Commun. Math. Phys. 104, 657–692 (1986)

Communications in
Mathematical
Physics

The Becker-Döring Cluster Equations: Basic Properties and Asymptotic Behaviour of Solutions

J. M. Ball[1], J. Carr[1], and O. Penrose[2]

[1] Department of Mathematics, Heriot-Watt University, Edinburgh, EH14 4AS, Scotland
[2] Faculty of Mathematics, The Open University, Milton Keynes

Abstract. Existence and uniqueness results are established for solutions to the Becker-Döring cluster equations. The density ϱ is shown to be a conserved quantity. Under hypotheses applying to a model of a quenched binary alloy the asymptotic behaviour of solutions with rapidly decaying initial data is determined. Denoting the set of equilibrium solutions by $c^{(\varrho)}$, $0 \leqq \varrho \leqq \varrho_s$, the principal result is that if the initial density $\varrho_0 \leqq \varrho_s$ then the solution converges strongly to $c^{(\varrho_0)}$, while if $\varrho_0 > \varrho_s$ the solution converges weak* to $c^{(\varrho_s)}$. In the latter case the excess density $\varrho_0 - \varrho_s$ corresponds to the formation of larger and larger clusters, i.e. condensation. The main tools for studying the asymptotic behaviour are the use of a Lyapunov function with desirable continuity properties, obtained from a known Lyapunov function by the addition of a special multiple of the density, and a maximum principle for solutions.

1. Introduction

Consider a system of a large number of clusters of particles that can coagulate to form larger clusters or fragment to form smaller ones. Becker and Döring (1935) proposed an infinite system of ordinary differential equations as a model for the time evolution of the distribution of cluster sizes for such a system. In its original form this system treated the number of one-particle clusters as fixed: it did not take into account the depletion of the number of one-particle clusters as larger clusters are formed. A modified version of these equations allowing for depletion, which we still refer to as the Becker-Döring equations, was described by Penrose and Lebowitz (1979). In this paper we make a rigorous study of some fundamental properties of solutions to the (modified) Becker-Döring equations, and in particular analyze aspects of the asymptotic behaviour of solutions as time $t \to \infty$.

If $c_r(t) \geqq 0$, $r = 1, 2, \dots$, denotes the expected number of r-particle clusters per unit volume at time t, then the Becker-Döring equations can be written in the form

$$\begin{aligned} \dot{c}_r &= J_{r-1}(c) - J_r(c), \qquad r \geqq 2, \\ \dot{c}_1 &= -J_1(c) - \sum_{r=1}^{\infty} J_r(c), \end{aligned} \tag{1.1}$$

where $c=(c_r)$,

$$J_r(c)=a_r c_1 c_r - b_{r+1} c_{r+1}, \tag{1.2}$$

and where the kinetic coefficients a_r, b_{r+1} $(r \geqq 1)$ are non-negative constants. The system (1.1)–(1.2) is a special case of the *discrete coagulation-fragmentation equations*

$$\begin{aligned} \dot{c}_r = & \frac{1}{2}\sum_{s=1}^{r-1} a_{s,r-s} c_s c_{r-s} - c_r \sum_{s=1}^{\infty} a_{r,s} c_s \\ & + \sum_{s=r+1}^{\infty} b_{s,r} c_s - \frac{c_r}{r}\sum_{s=1}^{r-1} s b_{r,s}, \qquad r \geqq 1, \end{aligned} \tag{1.3}$$

where the first and last sums are absent when $r=1$. The balance law (1.3) was first derived by Smoluchowski (1917) for the case of pure coagulation (all $b_{r,s}=0$). The form quoted of (1.3) is taken from Spouge (1984), his derivation following that of Melzak (1957) for an analogue in which the cluster size can be a continuous variable. For other derivations of similar equations see Friedlander (1960), Binder (1977) and the references cited in Drake (1972). These papers cite some of the numerous applications of coagulation-fragmentation equations in pure and applied science. To obtain (1.3) one assumes that clusters coagulate by binary collisions, the probability per unit time that a given r-cluster will collide with some s-cluster to form an $(r+s)$-cluster being given by $a_{r,s}c_s$ (with $a_{r,s}=a_{s,r}$); it then follows that the rate per unit volume at which r-clusters and s-clusters combine to form $(r+s)$-clusters is given by $a_{r,s}c_r c_s$ if $r \neq s$ and by $\frac{1}{2}a_{r,r}c_r^2$ if $r=s$. As regards fragmentation it is assumed that the rate per unit volume at which s-clusters are formed by fragmentation of r-clusters $(r>s)$ is given by $b_{r,s}c_r$; in the case of binary fragmentation we have $b_{r,s}=b_{r,r-s}$, and the last sum in (1.3) can be written as $-\frac{c_r}{2}\sum_{s=1}^{r-1} b_{r,s}$. To obtain (1.1) from (1.3) we assume that

$$\begin{aligned} a_r = a_{r,1} = a_{1,r}, \quad & b_{r+1} = b_{r+1,1} = b_{r+1,r}, \qquad r>1, \\ 2a_1 = a_{1,1}, \quad & 2b_2 = b_{2,1}, \end{aligned} \tag{1.4}$$

all other $a_{r,s}$, $b_{r,s}$ zero.

The Becker-Döring equations are thus intended to describe situations in which the evolution is dominated by clusters gaining or shedding just one particle. The validity of this assumption and of the underlying cluster picture has been discussed by Abraham (1974), Penrose and Lebowitz (1979), Kalos et al. (1978) and others. More recently Penrose and Buhagiar (1983) and Penrose et al. (1984) have compared the results of computer simulations with numerical solutions of (1.1) for a model of a binary alloy quenched from a high temperature, the clusters consisting of atoms of the minority component. The coefficients a_r, b_r used [see Eq. (5.8) below] were obtained by extrapolation from microscopic calculations carried out in Penrose and Buhagiar (1983) for small values of r, and have the property that $a_r \sim \text{const}\, r^{1/3}$, $b_r \sim \text{const}\, r^{1/3}$ as $r \to \infty$. In the computer simulations the atoms were confined to the vertices of a simple cubic lattice and moved by changes of nearest neighbour pairs. The solutions of (1.1) were computed by solving the finite-dimensional system obtained by truncating the equations at $r=N$ and setting $J_N=0$. The initial data satisfied the condition $c_r(0)=0$ if $r>r_0$, where r_0 was a small positive integer. For small values of the initial density $\sum_{r=1}^{\infty} rc_r(0)$, quite good agreement was found between the computer simulations and the numerical solutions. [As explained in Sect. 5 the coefficients a_r, b_r were allowed to depend on the density $\varrho=\sum_{r=1}^{\infty} rc_r(t)$, but since, as is shown in Corollary 2.6, ϱ is a conserved quantity the solution of (1.1) in this case can be reduced to that for constant coefficients.]

As far as we are aware there have been no general studies of the existence, uniqueness and continuous dependence on the initial data of solutions to (1.1). For (1.3) Spouge (1984) has proved existence, but under the hypothesis [apparently taken from Melzak (1957)] $\sup_{r\geq 0}\frac{1}{r}\sum_{s=1}^{r-1} sb_{r,s}<\infty$ which is violated by (1.4) when $\lim_{r\to\infty} b_r=\infty$. McLeod (1962a, b, c) has proved certain existence and uniqueness theorems for (1.3) in the pure coagulation case and Spouge refers to various other results on the pure coagulation case and its continuous analogue. We begin in Sect. 2 by proving a general existence theorem (Theorem 2.2) for solutions of (1.1), which in a special case (Corollary 2.3) implies global existence when the initial data has finite density and when $a_r=O(r)$. (Here and throughout the paper the o, O notation refers to behaviour as $r\to\infty$.) The absence of hypotheses on b_r stems from the identity (2.24), which enables estimates to be obtained independent of b_r and indicates that fragmentation can be thought of as a dissipative mechanism. If $\lim_{r\to\infty}\frac{a_r}{r}=\infty$, there is in general no solution of (1.1) even on a short time interval (Theorem 2.7). In Corollary 2.6 we show that the density is a conserved quantity for solutions. This crucial result is used frequently in the rest of the paper; it is not true in general for the discrete coagulation-fragmentation equations, for which density conservation can break down at a finite time, a phenomenon known as *gelation* (Leyvraz and Tschudi, 1981; Hendricks et al., 1983).

In Sect. 3 we study the differentiability of solutions with respect to t (Theorem 3.2) and the continuous dependence of solutions on the initial data both (Theorem 3.4) with respect to convergence in the Banach space X of sequences $c=(c_r)$ such that $\|c\|=\sum_{r=1}^{\infty} r|c_r|<\infty$ and (Theorem 3.5) with respect to weak * convergence in X. We also give two uniqueness theorems, the first (Theorem 3.6) assuming more about the initial data, and the second (Theorem 3.7) more about the coefficients a_r, b_r; as a consequence we give conditions under which the numerical scheme used in the papers cited above to solve (1.1) converges to a solution as the truncation mode N tends to infinity.

In Sects. 4, 5 we turn to the question of the asymptotic behaviour of solutions as $t\to\infty$. This is of considerable importance for applications; for example, in the binary alloy problem the essence of the phase transition lies in the formation of larger and larger clusters as t increases. The asymptotic behaviour is also of unusual mathematical interest, and in particular tests our understanding of the convergence to equilibrium of infinite-dimensional systems endowed with a Lyapunov function. In this introduction we describe our results for the case of the binary alloy problem; for other cases the reader is referred to the main text. For the binary alloy, as is well known, there is a critical density ϱ_s, $0<\varrho_s<\infty$, such that if $0\leqq\varrho\leqq\varrho_s$ there is a unique equilibrium state c^ϱ of (1.1) having density ϱ, while if $\varrho>\varrho_s$ there is no equilibrium state with density ϱ. For $0\leqq\varrho\leqq\varrho_s$,

$$c_r^\varrho=Q_r\cdot(c_1^\varrho)^r\,,\qquad r\geqq 1\,, \tag{1.5}$$

where $Q_1=1$, $\dfrac{Q_{r+1}}{Q_r}=\dfrac{a_r}{b_{r+1}}$ for $r\geqq 1$. The equilibrium c^ϱ is the unique minimizer of

$$V(c)=\sum_{r=1}^{\infty} c_r\left(\ln\left(\frac{c_r}{Q_r}\right)-1\right) \tag{1.6}$$

on the set $X_\varrho^+=\left\{c=(c_r)\colon c_r\geqq 0 \text{ for all } r,\ \sum_{r=1}^{\infty} rc_r=\varrho\right\}$. The "free-energy" function V is a Lyapunov function for (1.1), that is it is non-increasing along solutions, a fact observed formally by Buhagiar (1980) and proved in Theorems 4.7 and 4.8. [We remark that there is a similar Lyapunov function for (1.3) under additional hypotheses on the $a_{r,s}$, $b_{r,s}$; for a special case of the continuous analogue of (1.3) the appropriate V has been given by Aizenman and Bak (1979).]

Whenever an evolution equation possesses a Lyapunov function V it is helpful to consider the following question: for a typical solution, do the successive states of the system at a sequence of times $t_j\to\infty$ form a minimizing sequence for V? In anticipation of a positive answer, a useful first step in understanding the asymptotic behaviour of solutions is to characterize the behaviour of minimizing sequences. [For many problems, though fortunately not for (1.1), the situation is complicated by the need to consider "relative" as opposed to "absolute" minimizing sequences.] Such a characterization is provided for (1.1) by Theorem 4.4; if $c^{(j)}$ is an arbitrary minimizing sequence of V on X_ϱ^+ then, if $0\leqq\varrho\leqq\varrho_s$, $c^{(j)}$ converges to c^ϱ strongly in X as $j\to\infty$, while if $\varrho>\varrho_s$, $c^{(j)}$ converges to c^{ϱ_s} weak * in X, but not strongly. This type of behaviour of minimizing sequences occurs in a number of other variational problems in mechanics and physics. Perhaps the simplest analogy is with the problem of filling up a hole in flat ground with water; if the volume of water is less than or equal to that of the hole there is a unique minimizer of the potential energy, but if the volume of water exceeds that of the hole then the hole fills to the maximum level and the excess water runs away to infinity. In the Thomas-Fermi theory of atoms and molecules (Lieb, 1981) there is a competition between the repulsion of the electrons and their attraction to the nuclei; if the total electron charge exceeds a critical value then an electron cloud of the critical charge envelops the nucleus and the excess electrons disperse to infinity. Another example, a rudimentary model predicting a finite height of the atmosphere, is discussed briefly in Ball (1981). A useful framework in which to study such examples can be the theory of concentration compactness (Lions, 1984).

Our main result on asymptotic behaviour, Theorem 5.5, was motivated by, and confirms, the numerical evidence in Penrose et al. (1984). We show that for rapidly decaying initial data with density ϱ_0 the solution $c(t)$ to (1.1) is minimizing for V on $X^+_{\varrho_0}$ as $t\to\infty$, so that, for $0 \leqq \varrho_0 \leqq \varrho_s$, $c(t)\to c^{\varrho_0}$ strongly in X and, for $\varrho > \varrho_s$, $c(t) \overset{*}{\rightharpoonup} c^{\varrho_s}$ in X. The proof is accomplished in two stages. We show first in Theorem 5.5 that $c(t) \overset{*}{\rightharpoonup} c^{\varrho}$ as $t\to\infty$ for some ϱ, $0 \leqq \varrho \leqq \min(\varrho_0, \varrho_s)$. This is achieved by an application of the invariance principle for evolution equations endowed with a Lyapunov function. As was first emphasized by Hale (1969), to apply the principle in its simplest form one has to find a metric with respect to which the solution has appropriate continuous dependence on the initial data, the Lyapunov function is continuous, and the positive orbit of a solution is relatively compact. Since the only obvious global estimate is that given by density conservation, to achieve relative compactness of positive orbits we are almost obliged to choose for (1.1) the metric induced by the weak * topology on bounded subsets of X. As so often happens, V is not continuous in this preferred metric. However we are saved by a piece of remarkable good fortune. Because density is concerved the functional

$$V_z(c) = V(c) - \ln z \sum_{r=1}^{\infty} r c_r \tag{1.7}$$

is also a Lyapunov function for every $z>0$; it turns out (Proposition 4.5) that there is exactly one value of z, namely the radius of convergence z_s of the series $\sum_{r=1}^{\infty} rQ_r z^r$, such that V_z is sequentially weak * continuous. We are thus able to apply the invariance principle using V_{z_s}. It turns out that we can do this under weaker hypotheses on the coefficients a_r, b_r by applying a version of the principle (cf. Ball, 1978) that does not assume uniqueness of solutions. The idea of modifying V by adding a linear combination of conserved quantities has proved useful for different reasons in various fluids problems (Arnold, 1969; Holm et al., 1983). The second stage of the proof involves identifying ϱ by using a maximum principle for (1.1) to control the "tail" of the solution in the case $\varrho < \varrho_s$; it is at this stage that we have to make hypotheses on the decay of the initial data. We end the paper by studying the stability of the equilibria and by describing how the assumptions of the various theorems are verified for the binary alloy problem.

It would be interesting to find other infinite-dimensional examples in which an energy cascade as $t\to\infty$ into higher and higher modes in the presence of a Lyapunov function can be rigorously established. Possible candidates for examples are various non-linear partial differential equations arising in continuum mechanics as models of materials that may undergo phase transitions (cf. Ball, 1984). One such example, from non-linear viscoelasticity of rate type, where the corresponding free energy is a non-convex integral of the calculus of variations whose minimizing sequences may converge to generalized curves in the sense of Young (1969), has been studied by Andrews and Ball (1982); however, recent work of Pego (to appear) indicates that the viscoelastic damping mechanism there is too strong to allow the solution itself to tend to a generalized curve as $t\to\infty$.

Aside from technical refinements, two of the main tasks left open by our analysis are (a) to provide some detailed information on the asymptotic behaviour of solutions to (1.1) and in particular give a rigorous treatment of some of the standard methods of nucleation theory for the study of metastable states, and (b) to extend our analysis to the discrete coagulation-fragmentation equations (1.3) and give a satisfactory treatment of gelation.

2. Existence, Nonexistence, and Density Conservation

In order to study the existence and other properties of solutions to (1.1) we introduce the Banach sequence space

$$X=\{y=(y_r):\|y\|<\infty\}, \qquad \|y\|=\sum_{r=1}^{\infty} r|y_r|.$$

We write $y \geqq 0$ if $y_r \geqq 0$ for each $r=1,2,\ldots$, and set $X^+=\{y\in X:\ y\geqq 0\}$.

Definition. Let $0<T\leqq\infty$. A *solution* $c=(c_r)$ of (1.1) on $[0,T)$ is a function c: $[0,T)\to X$ such that

(i) $c(t)\geqq 0$ for all $t\in[0,T)$,

(ii) each c_r: $[0,T)\to\mathbb{R}$ is continuous, and $\sup_{t\in[0,T)}\|c(t)\|<\infty$,

(iii) $\int_0^t \sum_{r=1}^{\infty} a_r c_r(s)\,ds<\infty$, $\int_0^t \sum_{r=2}^{\infty} b_r c_r(s)\,ds<\infty$ for all $t\in[0,T)$, and

(iv)
$$\begin{aligned} c_r(t)&=c_r(0)+\int_0^t [J_{r-1}(c(s))-J_r(c(s))]\,ds, \qquad r\geqq 2,\\ c_1(t)&=c_1(0)-\int_0^t\left[J_1(c(s))+\sum_{r=1}^{\infty} J_r(c(s))\right]ds, \end{aligned} \tag{2.1}$$

for all $t\in[0,T)$, where $J_r=a_r c_r c_1-b_{r+1}c_{r+1}$.

Note that by (ii) each c_r is bounded on $[0,t]$ for any $t\in[0,T)$, so that by (iii) and (1.2) the integrals in (2.1) exist and are finite. It follows from (2.1) that if c is a solution then each c_r is absolutely continuous for $t\in[0,T)$, so that c satisfies (1.1) for a.e. $t\in[0,T)$.

In common with earlier work on related equations (Reuter and Ledermann, 1953; McLeod, 1962a; Spouge, 1984) we prove existence of solutions to (1.1) by taking a limit of solutions of the finite-dimensional system

$$\begin{aligned} &\dot c_r=J_{r-1}-J_r, \qquad 2\leqq r\leqq n-1,\\ &\dot c_1=-J_1-\sum_{r=1}^{n-1} J_r, \qquad \dot c_n=J_{n-1},\\ &c_r(0)\geqq 0,\ 1\leqq r\leqq n. \end{aligned} \tag{2.2}$$

Lemma 2.1. *The system* (2.2) *has a unique solution for* $t\geqq 0$ *with* $c_r(t)\geqq 0$, $1\leqq r\leqq n$, *and* $\sum_{r=1}^{n} rc_r(t)=\sum_{r=1}^{n} rc_r(0)$ *for* $t\geqq 0$.

Proof. Consider for $\varepsilon>0$ the solution $c_r^{(\varepsilon)}$ of the system obtained by adding ε to the right-hand sides of each of the equations for $\dot c_r$, $1\leqq r\leqq n$, in (2.2). By considering the sign of $\dot c_s^{(\varepsilon)}(t)$ for t such that $c_s^{(\varepsilon)}(t)=0$, $c_r^{(\varepsilon)}(\tau)\geqq 0$ for $0\leqq\tau\leqq t$, $r\neq s$, it is easily shown using standard results on ordinary differential equations that $c_r^{(\varepsilon)}(t)$ is non-negative and tends as $\varepsilon \to 0+$ to a unique non-negative solution $c_r(t)$ of (2.2) defined for all t in some interval $[0, t_0)$, $t_0 > 0$. The fact that $f(t) = \sum_{r=1}^{n} r c_r(t)$ is a constant of the motion follows by showing that $\dot{f}(t) = 0$, and the global existence then results from the bounds $0 \leqq c_r(t) \leqq r^{-1} \sum_{k=1}^{n} k c_k(0)$. □

Remark. The device of adding ε to the equations can be found in Hartman (1964, p. 25); the lemma may also be proved directly via appropriate positivity preserving successive approximations (cf. the proof of Theorem 4.6).

Theorem 2.2. *Let (g_r) be a positive sequence satisfying $g_{r+1} - g_r \geqq \delta > 0$, $r = 1, 2, \ldots$, for some constant δ. Assume that*

$$a_r(g_{r+1} - g_r) = O(g_r). \tag{2.3}$$

Let $c_0 = (c_{0r}) \geqq 0$ satisfy $\sum_{r=1}^{\infty} g_r c_{0r} < \infty$. Then there exists a solution c of (1.1) on $[0, \infty)$ with $c(0) = c_0$ and satisfying

$$\sup_{t \in [0, T]} \sum_{r=1}^{\infty} g_r c_r(t) < \infty, \qquad \int_0^T \sum_{r=1}^{\infty} (g_r - g_{r-1}) b_r c_r(t)\, dt < \infty. \tag{2.4}$$

for all $T > 0$.

Setting $g_r = r$ we obtain the following important corollary.

Corollary 2.3. *Assume that $a_r = O(r)$ and that $c_0 \in X^+$. Then there exists a solution c of (1.1) on $[0, \infty)$ with $c(0) = c_0$.*

The theorem also shows that if the initial data decays rapidly as $r \to \infty$ then there exists a solution with similar decay. The following easily proved proposition gives two examples of this property.

Proposition 2.4. *The hypotheses on a_r, g_r in Theorem 2.2 hold in the following cases:*
(i) $a_r = O(r)$; $g_r = r^\alpha$, $\alpha > 1$,
(ii) $a_r = O(r^\alpha)$, $0 \leqq \alpha < 1$; $g_r = \exp(\mu r^{1-\alpha})$, $\mu > 0$.

If $a_r > 0$ for all r the hypotheses of Theorem 2.2 imply that

$$a_r \delta \leqq D g_r, \qquad g_{r+1} \leqq g_r \left(1 + \frac{D}{a_r}\right),$$

and thus that

$$a_r \leqq C \prod_{j=1}^{r-1} \left(1 + \frac{D}{a_j}\right), \qquad r > 1, \tag{2.5}$$

for some constants $C > 0$, $D > 0$. Conversely, if (2.5) holds then the hypotheses of Theorem 2.2 are satisfied by taking $g_1 = 1$ and

$$g_r = r + \prod_{j=1}^{r-1} \left(1 + \frac{D}{a_j}\right), \qquad r > 1. \tag{2.6}$$

The example $a_{n^2} = n^4$, $n = 1, 2, \ldots$, $a_r = 1$ otherwise, shows that (2.5) does not in general imply that $a_r = O(r)$.

Proof of Theorem 2.2. Let $c^n(0)=(c_{01}, c_{02}, \dots, c_{0n})$. By Lemma 2.1 the system (2.2) has a unique solution c^n defined on $[0,\infty)$ with $c_r^n(t)\geqq 0$, $1\leqq r\leqq n$, and $\sum_{r=1}^{n} rc_r^n(t)=\sum_{r=1}^{n} rc_r^n(0)$ for all $t\geqq 0$. We regard $c^n(t)$ as an element of X by defining $c_r^n(t)=0$ if $r>n$. Thus $\|c^n(t)\|\leqq\|c_0\|$ and $0\leqq c_r^n(t)\leqq r^{-1}\|c_0\|$ for all $t\geqq 0$ and all r, n. Therefore, by (2.2),

$$|\dot{c}_r^n(t)|\leqq\left(\frac{a_{r-1}}{r-1}+\frac{a_r}{r}\right)\|c_0\|^2+\left(\frac{b_r}{r}+\frac{b_{r+1}}{r+1}\right)\|c_0\|\leqq M_r<\infty,$$

for $r\geqq 2$, where M_r is a constant. Therefore, for each $r\geqq 2$ the functions $\{c_r^n(\cdot)\}$ are equicontinuous on $[0,\infty)$. Applying the Arzela-Ascoli theorem and extracting a suitable diagonal subsequence $n_k\to\infty$, we deduce that for each $r\geqq 2$ there exists a continuous function $c_r:[0,\infty)\to\mathbb{R}$ with $c_r^{n_k}\to c_r$ uniformly on compact subsets of $[0,\infty)$ as $k\to\infty$. Note that $c_r\geqq 0$; also, since $\sum_{r=2}^{l} rc_r(t)=\lim_{k\to\infty}\sum_{r=2}^{l} rc_r^{n_k}(t)\leqq\|c_0\|$ we have that

$$\sum_{r=2}^{\infty} rc_r(t)\leqq\|c_0\| \quad \text{for all} \quad t\geqq 0. \tag{2.7}$$

Since we have made no growth hypothesis on the coefficients b_r we cannot bound $|\dot{c}_1^n(t)|$ in a simple fashion. However, since $|c_1^n(t)|\leqq\|c_0\|$ we can extract a further subsequence, again denoted n_k, such that

$$c_1^{n_k}\overset{*}{\rightharpoonup} c_1 \quad \text{in} \quad L^\infty(0,\infty) \quad \text{as} \quad k\to\infty$$

for some non-negative $c_1\in L^\infty(0,\infty)$; i.e. $\int_0^\infty [c_1^{n_k}(t)-c_1(t)]\,\phi(t)\,dt\to 0$ as $k\to\infty$ for each $\phi\in L^1(0,\infty)$. In order to pass to the limit in (2.2) we need further a priori estimates. From (2.2) we obtain for any $m\geqq 2$, $n_k>m$,

$$\begin{aligned}\frac{d}{dt}\sum_{r=m}^{n_k} g_r c_r^{n_k}+\sum_{r=m}^{n_k-1}(g_{r+1}-g_r)b_{r+1}c_{r+1}^{n_k}&=g_m(a_{m-1}c_1^{n_k}c_{m-1}^{n_k}-b_m c_m^{n_k})\\&\quad+\sum_{r=m}^{n_k-1}(g_{r+1}-g_r)a_r c_1^{n_k}c_r^{n_k}.\end{aligned}\tag{2.8}$$

Setting $m=2$ in (2.8) and using the bound on $c_1^{n_k}$ and the hypotheses on a_r, g_r we obtain

$$\begin{aligned}\sum_{r=2}^{n_k} g_r c_r^{n_k}(t)+\int_0^t\sum_{r=2}^{n_k-1}(g_{r+1}-g_r)b_{r+1}c_{r+1}^{n_k}(s)\,ds&\leqq\sum_{r=2}^{n_k} g_r c_{0r}\\&\quad+K\left(1+\int_0^t\sum_{r=2}^{n_k} g_r c_r^{n_k}(s)\,ds\right)\end{aligned}$$

for all $t\geqq 0$, where K is a constant independent of k. By Gronwall's inequality and the fact that $\sum_{r=1}^{\infty} g_r c_{0r}<\infty$, it follows that

$$\sum_{r=2}^{n_k} g_r c_r^{n_k}(t)+\int_0^t\sum_{r=2}^{n_k-1}(g_{r+1}-g_r)b_{r+1}c_{r+1}^{n_k}(s)\,ds\leqq Me^{Kt}, \tag{2.9}$$

for all $t \geqq 0$, where M is a constant independent of k. Writing the sums in (2.9) as $\sum_{r=2}^{l-1} + \sum_{r=l}^{n_k}$, $\sum_{r=2}^{l-1} + \sum_{r=l}^{n_k-1}$ and letting $k \to \infty$ and then $l \to \infty$ we deduce by the monotone convergence theorem that

$$\sum_{r=2}^{\infty} g_r c_r(t) + \int_0^t \sum_{r=2}^{\infty} (g_{r+1} - g_r) b_{r+1} c_{r+1}(s)\, ds \leqq M e^{Kt}, \tag{2.10}$$

for all $t \geqq 0$.

Replacing g_r by 1 in (2.8) and integrating we obtain

$$\sum_{r=m}^{n_k} c_r^{n_k}(t) - \sum_{r=m}^{n_k} c_{0r} = \int_0^t (a_{m-1} c_1^{n_k}(s)\, c_{m-1}^{n_k}(s) - b_m c_m^{n_k}(s))\, ds \tag{2.11}$$

for all $t \geqq 0$. Writing $c_1^{n_k}(s)\, c_{m-1}^{n_k}(s) = c_1^{n_k}(s)\, c_{m-1}(s) + c_1^{n_k}(s)\,(c_{m-1}^{n_k}(s) - c_{m-1}(s))$, we have that for $m > 2$ and any $t \geqq 0$,

$$\lim_{k \to \infty} \int_0^t (a_{m-1} c_1^{n_k}(s)\, c_{m-1}^{n_k}(s) - b_m c_m^{n_k}(s))\, ds = \int_0^t (a_{m-1} c_1(s)\, c_{m-1}(s) - b_m c_m(s))\, ds\,. \tag{2.12}$$

Furthermore,

$$\sum_{r=m}^{n_k} c_r^{n_k}(t) - \sum_{r=m}^{\infty} c_r(t) = \left(\sum_{r=m}^{l-1} + \sum_{r=l}^{\infty} \right) (c_r^{n_k}(t) - c_r(t)),$$

and since by (2.7)

$$\left| \sum_{r=l}^{\infty} (c_r^{n_k}(t) - c_r(t)) \right| \leqq l^{-1} \sum_{r=l}^{\infty} r(c_r^{n_k}(t) + c_r(t)) \leqq 2 l^{-1} \|c_0\|,$$

we deduce that

$$\lim_{k \to \infty} \sum_{r=m}^{n_k} c_r^{n_k}(t) = \sum_{r=m}^{\infty} c_r(t) \quad \text{for all} \quad t \geqq 0\,. \tag{2.13}$$

From (2.11)–(2.13) we obtain for $m > 2$

$$g_m \int_0^t (a_{m-1} c_1(s)\, c_{m-1}(s) - b_m c_m(s))\, ds = h_m(t) \quad \text{for all} \quad t \geqq 0\,, \tag{2.14}$$

where

$$h_m(t) \stackrel{\text{def}}{=} g_m \left(\sum_{r=m}^{\infty} c_r(t) - \sum_{r=m}^{\infty} c_{0r} \right).$$

Let $T > 0$. Note that

$$|h_m(t)| \leqq \sum_{r=m}^{\infty} g_r c_r(t) + \sum_{r=m}^{\infty} g_r c_{0r},$$

so that by (2.10),

$$\lim_{m \to \infty} h_m(t) = 0, \ |h_m(t)| \leqq \text{const} \quad \text{for all} \quad t \in [0, T]\,.$$

Thus $\lim_{m\to\infty}\int_0^T |h_m(t)|\,dt=0$, and so given $\varepsilon>0$ there exists $M>2$ such that

$$\int_0^T |h_M(t)|\,dt<\varepsilon \quad\text{and}\quad \sum_{r=M}^{\infty} g_r c_{0r}<\varepsilon\,. \tag{2.15}$$

By (2.14), (2.15),

$$g_M\int_0^t\int_0^s (a_{M-1}c_1(\tau)\,c_{M-1}(\tau)-b_M c_M(\tau))\,d\tau\,ds<\varepsilon \quad\text{for all}\quad t\in[0,T]\,.$$

By (2.12) and the bounded convergence theorem there exists k_0 such that

$$g_M\int_0^t\int_0^s (a_{M-1}c_1^{n_k}(\tau)\,c_{M-1}^{n_k}(\tau)-b_M c_M^{n_k}(\tau))\,d\tau\,ds<2\varepsilon$$
$$\text{for all}\quad t\in[0,T]\quad\text{and all}\quad k\geqq k_0\,. \tag{2.16}$$

Returning to (2.8), we have that

$$\sum_{r=M}^{n_k} g_r c_r^{n_k}(s)-\sum_{r=M}^{n_k} g_r c_{0r}+\int_0^s\sum_{r=M}^{n_k-1}(g_{r+1}-g_r)b_{r+1}c_{r+1}^{n_k}(\tau)\,d\tau$$
$$\leqq g_M\int_0^s (a_{M-1}c_1^{n_k}(\tau)\,c_{M-1}^{n_k}(\tau)-b_M c_M^{n_k}(\tau))\,d\tau+K_1\int_0^s\sum_{r=M}^{n_k} g_r c_r^{n_k}(\tau)\,d\tau$$
$$\text{for all}\quad s\geqq 0\,, \tag{2.17}$$

where K_1 is a constant independent of k and ε. Integrating (2.17) over $(0,t)$ and applying Gronwall's inequality, we deduce using (2.15), (2.16) that for all $k\geqq k_0$,

$$\int_0^t\sum_{r=M}^{n_k} g_r c_r^{n_k}(s)\,ds+\int_0^t\int_0^s\sum_{r=M}^{n_k-1}(g_{r+1}-g_r)\,b_{r+1}c_{r+1}^{n_k}(\tau)\,d\tau\,ds$$
$$<\frac{1}{K_1}(e^{K_1t}-1)\sum_{r=M}^{n_k} g_r c_{0r}+2\varepsilon e^{K_1t}$$
$$<\varepsilon(2+K_1^{-1})e^{K_1t}\quad\text{for all}\quad t\in[0,T]\,. \tag{2.18}$$

Since

$$\int_0^t\int_0^s\sum_{r=M}^{n_k-1}(g_{r+1}-g_r)\,b_{r+1}c_{r+1}^{n_k}(\tau)\,d\tau\,ds=\int_0^t (t-s)\sum_{r=M}^{n_k-1}(g_{r+1}-g_r)\,b_{r+1}c_{r+1}^{n_k}(s)\,ds\,,$$

it follows from (2.18) that for $k\geqq k_0$,

$$\int_0^t\sum_{r=M}^{n_k} g_r c_r^{n_k}(s)\,ds+\frac{t}{2}\int_0^{t/2}\sum_{r=M}^{n_k-1}(g_{r+1}-g_r)\,b_{r+1}c_{r+1}^{n_k}(s)\,ds$$
$$<\varepsilon(2+K_1^{-1})e^{K_1t}\quad\text{for all}\quad t\in[0,T]\,. \tag{2.19}$$

For $k,l\geqq k_0$ and $t\geqq 0$ we deduce from (2.2) that

$$c_1^{n_k}(t)-c_1^{n_l}(t)=-2\int_0^t [a_1(c_1^{n_k}(s)-c_1^{n_l}(s))\,(c_1^{n_k}(s)+c_1^{n_l}(s))$$
$$-b_2(c_2^{n_k}(s)-c_2^{n_l}(s))]\,ds-\int_0^t\left(\sum_{r=2}^{M-1}+\sum_{r=M}^{n_k-1}\right)[a_r(c_1^{n_k}(s)\,c_r^{n_k}(s)$$
$$-c_1^{n_l}(s)\,c_r^{n_l}(s))-b_{r+1}(c_{r+1}^{n_k}(s)-c_{r+1}^{n_l}(s))]\,ds\,. \tag{2.20}$$

Using the bounds on $c_r^{n_k}$, the uniform convergence of $c_r^{n_k}$ for $r \geqq 2$ and the weak * convergence of $c_1^{n_k}$, it is easily shown that the integrals of the $\sum_{r=2}^{M-1}$ term and of the $b_2(c_2^{n_k}(s) - c_2^{n_l}(s))$ term in (2.20) converge to zero as $k, l \to \infty$ uniformly for $t \in [0, T]$. On the other hand, putting $t = T$ in (2.19) and using $a_r = O(g_r)$, $g_{r+1} - g_r \geqq \delta > 0$, we see that the integral of the $\sum_{r=M}^{n_k - 1}$ term is bounded absolutely by $K_2 \varepsilon$, independently of $t \in [0, T/2]$, for some constant K_2. Therefore if k, l are sufficiently large,

$$|c_1^{n_k}(t) - c_1^{n_l}(t)| \leqq K_3 \left(\varepsilon + \int_0^t |c_1^{n_k}(s) - c_1^{n_l}(s)| \, ds \right), t \in [0, T/2],$$

for some constant K_3. It follows using Gronwall's inequality and the arbitrariness of ε that $c_1^{n_k}$ is a Cauchy sequence in $C([0, T/2])$, and since T is arbitrary c_1 has a continuous representative in $[0, \infty)$ with $c_1^{n_k} \to c_1$ as $k \to \infty$ uniformly on compact subsets of $[0, \infty)$.

We are finally in a position to pass to the limit in (2.2). For $r \geqq 2$ we have that for k sufficiently large

$$c_r^{n_k}(t) = c_{0r} + \int_0^t [J_{r-1}(c^{n_k}(s)) - J_r(c^{n_k}(s))] \, ds,$$

and the first equation in (2.1) follows from the uniform convergence of each $c_r^{n_k}$, $r \geqq 1$. Writing the c_1 equation in (2.2) in the form

$$c_1^{n_k}(t) = c_{01} - \int_0^t \left[a_1 c_1^{n_k}(s)^2 - b_2 c_2^{n_k}(s) + \left(\sum_{r=1}^{M-1} + \sum_{r=M}^{n_k - 1} \right) (a_r c_1^{n_k}(s) \, c_r^{n_k}(s) - b_{r+1} c_{r+1}^{n_k}(s)) \right] ds, \tag{2.21}$$

we note that by (2.19) there is a constant K_4 such that for $k \geqq k_0$,

$$\int_0^t \sum_{r=M}^{n_k - 1} (a_r c_1^{n_k}(s) \, c_r^{n_k}(s) + b_{r+1} c_{r+1}^{n_k}(s)) \, ds \leqq K_4 \varepsilon \quad \text{for all} \quad t \in [0, T/2]. \tag{2.22}$$

Writing the sum in (2.22) as $\sum_{r=M}^{l} + \sum_{r=l+1}^{n_k - 1}$ and using the uniform convergence of the $c_r^{n_k}$ we deduce from (2.22) that also

$$\int_0^t \sum_{r=M}^{\infty} (a_r c_1(s) \, c_r(s) + b_{r+1} c_{r+1}(s)) \, ds \leqq K_4 \varepsilon \quad \text{for all} \quad t \in [0, T/2]. \tag{2.23}$$

From (2.21)–(2.23) and the uniform convergence of the $c_r^{n_k}$ we deduce that

$$\left| c_1(t) - c_{01} + \int_0^t \left[J_1(c(s)) + \sum_{r=1}^{\infty} J_r(c(s)) \right] ds \right| \leqq 2K_4 \varepsilon \quad \text{for all} \quad t \in [0, T/2],$$

and since ε, T are arbitrary the second equation in (2.1) follows. The relations (2.4) and property (iii) in the definition of a solution are an immediate consequence of (2.10), while property (ii) follows from (2.7). □

Remark. The proof of existence is much easier in the special case $a_r = o(r)$, $b_r = o(r)$. Then $|\dot{c}_1^n(t)|$ is bounded from (2.2) and Lemma 2.1, so that we may suppose $c_1^{n_k}$ converges uniformly on compact subsets of $[0, \infty)$. Passage to the limit in the c_1 equation is then simple (cf. the proof of Theorem 3.5).

We now derive a priori estimates for solutions of (1.1) whose analogues for approximating solutions were used in the proof of Theorem 2.2. As a consequence we show that *any* solution of (1.1) conserves density; the corresponding statement for the general coagulation-fragmentation equations is false (Leyvraz and Tschudi, 1981).

Theorem 2.5. *Let (g_r) be a given sequence. Let c be a solution of (1.1) on some interval $[0, T)$, $0 < T \leqq \infty$. Suppose that $0 \leqq t_1 < t_2 < T$, $\int_{t_1}^{t_2} \sum_{r=1}^{\infty} |g_{r+1} - g_r| a_r c_r \, dt < \infty$ and either that $g_r = O(r)$ and $\int_{t_1}^{t_2} \sum_{r=1}^{\infty} |g_{r+1} - g_r| b_{r+1} c_{r+1} \, dt < \infty$ or that $\sum_{r=1}^{\infty} g_r c_r(t_i) < \infty$ for $i = 1, 2$ and $g_{r+1} \geqq g_r \geqq 0$ for sufficiently large r. Then for $m \geqq 2$,*

$$\sum_{r=m}^{\infty} g_r c_r(t_2) - \sum_{r=m}^{\infty} g_r c_r(t_1) + \int_{t_1}^{t_2} \sum_{r=m}^{\infty} (g_{r+1} - g_r) b_{r+1} c_{r+1} \, dt$$
$$= \int_{t_1}^{t_2} \sum_{r=m}^{\infty} (g_{r+1} - g_r) a_r c_1 c_r \, dt + \int_{t_1}^{t_2} g_m J_{m-1}(c(t)) \, dt. \tag{2.24}$$

Corollary 2.6. *Let c be a solution of (1.1) on some interval $[0, T)$, $0 < T \leqq \infty$. Then for all $t \in [0, T)$*

$$\sum_{r=1}^{\infty} r c_r(t) = \sum_{r=1}^{\infty} r c_r(0), \tag{2.25}$$

and for $m \geqq 2$

$$\sum_{r=m}^{\infty} r c_r(t) - \sum_{r=m}^{\infty} r c_r(0) = \int_0^t \sum_{r=m}^{\infty} J_r(c(s)) \, ds + m \int_0^t J_{m-1}(c(s)) \, ds, \tag{2.26}$$

$$\sum_{r=m}^{\infty} c_r(t) - \sum_{r=m}^{\infty} c_r(0) = \int_0^t J_{m-1}(c(s)) \, ds. \tag{2.27}$$

Proof of Theorem 2.5. From (2.1) we obtain for $n > m \geqq 2$

$$\sum_{r=m}^{n} g_r c_r(t_2) - \sum_{r=m}^{n} g_r c_r(t_1)$$
$$+ \int_{t_1}^{t_2} \sum_{r=m}^{n} (g_{r+1} - g_r) b_{r+1} c_{r+1} \, dt = \int_{t_1}^{t_2} \sum_{r=m}^{n} (g_{r+1} - g_r) a_r c_1 c_r \, dt$$
$$- \int_{t_1}^{t_2} g_{n+1} J_n(c(t)) \, dt + \int_{t_1}^{t_2} g_m J_{m-1}(c(t)) \, dt. \tag{2.28}$$

By properties (ii) and (iii) of a solution $\lim_{n \to \infty} \int_{t_1}^{t_2} J_n(c(t)) \, dt = 0$. Thus, setting $g_r = 1$ for all r in (2.28) and letting $n \to \infty$, we obtain

$$\sum_{r=m}^{\infty} c_r(t_2) - \sum_{r=m}^{\infty} c_r(t_1) = \int_{t_1}^{t_2} J_{m-1}(c(t)) \, dt. \tag{2.29}$$

Replacing m by $n+1$ in (2.29), and noting that for $i=1,2$ we have either that

$$\lim_{n\to\infty} |g_{n+1}| \sum_{r=n+1}^{\infty} c_r(t_i) \leqq \text{const} \lim_{n\to\infty} (n+1) \sum_{r=n+1}^{\infty} c_r(t_i)$$

$$\leqq \text{const} \lim_{n\to\infty} \sum_{r=n+1}^{\infty} r c_r(t_i) = 0\,,$$

or that

$$\lim_{n\to\infty} |g_{n+1}| \sum_{r=n+1}^{\infty} c_r(t_i) \leqq \text{const} \lim_{n\to\infty} \sum_{r=n+1}^{\infty} g_r c_r(t_i) = 0\,,$$

we deduce that

$$\lim_{n\to\infty} g_{n+1} \int_{t_1}^{t_2} J_n(c(t))\,dt = 0\,. \tag{2.30}$$

Since c_1 is bounded in $[t_1, t_2]$ and $\int_{t_1}^{t_2} \sum_{r=m}^{\infty} |g_{r+1} - g_r| a_r c_r\,dt < \infty$, we have

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \sum_{r=m}^{n} (g_{r+1} - g_r) a_r c_1 c_r\,dt = \int_{t_1}^{t_2} \sum_{r=m}^{\infty} (g_{r+1} - g_r) a_r c_1 c_r\,dt\,. \tag{2.31}$$

The theorem follows from (2.28), (2.30), (2.31), property (ii) of a solution and either the monotone or the dominated convergence theorem. □

Proof of Corollary 2.6. Equations (2.26), (2.27) are obtained by setting $g_r = r$, $g_r = 1$ in (2.24) respectively. Putting $m=2$ in (2.26) and adding the c_1 equation in (2.1) then gives (2.25). □

We now adapt a technique of Reuter and Ledermann (1953) to show that if $\lim_{r\to\infty} \frac{a_r}{r} = \infty$, then solutions to (1.1) do not in general exist for all initial data $c_0 \in X^+$.

Theorem 2.7. *Suppose* $\lim_{r\to\infty} \frac{a_r}{r} = \infty$ *and that* $b_r \leqq z a_r$ *for all sufficiently large* r, *where* $z \geqq 0$ *is a constant. Let* $c_0 = (c_{0r}) \in X$ *be such that* $c_{01} > z$ *and, for every* $\delta > 0$, $e^{\delta \gamma_m} \sum_{r=m+1}^{\infty} (r-m) c_{0r} \not\to 0$ *as* $m \to \infty$, *where* $\gamma_m \overset{\text{def}}{=} \min_{r \geqq m} \frac{a_r}{r}$. *Then there is no solution* c *of* (1.1), *defined on any interval* $[0, T)$, $T > 0$, *and with initial data* $c(0) = c_0$.

Proof. Let c be a solution on $[0, T)$ with $c(0) = c_0$. Then there exist $\varepsilon > 0$ and $\tau \in (0, T)$ such that $c_1(t) \geqq z + \varepsilon$ for all $t \in [0, \tau)$. For $t \in [0, \tau)$ and m sufficiently large we have by Corollary 2.6 and property (iii) of a solution,

$$\sum_{r=m+1}^{\infty} (r-m)\,c_r(t) = \sum_{r=m+1}^{\infty} (r-m) c_{0r} + \int_0^t \sum_{r=m}^{\infty} J_r(c(s))\,ds$$

$$= \sum_{r=m+1}^{\infty} (r-m) c_{0r} + \int_0^t \left[b_m c_m(s) + \sum_{r=m}^{\infty} (a_r c_1(s) - b_r)\,c_r(s) \right] ds$$

$$\geqq \sum_{r=m+1}^{\infty} (r-m) c_{0r} + \varepsilon \int_0^t \sum_{r=m}^{\infty} a_r c_r(s)\,ds\,.$$

Hence

$$\sum_{r=m}^{\infty} rc_r(t) \geqq \sum_{r=m+1}^{\infty} (r-m)c_{0r} + \varepsilon\gamma_m \int_0^t \sum_{r=m}^{\infty} rc_r(s)\,ds,$$

and therefore

$$\sum_{r=m}^{\infty} rc_r(t) \geqq e^{\varepsilon\gamma_m t} \sum_{r=m+1}^{\infty} (r-m)c_{0r}, \qquad t\in[0,\tau).$$

This contradicts $\lim\limits_{m\to\infty} \sum\limits_{r=m}^{\infty} rc_r(t)=0$. □

We remark that there are always $c_0 \in X^+$ satisfying the conditions of the theorem. In fact, since $\gamma_m \to \infty$ as $m\to\infty$, there is a sequence $m_j\to\infty$ such that $\sum\limits_{j=1}^{\infty} \frac{1}{\gamma_{m_j}} < \infty$. Let $c_{01} > z$, $c_{0,2m_j} = \frac{1}{\gamma_{m_j} m_j}$ for $j=1,2,\ldots$, $c_{0r}=0$ otherwise. Then $c_0 \in X^+$, but

$$e^{\delta\gamma_{m_j}} \sum_{r=m_j+1}^{\infty} (r-m_j)c_{0r} \geqq c^{\delta\gamma_{m_j}} m_j c_{0,2m_j} = \frac{e^{\delta\gamma_{m_j}}}{\gamma_{m_j}} \to \infty$$

as $j\to\infty$ for every $\delta>0$.

We end this section with some observations concerning the case when $\lim\limits_{r\to\infty} \frac{b_r}{r} = \infty$. In this case any solution c of (1.1) on $[0,T)$ with initial data $c_0 \in X^+$ satisfies, by property (iii), $\sum\limits_{r=1}^{\infty} b_r c_r(t) < \infty$ for a.e. $t\in(0,T)$, and hence has in general more rapid decay as $r\to\infty$ than c_0. Under additional hypotheses a slightly stronger result can be proved.

Proposition 2.8. *Assume that $b_{r+1} \geqq b_r$ for sufficiently large r, and that $(b_{r+1}-b_r)a_r = O(b_r)$. Let c be a solution of (1.1) on some interval $[0,T)$, $0<T\leqq\infty$. Then $\sum\limits_{r=1}^{\infty} b_r c_r(t)$ is an absolutely continuous function of t on compact intervals of $(0,T)$.*

Proof. Let $0<t_1<t_2<T$ with $\sum\limits_{r=1}^{\infty} b_r c_r(t_1) < \infty$, $\sum\limits_{r=2}^{\infty} b_r c_r(t_2) < \infty$. We apply Theorem 2.5 with $g_r = b_r$. Since $(b_{r+1}-b_r)a_r = O(b_r)$, and by properties (ii) and (iii) of a solution, we have that $\int_{t_1}^{t_2} \sum\limits_{r=1}^{\infty} (b_{r+1}-b_r)a_r c_r\,dt < \infty$. Hence, choosing m sufficiently large for $g_{r+1} \geqq g_r$ when $r\geqq m$, we have that

$$\sum_{r=m}^{\infty} b_r c_r(t_2) - \sum_{r=m}^{\infty} b_r c_r(t_1) \leqq K\left(1 + \int_{t_1}^{t_2} \sum_{r=m}^{\infty} b_r c_r(t)\,dt\right).$$

where the constant K does not depend on t_1 or t_2. By Gronwall's inequality we deduce that $\sum\limits_{r=m}^{\infty} g_r c_r(t)$ is finite for *all* $t\in[0,T)$. The absolute continuity now follows from (2.24). □

The hypotheses of Proposition 2.8 are satisfied, for example, if $b_r = r^\beta$, $a_r = r^\alpha$, where $\beta > 1 \geqq \alpha \geqq 0$. In fact in this case, by iterating the proof and using the fact that

$\int_{t_1}^{t_2} \sum_{r=1}^{\infty} (b_{r+1}-b_r)b_{r+1}c_{r+1}\,dt<\infty$, one can prove that $\sum_{r=1}^{\infty} r^{\gamma}c_r(t)$ is absolutely continuous on compact intervals of $(0, T)$ for any $\gamma>1$.

3. Differentiability, Continuous Dependence, and Uniqueness

We first consider the continuity of solutions in t.

Proposition 3.1. *Let c be a solution of (1.1) on some interval $[0, T)$, $0<T\leqq\infty$. Then $c:[0, T)\to X$ is continuous, and the series $\sum_{r=1}^{\infty} rc_r(t)$ is uniformly convergent on compact intervals of $[0, T)$.*

Proof. Let $f_n(t)=\sum_{r=1}^{n} rc_r(t)$. Then f_n is continuous, $f_{n+1}\geqq f_n$, and by Theorem 2.5 $\lim_{n\to\infty} f_n(t)=\|c(0)\|$ for all $t\in[0, T)$. The uniform convergence follows from Dini's theorem. The continuity of $c(\cdot)$ is then an obvious consequence of the continuity of c_r for each r. □

So as to study the differentiability of solutions with respect to t we introduce some notation. If $\lambda=(\lambda_r)$, $\mu=(\mu_r)$ are sequences we set $(\lambda\mu)_r=\lambda_r\mu_r$, $(\Delta\lambda)_r=\lambda_{r+1}-\lambda_r$, $(P\lambda)_r=\lambda_{r+1}$, $(P^{-1}\lambda)_r=\lambda_{r-1}$. Thus, for example,

$$\Delta(\lambda\mu)=(\Delta\lambda)P\mu+\lambda\Delta\mu, \quad (P\Delta)\lambda=(\Delta P)\lambda, (P^{-1}\Delta)\lambda=(\Delta P^{-1})\lambda. \tag{3.1}$$

Theorem 3.2. *Let k be a positive integer. Assume $(\Delta^j a)_r=O(r^{1-j})$, $(\Delta^j b)_r=O(r^{1-j})$ for $0\leqq j\leqq k-1$. Let c be a solution of (1.1) on some interval $[0, T)$, $0<T\leqq\infty$. Then c_r is C^k on $[0, T)$ for each $r=1, 2, \dots$.*

Proof. We first note that it suffices to prove that c_1 is C^k on $[0, T)$, since then an obvious induction using (2.1) shows that each c_r, $r\geqq 2$, is C^k. To carry out the successive differentiations of the c_1 equation in (2.1) we make use of Theorem 2.5. This leads to consideration of the operators

$$T_1 g=a\Delta g, T_2 g=b(P^{-1}\Delta g). \tag{3.2}$$

We note that if M^l is a product of l T_i's, $l\leqq k-1$, then by (3.1), (3.2) and our assumptions on a_r, b_r,

$$(M^l a)_r=O(r), (M^l b)_r=O(r). \tag{3.3}$$

Fix $m>k$. We prove by induction that for $s=1, \dots, k$ c_1 is C^s on $[0, T)$ and $\frac{d^s c_1}{dt^s}$ is expressible as a polynomial in finitely many of the c_r and the sums $\sum_{r=m}^{\infty} (M^l a)_r c_r$, $\sum_{r=m}^{\infty} (M^l b)_r c_r$, where M^l runs through all products of l T_i's, $0\leqq l\leqq s-1$. This is true for $s=1$ from the c_1 equation in (2.1), since the hypotheses $a_r, b_r=O(r)$ and Proposition 3.1 imply that $\sum_{r=m}^{\infty} a_r c_r$ and $\sum_{r=m}^{\infty} b_r c_r$ are continuous functions of t on

$[0, T)$. Suppose that the assertion is true for some $s \leqq k-1$; we prove it holds for $s+1$. We note that from the case $s=1$ and (2.1) each c_l is C^1 on $[0, T)$ with $\dot{c}_l$ expressible as a polynomial in finitely many c_r and the sums $\sum_{r=m}^{\infty} a_r c_r$, $\sum_{r=m}^{\infty} b_r c_r$. From the induction hypothesis it thus is enough to show that for any M^l, $0 \leqq l \leqq s-1$, the sums $\sum_{r=m}^{\infty} (M^l a)_r c_r$, $\sum_{r=m}^{\infty} (M^l b)_r c_r$ are C^1 on $[0, T)$ and have derivatives expressible as polynomials in finitely many c_r and the sums $\sum_{r=m}^{\infty} (M^l a)_r c_r$, $\sum_{r=m}^{\infty} (M^{l'} b)_r c_r$, $0 \leqq l' \leqq s$. But by (3.3) and Theorem 2.5 we have

$$\sum_{r=m}^{\infty} (M^l a)_r c_r(t) - \sum_{r=m}^{\infty} (M^l a)_r c_r(0) + \int_0^t \sum_{r=m}^{\infty} (T_2 M^l a)_r c_r \, ds$$
$$= \int_0^t c_1 \sum_{r=m}^{\infty} (T_1 M^l a)_r c_r \, ds + \int_0^t [(M^l a)_m J_{m-1}(c(s)) + (T_2 M^l a)_m c_m] \, ds, \; t \in [0, T),$$

where the integrands are continuous on $[0, T)$ by Proposition 3.1. Thus $\sum_{r=m}^{\infty} (M^l a)_r c_r$ $\left(\text{and similarly } \sum_{r=m}^{\infty} (M^l b)_r c_r\right)$ has the desired property. □

Note that if $a_r = O(r)$ but $\lim_{r \to \infty} \frac{b_r}{r} = \infty$, then $\dot{c}_1$ cannot be continuous up to zero if the initial data c_0 satisfies $\sum_{r=2}^{\infty} b_r c_{0r} = \infty$. However, in this case differentiability results for $t > 0$ can be proved by combining the methods of the theorem with those of Proposition 2.8 and the subsequent remark.

Before discussing the continuous dependence of solutions on the initial data we introduce some terminology.

Definition. We say that a sequence $\{y^{(i)}\}$ of elements of X converges weak * to $y \in X$ (symbolically $y^{(j)} \xrightarrow{*} y$) if

(i) $\sup_j \|y^{(j)}\| < \infty$, and

(ii) $y_r^{(j)} \to y_r$ as $j \to \infty$ for each $r = 1, 2, \ldots$.

To justify the terminology, we note that (cf. Dunford and Schwartz, 1958, p. 374) X can be identified with the dual of the space Y of sequences $y = (y_r)$ satisfying $\lim_{r \to \infty} r^{-1} y_r = 0$ with norm $\|y\|_Y = \max_r r^{-1} |y_r|$, and that weak * convergence as defined above is exactly weak * convergence in $X = Y^*$. (We thank M.G. Crandall for pointing this out to us.)

For $\varrho > 0$ let $B_\varrho = \{y \in X : \|y\| \leqq \varrho\}$. We make B_ϱ into a metric space by giving it the metric

$$d(y, z) = \sum_{r=1}^{\infty} |y_r - z_r| .$$

Clearly a sequence $\{y^{(j)}\} \subset B_\varrho$ converges weak * to $y \in X$ if and only if $y \in B_\varrho$ and $d(y^{(j)}, y) \to 0$ as $j \to \infty$. Also B_ϱ is compact; equivalently, any bounded sequence in X has a weak * convergent subsequence.

Let $E \subset X$. A function $\theta: E \to \mathbb{R}$ is *sequentially weak * continuous* if $\theta(y^{(j)}) \xrightarrow{*} \theta(y)$ whenever $y^{(j)}, y \in E$ with $y^{(j)} \xrightarrow{*} y$ as $j \to \infty$. For example, the function $\sum_{r=1}^{\infty} g_r y_r$ is well defined for all $y \in X$ if $|g_r| = O(r)$, but is sequentially weak *continuous if and only if $|g_r| = o(r)$.

We will make frequent use of the following elementary lemma, whose proof we include for the convenience of the reader.

Lemma 3.3. *If* $y^{(j)} \xrightarrow{*} y$ *in* X *and* $\|y^{(j)}\| \to \|y\|$, *then* $y^{(j)} \to y$ *in* X.

Proof. Define $z_r^{(j)} = |y_r^{(j)}| + |y_r| - |y_r^{(j)} - y_r| \geqq 0$. Then $z_r^{(j)} \to 2|y_r|$ as $j \to \infty$ for each r. Since for any m

$$\sum_{r=1}^{\infty} r z_r^{(j)} \geqq \sum_{r=1}^{m} r z_r^{(j)},$$

it follows that

$$\liminf_{j \to \infty} \sum_{r=1}^{\infty} r z_r^{(j)} \geqq 2 \sum_{r=1}^{\infty} r |y_r|.$$

Hence

$$\limsup_{j \to \infty} \|y^{(j)} - y\| = 2\|y\| - \liminf_{j \to \infty} \sum_{r=1}^{\infty} r z_r^{(j)} \leqq 0,$$

which proves the assertion. □

Definition. A *generalized flow* G on a metric space Y is a family of continuous mappings $\phi: [0, \infty) \to Y$ with the properties

(i) if $\phi \in G$ and $\tau \geqq 0$, then $\phi_\tau \in G$, where $\phi_\tau(t) \stackrel{\text{def}}{=} \phi(t+\tau)$, $t \in [0, \infty)$,

(ii) if $y \in Y$, there exists at least one $\phi \in G$ with $\phi(0) = y$, and

(iii) if $\phi_j \in G$ with $\phi_j(0)$ convergent in Y as $j \to \infty$, then there exist a subsequence ϕ_{j_k} of ϕ_j and an element $\phi \in G$ such that $\phi_{j_k}(t) \to \phi(t)$ in Y uniformly for t in compact intervals of $[0, \infty)$. A generalized flow G with the property that for each $y \in Y$ there is a *unique* $\phi \in G$ with $\phi(0) = y$ is called a *semigroup*; we then write $T(t)y = \phi(t)$, so that the mappings $T(t): Y \to Y$, $t \geqq 0$, satisfy

(i) $T(0) = \text{identity}$,

(ii) $T(s+t) = T(s)\,T(t)$ for all $s, t \geqq 0$,

(iii) the mapping $(t, y) \to T(t)y$ is continuous from $[0, \infty) \times Y \to Y$.

[For the purpose of this paper we have used a somewhat stronger definition of a generalized flow than in Ball (1978, p. 232).]

Theorem 3.4. *Assume* $a_r = O(r)$. *Let* G *denote the set of all solutions* c *of* (1.1) *on* $[0, \infty)$. *Then* G *is a generalized flow on the closed metric subspace* X^+ *of* X.

Proof. That each solution $c: [0, \infty) \to X^+$ is continuous was proved in Proposition 3.1. Property (i) in the definition of a generalized flow is obvious from (2.1), while property (ii) follows from Corollary 2.3. It thus remains to prove the upper semicontinuity property (iii). Let $c^{(j)}$ be a sequence of solutions of (1.1) on $[0, \infty)$ satisfying $c^{(j)}(0) \to c_0$ in X as $j \to \infty$. Repeating the proof of Theorem 2.2 with $g_r = r$, with $c^{(j)}$ playing the rôle of the approximating solutions, and using the relations

(2.25)–(2.27), we obtain a subsequence $c^{(j_k)}$ and a solution c such that $c_r^{(j_k)}(t) \to c_r(t)$ uniformly on $[0, T]$ for every $T > 0$ and $r = 1, 2, \ldots$. Also, by Corollary 2.6,

$$\sum_{r=1}^{\infty} r c_r^{(j_k)}(t) = \sum_{r=1}^{\infty} r c_r^{(j_k)}(0) \to \sum_{r=1}^{\infty} r c_r(0) = \sum_{r=1}^{\infty} r c_r(t)$$

as $k \to \infty$, for every $t \geqq 0$. Property (iii) follows by Lemma 3.3. □

For $\varrho > 0$ set $B_\varrho^+ = B_\varrho \cap X^+$. Clearly B_ϱ^+ is a closed metric subspace of B_ϱ (with metric d).

Theorem 3.5. *Assume* $a_r = o(r)$, $b_r = o(r)$. *For* $\varrho > 0$ *let* G_ϱ *denote the set of all solutions* c *of* (1.1) *on* $[0, \infty)$ *with* $c(0) \in B_\varrho^+$. *Then* G_ϱ *is a generalized flow on* B_ϱ^+.

Proof. We must check property (iii) in the definition of a generalized flow. Let $c^{(j)}$ be a sequence of solutions of (1.1) on $[0, \infty)$ with $c^{(j)}(0) \xrightarrow{*} c_0$ as $j \to \infty$. It follows from (2.1), (2.25) and Theorem 3.2 that $\dot{c}_r^{(j)}(t)$ exists for each $r = 1, 2, \ldots$ and is absolutely bounded independently of j and $t \geqq 0$. Hence by the Arzela-Ascoli theorem there exist a diagonal subsequence $c^{(j_k)}$ of $c^{(j)}$ and a function $c: [0, \infty) \to X^+$ such that $c_r^{(j_k)}(t) \to c_r(t)$ uniformly for t in compact subsets of $[0, \infty)$ for each r. This implies also that $d(c^{(j_k)}(t), c(t)) \to 0$ uniformly on compact subsets of $[0, \infty)$. Clearly c satisfies the first equation in (2.1) for $r \geqq 2$ and $t \geqq 0$. To pass to the limit in the c_1 equation we use the sequential weak * continuity of the functions $\sum_{r=1}^{\infty} a_r y_r$, $\sum_{r=2}^{\infty} b_r y_r$ and the bounded convergence theorem. Thus c is a solution. □

If *either* $a_r = 0$, $b_r = r$ for all r *or* $a_r = r$, $b_r = 0$ for all r the conclusion of Theorem 3.5 is false. For example, in the latter case any solution satisfies

$$\dot{c}_1 + \varrho c_1 + c_1^2 = 0,$$

where $\varrho = \sum_{r=1}^{\infty} r c_r$, and hence

$$c_1(t) = \frac{\varrho c_1(0)}{(\varrho + c_1(0)) e^{\varrho t} - c_1(0)}.$$

Therefore if $c^{(j)}(0) \xrightarrow{*} c_0$ with $\lim_{j \to \infty} \sum_{r=1}^{\infty} r c_r^{(j)}(0) = \bar{\varrho} > \varrho = \sum_{r=1}^{\infty} r c_{0r}$ and $c_{01} \neq 0$, then

$$\lim_{j \to \infty} c_1^{(j)}(t) = \frac{\bar{\varrho} c_{01}}{(\bar{\varrho} + c_{01}) e^{\bar{\varrho} t} - c_{01}} \neq \frac{\varrho c_{01}}{(\varrho + c_{01}) e^{\varrho t} - c_{01}}.$$

We have not found a general uniqueness theorem for solutions of (1.1). Instead we give two different uniqueness results; the first assumes more about the initial data, the second more about the coefficients a_r, b_r.

Theorem 3.6. *Assume that* $a_r = O(r)$. *Let* $c_0 = (c_{0r}) \geqq 0$ *satisfy* $\sum_{r=1}^{\infty} g_r c_{0r} < \infty$ *for some positive sequence* g_r *satisfying the conditions* $g_{r+1} - g_r \geqq \delta > 0$ *for some* $\delta > 0$, $a_r(g_{r+1} - g_r) = O(g_r)$ *and* $r a_r = O(g_r)$. *Let* $T > 0$. *Then there is exactly one solution* c *of* (1.1) *on* $[0, T)$ *satisfying* $c(0) = c_0$ *(that proved to exist in Theorem 2.2).*

Proof. For $\lambda \in \mathbb{R}$ define $\operatorname{sgn}\lambda$ to equal 1, 0 or -1 according as $\lambda>0$, $\lambda=0$ or $\lambda<0$. Note that if $\phi(\cdot)$ is an absolutely continuous function of t then so is $t \mapsto |\phi(t)|$, and $\frac{d}{dt}|\phi(t)| = (\operatorname{sgn}\phi(t))\frac{d\phi}{dt}(t)$ a.e.

Let c be the solution proved to exist in Theorem 2.2, let d be another solution with $d(0)=c_0$, and let $x(t)=c(t)-d(t)$. Let $N \geqq 2$. Then for a.e. $t \in [0, T)$ we have

$$\frac{d}{dt}\sum_{r=2}^{N} r|x_r| = \sum_{r=2}^{N} [J_r(c)-J_r(d)]\,[(r+1)\operatorname{sgn}x_{r+1} - r\operatorname{sgn}x_r] + 2\operatorname{sgn}x_2[J_1(c)-J_1(d)] - (N+1)\operatorname{sgn}x_{N+1}[J_N(c)-J_N(d)]. \tag{3.4}$$

Now

$$J_r(c)-J_r(d) = a_r(d_1x_r + x_1c_r) - b_{r+1}x_{r+1},$$

and so

$$\begin{aligned}[J_r(c)-J_r(d)]\,[(r+1)\operatorname{sgn}x_{r+1} - r\operatorname{sgn}x_r] &= d_1a_r|x_r|\,[(r+1)\operatorname{sgn}(x_rx_{r+1}) - r] \\ &\quad + x_1a_rc_r[(r+1)\operatorname{sgn}x_{r+1} - r\operatorname{sgn}x_r] - b_{r+1}|x_{r+1}|\,[r+1-r\operatorname{sgn}(x_rx_{r+1})] \\ &\leqq d_1a_r|x_r| + (2r+1)a_rc_r|x_1|.\end{aligned}$$

Integrating (3.4) we therefore obtain for $t \in [0, T)$

$$\sum_{r=2}^{N} r|x_r(t)| \leqq \int_0^t \left[d_1 \sum_{r=2}^{N} a_r|x_r| + |x_1| \sum_{r=2}^{N} (2r+1)a_rc_r + 2a_1|x_1|(c_1+d_1)\right] ds - (N+1)\int_0^t \operatorname{sgn}x_{N+1}[J_N(c)-J_N(d)]\,ds. \tag{3.5}$$

We now note that since $ra_r = O(g_r)$, we have by (2.4) that

$$\sup_{t\in[0,T)} \sum_{r=2}^{\infty} (2r+1)a_rc_r(t) < \infty. \tag{3.6}$$

Also, computing $\frac{d}{dt}\sum_{r=2}^{N}|x_r|$ as above and arguing as in Theorem 2.5, we deduce that

$$\lim_{N\to\infty} (N+1)\int_0^t \operatorname{sgn}x_{N+1}[J_N(c)-J_N(d)]\,ds = 0, \quad t \in [0, T). \tag{3.7}$$

By Corollary 2.6, $\sum_{r=1}^{\infty} rc_r(t) = \sum_{r=1}^{\infty} rd_r(t) = \sum_{r=1}^{\infty} rc_{0r}$, and thus

$$|x_1(t)| \leqq \sum_{r=2}^{\infty} r|x_r(t)|. \tag{3.8}$$

Using (3.5)–(3.8) and $a_r = O(r)$, we therefore obtain

$$\sum_{r=2}^{\infty} r|x_r(t)| \leqq \text{const}\int_0^t \sum_{r=2}^{\infty} r|x_r(s)|\,ds, \quad t \in [0, T).$$

By Gronwall's inequality $\sum_{r=2}^{\infty} r|x_r(t)| = 0$, $t \in [0, T)$, and thus [by (3.8)] $c=d$, as claimed. □

Theorem 3.7. *Let k be a non-negative integer and assume that*

$$(\Delta^l a)_r = O(r^{(k+1)/(k+2)-l}), \ (\Delta^l b)_r = O(r^{(k+1)/(k+2)-l}) \tag{3.9}$$

for $0 \leqq l \leqq k$. Let $c_0 \in X^+$ and let $T>0$. Then there is exactly one solution c of (1.1) *on $[0, T)$ satisfying $c(0)=c_0$ (that proved to exist in Corollary 2.3).*

Remark. The special cases $k=0, 1$ give uniqueness when (i) $a_r, b_r = O(r^{1/2})$ or (ii) $a_r, b_r = O(r^{2/3})$, $a_{r+1}-a_r = O(r^{-1/3})$, $b_{r+1}-b_r = O(r^{-1/3})$ respectively.

Proof of Theorem 3.7. Let $\alpha = \dfrac{k+1}{k+2}$, let c, d be two solutions of (1.1) on $[0, T)$ satisfying $c(0)=d(0)=c_0$, and set $x=c-d$. Fix $m>k+1$. Consider the functional

$$\theta(t) = \sum_{r=1}^{\infty} r^{1-\alpha}|x_r| + \Sigma_{k-1}\left[\left|\sum_{r=m}^{\infty} (M^l a)_r x_r\right| + \left|\sum_{r=m}^{\infty} (M^l b)_r x_r\right|\right], \tag{3.10}$$

where Σ_{k-1} indicates summation over all products M^l of l T_i's, $0 \leqq l \leqq k-1$, where T_1, T_2 are defined by (3.2). If $k=0$ we set $\Sigma_{k-1}[\ldots]=0$. Note that by (3.1), (3.9)

$$(M^l a)_r = O(r^{\alpha+l(\alpha-1)}), \quad (M^l b)_r = O(r^{\alpha+l(\alpha-1)}), \quad 0 \leqq l \leqq k, \tag{3.11}$$

so that by Proposition 3.1 θ is well defined (and continuous) on $[0, T)$. We will show that

$$\theta(t) \leqq \text{const} \int_0^t \theta(s)\, ds, \qquad t \in [0, T), \tag{3.12}$$

from which it follows by Gronwall's lemma that $\theta(t)=0$ and $c=d$. To this end we first note that for $N \geqq 2$ and a.e. $t \in [0, T)$,

$$\begin{aligned}
&\frac{d}{dt}\sum_{r=2}^{N} r^{1-\alpha}|x_r| \\
&\quad = \sum_{r=2}^{N} (a_r(d_1 x_r + x_1 c_r) - b_{r+1}x_{r+1})((r+1)^{1-\alpha}\operatorname{sgn} x_{r+1} - r^{1-\alpha}\operatorname{sgn} x_r) \\
&\qquad + 2^{1-\alpha}\operatorname{sgn} x_2(a_1 x_1(c_1+d_1) - b_2 x_2) - (N+1)^{1-\alpha}\operatorname{sgn} x_{N+1}[J_N(c) - J_N(d)].
\end{aligned} \tag{3.13}$$

But

$$\begin{aligned}
d_1 \sum_{r=2}^{N} a_r x_r((r+1)^{1-\alpha}\operatorname{sgn} x_{r+1} - r^{1-\alpha}\operatorname{sgn} x_r) &\leqq d_1 \sum_{r=2}^{N} a_r|x_r|((r+1)^{1-\alpha} - r^{1-\alpha}) \\
&\leqq \text{const} \sum_{r=2}^{N} |x_r|,
\end{aligned} \tag{3.14}$$

and

$$-\sum_{r=2}^{N} b_{r+1}x_{r+1}((r+1)^{1-\alpha}\operatorname{sgn} x_{r+1} - r^{1-\alpha}\operatorname{sgn} x_r) \leqq 0. \tag{3.15}$$

Also

$$x_1 \sum_{r=2}^{N} a_r c_r((r+1)^{1-\alpha}\operatorname{sgn} x_{r+1} - r^{1-\alpha}\operatorname{sgn} x_r) \leqq \text{const}|x_1| \sum_{r=2}^{N} r c_r \leqq \text{const}|x_1|. \tag{3.16}$$

Further, it follows easily from Proposition 3.1 that

$$\lim_{N\to\infty} \int_0^t (N+1)^{1-\alpha}[|J_N(c)|+|J_N(d)|]\,ds=0, \qquad t\in[0,T). \tag{3.17}$$

Combining (3.13)–(3.17) we deduce that

$$\sum_{r=2}^{\infty} r^{1-\alpha}|x_r(t)| \leqq \text{const} \int_0^t \sum_{r=1}^{\infty} |x_r(s)|\,ds, \qquad t\in[0,T). \tag{3.18}$$

From (2.1) we have that

$$\begin{aligned} |x_1(t)| &\leqq \int_0^t \left[2a_1(c_1+d_1)|x_1|+2b_2|x_2|+d_1\left|\sum_{r=2}^{\infty} a_r x_r\right| - \left|\sum_{r=3}^{\infty} b_r x_r\right| + |x_1| \sum_{r=2}^{\infty} a_r c_r \right] ds \\ &\leqq \text{const} \int_0^t \left[\sum_{r=1}^{m-1} |x_r(s)| + \left|\sum_{r=m}^{\infty} a_r x_r(s)\right| + \left|\sum_{r=m}^{\infty} b_r x_r(s)\right| \right] ds, \ t\in[0,T). \end{aligned} \tag{3.19}$$

If $k=0$, then

$$\left|\sum_{r=m}^{\infty} a_r x_r\right| + \left|\sum_{r=m}^{\infty} b_r x_r\right| \leqq \text{const} \sum_{r=m}^{\infty} r^{\alpha}|x_r| = \text{const} \sum_{r=m}^{\infty} r^{1-\alpha}|x_r|,$$

and (3.12) follows from (3.18), (3.19). [It is because a similar estimation cannot be directly carried out for $k\geqq 1$ that the extra terms in (3.10) are necessary.] So suppose $k\geqq 1$ and that $0\leqq l\leqq k-1$. For any product M^l of l T_i's we have by Theorem 2.5 that for $t\in[0,T)$,

$$\begin{aligned} \sum_{r=m}^{\infty} (M^l a)_r x_r(t) = \int_0^t \Bigg[&\sum_{r=m}^{\infty} (T_1 M^l a)_r (d_1 x_r + x_1 c_r) - \sum_{r=m}^{\infty} (T_2 M^l a)_r x_r \\ &+ (M^l a)_m (J_{m-1}(c) - J_{m-1}(d)) + (T_2 M^l a)_m x_m \Bigg] ds. \end{aligned} \tag{3.20}$$

It follows from (3.11), (3.20) that

$$\begin{aligned} \left|\sum_{r=m}^{\infty} (M^l a)_r x_r(t)\right| \leqq \text{const} \int_0^t \Bigg[&\sum_{r=1}^{m} |x_r(s)| + \left|\sum_{r=m}^{\infty} (T_1 M^l a)_r x_r(s)\right| \\ &+ \left|\sum_{r=m}^{\infty} (T_2 M^l a)_r x_r(s)\right| \Bigg] ds, \qquad t\in[0,T). \end{aligned} \tag{3.21}$$

If $l<k-1$ it follows immediately from (3.21) that

$$\left|\sum_{r=m}^{\infty} (M^l a)_r x_r(t)\right| \leqq \text{const} \int_0^t \theta(s)\,ds, \qquad t\in[0,T). \tag{3.22}$$

If $l=k-1$ we note that by (3.11)

$$\left|\sum_{r=m}^{\infty} (T_i M^{k-1} a)_r x_r\right| \leqq \text{const} \sum_{r=m}^{\infty} r^{\alpha+k(\alpha-1)}|x_r| \leqq \text{const} \sum_{r=1}^{\infty} r^{1-\alpha}|x_r|,$$

for $i=1,2$, so that (3.22) again holds. Clearly a similar inequality holds for $\left|\sum_{r=m}^{\infty} (M^l b)_r x_r\right|$. Combining (3.18), (3.19), (3.22) we therefore obtain (3.12), which completes the proof. □

Corollary 3.8. *Assume a_r, b_r satisfy* (3.9) *for some $k \geqq 0$. For $c_0 \in X^+$, define $T(t)c_0 = c(t)$, where $c(t)$ is the unique solution of* (1.1) *on* $[0, \infty)$ *satisfying $c(0) = c_0$. Then $\{T(t)\}_{t \geqq 0}$ is a semigroup on X^+ and on B_ϱ^+.*

We end this section by giving conditions under which the approximation scheme in Theorem 2.2 converges to a solution without extraction of a subsequence. For $c_0 = (c_{0r}) \in X^+$, we denote by $c^n : [0, \infty) \to X$ the solution of the truncated system (2.2) with initial data $(c_{01}, \ldots, c_{0n})$, the components c_r^n for $r > n$ being set to zero.

Theorem 3.9. *Let $c_0 \in X^+$, and suppose that either the hypotheses of Theorem 3.6 or those of Theorem 3.7 hold. Then as $n \to \infty$ $c^n(t) \to c(t)$ in X uniformly on compact intervals of* $[0, \infty)$, *where c denotes the unique solution of* (1.1) *on* $[0, \infty)$ *with* $c(0) = c_0$.

Proof. This follows by a contradiction argument, applying the proof of Theorem 2.2 to a subsequence of c^n assumed to lie outside some neighbourhood of c in $C([0, T]; X)$. By the proof and uniqueness of solutions, any such subsequence possesses a further subsequence, c^{n_j} say, such that $c_r^{n_j} \to c_r$ uniformly on compact intervals of $[0, \infty)$ for each r. Also, by Lemma 2.1 and Corollary 2.6 we have that

$$\begin{aligned} \sum_{r=1}^{\infty} r c_r^{n_j}(t) &= \sum_{r=1}^{n_j} r c_{0r}, \quad t \geqq 0, \\ \sum_{r=1}^{\infty} r c_r(t) &= \sum_{r=1}^{\infty} r c_{0r}, \quad t \geqq 0. \end{aligned} \tag{3.23}$$

It follows from Lemma 3.3 that $c^{n_j} \to c$ in $C([0, T]; X)$, and this contradiction proves the theorem. □

4. Equilibria and Lyapunov Functions

By an *equilibrium state* we mean a time-independent solution of (1.1) on $[0, \infty)$. From the definition of a solution it follows that c is an equilibrium state if and only if $c \in X^+$ with $J_r(c) = 0$ for all r. We consider three cases. The first is *pure fragmentation*, in which we assume that $a_r = 0$, $b_r > 0$ for all r; in this case the equilibria satisfy $b_r c_r = 0$ for $r \geqq 2$ and are thus given by $c = c^\varrho$, $0 \leqq \varrho < \infty$, where

$$c_1^\varrho = \varrho, \; c_r^\varrho = 0 \quad \text{for} \quad r \geqq 2. \tag{4.1}$$

The second is *pure coagulation*, in which we assume that $a_r > 0$, $b_r = 0$ for all r; in this case we must solve the equations $a_r c_1 c_r = 0$, $r \geqq 1$, and thus the equilibrium states are given by those $c \in X^+$ with $c_1 = 0$. Note that there are infinitely many equilibria with fixed density $\varrho = \sum_{r=1}^{\infty} r c_r$ for each $\varrho > 0$. The fact that there are non-zero equilibria in the pure coagulation case highlights the limitations of the Becker-Döring assumption (1.4). The third and most interesting case, which we study in the remainder of this section, is that of *coagulation-fragmentation*, in which we assume that

$$a_r > 0, \; b_r > 0 \quad \text{for all } r. \tag{4.2}$$

In this case the equilibria satisfy

$$\frac{c_{r+1}}{c_r} = \frac{a_r}{b_{r+1}} c_1, \qquad r \geqq 1, \tag{4.3}$$

and therefore have the form

$$c_r = Q_r c_1^r, \qquad r \geqq 1, \tag{4.4}$$

where the Q_r are defined by

$$Q_1 = 1; \frac{Q_{r+1}}{Q_r} = \frac{a_r}{b_{r+1}}, \qquad r \geqq 1. \tag{4.5}$$

In order for $c = (c_r)$ given by (4.4) to be an equilibrium state, c_1 must be chosen so that $c \in X^+$. For $z \geqq 0$ define

$$F(z) = \sum_{r=1}^{\infty} r Q_r z^r. \tag{4.6}$$

The radius of convergence z_s of this series is given by

$$z_s^{-1} = \limsup_{r \to \infty} Q_r^{1/r}. \tag{4.7}$$

We shall always assume that

$$\limsup_{r \to \infty} Q_r^{1/r} < \infty, \tag{4.8}$$

so that $0 < z_s \leqq \infty$. Note that F is smooth and strictly increasing for $0 \leqq z < z_s$. Define

$$\varrho_s = \sup_{0 \leqq z < z_s} F(z). \tag{4.9}$$

If $z_s = \infty$, then $\varrho_s = \infty$. If $0 < z_s < \infty$, then $0 < \varrho_s \leqq \infty$, and in the case when $0 < \varrho_s < \infty$, we have $\varrho_s = F(z_s)$. We thus obtain the following characterization of equilibria.

Proposition 4.1. *Let* (4.2) *hold.*

(i) *Let* $\varrho < \infty$, $0 \leqq \varrho \leqq \varrho_s$. *Then there is exactly one equilibrium state* c^ϱ *with density* ϱ, *and it is given by*

$$c_r^\varrho = Q_r z(\varrho)^r, \qquad r = 1, 2, \ldots, \tag{4.10}$$

where $z(\varrho)$ *is the unique root of* $F(z) = \varrho$.

(ii) *If* $\varrho_s < \varrho < \infty$, *there is no equilibrium state with density* ϱ.

For $c \in X^+$ let

$$V(c) = \sum_{r=1}^{\infty} c_r \left(\ln\left(\frac{c_r}{Q_r} \right) - 1 \right), \tag{4.11}$$

where the summand is defined to be zero when $c_r = 0$.

Lemma 4.2. *The function*

$$G(c) = \sum_{r=1}^{\infty} c_r (\ln c_r - 1) \tag{4.12}$$

*is finite and sequentially weak * continuous on* X^+.

Proof. Let $0<\varepsilon<\frac{1}{2}$. If $c=(c_r)\in X^+$ and $1\leqq m\leqq n$, then by Hölder's inequality

$$\sum_{r=m}^{n} c_r^{1-\varepsilon}\leqq\left(\sum_{r=m}^{n} rc_r\right)^{1-\varepsilon}\left(\sum_{r=m}^{n} r^{1-1/\varepsilon}\right)^{\varepsilon}. \tag{4.13}$$

In particular, setting $m=1$ and using the inequality

$$x|\ln x|\leqq \mathrm{const}(x^{1+\varepsilon}+x^{1-\varepsilon}), \qquad x>0, \tag{4.14}$$

it is easily seen that the series defining G is absolutely convergent. To prove the sequential weak * continuity, let $c^{(j)}\in X^+$ for $j=1,2,\dots$ with $c^{(j)}\overset{*}{\rightharpoonup} c$ as $j\to\infty$. Then

$$G(c^{(j)})=\left(\sum_{r=1}^{m-1}+\sum_{r=m}^{\infty}\right)c_r^{(j)}(\ln c_r^{(j)}-1),$$

and by (4.13) the second sum is bounded in absolute value by

$$\mathrm{const}\left(\frac{\|c^{(j)}\|^{1+\varepsilon}}{m^{1+\varepsilon}}+\frac{\|c^{(j)}\|}{m}+\|c^{(j)}\|^{1-\varepsilon}\left(\sum_{r=m}^{\infty} r^{1-1/\varepsilon}\right)^{\varepsilon}\right),$$

and therefore tends to zero as $m\to\infty$ uniformly in j. Since $c_r^{(j)}\to c_r$ for each r we obtain $\lim_{j\to\infty} G(c^{(j)})=G(c)$ as required. □

Note that it follows either directly from the proof of the lemma or from the sequential weak * continuity that G is bounded above and below on B_ϱ^+ for each $\varrho\geqq 0$.

We note that by (4.11), (4.12),

$$V(c)=G(c)-\sum_{r=1}^{\infty} rc_r\ln(Q_r^{1/r}). \tag{4.15}$$

It thus follows from (4.8) that V is bounded below on B_ϱ^+ for every $\varrho\geqq 0$. In general V may take the value $+\infty$, but if

$$0<\liminf_{r\to\infty} Q_r^{1/r}, \tag{4.16}$$

then V is bounded above on B_ϱ^+ for every $\varrho\geqq 0$.

For $0<z<\infty$, we define

$$V_z(c)=V(c)-\ln z\sum_{r=1}^{\infty} rc_r=\sum_{r=1}^{\infty} c_r\left(\ln\left(\frac{c_r}{Q_r z^r}\right)-1\right). \tag{4.17}$$

Proposition 4.3. *Let $\varrho<\infty$, $0\leqq\varrho\leqq\varrho_s$. Then*

$$V_z(c^\varrho)=\int_0^\varrho \ln\left(\frac{z(\sigma)}{z}\right)d\sigma. \tag{4.18}$$

Proof. Since

$$V_z(c^\varrho)=\varrho\ln\left(\frac{z(\varrho)}{z}\right)-\sum_{r=1}^{\infty} Q_r z(\varrho)^r,$$

we have that

$$\frac{d}{d\varrho} V_z(c^\varrho) = \ln\left(\frac{z(\varrho)}{z}\right) + \varrho \frac{z'(\varrho)}{z(\varrho)} - \varrho \frac{z'(\varrho)}{z(\varrho)} = \ln\left(\frac{z(\varrho)}{z}\right), \qquad 0 < \varrho < \varrho_s.$$

The result follows since $z(\varrho) \sim \varrho$ as $\varrho \to 0+$. $\square$

For $0 \leqq \varrho < \infty$, define $X_\varrho^+ = \left\{ c \in X^+ : \sum_{r=1}^{\infty} r c_r = \varrho \right\}$.

Theorem 4.4. *Assume* (4.2) *and* (4.8) *hold.*

(i) *Let $\varrho < \infty$, $0 \leqq \varrho \leqq \varrho_s$. Then c^ϱ is the unique minimizer both of V on X_ϱ^+ and of $V_{z(\varrho)}$ on X^+. Furthermore, every minimizing sequence $c^{(j)}$ of V on X_ϱ^+ converges to c^ϱ strongly in X.*

(ii) *Let $\varrho_s < \varrho < \infty$. Then*

$$\inf_{c \in X_\varrho^+} V(c) = V(c^{\varrho_s}) + \ln z_s (\varrho - \varrho_s), \tag{4.19}$$

$$\inf_{c \in X_\varrho^+} V_{z_s}(c) = V_{z_s}(c^{\varrho_s}), \tag{4.20}$$

*and every minimizing sequence $c^{(j)}$ of V or of V_{z_s} on X_ϱ^+ converges to c^{ϱ_s} weak *, but not strongly, in X.*

Proof. (i) It is easily verified that the function $c_r \to c_r \left(\ln\left(\frac{c_r}{Q_r z(\varrho)^r} \right) - 1 \right)$ attains a unique minimum at $c_r = Q_r z(\varrho)^r = c_r^\varrho$. Therefore c^ϱ is the unique minimizer of $V_{z(\varrho)}$ on X^+, and thus also of V on X_ϱ^+. Let $c^{(j)}$ be a minimizing sequence of V on X_ϱ^+. By the preceding argument $c_r^{(j)} \to c_r^\varrho$ as $j \to \infty$ for each r. Therefore $c^{(j)} \xrightarrow{*} c^\varrho$, and so by Lemma 3.3, $c^{(j)} \to c^\varrho$ strongly.

(ii) By the argument in part (i) we have that $V_{z_s}(c) \geqq V_{z_s}(c^{\varrho_s})$ for all $c \in X^+$. Define $c^{(j)}$ by

$$c_r^{(j)} = c_r^{\varrho_s} + \delta_{jr} \left(\frac{\varrho - \varrho_s}{j} \right), \qquad r = 1, 2, \dots,$$

where $\delta_{jr} = 1$ if $j = r$, $= 0$ otherwise. Then $c^{(j)} \in X_\varrho^+$ and

$$V_{z_s}(c^{(j)}) = V_{z_s}(c^{\varrho_s}) + \varepsilon_j,$$

where

$$\varepsilon_j \stackrel{\text{def}}{=} \left(c_j^{\varrho_s} + \frac{\varrho - \varrho_s}{j} \right) \left(\ln \left(\frac{c_j^{\varrho_s} + \frac{\varrho - \varrho_s}{j}}{c_j^{\varrho_s}} \right) - 1 \right) + c_j^{\varrho_s}.$$

It is easily verified that $\lim_{j \to \infty} \varepsilon_j = 0$, and it follows that (4.20) [and hence (4.19)] holds.

Now let $c^{(j)}$ be an *arbitrary* minimizing sequence of V on X_ϱ^+. By (4.20), $V_{z_s}(c^{(j)}) \to V_{z_s}(c^{\varrho_s})$ as $j \to \infty$, so that $c^{(j)}$ is also a minimizing sequence for V_{z_s} on X_ϱ^+. As in part (i) this implies that $c_r^{(j)} \to c_r^{\varrho_s}$ as $j \to \infty$ for each r. Therefore $c^{(j)} \xrightarrow{*} c^{\varrho_s}$. The convergence cannot be strong since $\|c^{(j)}\| = \varrho > \varrho_s = \|c^{\varrho_s}\|$. $\square$

By (4.15), (4.17) and Lemma 4.2, $V_z(\cdot)$ is sequentially weak *continuous on X^+ if and only if the functional $c \to \sum_{r=1}^{\infty} c_r \ln(Q_r z^r)$ is, that is, if and only if $\ln(Q_r z^r) = o(r)$. Since $r^{-1}\ln(Q_r z^r) = \ln(Q_r^{1/r} z)$ we have proved

Proposition 4.5. *$V_z(\cdot)$ is sequentially weak * continuous on X^+ if and only if $\lim_{r\to\infty} Q_r^{1/r}$ exists and $z = z_s$.*

Remark. Dickman and Schieve (1984) prove that for various lattice gas and continuous space models $\lim_{r\to\infty} Q_r^{1/r}$ exists and $0 < z_s < \infty$.

As a preliminary to showing that V is a Lyapunov function we prove a strict positivity result for solutions of (1.1).

Theorem 4.6. *Suppose* (4.2) *holds. Let c be a solution of* (1.1) *on some interval $[0, T)$, $0 < T \leqq \infty$, with $c(0) \neq 0$. Then $c_r(t) > 0$ for all $t \in (0, T)$ and all $r = 1, 2, \ldots$.*

Proof. Suppose for contradiction that $c_r(\tau) = 0$ for some r and some $\tau \in (0, T)$. If $r > 1$, then since

$$\dot{c}_r = a_{r-1} c_{r-1} c_1 + b_{r+1} c_{r+1} - \theta_r c_r,$$

where

$$\theta_r(t) \stackrel{\text{def}}{=} a_r c_1(t) + b_r,$$

we have that

$$0 = c_r(\tau) \exp\left(\int_0^\tau \theta_r(s)\, ds\right)$$
$$= c_r(0) + \int_0^\tau \exp\left(\int_0^t \theta(s)\, ds\right)(a_{r-1} c_{r-1}(t)\, c_1(t) + b_{r+1} c_{r+1}(t))\, dt.$$

Hence $a_{r-1} c_{r-1}(t)\, c_1(t) = 0$ for all $t \in [0, \tau]$, and thus either $c_{r-1}(\tau) = 0$ or $c_1(\tau) = 0$. By induction we deduce that in all cases $c_1(\tau) = 0$. Let

$$\phi(t) \stackrel{\text{def}}{=} a_1 c_1(t) + \sum_{r=1}^{\infty} a_r c_r(t), \qquad h(t) \stackrel{\text{def}}{=} b_2 c_2(t) + \sum_{r=2}^{\infty} b_r c_r(t).$$

By the definition of a solution $\phi \in L^1(0, \tau)$, $h \in L^1(0, \tau)$ and

$$\dot{c}_1(t) = -c_1(t)\,\phi(t) + h(t) \quad \text{a.e.} \quad t \in (0, T).$$

It follows easily that

$$c_1(\tau) \exp\left(\int_0^\tau \phi(s)\, ds\right) = c_1(0) + \int_0^\tau \exp\left(\int_0^t \theta(s)\, ds\right) h(t)\, dt.$$

Hence $c_1(0) = 0$ and $h(t) = 0$ a.e. $t \in (0, \tau)$. Since each c_r is continuous, we obtain $c_r(0) = 0$ for all $r \geqq 2$, and thus $c(0) = 0$, a contradiction. □

Theorem 4.7. *Suppose that* (4.2), (4.8), (4.16) *hold and that $a_r = O(r/\ln r)$, $b_r = O(r/\ln r)$. Let c be a solution of* (1.1) *on some interval $[0, T)$, $0 < T \leqq \infty$, with $c_m(0) > 0$ for some m. Then*

$$V(c(t)) + \int_0^t D(c(s))\, ds = V(c(0)) \quad \textit{for all} \quad t \in [0, T), \tag{4.21}$$

where

$$D(c) \stackrel{\text{def}}{=} \sum_{r=1}^{\infty} (a_r c_1 c_r - b_{r+1} c_{r+1})(\ln(a_r c_1 c_r) - \ln(b_{r+1} c_{r+1})). \tag{4.22}$$

Proof. For $n=2,3,\ldots$, define

$$V^{(n)}(c) = \sum_{r=1}^{n} c_r \left(\ln\left(\frac{c_r}{Q_r}\right) - 1 \right),$$

$$D_n(c) = \sum_{r=1}^{n} (a_r c_1 c_r - b_{r+1} c_{r+1})(\ln(a_r c_1 c_r) - \ln(b_{r+1} c_{r+1})).$$

By Theorem 4.6 and an easy computation we find that for a.e. $t \in (0, T)$,

$$\begin{aligned} \dot{V}^{(n)}(c(t)) &= -D_{n-1}(c) - (\ln c_1) \sum_{r=n}^{\infty} J_r - J_n \ln\left(\frac{c_n}{Q_n}\right) \\ &= -D_n(c) - (\ln c_1) \sum_{r=n+1}^{\infty} J_r - J_n \ln\left(\frac{c_{n+1}}{Q_{n+1}}\right). \end{aligned} \tag{4.23}$$

For sufficiently large n we have that $\ln c_n < 0$ on $(0, T)$, and hence

$$-J_n \ln c_n \leqq -a_n c_1 c_n \ln c_n, \qquad -J_n \ln c_{n+1} \geqq b_{n+1} c_{n+1} \ln c_{n+1}. \tag{4.24}$$

Note that by Proposition 3.1 we have that $nc_n \to 0$ as $n \to \infty$ uniformly on $(0, T)$, so that by our hypotheses on a_r, b_r the right-hand sides of (4.24) also tend to zero uniformly. Furthermore, by Theorem 4.6 and the definition of a solution,

$$\lim_{n\to\infty} \int_{\tau}^{t} (\ln c_1) \sum_{r=n}^{\infty} J_r \, ds = 0, \qquad 0 < \tau < t < T, \tag{4.25}$$

and by (4.8), (4.16), and (2.30) (with $g_r = r$)

$$\lim_{n\to\infty} \int_{\tau}^{t} J_n \ln Q_n \, ds = \lim_{n\to\infty} \int_{\tau}^{t} J_n \ln Q_{n+1} \, ds = 0, \qquad 0 < \tau < t < T. \tag{4.26}$$

Combining (4.23)–(4.26) we deduce that as $n \to \infty$,

$$\begin{aligned} \int_{0}^{t} D_{n-1}(c(s)) \, ds + o(1) &\leqq V^{(n)}(c(\tau)) - V^{(n)}(c(t)) \\ &\leqq \int_{\tau}^{t} D_n(c(s)) \, ds + o(1), \qquad 0 < \tau < t < T. \end{aligned} \tag{4.27}$$

Since

$$(x-y)(\ln x - \ln y) > 0 \quad \text{for} \quad x, y > 0, x \neq y, \tag{4.28}$$

we deduce from (4.27) and the monotone convergence theorem that

$$V(c(t)) + \int_{\tau}^{t} D(c(s)) \, ds = V(c(\tau)).$$

Since, by Proposition 3.1, $c: [0, T) \to X$ is continuous, and since, by Lemma 4.2, (4.8), and (4.16), $V: X^+ \to \mathbb{R}$ is continuous, the result follows from letting $\tau \to 0+$. □

Theorems 4.4, 4.7 suggest that V and V_z are thermodynamic free energy functions, a view supported by some formal calculations we have carried out for the case of a binary alloy. We do not know if the energy equation (4.21) holds without the supplementary hypotheses on the a_r, b_r; however, the following result can be proved.

Theorem 4.8. *Let the hypotheses of Theorem 2.2,* (4.2) *and* (4.8) *hold, and suppose further that* $c_0 \neq 0$, $V(c_0) < \infty$. *Then there exists a solution* c *of* (1.1) *on* $[0, \infty)$ *with* $c(0) = c_0$ *satisfying* (2.4) *and the energy inequality*

$$V(c(t)) + \int_0^t D(c(s))\, ds \leqq V(c(0)) \quad \text{for all} \quad t \geqq 0. \tag{4.29}$$

Sketch of Proof. For n sufficiently large the approximating solutions c^n defined in the proof of Theorem 2.2 satisfy, by the same argument as in Theorem 4.6, $c_r^n(t) > 0$ for all $t > 0$, $1 \leqq r \leqq n$, and hence, using the notation in the proofs of Theorem 2.2, 4.7,

$$V(c^n(t)) + \int_0^t D_{n_k-1}(c^{n_k}(s))\, ds = V(c^{n_k}(0)), \qquad t \geqq 0. \tag{4.30}$$

Since $V(c(0)) < \infty$, we have $\lim_{k\to\infty} V(c^{n_k}(0)) = V(c(0))$. Since $D_{n_k-1}(c^{n_k}) \geqq D_m(c^{n_k})$ for $n_k > m$, we have

$$\liminf_{k\to\infty} \int_0^t D_{n_k-1}(c^{n_k}(s))\, ds \geqq \int_0^t D(c(s))\, ds.$$

Finally, by (4.8), Lemma 4.2, and the fact that (cf. the proof of Theorem 3.9) $c^{n_k}(t) \to c(t)$ strongly in X,

$$\liminf_{k\to\infty} V(c^{n_k}(t)) \geqq V(c(t)).$$

The inequality (4.29) follows by passing to the limit in (4.30). □

5. Asymptotic Behaviour of Solutions and Stability

In this section we study the behaviour as $t \to \infty$ of solutions $c(t)$ of (1.1). We consider the same three cases as in Sect. 4, beginning with that of pure fragmentation.

Theorem 5.1. *Suppose that* $a_r = 0$, $b_r > 0$ *for all* r. *Let* c *be a solution of* (1.1) *on* $[0, \infty)$ *and let* $\varrho_0 = \sum_{r=1}^{\infty} rc_r(0)$. *Then* $c(t) \to c^{\varrho_0}$ *strongly in* X *as* $t \to \infty$, *where* c^{ϱ_0} *is defined by* (4.1).

Proof. By Corollary 2.6

$$\sum_{r=1}^{m} rc_r(t) - \sum_{r=1}^{m} rc_r(0) = \int_0^t \left[mb_{m+1}c_{m+1}(s) + \sum_{r=m+1}^{\infty} b_r c_r(s) \right] ds, \qquad t \geqq 0, \tag{5.1}$$

so that $\sum_{r=1}^{m} rc_r(t)$ is non-decreasing in t for each m. Therefore $c_r(t)$ tends to a limit $\bar{c}_r$ as $t \to \infty$ for each r. Writing $c_1(t+1) - c_1(t)$ as an integral using (2.1) and letting

$t \to \infty$, it is easily shown that $\bar{c}_r = 0$ for $r > 1$. Passing to the limit $t \to \infty$ in (5.1) we find $\bar{c}_1 \geqq \varrho_0$, and doing the same in (2.25) yields $\bar{c}_1 \leqq \varrho_0$. Since $\|c(t)\| = \|c^{\varrho_0}\|$, the result follows using Lemma 3.3. □

Next we consider pure coagulation.

Theorem 5.2. *Suppose that* $a_r > 0$, $b_r = 0$ *for all* r. *Let* c *be a solution of* (1.1) *on* $[0, \infty)$ *and let* $\varrho_0 = \sum_{r=1}^{\infty} r c_r(0)$. *Then* $c(t) \to c^{\varrho_0}$ *strongly in* X *as* $t \to \infty$ *for some equilibrium state* c^{ϱ_0} *with density* ϱ_0.

Proof. For $m = 1, 2, \ldots$ define

$$p_m(t) = \sum_{r=m}^{\infty} c_r(t).$$

Then by (2.27) $p_m(t)$ is nondecreasing in t for $m \geqq 2$, while by (2.1) $p_1(t)$ is nonincreasing in t. Since each $p_m(t)$ is bounded, it follows that $\lim_{t \to \infty} p_m(t) = \bar{p}_m$ exists, and since $c_r(t) = p_r(t) - p_{r+1}(t)$, we have also that $\lim_{t \to \infty} c_r(t) = \bar{c}_r$ exists. By the monotone convergence theorem it follows that

$$\varrho_0 = \sum_{r=1}^{\infty} r c_r(t) = \sum_{m=1}^{\infty} p_m(t) \to \sum_{m=1}^{\infty} \bar{p}_m = \sum_{r=1}^{\infty} r \bar{c}_r \quad \text{as} \quad t \to \infty.$$

Therefore, by Lemma 3.3, $c(t) \to \bar{c}$ strongly in X as $t \to \infty$. Since by (2.1) $\int_0^{\infty} a_1 c_1^2(t)\,dt < \infty$, we have $\bar{c}_1 = 0$, completing the proof. □

In order to handle the coagulation-fragmentation case we recall some facts concerning the asymptotic behaviour of generalized flows. Given a generalized flow on a metric space Y and some $\phi \in G$ we denote by $\mathcal{O}^+(\phi) = \bigcup_{t \geqq 0} \phi(t)$ the *positive orbit* of ϕ and by $\omega(\phi) = \{y \in Y : \phi(t_j) \to y$ for some sequence $t_j \to \infty\}$ the *ω-limit set* of ϕ. A subset $E \subset Y$ is said to be *quasi-invariant* (cf. Barbashin, 1948) if given any $y \in E$ and $t \geqq 0$ there exists $\psi \in G$ with $\psi(t) = y$ and $\mathcal{O}^+(\psi) \subset E$. The following result is standard, and we include the proof for the reader's convenience.

Proposition 5.3. *Let* G *be a generalized flow on* Y, *let* $\phi \in G$ *and suppose that* $\mathcal{O}^+(\phi)$ *is relatively compact. Then* $\omega(\phi)$ *is nonempty and quasi-invariant, and* $\operatorname{dist}(\phi(t), \omega(\phi)) \to 0$ *as* $t \to \infty$.

Proof. We prove the quasi-invariance, the other assertions being obvious. Let $y \in \omega(\phi)$, so that $\phi(t_j) \to y$ for some sequence $t_j \to \infty$. Let $t \geqq 0$ and consider the sequence $\phi(t_j - t)$. Since $\mathcal{O}^+(\phi)$ is relatively compact there is a subsequence t_{j_k} such that $\phi(t_{j_k} - t) = \phi_{t_{j_k} - t}(0)$ is convergent. By property (iii) in the definition of a generalized flow there exist a further subsequence, again denoted t_{j_k}, and an element $\psi \in G$ such that $\phi_{t_{j_k} - t}(s) = \phi(t_{j_k} - t + s) \to \psi(s)$ as $k \to \infty$ uniformly for s in compact intervals of $[0, \infty)$. Clearly $\mathcal{O}^+(\psi) \subset \omega(\phi)$ and $\psi(t) = y$. □

A function $\mathscr{V} : Y \to \mathbb{R}$ is called a *Lyapunov function* if $t \to \mathscr{V}(\phi(t))$ is nonincreasing on $[0, \infty)$ for each $\phi \in G$. For generalized flows the simplest form of the "invariance principle" consists of the following immediate consequence of Proposition 5.3. If $\mathscr{V}$ is a continuous Lyapunov function and if $\mathcal{O}^+(\phi)$ is relatively compact, then $\omega(\phi)$ consists of complete orbits along which $\mathscr{V}$ has the constant value $\mathscr{V}^\infty = \lim_{t\to\infty} \mathscr{V}(\phi(t))$. This information may determine $\omega(\phi)$. For a complete bibliography and more details see Ball (1978).

We begin by studying a case in which $z_s = \infty$.

Theorem 5.4. *Assume* $a_r > 0$, $b_r > 0$ *for all* r, $a_r = O(r)$ *and*

$$\lim_{r\to\infty} Q_r^{1/r} = 0. \tag{5.2}$$

Let c *be any solution of* (1.1) *on* $[0, \infty)$ *satisfying* $c(0) \neq 0$, $V(c(0)) < \infty$, *and the energy inequality* (4.29). *Let* $\varrho_0 = \sum_{r=1}^{\infty} rc_r(0)$. *Then* $c(t) \to c^{\varrho_0}$ *strongly in* X *as* $t \to \infty$, *where* c^{ϱ_0} *is the unique equilibrium state with density* ϱ_0 *(given by* (4.10)*).*

Proof. By Theorem 3.4 the set G of all solutions of (1.1) on $[0, \infty)$ is a generalized flow on X^+. By (4.29) $V(c(t)) \leqq V(c(0))$ for all $t \geqq 0$, and by (4.15), Corollary 2.6 it follows that

$$-\sum_{r=1}^{\infty} rc_r(t) \ln(Q_r^{1/r}) \leqq M < \infty \quad \text{for all} \quad t \geqq 0. \tag{5.3}$$

As is easily shown, (5.2) and (5.3) imply that $\mathcal{O}^+(c)$ is relatively compact in X^+. Since V is not continuous on X^+ we cannot apply the invariance principle directly to determine $\omega(c)$. Instead we note by (4.29) that for any n, any $T > 0$ and any sequence $t_j \to \infty$,

$$\lim_{j\to\infty} \int_0^T D_n(c(t_j + s))\, ds = \lim_{j\to\infty} \int_{t_j}^{t_j+T} D_n(c(s))\, ds = 0. \tag{5.4}$$

Let $\bar{c} \in \omega(c)$, so that $c(t_j) \to \bar{c}$ in X for some sequence $t_j \to \infty$. By the proof of Proposition 5.3 there is a subsequence, again denoted t_j, and a solution d of (1.1) on $[0, \infty)$ such that $d(0) = \bar{c}$ and $c(t_j + \cdot) \to d(\cdot)$ in $C([0, T]; X)$. Since $\sum_{r=1}^{\infty} rd_r(t) = \varrho_0 > 0$, we have by Theorem 4.6 that $d_r(t) > 0$ for all r and all $t > 0$. Thus by (5.4) and Fatou's lemma,

$$\int_0^T D_n(d(s))\, ds = 0. \tag{5.5}$$

Since n is arbitrary, it follows from (5.5), (4.28) and the continuity of each $d_r(\cdot)$ that

$$d_r(s) = Q_r d_1(s)^r, \quad r \geqq 1, s \in [0, T].$$

In particular, $\bar{c}_r = Q_r \bar{c}_1^r$ for $r \geqq 1$, and since $\sum_{r=1}^{\infty} r\bar{c}_r = \varrho_0$, this implies that $\bar{c} = c^{\varrho_0}$. Hence $\omega(c) = \{c^{\varrho_0}\}$ and the result follows from Proposition 5.3. □

We now discuss the case $0 < z_s < \infty$. This is more difficult because if $\varrho_0 = \sum_{r=1}^{\infty} rc_r(0) > \varrho_s$, then the positive orbit of c is never relatively compact in X.

Theorem 5.5. *Assume $a_r>0$, $b_r>0$ for all r, $a_r=O(r/\ln r)$, $b_r=O(r/\ln r)$, and that $\lim_{r\to\infty} Q_r^{1/r}=\frac{1}{z_s}$ exists with $0<z_s<\infty$. Let c be a solution of (1.1) on $[0,\infty)$ and let $\varrho_0=\sum_{r=1}^{\infty} rc_r(0)$. Then $c(t)\overset{*}{\rightharpoonup} c^{\varrho}$ as $t\to\infty$ for some ϱ with $0\leqq\varrho\leqq\min(\varrho_0,\varrho_s)$.*

Proof. The case $\varrho_0=0$ being trivial, we suppose $\varrho_0>0$. By Theorem 3.5 G_{ϱ_0} is a generalized flow on $B^+_{\varrho_0}$. By Proposition 4.5 V_{z_s} is continuous on $B^+_{\varrho_0}$, and by Theorem 4.7, Corollary 2.6

$$V_{z_s}(c(t))+\int_0^t D(c(s))\,ds=V_{z_s}(c(0)),\qquad t\geqq 0. \tag{5.6}$$

Since $\sum_{r=1}^{\infty} rc_r(t)$ is bounded, $\mathcal{O}^+(c)$ is relatively compact in $B^+_{\varrho_0}$. By the invariance principle $\omega(c)$ is nonempty and consists of solutions $c(\cdot)$ along which V_{z_s} has the constant value $V_{z_s}^{\infty}=\lim_{t\to\infty} V_{z_s}(c(t))$. Applying (5.6) to a nonzero such solution $\bar{c}(\cdot)$ we see that necessarily $\bar{c}_r(t)=Q_r\bar{c}_1(t)^r$, $r\geqq 1$, and by density conservation it follows that $\bar{c}$ is an equilibrium. Hence $\omega(c)$ consists of equilibria c^{ϱ} with $0\leqq\varrho\leqq\min(\varrho_0,\varrho_s)$ and $V_{z_s}(c^{\varrho})=V_{z_s}^{\infty}$. But by Proposition 4.3 $V_{z_s}(c^{\varrho})$ is strictly decreasing in ϱ, and thus $\omega(c)=\{c^{\varrho}\}$ for a unique ϱ, $0\leqq\varrho\leqq\min(\varrho_0,\varrho_s)$. The result follows from Proposition 5.3. □

In order to determine the density ϱ of the limiting equilibrium in Theorem 5.5, and thus show that $c(t_j)$ is a minimizing sequence of V on $X^+_{\varrho_0}$ for $t_j\to\infty$, we need to control the "tail" of a solution. At the expense of making further hypotheses, this can be done using a maximum principle.

Theorem 5.6. *In addition to the hypotheses of Theorem 5.5 assume that there is a constant M such that*

$$M\geqq b_{r+1}-a_r z_s\geqq 0,\, b_r-a_r z_s\geqq 0, \tag{5.7}$$

for r sufficiently large. Assume further that $\sum_{r=1}^{\infty}\frac{c_r(0)}{Q_r z_s^r}<\infty$, and that, in the case $\varrho_s=\infty$, $\lim_{r\to\infty} rQ_r z_s^r=0$. Suppose finally that c is the only solution of (1.1) on $[0,\infty)$ with initial data $c(0)$. Then

(i) *if $0\leqq\varrho_0\leqq\varrho_s$, $c(t)\to c^{\varrho_0}$ strongly in X as $t\to\infty$, and*

$$\lim_{t\to\infty} V(c(t))=V(c^{\varrho_0}),$$

(ii) *if $\varrho_0>\varrho_s$, $c(t)\overset{*}{\rightharpoonup} c^{\varrho_s}$ as $t\to\infty$, and*

$$\lim_{t\to\infty} V(c(t))=V(c^{\varrho_s})+(\varrho_0-\varrho_s)\ln z_s.$$

Proof. We will show that if $c(t)\overset{*}{\rightharpoonup} c^{\varrho}$ as $t\to\infty$ for some $\varrho<\varrho_s$ then $c(t)\to c^{\varrho}$ strongly, so that by density conservation $\varrho=\varrho_0$. The assertions in the theorem for $\varrho_0<\varrho_s$ and $\varrho_0>\varrho_s$ follow immediately from Theorem 5.5. If $\varrho_0=\varrho_s$, then we deduce that $\varrho=\varrho_s$ and the strong convergence follows by Lemma 3.3. The statements concerning $\lim_{t\to\infty} V(c(t))$ follow from Proposition 4.5.

For K chosen sufficiently large define

$$g_r = Kr + \frac{1}{Q_r z_s^r}.$$

Then

$$g_{r+1} - g_r = K + \frac{1}{a_r Q_r z_s^{r+1}} (b_{r+1} - a_r z_s) \geqq \delta > 0$$

for all r and some constant δ, and since $a_r = O(r)$, it is easily shown that $a_r(g_{r+1} - g_r) = O(g_r)$. It follows from Theorem 2.2 and the uniqueness assumption that $\sum_{r=1}^{\infty} x_r(t) < \infty$ for $t \geqq 0$, where $x_r(t) \stackrel{\text{def}}{=} \frac{c_r(t)}{Q_r z_s^r}$, and by Theorem 2.5 and the same proof as Proposition 3.1 the series is uniformly convergent on compact intervals of $[0, \infty)$.

Let (5.7) hold for $r > r_0$. Since $c(t) \stackrel{*}{\rightharpoonup} c^{\varrho}$ with $\varrho < \varrho_s$, there exists t_1 such that $x_r(t) < 1$ for all $t \geqq t_1$ and $1 \leqq r \leqq r_0$. Let $K = \sup\{x_r(t_1) : r \geqq 1\}$. We show that $x_r(t) \leqq K+1$ for all $t \geqq t_1$ and $r \geqq 1$. If not, by the uniform convergence of $\sum_{r=1}^{\infty} x_r(t)$, there exist $t_2 \geqq t_1$ and a minimal $n > r_0$ such that $x_n(t_2) = K+1 \geqq x_r(t_2)$, $r \neq n$, and $x_n(t) \leqq x_n(t_2)$ for $t \in [t_1, t_2]$. But

$$\dot{x}_n = (x_1 - 1)(b_n - a_n z_s) x_n + b_n x_1 (x_{n-1} - x_n) + a_n z_s (x_{n+1} - x_n).$$

Supposing without loss of generality that $c \not\equiv 0$, we have by Theorem 4.6 that $x_1(t_2) > 0$. By (5.7) and $x_{n-1}(t_2) < x_n(t_2)$, $x_{n+1}(t_2) \leqq x_n(t_2)$ we deduce that $\dot{x}_n(t_2) < 0$. This contradiction proves that

$$c_r(t) \leqq (K+1) Q_r z_s^r \quad \text{for all} \quad t \geqq t_1, r \geqq 1.$$

Since $\lim_{r\to\infty} r Q_r z_s^r = 0 \left(\text{in the case } \varrho_s < \infty \text{ because } \sum_{r=1}^{\infty} r Q_r z_s^r < \infty \right)$ it follows easily that $\mathcal{O}^+(c)$ is relatively compact in X^+, and hence $c(t) \to c^{\varrho}$ strongly as claimed. □

For brevity we study the Lyapunov stability of the equilibria just when $0 < z_s < \infty$; the case $z_s = \infty$ can be treated similarly.

Theorem 5.7. *Assume $a_r > 0$, $b_r > 0$ for all r, $a_r = O\left(\frac{r}{\ln r}\right)$, $b_r = O\left(\frac{r}{\ln r}\right)$ and that $\lim_{r\to\infty} Q_r^{1/r} = \frac{1}{z_s}$ exists with $0 < z_s < \infty$.*

(i) *Let $\varrho < \infty$, $0 \leqq \varrho \leqq \varrho_s$. Then c^{ϱ} is stable in X_{ϱ}^+; i.e. given $\varepsilon > 0$, there exists $\delta > 0$ such that any solution c of (1.1) on $[0, \infty)$ with $\|c(0) - c^{\varrho}\| < \delta$, $\|c(0)\| = \varrho$ satisfies $\|c(t) - c^{\varrho}\| < \varepsilon$ for all $t \geqq 0$.*

(ii) *Let $\varrho_s < \varrho < \infty$. Then c^{ϱ_s} is weak * stable in X_{ϱ}^+; i.e. given $\varepsilon > 0$ there exists $\delta > 0$ such that any solution c of (1.1) on $[0, \infty)$ with $d(c(0), c^{\varrho_s}) < \delta$, $\|c(0)\| = \varrho$ satisfies $d(c(t), c^{\varrho_s}) < \varepsilon$ for all $t \geqq 0$.*

Proof. (i) Since V is a continuous Lyapunov function on X_{ϱ}^+, standard results on Lyapunov stability [convenient references are Knops and Payne (1978) and Ball and Marsden (1984)] imply that we need only show that c^{ϱ} lies in a *potential well*, that is for $\delta > 0$ sufficiently small

$$\inf\{V(c): c \in X_\varrho^+, \|c - c^\varrho\| = \delta\} > V(c^\varrho).$$

But this follows immediately from Theorem 4.4.

(ii) Since V_{z_s} is a sequentially weak *continuous Lyapunov function on X_ϱ^+, we similarly need only show that for $\delta > 0$ sufficiently small,

$$\inf\{V_{z_s}(c): c \in X_\varrho^+, d(c, c^{\varrho_s}) = \delta\} > V_{z_s}(c^{\varrho_s}).$$

This also follows from Theorem 4.4. □

We end this paper by indicating how the hypotheses of the various theorems are satisfied for the binary alloy problem discussed by Kalos et al. (1978), Penrose and Buhagiar (1983), and Penrose et al. (1984).

In the zero-density limit the corresponding kinetic coefficients for temperatures less than but not too close to the critical temperature are strictly positive, can be calculated for small r, and extrapolated for large r using the formulae

$$\begin{aligned} a_r &= D(k_0 + k_1 r)^{1/3}, \qquad r \geqq 1, \\ w_r &= \zeta_s\left(1 + \frac{k_2}{(r-2)^{1/3}}\right), \qquad r \geqq 3, \end{aligned} \tag{5.8}$$

where $w_r \stackrel{\text{def}}{=} \frac{b_{r+1}}{a_r}$, and the constants $D, k_0, k_1, k_2, \zeta_s$ are strictly positive. The $r^{1/3}$ power is motivated by the work of Lifshitz and Slyozov (1961). Since, by (5.8),

$$a_r = O(r^{1/3}), \; b_r = O(r^{1/3}),$$

it follows from Theorem 2.2, Corollary 3.8 that (1.1) generates a semigroup on X^+ and B_ϱ^+, $\varrho \geqq 0$. Both the cases in Proposition 2.4 hold, guaranteeing appropriate decay of $c_r(t)$ as $r \to \infty$ for each $t \geqq 0$. Furthermore, by Theorem 3.9 the approximation scheme in Theorem 2.2 converges to a solution without extraction of a subsequence. It is also not hard to check that the hypotheses of Theorem 3.2 hold for all $k \geqq 1$, so that each $c_r(t)$ is C^∞ in t. Taking logarithms in (4.5) we see that as $r \to \infty$

$$Q_r = \text{const}\, \zeta_s^{-r} \exp\left(-\frac{3k_2}{2} r^{2/3}(1 + o(1))\right), \tag{5.9}$$

and it follows that

$$\lim_{r \to \infty} Q_r^{1/r} = \frac{1}{\zeta_s},$$

and thus that $z_s = \zeta_s$, $\varrho_s < \infty$. Hence Theorems 5.5, 5.7 hold. The inequalities (5.7) also follow from (5.8), so that the conclusions of Theorem 5.6 hold provided the initial data satisfies the decay estimate

$$\sum_{r=1}^{\infty} \exp(\mu r^{2/3})\, c_r(0) < \infty \tag{5.10}$$

for some $\mu > \frac{3k_2}{2}$.

The authors cited above also consider low density corrections to (5.8) in which the coefficients a_r, b_r are allowed to depend on the density ϱ. In one such correction $a_r(\varrho)$ is defined by the first equation in (5.8) with $D=D(\varrho)>0$, and $\frac{b_{r+1}(\varrho)}{a_r(\varrho)}=w_r(\varrho)$, where for sufficiently small ϱ,

$$w_1(\varrho)=(1-\varrho)^2 w_1, \quad w_r(\varrho)=(1-\varrho)^3 w_r \quad \text{for} \quad r\geqq 2, \tag{5.11}$$

with w_r as in (5.8). Since the $a_r(\varrho)$, $b_r(\varrho)$ depend on $\varrho=\sum_{r=1}^{\infty} rc_r$, the resulting Becker-Döring equations are not of the form considered in this paper. However, with an appropriate definition of solution one can reprove Corollary 2.6 and hence reduce the problem to that for constant coefficients. Note that Q_r given by (5.9) is now replaced by

$$Q_r(\varrho)=(1-\varrho)^{4-3r}Q_r,\ r\geqq 2, \quad Q_1(\varrho)=1, \tag{5.12}$$

so that the radius of convergence z_s^ϱ of $\sum_{r=1}^{\infty} rQ_r(\varrho)z^r$ is given by

$$z_s^\varrho=(1-\varrho)^3\zeta_s. \tag{5.13}$$

Define

$$\varrho_s(\varrho)=\sum_{r=1}^{\infty} rQ_r(\varrho)(z_s^\varrho)^r.$$

Then

$$\varrho_s(\varrho)=(1-\varrho)^4\varrho_s+\varrho(1-\varrho)^3\zeta_s,$$

where $\varrho_s=\sum_{r=1}^{\infty} rQ_r\zeta_s^r$ is as above. Given $0\leqq\varrho<1$, the Becker-Döring equations with constant coefficients $a_r(\varrho)$, $b_r(\varrho)$ thus have a unique equilibrium $c_\varrho^{\bar{\varrho}}$ corresponding to every density $\bar{\varrho}$ with $0\leqq\bar{\varrho}\leqq\varrho_s(\varrho)$. Applying Theorem 5.6 we see that provided (5.10) holds and $\sum_{r=1}^{\infty} rc_r(0)=\varrho<1$ the solution has the properties

(i) if $0\leqq\varrho\leqq\varrho_s(\varrho)$, $c(t)\to c_\varrho^\varrho$ strongly in X as $t\to\infty$, and

(ii) if $\varrho>\varrho_s(\varrho)$, $c(t)\xrightarrow{*} c_\varrho^{\varrho_s(\varrho)}$ as $t\to\infty$.

Note that by (5.12), (5.13),

$$(c_\varrho^{\varrho_s(\varrho)})_1=(1-\varrho)^3\zeta_s;\ (c_\varrho^{\varrho_s(\varrho)})_r=(1-\varrho)^4 Q_r\zeta_s^r, \quad r\geqq 2. \tag{5.14}$$

Acknowledgements. The research of J.M.B. was supported by an S.E.R.C. Senior Fellowship. Part of the work on the project was also carried out while J.M.B. was visiting the Institute of Mathematics and its Applications at the University of Minnesota in 1985.

References

Abraham, F.: Homogeneous nucleation theory. New York: Academic Press 1974

Aizenman, M., Bak, T.A.: Convergence to equilibrium in a system of reacting polymers. Commun. Math. Phys. **65**, 203–230 (1979)

Andrews, G., Ball, J.M.: Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity. J. Differ. Equations **44**, 306–341 (1982)

Arnold, V.I.: On an a priori estimate in the theory of hydrodynamic stability. Am. Math. Soc. Transl. **79**, 267–269 (1969)
Ball, J.M.: On the asymptotic behaviour of generalized processes, with applications to nonlinear evolution equations. J. Differ. Equations **27**, 224–265 (1978); Existence of solutions in finite elasticity. In: Proc. IUTAM symp. on finite elasticity. Carlson, D.E., Shield, R.T. (eds.). The Hague: Martinus Nijhoff 1981; Material instabilities and the calculus of variations. In: Phase transformations and material instabilities in solids. Gurtin, M.E. (ed.). New York: Academic Press 1984
Ball, J.M., Marsden, J.E.: Quasiconvexity at the boundary, positivity of the second variation and elastic stability. Arch. Rat. Mech. Anal. **86**, 251–277 (1984)
Barbashin, E.A.: On the theory of generalized dynamical systems. Moskov. Gos. Ped. Inst. Učen. Zap. **2**, 110–133 (1948); English translation by U.S. Department of Commerce, Office of Technical Services, Washington D.C. 20235
Becker, R., Döring, W.: Kinetische Behandlung der Keimbildung in übersättigten Dämpfern. Ann. Phys. (Leipzig) **24**, 719–752 (1935)
Binder, K.: Theory for the dynamics of "clusters". II. Critical diffusion in binary systems and the kinetics of phase separation. Phys. Rev. B **15**, 4425–4447 (1977)
Buhagiar, A.: Ph. D Thesis, Open University 1981
Dickman, R., Schieve, W.C.: Proof of the existence of the cluster free energy. J. Stat. Phys. **36**, 435–446 (1984)
Drake, R.: In: Topics in current aerosol research. International reviews in aerosol physics and chemistry, Vol. 2. Hidy, G.M., Brock, J.R. (eds.). Oxford: Pergamon Press 1972
Dunford, N., Schwartz, J.T.: Linear operators, Part I. New York: Interscience 1958
Friedlander, S.K.: On the particle size spectrum of a condensing vapour. Phys. Fluids **3**, 693–696 (1960)
Hale, J.K.: Dynamical systems and stability. J. Math. Anal. Appl. **26**, 39–59 (1969)
Hendricks, E.M., Ernst, M.H., Ziff, R.M.: Coagulation equations with gelation. J. Stat. Phys. **31**, 519–563 (1983)
Holm, D.D., Marsden, J.E., Ratiu, T., Weinstein, A.: Non-linear stability conditions and a priori estimates for barotropic hydrodynamics. Phys. Lett. **98** A, 15–21 (1983)
Hartman, P.: Ordinary differential equations. New York: Wiley 1964; reprinted Boston: Birkhäuser 1982
Kalos, M., Lebowitz, J.L., Penrose, O., Sur, A.: Clusters, metastability, and nucleation: kinetics of first-order phase transitions. J. Stat. Phys. **18**, 39–52 (1978)
Knops, R.J., Payne, L.E.: On potential wells and stability in nonlinear elasticity. Math. Proc. Camb. Phil. Soc. **84**, 177–190 (1978)
Leyvraz, F., Tschudi, H.R.: Singularities in the kinetics of coagulation processes. J. Phys. A: Math. Gen. **14**, 3389–3405 (1981)
Lieb, E.: Thomas-Fermi and related theories of atoms and molecules. Rev. Mod. Phys. **53**, 603–641 (1981)
Lifshitz, I.M., Slyozov, V.V.: J. Phys. Chem. Solids **19**, 35–50 (1961)
Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case, Part 1. Ann. Inst. Henri Poincaré **1**, 109–145 (1984)
McLeod, J.B.: On an infinite set of non-linear differential equations. Q. J. Math. Oxf. Ser. (2) **13**, 119–128 (1962a); On an infinite set of non-linear differential equations. Q. J. Math. Oxf. Ser. (2) **13**, 193–205 (1962b); On a recurrence formula in differential equations. Q. J. Math. Oxf. Ser. (2) **13**, 283–284 (1962c)
Melzak, Z.: A scalar transport equation. Trans. Am. Math. Soc. **85**, 547–560 (1957)
Pego, R.: Phase transitions: stability and admissibility in one dimensional nonlinear viscoelasticity. Arch. Rat. Mech. Anal. (to appear)
Penrose, O., Buhagiar, A.: Kinetics of nucleation in a lattice gas model: Microscopic theory and simulation compared. J. Stat. Phys. **30**, 219–241 (1983)
Penrose, O., Lebowitz, J.L.: Towards a rigorous theory of metastability. In: Studies in statistical mechanics, Vol. VII. Fluctuation phenomena. Montroll, E.W., Lebowitz, J.L. (eds.). Amsterdam: North-Holland 1979

Penrose, O., Lebowitz, J., Marro, J., Kalos, M., Tobochnik, J.: Kinetics of a first-order phase transition: Computer simulations and theory. J. Stat. Phys. **34**, 399–426 (1984)
Reuter, G.E.H., Ledermann, W.: On the differential equations for the transition probabilities of Markov processes with enumerably many states. Proc. Camb. Phil. Soc. **49**, 247–262 (1953)
von Smoluchowski, M.: Versuch einer mathematischen Theorie der kolloiden Lösungen. Z. Phys. Chem. **92**, 129–168 (1917)
Spouge, J.L.: An existence theorem for the discrete coagulation-fragmentation equations. Math. Proc. Camb. Phil. Soc. **96**, 351–357 (1984)
Young, L.C.: Lectures on the calculus of variations and optimal control theory. Philadelphia: Saunders reprinted by Chelsea 1969

Communicated by J. L. Lebowitz

Received August 23, 1985