\frac{1}{t} e^{A_1 t} = \frac{1}{t} I + A_1 + t \sum_{k=2}^{\infty} \frac{t^{k-2}}{k!} A_1^k$$

it follows immediately that for $t \leq r - r_0$ sufficiently small the element in the ith row and jth column of the right-hand side matrix remains negative. This is a contradiction to $e^{A_1 t} \in \mathbb{R}^{n\times n}_+$ for every $t \in [0, r - r_0]$ and hence A_1 is a Metzler matrix.

Next, choose $\varphi \in \mathscr{C}([-r, 0], \mathbb{R}^n_+)$ such that $\varphi(0) = 0$. Then the positivity of system (4.2) implies that

$$\frac{1}{t} \int_0^t e^{A_1(t-\tau)} A_2 z(\tau - r)\, d\tau \in \mathbb{R}^n_+, \qquad t > 0. \tag{4.4}$$

It is easily verified that the expression in (4.4) tends to $A_2 z(-r) \in \mathbb{R}^n_+$ if $t \to 0$. This holds for arbitrary $z(-r) = \varphi(-r) \in \mathbb{R}^n_+$. Hence $A_2 \in \mathbb{R}^{n\times n}_+$.

In order to prove the converse implication we argue as follows. Since A_1 is a Metzler matrix there exists $c > 0$ such that $cI + A_1 \in \mathbb{R}^{n\times n}_+$, that is, $e^{(cI+A_1)t} = e^{ct} e^{A_1 t} \in \mathbb{R}^{n\times n}_+$ for every $t \in \mathbb{R}_+$. Therefore, we obtain $e^{A_1 t} \in \mathbb{R}^{n\times n}_+$ and the assertion immediately follows from (4.3). ∎

A characterization of positive systems in a more general setting than system (4.2) can be found on pp. 224 and 225 of [N]. Using the previous proposition it is obvious that the positivity of system (4.1), that is, $A \in \mathscr{L}^+(X)$, is equivalent to the positivity of system (4.2).

Stability

For system (4.1) it is shown on p. 7 of [P2] that

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(\lambda I - e^{(A_1 + 1/\lambda A_2)r}) = 0\} \cup \{0\}.$$

In particular, every nonzero $\lambda \in \sigma(A)$ is an eigenvalue of A. Recall that system (4.2) is exponentially stable if and only if $\{\mu \in \mathbb{C} \mid \det(\mu I - A_1 - A_2 e^{-\mu r}) = 0\} \subseteq \mathbb{C}_-$, see p. 190 of [BC].

For $\lambda = e^{\mu r}$ we have that $|\lambda| < 1$ is equivalent to $e^{(\mathrm{Re}\,\mu)r} < 1$ and this is true if and only if $\mathrm{Re}\,\mu < 0$. Hence, by using the Spectral Mapping Theorem, we obtain

$$\begin{aligned}
\{\lambda \in \mathbb{C} | \det(\lambda I - e^{(A_1 + (1/\lambda)A_2)r}) = 0\} \subseteq \mathbb{D} \quad &\Leftrightarrow \quad \{\lambda \in \mathbb{C} | \lambda \in \sigma(e^{(A_1 + (1/\lambda)A_2)r})\} \subseteq \mathbb{D} \\
&\Leftrightarrow \quad \{e^{\mu r} \in \mathbb{C} \mid e^{\mu r} \in e^{\sigma((A_1 + A_2 e^{-\mu r})r)}\} \subseteq \mathbb{D} \\
&\Leftrightarrow \quad \{\mu \in \mathbb{C} \mid \mu \in \sigma(A_1 + A_2 e^{-\mu r})\} \subseteq \mathbb{C}_- \\
&\Leftrightarrow \quad \{\mu \in \mathbb{C} \mid \det(\mu I - A_1 - A_2 e^{-\mu r}) = 0\} \subseteq \mathbb{C}_-.
\end{aligned}$$

This shows that system (4.1) is exponentially stable if and only if system (4.2) is exponentially stable.

Computing the Complex Stability Radius

We compute the complex stability radius of system (4.1) with respect to the unstructured perturbation form $A \mapsto A + \Delta$, that is, $D = E = I$. We only consider the scalar case, that is, $n = 1$. Setting $A_1 = a_1$ and $A_2 = a_2$ we obtain

$$(Ax)(t) = e^{a_1 t} x(r) + \int_0^t e^{a_1(t-\tau)} a_2 x(\tau)\, d\tau, \qquad t \in [0, r].$$

Conditions on a_1 and a_2 for the exponential stability of the operator A can be found on p. 444 of [BC] where a complete characterization of the exponential stability of scalar systems (4.2) is derived. Furthermore, from Proposition 4.2 we know how to choose a_1 and a_2 for A to be positive. From this it follows that A is exponentially stable *and* positive if and only if $0 \leq a_2 < -a_1$. Assume in the following that this condition on a_1 and a_2 is satisfied. This implies that $(I - A)^{-1}$ exists and according to Theorem 3.10 we have to compute $\|G(1)\| = \|(I - A)^{-1}\|$. Obviously,

$$\|(I - A)^{-1}\| = \sup_{\substack{y \in \mathscr{C}^1([0,r],\mathbb{R}) \\ \|y\|_\infty \leq 1}} \|(I - A)^{-1} y\|_\infty.$$

Now let $x \in \mathscr{C}^1([0, r], \mathbb{R})$. Then differentiation of $y := (I - A)x \in \mathscr{C}^1([0, r], \mathbb{R})$ yields

$$\dot{x}(t) - (a_1 + a_2)x(t) = \dot{y}(t) - a_1 y(t). \tag{4.5}$$

Solving (4.5) by variation of constants we obtain

$$x(t) = e^{(a_1+a_2)t}\left(x(0) + \int_0^t e^{-(a_1+a_2)\tau}\left(\frac{d}{d\tau}y(\tau) - a_1 y(\tau)\right)d\tau\right) = ((I-A)^{-1}y)(t). \tag{4.6}$$

Since $x(0) - y(0) = (Ax)(0) = x(r)$, equality (4.6) shows

$$\begin{aligned} x(r) &= e^{(a_1+a_2)r}\left(x(0) + \int_0^r e^{-(a_1+a_2)\tau}\left(\frac{d}{d\tau}y(\tau) - a_1 y(\tau)\right)d\tau\right) \\ &= e^{(a_1+a_2)r}\left(x(0) + \int_0^r \frac{d}{d\tau}(e^{-(a_1+a_2)\tau}y(\tau))\,d\tau + a_2\int_0^r e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right) \\ &= e^{(a_1+a_2)r}\left(x(0) + e^{-(a_1+a_2)r}y(r) - y(0) + a_2\int_0^r e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right) \\ &= e^{(a_1+a_2)r}\left(x(r) + e^{-(a_1+a_2)r}y(r) + a_2\int_0^r e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right). \end{aligned}$$

Hence

$$x(r) = \frac{1}{1-e^{(a_1+a_2)r}}\left(y(r) + a_2 e^{(a_1+a_2)r}\int_0^r e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right). \tag{4.7}$$

Using (4.7) the norm of $(I-A)^{-1}y$ can be computed in the following way:

$$\begin{aligned} &\|(I-A)^{-1}y\|_\infty \\ &= \max_{0\le t\le r}\left|e^{(a_1+a_2)t}\left(x(0) + \int_0^t e^{-(a_1+a_2)\tau}\left(\frac{d}{d\tau}y(\tau) - a_1 y(\tau)\right)d\tau\right)\right| \\ &= \max_{0\le t\le r}\left|e^{(a_1+a_2)t}\left(x(0) + \int_0^t \frac{d}{d\tau}(e^{-(a_1+a_2)\tau}y(\tau))\,d\tau + a_2\int_0^t e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right)\right| \\ &= \max_{0\le t\le r}\left|e^{(a_1+a_2)t}\left(x(0) + e^{-(a_1+a_2)t}y(t) - y(0) + a_2\int_0^t e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right)\right| \\ &= \max_{0\le t\le r}\left|e^{(a_1+a_2)t}\left(\frac{y(r)}{1-e^{(a_1+a_2)r}} + a_2\frac{e^{(a_1+a_2)r}}{1-e^{(a_1+a_2)r}}\int_0^r e^{-(a_1+a_2)\tau}y(\tau)\,d\tau \right.\right. \\ &\qquad \left.\left. + e^{-(a_1+a_2)t}y(t) + a_2\int_0^t e^{-(a_1+a_2)\tau}y(\tau)\,d\tau\right)\right|. \end{aligned}$$

Thus, for $y \equiv 1$ we obtain

$$\begin{aligned} \|(I-A)^{-1}\| &= \max_{0\le t\le r}\left|e^{(a_1+a_2)t}\left(\frac{1}{1-e^{(a_1+a_2)r}} + a_2\frac{e^{(a_1+a_2)r}}{1-e^{(a_1+a_2)r}}\int_0^r e^{-(a_1+a_2)\tau}\,d\tau \right.\right. \\ &\qquad \left.\left. + e^{-(a_1+a_2)t} + a_2\int_0^t e^{-(a_1+a_2)\tau}\,d\tau\right)\right| \\ &= \max_{0\le t\le r}\left|\frac{e^{(a_1+a_2)t}}{1-e^{(a_1+a_2)r}} - \frac{a_2}{a_1+a_2} + 1\right| \\ &= \frac{1}{1-e^{(a_1+a_2)r}} + \frac{a_1}{a_1+a_2}, \end{aligned}$$

and, finally,

$$r_{\mathbb{C}}(A;I,I) = \left(\frac{1}{1-e^{(a_1+a_2)r}} + \frac{a_1}{a_1+a_2}\right)^{-1}.$$

5. Concluding Remarks

Future research is necessary to extend the results of this note to the more general class of multiperturbations $A \mapsto A + \sum_{i=1}^{n} D_i \Delta_i E_i$ as considered in μ-analysis. For finite-dimensional systems this has been elaborated in [HS]. The continuous-time counterparts of Theorems 3.6 and 3.10 are subject of current research, see [F].

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