Econ. Theory 2, 191–196 (1992)

Economic Theory

Second best equilibria for games of joint exploitation of a productive asset

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Received: February 16, 1991

1 Introduction

In a dynamic game the payoffs at equilibrium for the players will not in general be on the efficiency frontier. It is therefore important to determine if a *constrained* efficiency can be achieved: i.e. if there exist equilibria of the game which give efficient outcomes over the equilibrium payoffs. We analyze this question in the context of games of joint exploitation of a productive asset, as in [2].

This note has also a second purpose. In the analysis of the existence of both first and second best for this class of games there is a natural requirement on the space of controls that is imposed in [2]: the productive asset must remain zero forever after taking the value zero once. When this additional restriction is introduced, the set of admissible paths is not closed, and the use of standard weak compactness results requires some additional care. We put in this extra care, proving the existence for both the first and second best under this additional restriction. The problem of existence of equilibria in stationary strategies for this type of game is a related issue, still topic of current research: see [5] and [6] for some recent contributions. No symmetry conditions will be used in this paper.

2 The model

Our assumptions are the same as in [2]. We have two players with instantaneous concave utility functions u_i, $i = 1, 2$, both continuous at zero, with $u_i(0) = 0$, $i = 1, 2$. The players have a common discount rate ρ; the measure with density $\rho e^{-\rho s}$ is denoted μ. If we denote by c_1 and c_2 the consumption rates of the two players, the productive asset y reproduces at a rate $\dot{y} = m(y(t)) - c_1(t) - c_2(t)$, with m concave, lipschitzian of norm M, and $m(0) = 0$. The Lipschitz condition is required to derive continuity of the value function at 0, as required in [2]: a Cobb-Douglas production function m is enough to show that this condition cannot be weakened.

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* The author wishes to thank Jess Benhabib, Prajit Dutta, Roy Radner and Raghu Sundaram for helpful discussions, an anonymous referee for his comments, and blames himself for all the errors.

The two players have an upper bound on the consumption rate $\bar{c}_i$, $i=1,2$. This bound is assumed to be large. In particular we require that $\sup_{y\geqq 0} m(y) < \min(\bar{c}_1, \bar{c}_2)$, and that the pair of *fast consumption strategies*, given by $c_i(y) = \bar{c}_i$, $i=1,2$ for every y, is a subgame perfect equilibrium. For the case they analyse (linear utility) Benhabib and Radner prove that this is the case in [2]. For our, more general, framework an easily checked sufficient condition is presented in Sect. 4 below.

The action of each player is perfectly observable after τ units of time, the *detection delay*. For any measurable pair of consumption paths $c=(c_1,c_2)$ and initial capital stock y_0 we denote by $y(\cdot, y_0, c)$ the solution of

$$\dot{y}(y) = m(y(t)) - c_1(t) - c_2(t), \quad y(0) = y_0. \tag{2.1}$$

As usual, (2.1) is assumed to hold in the almost everywhere sense. The economic interpretation of it requires that we impose the additional restriction:

$$\text{if } y(t_1, y_0, c) = 0 \text{ for some } t_1 \geqq 0, \text{ then } y(t, y_0, c) = 0 \text{ for } t \geqq t_1. \tag{2.2}$$

The restriction (2.2) is important. It is not implied by the condition $m(0)=0$ and (2.1) above: so it has to be introduced to rule out paths of the asset which are not acceptable. Unfortunately it introduces a discontinuity which has to be treated carefully. To clarify the nature of the discontinuity, and explain how we deal with it, we introduce the stopping time

$$t_c(y_0) = \min\{t \geqq 0 \colon y(t, y_0, c) = 0\}. \tag{2.3}$$

Notice that the integration of the instantaneous utility in the evaluation of the payoff is only allowed over the interval $[0, t_c(y_0)]$, which depends on c. But this dependence is not upper semicontinuous: we shall see later that the functional $c \mapsto t_c(y_0)$ is only lower semicontinuous. This precludes a direct application of standard upper semicontinuity arguments (we use these results in our proof: see [1] for a reference.)

A pair of consumption paths $c=(c_1,c_2)$ is *admissible* if a solution of (2.1) exists, $\bar{c}_i \geqq c_i \geqq 0$, $i=1,2$, and $y(t, y_0, c) \geqq 0$ for $t \geqq 0$. We denote by $\mathscr{C}$ the set of admissible pairs. Any admissible consumption pair belongs to the product space $L^2_\mu \times L^2_\mu$ of pairs of measurable functions, μ-square integrable. The set $\mathscr{C}$ is weakly compact in this space. We denote as usual with $c_n \rightharpoonup c$ the weak convergence of (c_n) to c. Information about the weak topology can be found in [1]. The reader only needs to know what is presented in [2]. The *First Best* is in this context the solution of

$$\sup \int_0^{t_c(y_0)} (\alpha u_1(c_1) + (1-\alpha) u_2(c_2))\, d\mu \tag{2.4}$$

over the admissible consumption pairs; here $\alpha \in [0,1]$.

The *Second Best* problem is the problem of determining the supremum in (2.4) over all consumption pairs which are outcome of subgame perfect equilibria. It is easy to prove that any such consumption pair can be supported as an outcome of the following *trigger strategies* equilibrium: player i consumes according to c_i until a defection is detected, and then consumes $\bar{c}_i$ forever. For a proof of this statement see, if necessary, [3].

Since trigger strategies are enough to support a second best outcome, the second best problem itself can be reduced to a maximization problem subject to the constraint that the value of defection from the second best pair is everywhere lower than its continuation value for each player.

If at time t player 1 defects from the agreed consumption path c_1 to the maximum consumption $\bar{c}_1$, this will produce a new pair $\tilde{c} = (\tilde{c}_1, \tilde{c}_2)$ defined by $\tilde{c}_1(s) = c_1(s)$ if $s < t$, $\bar{c}_1$ if $s \geqq t$, and $\tilde{c}_2(s) = c_2(s)$ if $s < t + \tau$, $\bar{c}_2$ if $s \geqq t + \tau$. Clearly the map from c to $\tilde{c}$ is continuous in the weak topology. Now the value of defecting from c at time t for player 1 is

$$v_D^1(c, t) \equiv \int_0^{t_{\tilde{c}}((y(t, y_0, c))} u_1(\tilde{c}_1)\, d\mu. \tag{2.5}$$

3 The existence result

From the assumptions on m we derive that

$$y(t, y_0, c) \leqq e^{Mt} y_0 \text{ for every } c, \text{ and for every } t \geqq 0. \tag{3.1}$$

We have immediately that if $c_n \to c$ in $\mathscr{C}$ then

$$\lim_n y(t, y_0, c_n) = y(t, y_0, c) \text{ for every } t \geqq 0; \tag{3.2}$$

$$t_c \leqq \liminf_n t_{c_n}; \tag{3.3}$$

$$v_D^i(c, t) \leqq \liminf_n v_D^i(c_n, t) \quad \text{for } i = 1, 2. \tag{3.4}$$

The inequality (3.4) is an immediate consequence of (3.3). In turn, (3.3) follows from (3.2) and the fact that $|\dot{y}| \leqq 2 \max(\bar{c}_1, \bar{c}_2)$. Finally to prove (3.2) notice that if $t \leqq t_c(y_0)$ then condition (2.2) is not relevant, and (3.2) follows from weak convergence of c_n. If $t \geqq t_c(y_0)$, then from (3.1) above $y(t, y_0, c_n) \leqq e^{M(t - t_c(y_0))} y(t_c(y_0), y_0, c_n)$; now use the previous result to establish (3.2).

Consider now any function U: $[0, \bar{c}_1] \times [0, \bar{c}_2] \to R^+$, increasing, concave, continuous at 0 with $U(0,0) = 0$. For a fixed interval I the functional

$$c \mapsto \int_I U(c)\, d\mu$$

is weakly uppersemicontinuous.

We can now prove:

Lemma 3.1. *The functional* $c \mapsto \int_t^{t_c(y_0)} U(c)\, d\mu \equiv V(c, t)$ *is weakly upper semicontinuous for every* $t \geqq 0$.

Proof. Let t_ε be defined by $\int_{t_\varepsilon} U(\bar{c}_1, \bar{c}_2)\, d\mu = \varepsilon$ and $\bar{t} = \limsup_n t_{c_n}$. From (3.3) $\bar{t} \geqq t_c$.

Consider first the case $U(c) = c_1 + c_2$. Then we have

$$\limsup_n V(c_n, t) - V(c, t)$$

$$\leqq \limsup_n \int_{t_c}^{t_{c_n}} U(c_n)\,d\mu$$

$$\leqq \limsup_n \int_{t_c}^{\min(t,t_\varepsilon)} (m(y(t, y_0, c_n)) - \dot{y}(t, y_0, c_n))\,d\mu + \varepsilon$$

$$\leqq Cy(t, y_0, c_n) + \varepsilon$$

for some positive number C. The last inequality follows integrating by parts and using (3.1) above. Since $y(t, y_0, c_n) \to 0$ as $c_n \to c$, the claim is proved in this case. In the general case the estimate $U(c) \leqq a + bc$ for some $a \in R_+, b \in R_+^2$ and the previous argument give

$$\limsup_n V(c_n, t) - V(c, t) \leqq \varepsilon + a(\min(\bar{t}, t_\varepsilon) - t_c)$$

independently of b. Since a can be chosen arbitrarily small, the proof is complete. Q.E.D.

Now define $v^i(c, t) \equiv \int_0^{t_c(y_0)} u_i(c_i)\,d\mu$, the value for player i of the equilibrium pair (c_1, c_2) and the set

$$\mathscr{I} \equiv \{c \in C: v^i(c, t) - v_D^i(c, t) \geqq 0, \text{ for every } t \geqq 0, \text{ and } i = 1, 2\}.$$

Then we have:

Lemma 3.2. *$\mathscr{I}$ is weakly compact.*

Proof. From (3.4) and (3.1), the functional $c \mapsto v^i(c, t) - v_D^i(c, t)$ is weakly upper semicontinuous for every fixed t and $i = 1, 2$. So $\mathscr{I}$ is a weakly closed subset of $\mathscr{C}$. Q.E.D.

From the previous results we now have:

Theorem 3.3. *First Best and Second Best solutions exist for the model described in sect. 2.*

Proof. For the First Best existence, choose U in (3.1) as $U = \alpha u_1 + (1 - \alpha) u_2$. For the Second Best, choose the same U and use weak compactness of $\mathscr{I}$. Q.E.D.

Remark 3.4. The Lipschitz conditon on m is not needed for the First Best existence: this condition can be for instance removed by considering the function $m_A(y) \equiv \min(Ay, m(y))$, applying the previous result and using the fact that an optimal solution is interior.

4 Fast consumption equilibria

We now turn to the issue, left open in sect. 2 above, of conditions under which the fast consumption strategies are a subgame perfect equilibrium. A first instance of such conditions is provided in [2]: if the utility function is linear and $m(y) - \bar{c}_2 \leqq 0$ for every y, then the *prodigal policy* (defined by $c_i(y) = \bar{c}_i$, $i = 1, 2$) is optimal for player 1 and *viceversa.*

It is reassuring to know that a similar conclusion holds for a larger class of utility functions. Some condition on the utility function is needed: it is enough to think of the example of a utility function which is satiated for values of consumption larger than some finite quantity. The same example shows that the form of the utility function around zero is not enough by itself to guarantee the result. Consider first the problem of the best response of the first player. He solves:

$$\max_{(T,c)} \int_0^T u_1(c(t))d\mu(t) \tag{4.1}$$

subject to

$$\dot{y}(t) = m(y(t)) - \bar{c}_2 - c(t)$$

$$y(0) = y_0$$

where

$$T = \sup_{t \geqq 0} \{y(t) > 0\}.$$

Since $m(y) - \bar{c}_2 < 0$ for every y, it follows that T is finite for any control path c. In this framework, the optimal consumption path is characterized by the solution of the usual system of state and costate variables

$$\dot{y}(t) = m(y(t)) - \bar{c}_2 - c(t) \tag{4.2}$$

$$\dot{q}(t) = [\rho - m'(y(t))]q(t)$$

$$y(0) = y_0$$

together with the transversality condition

$$u_1(c(T)) + q(T)[m(0) - \bar{c}_2 - c(T)] = 0$$

which gives the boundary condition on the costate variable

$$q(T) = \frac{u_1(c(T))}{\bar{c}_2 + c(T) - m(0)}. \tag{4.3}$$

The optimal consumption is given by:

$$\hat{c} \equiv \arg\max_{\bar{c}_1 \geqq c \geqq 0} \{u_1(c) + q(m(y) - \bar{c}_2 - c)\}. \tag{4.4}$$

From Eq. (4.2) it is clear that the maximum value of the costate variable q, $\bar{q}$ say, is achieved at the value of y that solves

$$m'(y) = \rho.$$

We call $\hat{y}$ this value. We now provide a simple estimate on $\bar{q}$. The stable manifold in the (q, y) space, for $0 \leqq y \leqq \hat{y}$, is described by the function $\tilde{q}$ which solves the differential equation

$$\frac{d\tilde{q}}{dy} = \frac{\dot{q}}{\dot{y}} = \frac{(\rho - m'(y))q}{m(y) - \bar{c}_2 - \bar{c}_1}, \quad \tilde{q}(0) = \frac{u_1(\bar{c}_1)}{\bar{c}_1 + \bar{c}_2 - m(0)}. \tag{4.5}$$

Since

$$\frac{d\tilde{q}}{dy} \leqq \frac{m'(0)}{\bar{c}_1 + \bar{c}_2} q,$$

the solution Q of

$$\frac{dQ}{dy} = \frac{m'(0)}{\bar{c}_1 + \bar{c}_2} Q, \quad Q(0) = \tilde{q}(0) \tag{4.6}$$

satisfies $Q(\hat{y}) \geqq \tilde{q}(\hat{y})$. (See for instance Hale, J. [4, Theorem 1.6.1].)

Solving (4.6) yields

$$Q(\hat{y}) = \exp\left(\frac{m'(0)}{\bar{c}_1 + \bar{c}_2} \hat{y} \right) \frac{u_i(\bar{c}_1)}{\bar{c}_1 + \bar{c}_2 - m(0)}.$$

The value of $Q(\hat{y})$ for the second player is similar: we write $Q_i(\hat{y})$ to distinguish among the two.

We can now conclude

Proposition 4.1 *The pair of fast consumptions $(\bar{c}_1, \bar{c}_2)$ is a subgame perfect equilibrium if*

$$Q_i(\hat{y}) \leqq u_i'(\bar{c}_i) \quad i = 1, 2. \tag{4.7}$$

Proof. Given the estimate of $\tilde{q}(\hat{y})$ the optimal consumption for the fast player in Eq. (4.2) is always achieved at $\bar{c}_1$; the same goes for the second player. Q.E.D.

Remark 4.2. *Let us discuss briefly some instance in which the condition of the proposition holds. If $u_i'(\bar{c}_i) > 0$, $i = 1, 2$, then* (4.7) *holds for large ρ's. If $u_i'(c) \geqq \varepsilon > 0$ for every c, then the choice of a large $\bar{c}_i$ will be enough. But also in the case, say, of $u_i(c) = c^\alpha$ if $\alpha > 1/2$ and $m(0) = 0$, it will be enough to choose $\bar{c}_i$ to be sufficiently large.*

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