Commun. Math. Phys. 92, 137 – 162 (1983)

Communications in Mathematical Physics

Convergence of $U(1)_3$ Lattice Gauge Theory to its Continuum Limit

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Abstract. It is shown that in three space-time dimensions the pure U(1) lattice gauge theory with Villain action and fixed coupling constant converges to the free electromagnetic field as the lattice spacing approaches zero. The same holds for the Wilson action on the electric sector.

1. Introduction

The lattice approximation to gauge field theories [22] beautifully preserves those differential geometric structures of the continuum theory which are important for physics. In addition to its value as a computational tool it provides a potential mechanism for proving the existence of continuum gauge theories.

Balian et al. [1] have shown that for pure gauge theories in two dimensions the compact lattice version is solvable, which yields, by explicit computation, a proof that the lattice theory indeed converges to a continuum limit in two dimensions. But in three or more dimensions compact lattice gauge theories are not solvable and no such model has heretofore been shown to have a continuum limit.

In three dimensions the compact pure gauge model with gauge group U(1) (= circle group) is expected to converge in the continuum limit to the free electromagnetic field. See for example [2]. In their deep paper on confinement in the $U(1)_3$ lattice gauge theory Göpfert and Mack [10] showed that the integer scalar field naturally associated to the dual model converges upon suitable normalization to a free scalar field if the coupling constant g is allowed to go to infinity at an appropriate rate as the lattice spacing goes to zero. They conjectured also that in the canonical limit in which g is held fixed the lattice gauge field itself should converge to the free electromagnetic field at the level of the field variables F_{uv} or the Wilson loops, at least for the Villain action.

* This research was supported in part by N.S.F. Grant MCS 81-02147, in part by the Institute for Mathematics and its Applications at the University of Minnesota and in part by the Institute for Advanced Study in Princeton, NJ, USA

We show that the compact U(1) lattice gauge theory in three dimensions converges to the free electromagnetic field for the Villain action with fixed coupling constant in the sense of convergence of the characteristic functional of the field variables, F_{uv}. We also show that for the Wilson action or any other action satisfying natural minimal conditions the F_{uv} converge to the free electromagnetic field on the electric sector. Thus we leave unsettled the question as to whether for the Wilson action monopoles survive in the continuum limit. If they do we show that they are independent of the free electromagnetic field. But we also give strong evidence using a spin wave approximation that there are no continuum monopoles for the Wilson action either (in dimension three).

DeAngelis and DeFalco [7] and Bellisard and DeAngelis [3] have shown that the U(1) gauge model as well as the XY model with either Wilson or Villain actions converge in a certain sense to the corresponding lattice Gaussian model as the temperature goes to zero. Although they do not prove the existence of a continuum limit they give evidence for its existence in the case of the XY model [7].

Informally the free Euclidean electromagnetic field on R^d is given by a measure on a space of 1-forms $A=\{A_\nu(x)\}_{\nu=1}^d$ by the expression

$$Z^{-1}\exp[-(2g^2)^{-1}\sum_{\mu<\nu}\int_{R^d} F_{\mu\nu}(x)^2\,dx]\mathscr{D}A, \tag{1.1}$$

where $F_{\mu\nu}(x)=\partial A_\nu/\partial x^\mu-\partial A_\mu/\partial x^\mu$, and $\mathscr{D}A$ is the infinite dimensional "Lebesgue measure": $\mathscr{D}A=\prod_{\nu=1}^{d}\prod_{x\in R^d} dA_\nu(x)$. Although this expression suffers, even at an informal level, from the fact that the quadratic form in the exponential factor has a large nullspace as a function of A, it is well known that by any of a number of gauge fixing mechanisms the integral of gauge invariant functions is well defined and in particular the integral of functions of the $F_{\mu\nu}$. We refer the reader to [21] for description of the free electromagnetic field in the relativistic region and to [11] and [13] for a description in the Euclidean region.

By a test r-form $\phi=\{\phi_{i_1,\ldots,i_r}(x)\}$ on R^d, we shall mean an r-form which is infinitely differentiable and has compact support. The basic objects we study in this paper are the lattice versions of the smoothed field variables which in the continuum are given by the informal expression

$$F(\phi)=\sum_{\mu<\nu}\int_{R^d} F_{\mu\nu}(x)\phi_{\mu\nu}(x)dx,$$

where ϕ is a test 2-form. An informal computation of Gaussian integrals starting with (1.1) shows that the integral of $\exp[iF(\phi)]$ is given by

$$\langle\exp[iF(\phi)]\rangle=\exp[-g^2((-\Delta)^{-1}\delta\phi,\delta\phi)/2], \tag{1.2}$$

where Δ denotes the Laplacian on R^d and $(\delta\phi)_\nu(x)=\sum_\mu \partial\phi_{\mu\nu}(x)/\partial x^\mu$ is the divergence of ϕ. We shall use d for the exterior derivative (= curl) and $\delta=d^*$ for its adjoint ($\delta=$ div.). $F(\phi)$ should be thought of, for fixed ϕ, as a measurable function of a point F in some measure space. In fact it is a simple consequence of Minlos' theorem [19] that there exists a unique probability measure on the dual space of the space of test 2-forms satisfying (1.2). Equation (1.2) contains all the information we shall need about the free Euclidean electromagnetic field, which may in fact be defined as the stochastic process $\phi \to F(\phi)$ satisfying (1.2).

Recall that Maxwell's equations may be written in 4-dimensional Minkowski space-time in the form $dF = 0$, $\delta F = J$, where $F = \{F_{\mu\nu}(x)\}$ is the field strength, J is the charge-current 1-form and the zero on the right of the first equation reflects the absence of magnetic charge (no monopoles). In the Euclidean region Eq. (1.2) shows there are also no monopoles, since for any test 3-form ψ $(dF)(\psi)$, which is to be understood in the usual distribution sense as $F(\delta\psi)$, is zero. Indeed, using the identity $\delta^2 = 0$ and substituting $\phi = t(\delta\psi)$ in (1.2) shows that $\langle e^{itF(\delta\psi)} \rangle = 1$ for all real t, so that $F(\delta\psi) = 0$ for all test 3-forms ψ. That is, $dF = 0$ for the free Euclidean electromagnetic field.

The discrete analog of the Bianchi identity $dF = 0$ fails in the lattice theory however. We shall show that starting with the Villain action on the lattice the Bianchi identity is recovered in the continuum limit. A proof of this for the Wilson action seems to require other techniques. On the electric sector however our results are quite complete. We show that for any action satisfying the expected minimal conditions (see conditions A1, A2, A3 in Sect. 2) the lattice analog of δF converges to the free electromagnetic field both in the sense of convergence of the characteristic functional and convergence of the Schwinger functions. The proof uses only the Schwinger–Dyson equations and the asymptotics of the single plaquette energy distribution near zero temperature. Duality arguments are not required anywhere. We also prove Gaussian domination bounds on the Laplace transform and on moments of δF in any dimension. These may be useful in four dimensions. For general background on lattice gauge theory and for a survey of some deep results on convergence of noncompact lattice models by Balaban, Brydges, Fröhlich and Seiler, see [18].

2. Notation and Statement of Results

We consider lattice gauge models determined by a real-valued energy function h on the real line R, satisfying some of the following conditions:

A1. $h: R \to R$ is even, periodic with period 2π, and has two continuous derivatives.

A2. $\min h = h(0) = 0$.

A3. $h''(x) = 1$ whenever $h(x) = 0$.

Examples 1 Wilson action: $h(x) = 1 - \cos x$. More generally, the energy function can have several zeros in the periodicity interval $(-\pi, \pi]$. For example $h(x) = m^{-2}(1 - \cos mx)$, $m = 1, 2, 3, \ldots$, also satisfies A1, A2 and A3, as does finite convex sums of these.

2. Villain action: For each real number $\beta > 0$, define $h_\beta(x)$ by

$$e^{-\beta h_\beta(x)} = c_\beta \sum_{n=-\infty}^{\infty} \exp[-\beta(x - 2\pi n)^2/2], \tag{2.1}$$

where c_β is a constant chosen so that the right side is one at $x = 0$. Then h_β satisfies A1 and A2. We shall see in Sect. 3 that $h''_\beta(0)$ converges to one as $\beta \to \infty$.

Let Λ be a cube in Z^d. That is, $\Lambda = \{k \in Z^d: m \leqq k_i \leqq M,\ i = 1,\dots,d\}$ for some choice of integers m and M. Denote by $\{\Lambda^{(j)}\}_{j=0}^d$ a complex built on Λ. $\Lambda^{(j)}$ is a set of oriented j-cells whose vertices are in Λ. We shall be interested for the most part in open or closed complexes. We recall that a complex is *open* if for any j cell b in $\Lambda^{(j)}$ and any $j+1$ cell c, $b \in \partial c$ implies $c \in \Lambda^{(j+1)}$. The complex is *closed* if for each $(j+1)$ cell c in $\Lambda^{(j+1)}$ any j cell $b \in \partial c$ is in $\Lambda^{(j)}$. Open complexes correspond to a choice of Dirichlet boundary conditions while closed complexes correspond to free boundary conditions. See [9 or 14] for a discussion of this basic machinery of lattice gauge theory. If ψ is an r-cochain over $\{\Lambda^{(j)}\}_{j=0}^d$, then we use the notation $(d\psi)(c) = \sum_{b \in \partial c} \psi(b)$ for the discrete exterior derivative. The sum extends only over those b in $\Lambda^{(r)}$ for which b lies in the boundary of c and has consistent orientation. Similarly $(\delta\psi)(c) = \sum_{b: c \in \partial b} \psi(b)$ is defined for c in $\Lambda^{(r-1)}$ and the sum extends over those b in $\Lambda^{(r)}$ whose boundary contains c with the proper orientation. If ψ and ψ' are two r-cochains over $\{\Lambda^r\}$, then we put $(\psi,\psi') = \sum_{c \in \Lambda^{(r)}} \psi(c)\psi'(c)$. All cochains are real-valued. The last sum is over unoriented r cells c. That is, c or $-c$ occurs but not both. It doesn't matter which since the sum is unaffected. This convention will be used for all sums or products over r-cells, as in (2.2) and (2.3). It should be emphasized again that a statement $b \in \partial c$ means that b occurs in the boundary of c with consistent orientation. We shall use the well known identities $d^2 = 0$, $\delta^2 = 0$ and $(d\psi,\psi') = (\psi,\delta\psi')$, when ψ is an r-cochain and ψ' is an $(r+1)$-cochain.

If ψ is a C^∞ r-form on R^d, we denote by $d\psi$ its exterior derivative and by $\delta\psi$ its coderivative; $\delta = d^*$. The two uses of the symbol d are related. Let $a > 0$ and let $aZ^d = \{ak: k \in Z^d\}$. If ψ is a test r-form on R^d, then there is a naturally induced r-cochain ψ_a on aZ^d. For example if $r=1$ and $\psi(x) = \sum_{j=1}^d \psi_j(x)\,dx^j$, and if b is a bond (oriented 1-cell) parallel to the j axis and extending from x (in aZ^d) to $x + ae_j$, where $(e_1,\dots,e_d)$ is the standard orthonormal basis of R^d, then $\psi_a(b) = \psi_j(x)$. More generally, if c is an r-cell in aZ^d, then we define $\psi_a(c) = \psi_{j_1\dots j_r}(x)$, where $\psi(x) = \sum_{j_1<\dots<j_r} \psi_{j_1,\dots,j_r}(x)\,dx^{j_1} \wedge \dots \wedge dx^{j_r}$, and where x is the "lower left" hand corner of c, c has edges $\langle x, x + ae_{j_i}\rangle$, $i = 1,\dots,r$, and c is properly oriented. For example if $r=2$, then the correct orientation of c is that which makes the bond $b = \langle x, x + ae_{j_1}\rangle$ positively oriented. If $r=3$ then the correct orientation is that which makes the face with edges $\langle x, x+ae_{j_1}\rangle$, $\langle x, x+ae_{j_2}\rangle$ positively oriented when this face is oriented as in the preceding sentence, and so on.

If ψ is an r-cochain on Z^d, we define $d_a\psi = a^{-1}\,d\psi$ and if ψ is an r-form on R^d, define $d_a\psi = a^{-1}\,d(\psi_a)$, which is an $(r+1)$-cochain. Similarly $\delta_a\psi = a^{-1}\,\delta\psi$ if ψ is a cochain and $\delta_a\psi = a^{-1}\,\delta\psi_a$ if ψ is an r-form.

Finally we write

$$(\psi,\psi') = \sum_{j_1<\dots<j_r} \int_{R^d} \psi_{j_1\dots j_r}(x)\psi'_{j_1\dots j_r}(x)\,dx$$

for two test r-forms on R^d. For two such forms one verifies easily using standard

Riemann integral techniques that

$$\lim_{a\downarrow 0} a^d(\psi_a,\psi'_a)=(\psi,\psi').$$

Using the mean value theorem judiciously, one can verify (easily for $r=0$, with care for $r=1$) that

$$\lim_{a\downarrow 0} a^d(d_a\psi,d_a\psi')=(d\psi,d\psi').$$

We shall need this for $r=1$.

Let $T=\{e^{i\theta}:-\pi<\theta\leqq\pi\}$. We identify functions on the circle, T, with functions on the real line, R, which are periodic with period 2π.

The finite volume Gibbs state in a cube $\Lambda\subset Z^d$, at inverse temperature β and with energy function h satisfying A1 is given by

$$\langle f\rangle_{\Lambda,\beta}=Z^{-1}\int_{T^{\Lambda^{(1)}}} d\theta f(\theta)\prod_{p\in\Lambda^{(2)}} e^{-\beta h(d\theta(p))}. \tag{2.2}$$

Here $\theta=\{\theta_b\}_{b\in\Lambda^{(1)}}$ is a 1-cochain on Λ with values in $(-\pi,\pi]$, $d\theta=\prod_{b\in\Lambda^{(1)}} d\theta_b$ is the product of Lebesgue measures on T, and $d\theta(p)=\sum_{b\in\partial p}\theta_b$ for each plaquette $p(=2\text{-cell})$, in $\Lambda^{(2)}$. For the open complex built on Λ, if ∂p has an edge, b, in the boundary of Λ then b is not in $\Lambda^{(1)}$ and is omitted from the sum defining $d\theta(p)$. This is equivalent to setting such a θ_b equal to zero (Dirichlet boundary condition). Z is the normalization constant and f is a function of the bond variables θ_b.

Since T is compact, infinite volume limits always exist. They satisfy the DLR equations and in fact there always exists at least one translation invariant infinite volume Gibbs state. We denote by $\langle\,\rangle_\beta$ any such translation invariant infinite volume Gibbs state. See [4, 5, 6, 8] for a discussion of uniqueness. If ϕ is a 2-cochain on Z^d with finite support and $a>0$, put

$$F_a(\phi)=a^d\sum_{p\in(Z^d)^{(2)}}\phi(p)a^{-2}h'(d\theta(p)). \tag{2.3}$$

Then $F_a(\phi)$ is a function of the infinite volume configuration $\theta\in T^{(Z^d)^{(1)}}$.

It is Wilson's proposal [22] that $a^{-2}h'(d\theta(p))$ is an approximation to the free electromagnetic field $F_{\mu\nu}(x)$, when $h(x)=1-\cos x$. Thus $F_a(\phi)$ is the natural smoothed version of the lattice approximation to $F_{\mu\nu}(x)$.

Theorem 2.1. *Let j be a test 1-form on R^3. Assume that h satisfies* A1, A2 *and* A3. *Let g be a strictly positive real number and put $\beta^{-1}=ag^2$. Then in dimension $d=3$*

$$\lim_{a\downarrow 0}\langle e^{iF_a(d_a j)}\rangle_\beta=e^{-g^2\|dj\|^2/2}. \tag{2.4}$$

Moreover, this equality holds if h is replaced throughout by the Villain action h_β. The right sides of (2.4) and (1.2) are equal when $\phi=dj$.

Theorem 2.2. (Gaussian Domination). *Assume h satisfies* A1, A2 *and* A3. *Let α be a positive real number satisfying $h''(x)\leqq\alpha$ for all x. Let ϕ be an exact 2-cochain of* *finite support on* Z^d. *Then for any* $a>0$ *and* $\beta>0$

$$\langle e^{F_a(\phi)}\rangle_\beta \leqq \exp[\alpha(a^{d-4}\beta^{-1})a^d\|\phi\|^2/2]. \qquad (2.5)$$

If h *is replaced throughout by the Villain action* h_β *then* (2.5) *holds with* $\alpha=1$.

Theorem. 2.3. I. (*Orthogonality of* dF *and* δF *for fixed lattice spacing in dimension* d.) *Assume* h *satisfies* A1. *Let* ψ *and* j *be finitely supported cochains on* Z^d *of degree* 3 *and* 1 *respectively. Then for any* $\beta>0$, $F_a(\delta\psi)$ *and* $F_a(dj)$ *are mutually orthogonal with respect to any Gibbs state* $\langle\ \rangle_\beta$ *determined by* h *if the state* $\langle\ \rangle_\beta$ *is invariant under translation and* 90° *rotations.*

II. (*Independence of* dF *and* δF *in dimension* 3). *Assume that* h *satisfies* A1, A2 *and* A3. *Let* $\langle\ \rangle_\beta$ *be a corresponding translation invariant Gibbs state. Assume further that for all test* 3-*forms* ψ *and test* 1-*forms* j, $F_a(\delta_a\psi+d_aj)$ *converges in distribution as* $a\to 0$ *with* $\beta^{-1}=ag^2$. *Then the limits* $F(\delta\psi)$ *and* $F(dj)$ *are mutually independent.*

The proofs of these three theorems are in Sect. 4.

Remark 2.4. It is expected that in dimension three $F(\delta\psi)=0$. We shall prove this for the Villain action (Theorem 2.6) and give evidence for this for the Wilson action (Theorem 8.1). Thus the second part of Theorem 2.3 should eventually prove to be vacuous.

Theorem 2.5. I. (*Moment domination in dimension* d). *Assume* h *satisfies* A1 *and that* $h''(x)\leqq\alpha$ *for some number* α *and for all* x. *Let* ϕ *be an exact* 2-*cochain of finite support on* Z^d. *Then*

$$\langle F_a(\phi)^{2n}\rangle_\beta \leqq 1.3\ldots(2n-1)(a^{d-4}\beta^{-1})^n\alpha^n(a^d\|\phi\|^2)^n. \qquad (2.6)$$

II. (*Convergence of moments in dimension 3*). *Assume* h *satisfies* A1, A2 *and* A3 *or is the Villain action* h_β. *If* j *is a* C_c^∞ 1-*form on* R^3 *and* $\beta^{-1}=ag^2$, *then*

$$\lim_{a\downarrow 0}\langle F_a(dj)^{2n}\rangle_\beta = 1.3\ldots(2n-1)g^{2n}\|dj\|^{2n}. \qquad (2.7)$$

The proof is in Sect. 5.

The preceding theorems all deal with the behavior of the lattice electric currents, $J_a=\delta F_a$. That is, they give information about $F_a(\phi)$ when ϕ is exact. The next theorem, for the Villain action, is our only result on the monopole sector for compact models. It asserts that in the continuum limit, starting with the Villain action, there are no monopoles.

Theorem 2.6. *Let* ψ *be a test* 3-*form on* R^3. *For the Villain action and for any* r *in* $[1,\infty)$.

$$\lim_{a\to 0}\langle |F_a(\delta_a\psi)|^r\rangle_{(ag^2)^{-1}}=0,$$

where $\langle\ \rangle_\beta$ *is any translation invariant Gibbs state which is a limit of finite volume Gibbs states with Dirichlet or periodic or free boundary conditions.*

The proof is given in Sect. 6.

In Sect. 7 we discuss the possibilities of replacing h' in (2.3) by some other function u. For example if $h(x)=0$ only at integer multiples of 2π, then h' can be replaced by an odd, periodic function u without altering the continuum limit if u and h' agree up to fourth order at $x=0$.

In Sect. 8 we show that in the spin wave approximation to the Wilson action there are no monopoles in the continuum limit. The spin wave approximation consists in replacing $\cos x$ by $-x^2/2$ in the Gibbs measure while continuing to use $\sin x$ to define the field variables as in (2.3). Sections 7 and 8 taken together strongly suggest that the continuum limit of the Wilson action has no monopoles.

3. Rates of Concentration of Gibbs State Mass near Minimum Energy

Lemma 3.1. *Let f be a bounded measurable function on a probability space (Ω,μ). Assume that for some real numbers b, c, γ and K*

a. $\mu(\{x:f(x)\leqq b\})\geqq\gamma>0$,
b. $\|f\|_\infty\leqq K$.

If $b<c$, then

$$(\int e^{-tf}d\mu)^{-1}(\int fe^{-tf}d\mu)\leqq c+(K/\gamma)e^{-t(c-b)}. \tag{3.1}$$

Proof. $e^{-tb}\leqq\gamma^{-1}\int\limits_{f\leqq b}e^{-tf}\leqq\gamma^{-1}\int e^{-tf}$.

Hence, if $A=\{x: f(x)<c\}$, then

$$\begin{aligned}\int fe^{-tf}&=\int\limits_A fe^{-tf}+\int\limits_{A^c}fe^{-tf}\\ &\leqq c\int\limits_A e^{-tf}+K\int\limits_{A^c}e^{-tf}\\ &\leqq c\int e^{-tf}+Ke^{-tc}\\ &\leqq c\int e^{-tf}+(K/\gamma)e^{-t(c-b)}\int e^{-tf}\end{aligned}$$

By decreasing $c-b$ in the exponent one can remove the condition that f be bounded. But this is not necessary for our purposes.

Lemma 3.2. *Let h be a continuous real-valued non-negative function on the real line which is periodic with period 2π and satisfying $h(0)=0$. Let Λ be a (large) cube in Z^d and put*

$$\mathscr{A}_\Lambda(\theta|\bar\theta)=\sum_{p\in\Lambda^{(2)}}h(d\theta(p)). \tag{3.2}$$

Here $\{\Lambda^{(j)}\}_{j=0}^d$ denotes the largest open complex with vertices in Λ. $\bar\theta_b$ denotes the value of θ_b when b lies in $\partial\Lambda^{(1)}$.

$$Z_{\Lambda,\beta}(\bar\theta)=\int\limits_{T^{\Lambda(1)}}e^{-\beta\mathscr{A}_\Lambda(\theta|\bar\theta)}d\theta,$$

where $d\theta=\prod\limits_{b\in\Lambda^{(1)}}d\theta_b$. Only those bonds enter this product whose vertics do not lie entirely in $\partial\Lambda$. However one endpoint of b may lie in $\partial\Lambda$. Let $M=\max h$. Then *there exists a real number* $\gamma > 0$ *depending on* Λ *and* h *such that for all* $\bar{\theta}$

$$\frac{1}{Z_{\Lambda,\beta}(\bar{\theta})} \int_{T^{\Lambda^{(1)}}} \mathscr{A}_\Lambda(\theta|\bar{\theta}) e^{-\beta \mathscr{A}_\Lambda(\theta|\theta)} d\theta \leqq 2Md|\partial\Lambda| + 4d^2 M|\Lambda|\gamma^{-1} e^{-\beta M d|\partial\Lambda|}. \tag{3.3}$$

Proof. Define $\hat{\theta}$ by $\hat{\theta}_b = 0$ if $b \in \Lambda^{(1)}$ and $\hat{\theta}_b = \bar{\theta}_b$ if $b \in \partial\Lambda^{(1)}$. Then $d\hat{\theta}(p) = 0$ if ∂p does not intersect $\partial\Lambda^{(1)}$. For such p, $h(d\hat{\theta}(p)) = 0$. If $p \in \Lambda^{(2)}$ and ∂p intersects $\partial\Lambda^{(1)}$ then $h(d\hat{\theta}(p)) \leqq M$. The number of such p is at most $|\partial\Lambda^{(1)}|$ (cardinality without regard to order) since at least one bond in ∂p lies in $\partial\Lambda^{(1)}$. But $|\partial\Lambda^{(1)}| \leqq (d-1)|\partial\Lambda|$. Hence,

$$\sum_{\substack{p \in \Lambda^{(2)} \\ \partial p \cap \partial\Lambda^{(1)} \neq \emptyset}} h(d\hat{\theta}(p)) \leqq M(d-1)|\partial\Lambda|. \tag{3.4}$$

Since h is continuous there is a neighbourhood U of $\hat{\theta}|\Lambda^{(1)}$ in $T^{\Lambda^{(1)}}$ such that $\sum h(d\theta(p)) \leqq M|\partial\Lambda|$ for θ in U. The last sum is over those $p \in \Lambda^{(2)}$ such that $\partial p \cap \partial\Lambda^{(1)} = \emptyset$. Let γ be the normalized $d\theta$ measure of U. Then $\gamma > 0$. Adding (3.4) to the last inequality we get

$$\sum_{p \in \Lambda^{(2)}} h(d\theta(p)) \leqq Md|\partial\Lambda| \tag{3.5}$$

uniformly in $\bar{\theta}$ for $\theta|\Lambda^{(1)}$ in U.

Apply Lemma 3.1 to $\mathscr{A}_\Lambda(\theta|\bar{\theta})$ with $b = Md|\partial\Lambda|$ and $c = 2Md|\partial\Lambda|$ to conclude that the left side of (3.3) is at most

$$2Md|\partial\Lambda| + (K/\gamma)e^{-\beta Md|\partial\Lambda|},$$

where $K = \sup \mathscr{A}_\Lambda(\theta|\bar{\theta})$, which is at most $M|\Lambda^{(2)}|$. Since each point in Λ is a corner of at most $(2d)(2d-1)$ plaquettes we have $K \leqq 4d^2 M|\Lambda|$, which establishes (3.3.).

Corollary 3.3. *Denote by* $\langle \mathscr{A}_\Lambda(\theta|\bar{\theta}) \rangle_{\Lambda,\bar{\theta},\beta}$ *the left side of inequality (3.3). Under the hypothesis of Lemma 3.2.*

$$\overline{\lim_{\beta\to\infty}} \sup_{\bar{\theta}} \langle \mathscr{A}_\Lambda(\theta|\bar{\theta}) \rangle_{\Lambda,\bar{\theta},\beta} \leqq 2Md|\partial\Lambda|. \tag{3.6}$$

Proof. Inequality (3.6) follows from (3.3).

Proposition 3.4. *Let* h *be a continuous real-valued, non-negative, even, periodic function on the line satisfying* $h(0) = 0$. *Then for any translation invariant Gibbs state* $\langle\ \rangle_\beta$ *on* Z^d *and any plaquette* p

$$\lim_{\beta\to\infty} \langle h(d\theta(p)) \rangle_\beta = 0. \tag{3.7}$$

Proof. There are $d(d-1)/2$ pairs of coordinate axes in Z^d. Pick plaquettes $p_1, \ldots, p_n$, $n = d(d-1)/2$ with one lying in each coordinate plane and with $p_1 = p$. Let Λ be a large cube. Then by translation invariance

$$\begin{aligned}\sum_{j=1}^{n} \langle h(d\theta(p_j)) \rangle_\beta &= \frac{d(d-1)}{2|\Lambda^{(2)}|} \left\langle \sum_{p \in \Lambda^{(2)}} h(d\theta(p)) \right\rangle_\beta \\ &= \frac{d(d-1)}{2|\Lambda^{(2)}|} \langle \langle \mathscr{A}_\Lambda(\theta|\bar{\theta}) \rangle_{\Lambda,\theta,\beta} \rangle_\beta .\end{aligned}$$

The last equality follows from the DLR equations [12, 15, 17]. By (3.6) we have

$$\overline{\lim_{\beta\to\infty}} \sum_{j=1}^{n} \langle h(d\theta(p_j))\rangle_\beta \leqq Md^2(d-1)|\partial\Lambda|/|\Lambda^{(2)}|. \tag{3.8}$$

But $|\Lambda^{(2)}|$ is proportional to $|\Lambda|$, so that $|\partial\Lambda|/|\Lambda^{(2)}|\to 0$ as $|\Lambda|\to\infty$. Since Λ is arbitrary the left side of (3.8) is zero. Each summand on the left of (3.8) is nonnegative. So (3.7) follows.

In order to get more detailed information about the rate at which the internal energy per site converges toward its minimum value we shall study the distribution of a single plaquette variable, $d\theta(p)$, when the function $\phi_\beta(x)=\exp(-\beta h(x))$ is positive definite.

Lemma 3.5. *Let Λ be a cube in Z^d and let ϕ be a positive definite even periodic function of period 2π on $(-\infty,\infty)$. Denote by $\{\Lambda^{(j)}\}$ the complex for periodic, free, or Dirichlet boundary conditions. Let p be a plaquette in the interior of $\Lambda^{(2)}$ and put*

$$u(\theta_{\partial p}) = \int \prod_{\substack{b\in\Lambda^{(1)}\\ b\notin\partial p}} d\theta_b \prod_{\substack{q\in\Lambda^{(2)}\\ q\neq p}} \phi(d\theta(q)), \tag{3.9}$$

where $\theta_{\partial p}$ denotes the four bond variables $\{\theta_b\}_{b\in\partial p}$. Then u depends only on $d\theta(p)$; $u(\theta_{\partial p})=Z(d\theta(p))$, where Z is periodic and positive definite.

Proof. Although the fact that g depends only on $d\theta(p)$ follows easily from gauge invariance (see Remark 3.1) we shall derive it from a Fourier representation of g which we need anyway.

We may write

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad a_n \geqq 0.$$

Substituting into (3.9) gives

$$u(\theta_{\partial p}) = \int \prod_{b\notin\partial p} d\theta_b \sum_{n(\cdot)} \left(\prod_{q\neq p} a_{n(q)}\right) \exp\left[i \sum_{q\neq p} n(q)\, d\theta(q)\right],$$

where $\sum_{n(\cdot)}$ is a sum over all integer-valued functions $n(.)$ on $\Lambda^{(2)}-\{p\}$. We note by the way that the closed complex $\{\Lambda^{(j)}\}_{j=1}^{d}$ built on Λ corresponds to free boundary conditions. If we define $n(p)=0$, then the sum in the exponential factor can be extended over all q in $\Lambda^{(2)}$ and then rewritten as $\sum_{b\in\Lambda^{(1)}} \theta_b(\delta n)(b)$. A θ_b integral therefore gives zero unless $(\delta n)(b)=0$. Carrying out all the indicated integrations yields

$$u(\theta_{\partial p}) = \sum_{\substack{\delta n(b)=0\\ b\notin\partial\pm p}} (2\pi)^{|\Lambda^{(1)}|-4} \left(\prod_{q\neq p} a_{n(q)}\right) \exp\left[i \sum_{b\in\partial p} \theta_b(\delta n)(b)\right].$$

Now let j be a vertex of p and let b_1 and b_2 be in ∂p with b_1 pointing toward j and b_2 pointing away from j. If $(\delta n)(b)=0$ for all $b\notin\pm\partial p$ then

$$0=(\delta^2 n)(j) = \sum_{b:\partial b\ni j} \delta n(b) = (\delta n)(b_1)-(\delta n)(b_2).$$

Hence $m \equiv (\delta n)(b)$ is the same for all $b \in \partial p$. Using $\sum_{b \in \partial p} \theta_b m = m\, d\theta(p)$ gives finally

$$u(\theta_{\partial p}) = \sum_{m=-\infty}^{\infty} e^{imd\theta(p)} \sum_{\substack{(\delta n)(b)=0, b \notin \partial p \\ (\delta n)(b)=m, b \in \partial p}} (2\pi)^{|\Lambda^{(1)}|} \prod_{q \neq p} a_n(q).$$

Thus the Fourier coefficients of $Z(d\theta(p))$ are nonnegative.

Remark 3.1. The fact that u depends only on $d\theta(p)$ follows immediately from gauge invariance, and in the nonabelian case as well. For if ϕ is a central function on a compact group G, and if 1, 2, 3, 4 labels the vertices of p in the order of the given orientation, then the gauge transformation $g_{xy} \to \gamma(x) g_{xy} \gamma(y)^{-1}$ takes $u(g_{\partial p})$ to $u(1, 1, 1, dg(p))$ if $\gamma(x) = 1$ for all x not in $\{2, 3, 4\}$ and $\gamma(j+1) = (g_{j,j-1} \cdots g_{2,1})^{-1}$ if $j = 2, 3$ or 4, while at the same time translation invariance of the Haar measure shows that u doesn't change.

Remark 3.2. One can prove Lemma 3.5 without using the Fourier representation of ϕ as follows. Using the definition of positive definiteness; $\sum c_i \bar{c}_j \phi(\theta_i - \theta_j) \geqq 0$ for all $c_1, \ldots, c_n, \theta_1, \ldots, \theta_n$, one sees easily that $\phi(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ is positive definite on $U(1)^4$ and that $\phi(d\theta(q))$ is therefore positive definite on $U(1)^{\Lambda^{(1)}}$ for each q. Hence the integrand in (3.9) is positive definite. But if $v(g, k)$ is a positive definite function on a product, $G \times K$ of compact groups then $g \to \int_K v(g, k) dk$ is also positive definite if dk is Haar measure on K. By Remark 3.1 we may put $\theta_b = 0$ for three bonds $b \in \partial p$ to see that Z is positive definite.

Theorem 3.6. *Assume that for each $\beta > 0$, h_β satisfies* A1 *and* A2. *Assume further that for some real number α and all $\beta > 0$:*

A4. $d^2 h_\beta(x)/dx^2 \leqq \alpha$ *for all* $x \in R$,

and

A5. $e^{-\beta h_\beta}$ *is positive definite.*

Let $\langle \ \rangle_\beta$ be an infinite volume Gibbs state on Z^d which is a limit of finite volume Gibbs states with Dirichlet or periodic or free boundary conditions with action $\mathscr{A} = \beta \sum_q h_\beta(d\theta(q))$. Then for any plaquette p and $c > 0$

$$\mathrm{prob}_\beta(h_\beta(d\theta(p)) \geqq c) \leqq 2\pi(\alpha\beta d)^{1/2} \exp(-\beta c) \tag{3.10}$$

for sufficiently large β and

$$\overline{\lim_{\beta \to \infty}} \beta^n \langle [h'_\beta(d\theta(p))]^{2n} \rangle_\beta \leqq \alpha^n 1.3 \ldots (2n-1) \overline{\lim_{\beta \to \infty}} (\alpha\beta d)^{1/2} \int_{-\pi}^{\pi} \exp[-\beta h_\beta(x)] dx. \tag{3.11}$$

The right side of the inequality is finite if h_β is the Villain action or is independent of β and satisfies A3. *The Villain action satisfies* A1, A2, A4, A5 *with $\alpha = 1$.*

Lemma 3.7. *Put $\phi(x) = \exp(-\beta h_\beta(x))$ in Lemma 3.5 and write $Z_\beta(x)$ for $Z(x)$. Then under the hypotheses of the Theorem*

$$Z_\beta(0) \exp[-\beta(2d-1)\alpha x^2/2] \leqq Z_\beta(x) \leqq Z_\beta(0) \quad \text{for} \quad |x| \leqq \pi. \tag{3.12}$$

Proof. The second inequality in (3.12) follows from Lemma 3.5 because a real-valued positive definite function achieves its maximum at zero. This is the only place where A5 is used.

To prove the first inequality in (3.12) fix β and choose a bond b in ∂p. If we put $\theta_b = x$ and $\theta_{b'} = 0$ for each of the other three bonds $b' \in \partial p$ then

$$Z_\beta(x) = \int \exp\left[-\beta \sum_{\substack{q\in\Lambda^{(2)}\\ q\neq p}} h_\beta(d\theta(q)) \right] d\theta, \tag{3.13}$$

where $d\theta = \prod_{b'\in\Lambda^{(1)}, b'\notin\partial p} d\theta_{b'}$. Exactly $2d-1$ of the summands on the right side of (3.13) depend on x and we may suppose that the orientations of these plaquettes q agree with b. Put $f(x) = -\beta^{-1}\log Z_\beta(x)$. Then

$$\begin{aligned} f'(x) &= \frac{1}{Z_\beta(x)} \int (\delta h')(b) \exp\left[-\beta \sum h_\beta(d\theta(q))\right] d\theta \\ &\equiv \langle (\delta h')(b) \rangle_x, \end{aligned}$$

where $(\delta h')(b) = \sum' h'_\beta(d\theta(q))$ and the sum $\sum'$ extends over those q with $b \in \partial q$ but excluding $q = p$. Since h is even and $h'(\theta)$ is odd one sees that $f'(0) = 0$. Moreover the second derivative of f is

$$f''(x) = \langle \delta h''(b) \rangle_x - \beta\{\langle [(\delta h')(b)]^2 \rangle_x - \langle (\delta h')(b) \rangle_x^2\}.$$

The coefficient of β is positive by Schwarz's inequality. Hence $f''(x) \leqq \langle \delta h'')(b) \rangle_x$. But $(\delta h'')(b) = \sum' h''(d\theta(q)) \leqq (2d-1)\alpha$ for all x and θ. Thus $f''(x) \leqq (2d-1)\alpha$. Since $f'(0) = 0$ we have $f(x) \leqq f(0) + (2d-1)\alpha x^2/2$. Hence $Z_\beta(x) \geqq Z_\beta(0) \times \exp[-\beta(2d-1)\alpha x^2/2]$.

Proof of Theorem 3.6. Using the notation of the previous two lemmas put

$$Z(\beta) = \int_{-\pi}^{\pi} \exp[-\beta h_\beta(x)] Z_\beta(x)\, dx.$$

Then for any bounded periodic function v the finite volume Gibbs state expectation of $v(d\theta(p))$ is given by

$$\langle v(d\theta(p)) \rangle_{\Lambda,\beta} = Z(\beta)^{-1} \int_{-\pi}^{\pi} v(x) \exp[-\beta h_\beta(x)] Z_\beta(x)\, dx. \tag{3.14}$$

The four integrals over the bond variables θ_b, $b \in \partial p$ have been replaced by an integral over one of them—denoted x—because both the density $u(\theta_{\partial p})$ in Lemma 3.5 and the integrand $v(d\theta(p))$ depend periodically on $d\theta(p)$ only. A factor $(2\pi)^3$ cancels.

Now $h_\beta(0) = 0$ and since $x = 0$ is a local minimum of $h_\beta(x)$ we also have $h'_\beta(0) = 0$. Thus, since $h''_\beta(x) \leqq \alpha$ for all x, we have $h_\beta(x) \leqq \alpha x^2/2$ for $|x| \leqq \pi$. By the first inequality in (3.12) we therefore have

$$\exp[-\beta h_\beta(x)] Z_\beta(x) \geqq \exp[-\beta(2d)\alpha x^2/2] Z_\beta(0).$$

Hence

$$Z(\beta) \geqq Z_\beta(0) \int_{-\pi}^{\pi} \exp[-\beta(2d)\alpha x^2/2]\, dx. \tag{3.15}$$

But

$$\int_{-\pi}^{\pi} e^{-tx^2/2}\,dx = (2\pi/t)^{1/2} - 2\int_{\pi}^{\infty} e^{-tx^2/2}\,dx$$

$$\geqq (2\pi/t)^{1/2} - (2/\pi)\int_{\pi}^{\infty} xe^{-tx^2/2}\,dx$$

$$= (2\pi/t)^{1/2} - (2/\pi t)e^{-t\pi^2/2} \geqq (2/t)^{1/2}$$

for sufficiently large t, say $t \geqq 2t_0$. Therefore

$$Z(\beta) \geqq Z_\beta(0)(\alpha\beta d)^{-1/2} \tag{3.16}$$

for $\alpha\beta d \geqq t_0$.

To prove (3.10) put $v(x) = 1$ if $h_\beta(x) \geqq c$ and zero otherwise. By Lemma 3.6 $Z_\beta(x) \leqq Z_\beta(0)$. So by (3.14)

$$\langle v(d\theta(p))\rangle_{A,\beta} \leqq Z(\beta)^{-1} Z_\beta(0) \int_{-\pi}^{\pi} v(x)\exp[-\beta h_\beta(x)]\,dx$$

$$\leqq (\alpha\beta d)^{1/2} 2\pi \exp[-\beta c]$$

if $\alpha\beta d \geqq t_0$.

To prove (3.11) note that by an integration by parts

$$\int_{-\pi}^{\pi} h_\beta'(x)^{2n} e^{-\beta h_\beta(x)}\,dx = (2n-1)\beta^{-1} \int_{-\pi}^{\pi} h_\beta''(x) h_\beta'(x)^{2n-2} e^{-\beta h_\beta(x)}\,dx$$

$$\leqq \alpha(2n-1)\beta^{-1} \int_{-\pi}^{\pi} h_\beta'(x)^{2n-2} e^{-\beta h_\beta(x)}\,dx.$$

Boundary terms cancel by periodicity. By induction

$$\int_{-\pi}^{\pi} h_\beta'(x)^{2n} e^{-\beta h_\beta(x)}\,dx \leqq \alpha^n \beta^{-n} 1.3\ldots(2n-1) \int_{-\pi}^{\pi} (2n-1) \int_{-\pi}^{\pi} e^{-\beta h_\beta(x)}\,dx. \tag{3.17}$$

Hence by (3.14) and Lemma 3.6

$$\beta^n \langle [h_\beta'(d\theta(p))]^{2n}\rangle_{A,\beta} \leqq Z(\beta)^{-1} Z_\beta(0) \int_{-\pi}^{\pi} h_\beta'(x)^{2n} \exp[-\beta h_\beta(x)]\,dx$$

$$\leqq Z(\beta)^{-1} Z_\beta(0) \int_{-\pi}^{\pi} e^{-\beta h_\beta(x)}\,dx\, \alpha^n 1.3\ldots(2n-1).$$

But $Z(\beta)^{-1} Z_\beta(0) \leqq (\alpha\beta d)^{1/2}$ for $\alpha\beta d \geqq t_0$, where t_0 is independent of Λ. Thus we may let $\Lambda \uparrow Z^d$ and then let $\beta \to \infty$ to get (3.11).

It remains only to prove that the right side of (3.11) is finite when h_β is independent of β or is the Villain action.

Lemma 3.8. *Assume that the function h_β in Theorem 3.6 does not depend on β and satisfies* A1, A2 *and* A3. *Then the right side of (3.11) is finite.*

Proof. h can have only a finite number of zeros in $[-\pi, \pi]$ because at any accumulation point h'' must be zero, contradicting A3. If $h(x_0) = 0$, then by A1 and

A3 there is a number $\varepsilon > 0$ such that $h''(x) \geqq \frac{1}{2}$ when $|x - x_0| \leqq \varepsilon$. Thus $h(x) \geqq (x - x_0)^2/4$ for $|x - x_0| < \varepsilon$ and therefore

$$\beta^{1/2} \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-\beta h(x)}\, dx \leqq \beta^{1/2} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \exp[-\beta(x - x_0)^2/4]\, dx \leqq 2\pi^{1/2}.$$

If there are n zeros in $(-\pi, \pi]$ then the lim sup on the right of (3.11) is therefore at most $2n\pi^{1/2}(\alpha d)^{1/2}$, since the integral of $e^{-\beta h}$ over any closed interval not containing a zero of h goes to zero exponentially as $\beta \to \infty$.

Lemma 3.9. (*Villain action.*) *Define* h_β *as in* (2.1). *Then* h_β *satisfies* A1, A2, A4 *and* A5 *with* $\alpha = 1$. *Moreover* $h_\beta(x) > 0$ *unless* x *is an integer multiple of* 2π. $h'_\beta(x)$ *is uniformly bounded in* x *and* β *for* $\beta \geqq 1$. *If* $0 < \gamma < \pi$, *then*

$$\sup\{|h'_\beta(x) - x| : |x| \leqq \gamma\} = 0(e^{-\beta\zeta}) \quad as \quad \beta \to \infty \tag{3.18}$$

and

$$\sup\{|h''_\beta(x) - 1| : |x| \leqq \gamma\} = 0(\beta e^{-\beta\zeta}) \quad as \quad \beta \to \infty, \tag{3.19}$$

where $\zeta = (2\pi - \gamma)^2 - \gamma^2$. *Also*

$$\sup\{|h''_\beta(x)| : x \in R\} = 0(\beta) \quad as \quad \beta \to \infty. \tag{3.20}$$

The right side of (3.11) *is finite.*

Proof. It is clear from the definition (2.1) that h_β is even, periodic with period 2π, infinitely differentiable, and that $h_\beta(0) = 0$. It is well known that $e^{-\beta h_\beta(\cdot)}$ is positive definite (see e.g. [14]), from which it follows that $\exp[-\beta h_\beta(x)]$ has a maximum at $x = 0$. If f is a function on the integers, put

$$\langle f \rangle_{x,\beta} = Z(x,\beta)^{-1} \sum_{n=-\infty}^{\infty} f(n) \exp[-\beta(x - 2\pi n)^2/2]$$

with $Z(x,\beta)$ chosen so that $\langle 1 \rangle_{x,\beta} = 1$. Then since

$$h_\beta(x) = -\beta^{-1} \log\left[c_\beta \sum_{n=-\infty}^{\infty} \exp\{-\beta(x - 2\pi n)^2/2\}\right],$$

one computes easily the first and second derivative of h_β to find

$$h'_\beta(x) = \langle x - 2\pi n \rangle_{x,\beta}, \tag{3.21}$$

and

$$h''_\beta(x) = 1 - \beta\{\langle (x - 2\pi n)^2 \rangle_{x,\beta} - \langle x - 2\pi n \rangle_{x,\beta}^2\}. \tag{3.22}$$

Note first that the expression in braces, $\{\ \}$, in (3.22) is nonnegative by by Schwarz's inequality. Hence $h''_\beta(x) \leqq 1$ for all x. Now we may rewrite (3.21) and (3.22) as $h'_\beta(x) - x = 2\pi\langle n \rangle_{x,\beta}$ and

$$h''_\beta(x) - 1 = -4\pi^2\beta\{\langle n^2 \rangle_{x,\beta} - \langle n \rangle_{x,\beta}^2\}.$$

For $0 \leqq x \leqq \pi$ the behavior of $Z(x,\beta)$ as $\beta \to \infty$ is determined by the term $n = 0$ in the rapidly convergent series expression for it: $Z(x,\beta) = \sum_{n=-\infty}^{\infty} \exp[-\beta(x - 2\pi n)^2/2]$. Consequently $\langle n \rangle_{x,\beta} = 0(e^{-\beta\zeta})$ uniformly for $|x| \leqq \gamma < \pi$

and the same behavior is valid for $\langle n^2\rangle_{x,\beta}$. Equations (3.18) and (3.19) follow from this. Since h_β is periodic and even it suffices for proving (3.20) to prove $\langle n^2\rangle_{x,\beta}$ and $\langle n\rangle^2_{x,\beta}$ are bounded uniformly for x in $[0,\pi]$ and uniformly in β for $\beta \geqq 1$. But

$$\langle n^2\rangle_{x,\beta} \leqq \left[\sum_{n=0}^{1} e^{-\beta(x-2\pi n)^2/2}\right]^{-1}\left[0+e^{-\beta(x-2\pi)^2/2}+e^{-\beta(x+2\pi)^2/2}\right.$$
$$\left.+2\sum_{n>2} n^2 e^{-\beta\pi^2(2n-1)^2/2}\right].$$

In the second bracket we have separated the terms $n=0$, $n=1$ and $n=-1$ and estimated the rest of the series. Looking at the three non-zero terms separately one sees that each remains bounded after division by the indicated estimate of $Z(x,\beta)$, uniformly for x in $[0,\pi]$ and uniformly in β for $\beta \geqq 1$. (It may be illuminating to note also that $\lim_{\beta\to\infty}\langle n^2\rangle_{\pi,\beta} = \lim_{\beta\to\infty}\langle n\rangle_{\pi,\beta} = \frac{1}{2}$.)

Finally, to see that the right side of (3.11) is finite note that

$$\int_{-\pi}^{\pi} \exp[-\beta h_\beta(x)]\,dx = c_\beta \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} \exp[-\beta(x-2\pi n)^2/2]\,dx$$
$$= c_\beta \int_{\infty}^{\infty} \exp[-\beta x^2/2]\,dx = c_\beta(2\pi/\beta)^{1/2},$$

while $c_\beta^{-1} = \sum_{n=-\infty}^{\infty} \exp[-\beta(2\pi n)^2/2]$ which converges to one as $\beta\to\infty$.

Remark 3.10. For the Villain action $h_\beta(x)$ converges uniformly in x to the periodic extension of $x^2/2$ on $(-\pi,\pi]$. This follows directly from the definition (2.1) as well as from (3.18). For example, factoring $\exp[-\beta x^2/2]$ out of (2.1) and taking logs yields

$$h_\beta(x)-x^2/2 = -\beta^{-1}\log c_\beta - \beta^{-1}\log\left[1+\sum_{n\neq 0}\exp[-\beta((x-2\pi n)^2-x^2)/2]\right].$$

But $c_\beta \to 1$ as $\beta\to\infty$ and the argument of the log function is easily seen to be bounded on $[-\pi,\pi]$ by its value at $x=\pi$.

Remark 3.11. Information about decay of $\langle h(d\theta(p))\rangle_\beta$ as $\beta\to\infty$ can also be gotten from correlation inequalities when these are applicable. To illustrate, consider the Wilson action $h(x)=1-\cos x$. De Angelis'et al. [4–7] have pointed out that Griffiths inequalities apply to U(1) gauge models. Let

$$\langle f\rangle_{s,t} = Z(s,t)^{-1}\int_{T^{A(1)}} d\theta\, f(\theta)\exp\left[s\cos d\theta(p) + \sum_{\substack{q\in A^{(2)}\\ q\neq p}} t\cos d\theta(q)\right].$$

One may conclude from Griffith's inequalities that $\langle \cos d\theta(p)\rangle_{s,t}$ is an increasing function of s and t for $s \geqq 0$ and $t \geqq 0$. Hence

$$\langle \cos d\theta(p)\rangle_{\beta,0} \leqq \langle \cos d\theta(p)\rangle_{\beta,\beta}.$$

Thus, $\langle 1-\cos d\theta(p)\rangle_{\beta,\beta} \leqq \langle 1-\cos d\theta(p)\rangle_{\beta,0}$. But the last expectation reduces to a one dimensional integral which the reader can verify is $0(\beta^{-1})$. Therefore, for the infinite volume limit,

$$\langle 1-\cos d\theta(p)\rangle_\beta = 0(\beta^{-1}). \tag{3.23}$$

After this manuscript was submitted Jean Bricmont kindly informed me that inequality (3.10) can also be deduced from correlation inequalities, as well as from reflection positivity, by the methods of [23].

4. Proofs of Theorems 2.1, 2.2, and 2.3

Lemma 4.1. *(Schwinger–Dyson Equations). Assume h satisfies* A1. *Let* $\langle\ \rangle_\beta$ *be any infinite volume Gibbs state for the action*

$$\mathscr{A} = \beta \sum_p h(d\theta(p)). \tag{4.1}$$

The sum is over plaquettes, p, in Z^d. *Let v be a differentiable periodic function of finitely many bond variables. Then for any bond b*

$$\langle \partial v/\partial\theta_b\rangle_\beta = \beta\langle v(\delta h')(b)\rangle_\beta, \tag{4.2}$$

where

$$(\delta h')(b) = \sum_{p:\partial p \ni b} h'(d\theta(p)). \tag{4.3}$$

Proof. v depends on the bond variables $\theta_{b'}$ for b' in some finite set $S \subset (Z^d)^{(1)}$. Let Λ be a large cube such that S is contained in the open complex $\Lambda^{(1)}$, and such that the given bond b is also in $\Lambda^{(1)}$. Put $\mathscr{A}_\Lambda(\theta) = \beta \sum_{p\in\Lambda^{(2)}} h(d\theta(p))$ and fix arbitrary boundary values $\bar{\theta}_{b'}$ on the boundary of $\Lambda^{(1)}$. Then by an integration by parts

$$\begin{aligned}\int d\theta\, e^{-\mathscr{A}_\Lambda(\theta)}\partial v/\partial\theta_b &= \beta \sum_{\substack{p\in\Lambda^{(2)}\\ \partial p\ni b}} \int d\theta\, h'(d\theta(p))e^{-\mathscr{A}_\Lambda(\theta)}v(\theta)\\ &= \beta\int d\theta(\delta h')(b)e^{-\mathscr{A}_\Lambda(\theta)}v(\theta).\end{aligned}$$

Here $d\theta = \prod_{b'\in\Lambda^{(1)}} d\theta_{b'}$. The boundary terms cancel by periodicity. Division by the normalization constant $Z(\bar{\theta})$ yields the identity (4.2) for the finite volume Gibbs state in Λ with arbitrary boundary conditions. Equation (4.2) now follows from the DLR equations.

Lemma 4.2. *Assume h satisfies* A1, A2, A3. *Let* $\langle\ \rangle_\beta$ *be a translation invariant infinite volume Gibbs state for the action (4.1). Then for any plaquette p in* Z^d *and* $1 \leqq r < \infty$

$$\lim_{\beta\to\infty} \langle |1-h''(d\theta(p))|^r\rangle_\beta = 0.$$

This holds also if h is replaced throughout by the Villain action h_β.

Proof. As noted in the proof of Lemma 3.8, h can have only finitely many zeros in the periodicity interval $(-\pi,\pi]$. It follows from A1, A2 and A3 that for any number $\varepsilon>0$ there is a number $\delta>0$ such that $|1-h''(x)|<\varepsilon$ whenever $h(x)<\delta$.

Hence, if $M \geqq 1 + \max |h''(x)|$, then

$$\langle |1 - h''(d\theta(p))|^r \rangle_\beta \leqq \varepsilon^r \operatorname{Prob}(h(d\theta(p)) < \delta) + M^r \operatorname{Prob}(h(d\theta(p)) \geqq \delta)$$
$$\leqq \varepsilon^r + M^r \delta^{-1} \langle h(d\theta(p)) \rangle_\beta .$$

So by Proposition 3.4 $\lim \sup_{\beta \to \infty} \langle |1 - h''(d\theta(p))|^r \rangle_\beta \leqq \varepsilon^r$. The first part of the Lemma follows from the arbitrariness of ε.

For the Villain action $|h''_\beta|$ is not uniformly bounded, so the previous proof must be modified slightly. By remark 3.10 there is a number $c > 0$ such that for all large β, $h_\beta(x) < c$ only if x is within $\pi/2$ of some integer multiple of 2π. Thus by (3.19) there is a number β_0 such that $|1 - h''_\beta(x)| < \varepsilon$ if $h_\beta(x) < c$ and $\beta > \beta_0$. As before

$$\langle |1 - h''_\beta(d\theta(p))|^r \rangle_\beta \leqq \varepsilon^r + \left(1 + \sup_x |h''_\beta(x)|\right)^r \operatorname{Prob}(h_\beta(d\theta(p)) \geqq c)$$
$$\leqq \varepsilon^r + 0(\beta^r \beta^{1/2} e^{-\beta c})$$

by (3.20) and (3.10). We may now take $\lim \sup_{\beta \to \infty}$ as before.

Proof of Theorem 2.1. Although our main result is only for dimension 3 we keep track of the dimension dependence of powers of the lattice spacing, a, for Theorem 2.2. and for other anticipated future use. Thus we consider a test 1-form j on R^d and put $\phi_a(p) = (d_a j)(p)$ for the two co-chain defined in Sect. 2. Put

$$F_a(\theta) = a^d \sum_p \phi_a(p) a^{-2} h'(d\theta(p)). \tag{4.4}$$

The sum runs over all plaquettes in $(aZ^d)^{(2)}$, but it is a finite sum for each a because j has compact support. Define, for $\beta > 0$.

$$u_a(s) = \langle e^{isF_a(\theta)} \rangle_\beta . \tag{4.5}$$

Then

$$\begin{aligned} du_a(s)/ds &= i \langle F_a e^{isF_a} \rangle_\beta \\ &= i a^d \sum_p \phi_a(p) a^{-2} \langle h'(d\theta(p)) e^{isF_a} \rangle_\beta \\ &= i a^{d-3} \sum_b j(b) \langle (\delta h')(b) e^{isF_a} \rangle_\beta \\ &= i a^{d-3} \sum_b j(b) \beta^{-1} \langle \partial e^{isF_a} / \partial \theta_b \rangle_\beta \\ &= i a^{d-3} \sum_b j(b) \beta^{-1} is \langle a^{d-2} \sum_{p:\partial p \ni b} \phi_a(p) h''(d\theta(p)) e^{isF_a} \rangle_\beta \\ &= i^2 s a^{2d-5} \beta^{-1} \sum_b j(b) \sum_{p:\partial p \ni b} \phi_a(p) \langle h''(d\theta(p)) e^{isF_a} \rangle_\beta \\ &= i^2 s a^{2d-5} \beta^{-1} \sum_p \sum_{b \in \partial p} j(b) \phi_a(p) \langle h''(d\theta(p)) e^{isF_a} \rangle_\beta \\ &= i^2 s a^{2d-4} \beta^{-1} \sum_p \phi_a(p)^2 \langle h''(d\theta(p)) e^{is \cdot F_a} \rangle_\beta . \end{aligned} \tag{4.6}$$

In the fourth line Lemma 4.1 was used and in the last line we used $a^{-1}\sum_{b\in\partial p} j(b) = \phi_a(p)$. Now we put $d=3$ and $\beta^{-1}=ag^2$ to get

$$du_a/ds = -sg^2a^3\sum_p \phi_a(p)^2\langle h''(d\theta(p))e^{isF_a}\rangle_\beta.$$

Put $\phi = dj$. Then

$$\begin{aligned} d(u_a(s)e^{s^2g^2\|\phi\|^2_{L^2}/2})/ds &= sg^2\left(\|\phi\|^2_{L^2} - a^3\sum_p \phi_a(p)^2\right)u_a(s) \\ &\quad + sg^2a^3\sum_p \phi_a(p)^2\langle(1-h''(d\theta(p)))e^{isF_a}\rangle_\beta \end{aligned} \tag{4.7}$$

Now $|u_a(s)| \leqq 1$. So the first term on the right of (4.7) goes to zero as $a\to 0$, uniformly for s in any bounded interval. Moreover,

$$|\langle(1-h''(d\theta(p)))e^{isF_a}\rangle_\beta| \leqq \langle|1-h''(d\theta(p)))|\rangle_\beta$$

goes to zero uniformly in s and uniformly in p as $a\to 0$ by Lemma 4.2, since $\beta\to\infty$. Since $a^3\sum_p\phi_a(p)^2$ is bounded the second term on the right of (4.7) also goes to zero as $a\to 0$ and does so uniformly on bounded s intervals. Hence the derivative of $u_a(s)e^{s^2g^2\|\phi\|^2_{L^2}/2}$ converges to zero uniformly on bounded s intervals as a approaches zero. But for all a this function is equal to one at $s=0$. It follows that $u_a(s)e^{s^2g^2\|\phi\|^2_{L^2}/2}$ converges to one as a approaches zero. Put $s=1$ to get Theorem 2.1.

For the Villain action replace h by h_β throughout the proof. This concludes the proof of Theorem 2.1.

Proof of Theorem 2.2. We use the same notation as in the proof of Theorem 2.1 except that we replace ϕ_a by an exact 2-cochain ϕ (i.e., $\phi = dj$ for some 1-cochain j) and we now set $u_a(s) = \langle e^{sF_a(\theta)}\rangle_\beta$. The equations (4.6) for the derivative of u_a are the same except for the absence of the factor i^2. Thus

$$du_a(s)/ds = sa^{2d-4}\beta^{-1}\sum_p \phi(p)^2\langle h''(d\theta(p))e^{sF_a}\rangle_\beta.$$

e^{sF_a} is a bounded positive function for fixed a. Using $h''(x)\leqq\alpha$ for all x we get

$$du_a(s)/ds \leqq s(\alpha a^{d-4}\beta^{-1})\left(a^d\sum_p \phi(p)^2\right)u_a(s).$$

Since $u_a(0)=1$ it follows (e.g., divide by u_a and integrate) that

$$u_a(s) \leqq \exp[s^2(\alpha a^{d-4}\beta^{-1})a^d\|\phi\|^2_{l^2}/2].$$

Put $s=1$ to get (2.5). For the Villain action replace h by h_β as before and use $h''_\beta(x)\leqq 1$ from Lemma 3.9.

Proof of Theorem 2.3. If ψ is a 3-cochain on Z^d and j is a 1-cochain on Z^d, both finitely supported, then

$$\begin{aligned} \langle F_a(\delta\psi)F_a(dj)\rangle_\beta &= a^{2d-4}\sum_{p,q}(\delta\psi)(p)(dj)(q)\langle h'(d\theta(p))h'(d\theta(q))\rangle_\beta \\ &= a^{2d-4}\sum_p(\delta\psi)(p)\sum_b j(b)\langle h'(d\theta(p))(\delta h')(b)\rangle_\beta \\ &= a^{2d-4}\sum_p(\delta\psi)(p)\sum_b j(b)\langle\partial h'(d\theta(p))/\partial\theta_b\rangle_\beta. \end{aligned}$$

In the last line we have used the Schwinger–Dyson equations. For a fixed plaquette p $\partial h'(d\theta(p))/\partial\theta_b$ is zero if $\pm b$ is not in ∂p. In the sum over b we may use those orientations which are consistent with p. In this case the integrand in the last line is $h''(d\theta(p))$. But $\langle h''(d\theta(p))\rangle_\beta$ is independent of p by the assumed translation and rotation invariance. Write $\alpha_1 = \langle h''(d\theta(p))\rangle_\beta$. Then

$$\begin{aligned}\langle F_a(\delta\psi)F_a(dj)\rangle_\beta &= a^{2d-4}\alpha_1 \sum_p (\delta\psi)(p) \sum_{b\in\partial p} j(b)\\ &= a^{2d-4}\alpha_1(\delta\psi, dj)\\ &= a^{2d-4}\alpha_1(\psi, d^2 j)\\ &= 0.\end{aligned}$$

For the second part of the theorem we take ψ to be a test 3-form on R^3 and j to be a test 1-form on R^3. Let

$$v_a(s) = \langle \exp[iF_a(sd_a j + \delta_a \psi)]\rangle_\beta$$

with $\beta^{-1} = ag^2$, for $a > 0$. By assumption $v_a(s)$ converges as $a \to 0$ for each s. Denote the limit by $v(s)$. In order to prove the asserted independence of the limits $F(dj)$ and $F(\delta\psi)$ it suffices to prove that

$$\langle e^{iF(dj+\delta\psi)}\rangle = \langle e^{iF(dj)}\rangle\langle e^{iF(\delta\psi)}\rangle$$

for all such j and ψ. But the first factor on the right is known from Theorem 2.1 and the second factor is $v(0)$. The right side is thus $\omega(1)$ if $\omega(s)$ is given by

$$\omega(s) = \exp[-s^2 g^2 \|dj\|^2/2]\cdot v(0).$$

Now $\omega(s)$ is the unique solution of the differential equation

$$d\omega/ds = -sg^2\|dj\|^2\omega(s), \omega(0) = v(0). \tag{4.8}$$

It suffices therefore to show that $v(s)$ also satisfies this equation. Put $\phi = sd_a j + \delta_a\psi$ and put $G = F_a(\phi)$. Then ($d = 3$ below).

$$\begin{aligned}dv_a(s)/ds &= i\langle F_a(d_a j)e^{iG}\rangle_\beta\\ &= ia^d \sum_p (d_a j)(p)\langle a^{-2}h'(d\theta(p))e^{iG}\rangle_\beta\\ &= ia^{d-3}\sum_b j(b)\langle(\delta h')(b)e^{iG}\rangle_\beta\\ &= ia^{d-3}\sum_b j(b)\beta^{-1}\langle \partial e^{iG}/\partial\theta_b\rangle_\beta\\ &= i^2 a^{d-3}\sum_b j(b)\beta^{-1}a^d \sum_p \phi(p)a^{-2}\langle(\partial h'(d\theta(p))/\partial\theta_b)e^{iG}\rangle_\beta.\end{aligned}$$

As noted in the first part of this proof $\partial h'(d\theta(p))/\partial\theta_b$ is 0 or $\pm h''(d\theta(p))$. Hence

$$\begin{aligned}dv_a(s)/ds &= i^2 a^{2d-5}\beta^{-1}\sum_b j(b) \sum_{p:\partial p\ni b} \phi(p)\langle h''(d\theta(p))e^{iG}\rangle_\beta\\ &= -a^{2d-4}\beta^{-1}\sum_p a^{-1}\sum_{b\in\partial p} j(b)\phi(p)\langle h''(d\theta(p))e^{iG}\rangle_\beta\\ &= -g^2 a^d \sum_p \phi(p)(d_a j)(p)\langle h''(d\theta(p))e^{iG}\rangle_\beta. \end{aligned} \tag{4.9}$$

Now

$$\langle h''(d\theta(p))e^{iG}\rangle_\beta = v_a(s) + \langle (h''(d\theta(p)) - 1)e^{iG}\rangle_\beta.$$

The second term converges to zero uniformly in p as $a \to 0$, as we saw in the proof of Theorem 2.1. Moreover the 2-cochains ϕ and $d_a j$ are bounded in l^2 norm uniformly in a as $a \to 0$ and uniformly for s in a bounded interval. Further,

$$\begin{aligned} a^d \sum_p \phi(p)(d_a j)(p) &= a^d s \sum_p (d_a j)(p)^2 \\ &\quad + a^d \sum_p (\delta_a \psi)(p)(d_a j)(p) \\ &= a^d s \sum_p (d_a j)(p)^2, \end{aligned}$$

which converges to $s\|dj\|^2$. Hence the right side of (4.9) converges to $-sg^2\|dj\|^2 v(s)$ as a goes to zero.

Upon taking the integral of both sides of (4.9), from 0 to s, then letting a go to zero, we see that $v(s)$ satisfies an integral equation equivalent to the differential equation (4.8). This concludes the proof.

5. Domination and Convergence of Moments

Proof of Theorem 2.5. I. By assumption there is a 1-cochain j on Z^d with finite support such that $\phi = dj$. Then

$$\begin{aligned} \langle F_a(\phi)^{2n}\rangle_\beta &= a^{d-2} \sum_p \phi(p)\langle h'(d\theta(p))F_a(\phi)^{2n-1}\rangle_\beta \\ &= a^{d-2} \sum_b j(b)\langle (\delta h')(b)F_a(\phi)^{2n-1}\rangle_\beta \\ &= a^{d-2}\beta^{-1}(2n-1)\sum_b j(b)\langle \partial F_a(\phi)/\partial\theta_b F_a(\phi)^{2n-2}\rangle_\beta \\ &= a^{2d-4}\beta^{-1}(2n-1)\sum_b j(b)\sum_p \phi(p)\langle \partial h'(d\theta(p))/\partial\theta_b F_a(\phi)^{2n-2}\rangle_\beta \\ &= a^{2d-4}\beta^{-1}(2n-1)\sum_b j(b)\sum_{\partial p \ni b} \phi(p)\langle h''(d\theta(p))F_a(\phi)^{2n-2}\rangle_\beta \\ &= a^{2d-4}\beta^{-1}(2n-1)\sum_p \phi(p)^2\langle h''(d\theta(p))F_a(\phi)^{2n-2}\rangle_\beta \\ &\leq (a^{d-4}\beta^{-1})(2n-1)a^d \sum_p \phi(p)^2 \alpha\langle F_a(\phi)^{2n-2}\rangle_\beta. \end{aligned}$$

The transition to the sixth line follows from reversing the sum on b and p as in the proof of Theorem 2.1. The inequality (2.6) now follows by induction on n.

II. With $\beta^{-1} = ag^2$ and $d = 3$ the sixth line above reads

$$\begin{aligned} \langle F_a(\phi)^{2n}\rangle_\beta &= g^2(2n-1)\left(a^d \sum_p \phi(p)^2\right)[\langle F_a(\phi)^{2n-2}\rangle_\beta \\ &\quad + \langle (h''(d\theta(p)) - 1)F_a(\phi)^{2n-2}\rangle_\beta]. \end{aligned} \tag{5.1}$$

Now if we put $\phi = d_a j$, then $a^d \sum_p \phi(p)^2$ converges to $\|dj\|^2$ as $a \to 0$. Hence by (2.6) with $\beta^{-1} = ag^2$ and $d=3$, $F_a(d_a j)$ is bounded in L^r norm uniformly in a for each $r < \infty$. Thus by Lemma 4.2 the last term on the right of (5.1) converges to zero. Equation (2.7) now follows by induction on n.

Remark 5.1. Theorem 2.2 can be deduced as a consequence of (2.6).

6. Absence of Continuum Monopoles for Villain Action

In this section h_β will denote the Villain energy function defined by (2.1), while $\langle\ \rangle_\beta$ denotes any translation invariant infinite volume Gibbs state which is a limit of finite volume Gibbs states with Dirichlet or periodic or free boundary conditions.

Lemma 6.1. *In d dimensions, let c be a 3-cell (i.e. an oriented three dimensional cube.) Let $1 \leqq r < \infty$, and put*

$$dh'_\beta(c) = \sum_{p \in \partial c} h'_\beta(d\theta(p)).$$

There exists a constant $k > 0$, independent of c, such that

$$\langle |dh'_\beta(c)|^r \rangle_\beta^{1/r} = 0(e^{-\beta k}) \quad as \quad \beta \to \infty.$$

Proof. Since $h_\beta(x)$ converges uniformly on $[-\pi, \pi]$ to $x^2/2$, there is a number β_0 and a small number $\gamma > 0$ such that $h_\beta(x) \leqq \gamma$ implies $|x| \leqq 2\pi/7$ if $\beta \geqq \beta_0$ and $|x| \leqq \pi$.

Let $\theta = \{\theta_b\}$ be an infinite volume configuration, fix $\beta \geqq \beta_0$, and assume that for each plaquette p in the boundary of the given cube c there holds $h_\beta(d\theta(P)) \leqq \gamma$. Each θ_b is in $(-\pi, \pi]$ so that $d\theta(p) \in (-4\pi, 4\pi]$. Since h_β is periodic it follows from the choice of β and γ that for each $p \in \partial c$, $d\theta(p)$ is within $2\pi/7$ of some integer multiple of 2π. That is, there is an integer k_p such that

$$|d\theta(p) - 2\pi k_p| \leqq 2\pi/7 \quad \text{for all} \quad p \in \partial c.$$

But $d^2\theta(c) = 0$, i.e. $\sum_{p \in \partial c} d\theta(p) = 0$. Hence

$$\left| \sum_{p \in \partial c} (-2\pi k_p) \right| = \left| \sum_{p \in \partial c} (d\theta(p) - 2\pi k_p) \right| \leqq 2\pi \cdot \tfrac{6}{7}.$$

Dividing by 2π shows that the integer $\left| \sum_{p \in \partial c} k_p \right|$ is at most $\frac{6}{7}$ and must therefore be zero. So

$$\sum_{p \in \partial c} (d\theta(p) - 2\pi k_p) = 0. \tag{6.1}$$

Let $\zeta = (2\pi - 2\pi/7)^2 - (2\pi/7)^2$. Then $\zeta > 0$. From (3.18) and the periodicity of h'_β we see that there is a number $\beta_1 > \beta_0$ such that for any integer k_p

$$|h'_\beta(x) - (x - 2\pi k_p)| \leqq Ce^{-\beta\zeta}, \quad \beta > \beta_1,$$

for some constant C, if $|x-2\pi k_p|<2\pi/7$. Thus

$$\begin{aligned}|dh'_\beta(c)| &= \left|\sum_{p\in\partial c} h'_\beta(d\theta(p))\right| \\ &= \left|\sum_{p\in\partial c}[h'_\beta(d\theta(p))-(d\theta(p)-2\pi k_p)]\right| \\ &\leqq 6Ce^{-\beta\zeta}\end{aligned}$$

if $\beta>\beta_1$ and $h_\beta(d\theta(p))\leqq\gamma$ for all $p\in\partial c$.

If A is the set of configurations θ for which $h(d\theta(p))\leqq\gamma$ for all $p\in\partial c$ and C_A is its characteristic function, then

$$\begin{aligned}\langle|dh'_\beta(c)|^r\rangle_\beta &= \langle|dh'_\beta(c)|^r C_A(\theta)\rangle_\beta + \langle|dh'_\beta(c)|^r(1-C_A)\rangle_\beta \\ &\leqq (6C)^r e^{-r\beta\zeta} + (6\max|h'_\beta|)^r < \langle 1-C_A\rangle_\beta .\end{aligned} \tag{6.2}$$

But $|h'_\beta(x)|$ is uniformly bounded in x and β by Lemma 3.9, for $\beta\geqq\beta_1$. Moreover it follows from (3.10) that

$$\langle 1-C_A\rangle_\beta \leqq \sum_{p\in\partial c}\mathrm{Prob}_\beta(h_\beta(d\theta(p))>\gamma)\leqq \text{constant}\,\beta^{1/2}e^{-\beta\gamma}$$

for large β. The lemma now follows with $k=\min(\zeta,\gamma/4)$.

Proof of Theorem 2.6. Let ψ be a test 3-form on R^3. Then

$$\begin{aligned}\|F_a(\delta_a\psi)\|_{L^r(\beta)} &= \left\|a^3\sum_p(\delta_a\psi)(p)a^{-2}h'_\beta(d\theta(p))\right\|_{L^r(\beta)} \\ &= \left\|a^{3-2-1}\sum_c\psi_a(c)dh'_\beta(c)\right\|_{L^r(\beta)} \\ &\leqq \sum_c|\psi_a(c)|\,\|dh'_\beta(c)\|_{L^r(\beta)} \\ &\leqq \left(a^3\sum_c|\psi_a(c)|\right)(a^{-3}Ce^{-\beta k}).\end{aligned}$$

This first factor converges to $\int_{R^3}|\psi(x)|dx$ as $a\to 0$ while the second factor converges to zero for $\beta^{-1}=ag^2$, and in fact exponentially fast in the lattice spacing a.

Remark 6.2. Combining Theorems 2.1 and 2.6 we derive easily that for any test one form j and test 3-form ψ

$$\lim_{a\to 0}\langle\exp[iF_a(\delta_a\psi+d_aj)]\rangle_\beta=\exp[-g^2\|dj\|^2/2]$$

for the Villain action h_β with the canonical value $\beta^{-1}=ag^2$.

7. Change of Field Function

Let $u(x)$ be a periodic real-valued continuous function with period 2π, and define

$$F^u_a(\phi)=a^d\sum_p\phi(p)a^{-2}u(d\theta(p))$$

for any 2-cochain ϕ of finite support. The choice $u(x)=h'(x)$ coincides with the field variables that we have used so far (cf. Eq. (2.3)). This is a natural choice because of the way it arises in the Schwinger–Dyson equations, Eq. (4.2). But it is illuminating to understand to what extent one can alter this choice. Specificallly, the question of the existence of monopoles in the continuum limit for the Wilson lattice action is not settled in this paper. This section shows that it is only the cubic terms in the expansion of $\sin x$ about $x=0$ which can possibly contribute to the survival of monopoles in the continuum limit. In the next section we shall show that in the Gaussian lattice model with field function $u(x)=\sin x$ there are no monopoles in the continuum limit ($d=3$), showing that even the cubic terms don't contribute in that case. But that proof depends on information about correlations which we have not used anywhere so far and which is not presently available for the Wilson action.

Theorem 7.1. *Assume that h satisfies* A1, A2 *and* A3 *and that* $e^{-\beta h}$ *is positive definite for all* $\beta>0$. *Assume further that h has six continuous derivatives. Let u be a five times continuously differentiable odd function on the line which is periodic with period* 2π. *If* $u^{(j)}(x)=h^{(j+1)}(x)$ *for* $j=0,1,2,3,4$ *whenever* $h(x)=0$, *then for any test 2-form* ϕ *on* R^3

$$\lim_{a\downarrow 0}\langle (F_a(\phi_a)-F_a^u(\phi_a))^2\rangle_\beta=0. \tag{7.1}$$

In particular if $F_a(\phi_a)$ *converges in distribution so does* $F_a^u(\phi_a)$, *and they have the same limit.*

Proof. Put $v(x)=h'(x)-u(x)$. Then

$$\begin{aligned}\langle [F_a(\phi_a)-F_a^u(\phi_a)]^2\rangle_\beta^{1/2}&=\langle [a^3\sum_p \phi_a(p)a^{-2}v(d\theta(p))]^2\rangle_\beta^{1/2}\\&\leqq a\sum_p |\phi_a(p)| < v(d\theta(p))^2\rangle_\beta^{1/2}.\end{aligned}\tag{7.2}$$

If $h(x_0)=0$, then by assumption v and its first four derivatives are zero at $x=x_0$, so that $v(x)=O(|x-x_0|^5)$ as $x\to x_0$. But $h''(x_0)=1$, so that in some small neighbourhood around x_0 we have $h''(x)\geqq 1/2$. Thus $|h'(x)|\geqq |x-x_0|/2$ in this neighbourhood, and $|v(x)|\leqq \text{const}\,|h'(x)|^5$ there. There are only a finite number of such points x_0 in $(-\pi,\pi]$, and the union of the above described neighbourhoods contains $\{x\in(-\pi,\pi]:h(x)<c\}$ for some sufficiently small c. Hence, using periodicity, $|v(x)|\leqq C_1|h'(x)|^5$ whenever $h(x)<c$. Thus

$$\langle v(d\theta(p))^2\rangle_\beta \leqq C_1\ \langle h'(d\theta(p))^{10}C_{\{h(d\theta(p))<c\}}\rangle_\beta+(\sup v^2)\langle C_{\{h(d\theta(p))\geqq c\}}\rangle_\beta.$$

By (3.11) the first term is $O(\beta^{-5})$ as $\beta\to\infty$ while the second term is, by (3.10) $O(\beta^{1/2}e^{-\beta c})$. Hence $\langle v(d\theta(p))^2\rangle_\beta=O(\beta^{-5})$ as $\beta\to\infty$, uniformly in p.

Thus with $\beta^{-1}=ag^2$ we see that the right side of (7.2) is at most a constant times $a(ag^2)^{5/2}\sum_p|\phi_a(p)|$, which equals $a^{1/2}g^5\left(a^3\sum_p|\phi_a(p)|\right)$. This converges to zero like $a^{1/2}$.

The final statement of the theorem means that the two sequences of characteristic functionals have the same limit. And in fact

$$|\langle e^{iF_a(\phi_a)}\rangle_\beta - \langle e^{iF_a^u(\phi_a)}\rangle_\beta| \leqq \langle |F_a(\phi_a) - F_a^u(\phi_a)|\rangle_\beta,$$

which converges to zero by Schwarz's inequality and (7.1).

Remark 7.2. A similar theorem holds with h replaced by the Villain action h_β. In this case the only zero if $h_\beta(x)$ in the periodicity interval $(-\pi, \pi]$ is at $x=0$. The conclusions of Theorem 7.1 hold if u is an odd periodic function in $C^5(R)$ satisfying $u'(0)=1$ and $u'''(0)=0$. The second and fourth derivatives are automatically zero because u is odd. This result for the Villain action doesn't seem very important at the present time because the absence of monopoles in the continuum limit for the Villain action has already been established in the preceding section. The condition $u'''(0)=0$ could probably be dropped if one had information about correlations. This is suggested by the results of the next section.

8. Gaussian Model with Nonlinear Field (Spin Wave Theory)

The noncompact version of the U(1) model is given in a cube $\Lambda \subset Z^3$ by the Gaussian measure.

$$Z^{-1}\exp\left[-(\beta/2)\left\{\sum_{p\in\Lambda^{(2)}}(d\theta(p))^2 + m^2\sum_{x\in\Lambda^{(0)}}(\delta\theta(x))^2\right\}\right]d\theta \tag{8.1}$$

on $R^{\Lambda^{(1)}}$, where $d\theta = \prod_{b\in\Lambda^{(1)}} d\theta_b$. We take $\{\Lambda^{(j)}\}_{j=0}^3$ to be the open complex formed over Λ. With $-\Delta_m = \delta d + m^2 d\delta$ (which automatically incorporates Dirichlet boundary conditions because the complex is open) one verifies easily that

$$\langle e^{i\sum_{p\in\Lambda^{(2)}} d\theta(p)\phi(p)}\rangle_{\Lambda,\beta} = \exp[-(2\beta)^{-1}((-\Delta_m)^{-1}\delta\phi, \delta\phi)] \tag{8.2}$$

for any 2-cochain ϕ over $\Lambda^{(2)}$. Informally the noncompact version of Eq. (2.2) is given by (8.1) with $m=0$. However the quadratic form in the exponent is degenerate when $m=0$ and insertion of the gauge fixing term $m^2\sum_x(\delta\theta(x))^2$ is one of the standard ways of making the measure finite. The integral of functions which depend only on the variables $d\theta(p)$, $p\in\Lambda^{(2)}$ is independent of m as the reader can verify for exponentials from (8.2). Henceforth we take $m=1$. We are concerned only with functions of the $d\theta(p)$. It is clear from (8.2) that the thermodynamic limit $\Lambda \nearrow Z^3$ exists on these functions, and if ϕ is a 2-cochain of finite support, then

$$\langle e^{i\sum_p d\theta(p)\phi(p)}\rangle_\beta = \exp[-(2\beta)^{-1}((-\Delta)^{-1}\delta\phi, \delta\phi)], \tag{8.3}$$

where $-\Delta = d\delta + \delta d$ acting on l^2 cochains.

Let $C_p(q)=1$ if $q=p$, -1 if $q=-p$ and zero otherwise for each plaquette p. Then putting $\phi(q') = sC_p(q') + tC_q(q')$ into (8.3) and differentiating once with respect to each of the real parameters s and t at $s=t=0$ gives

$$\langle d\theta(p)d\theta(q)\rangle_\beta = \beta^{-1}((-\Delta)^{-1}\delta C_p, \delta C_q) \equiv \beta^{-1}G(p,q).$$

Note that $G(p,q)$ is independent of β.

The linear fields $d\theta(p)$ are well understood Gaussian random variables and Eq. (8.3) implies without much difficulty that the characteristic functional of the field variables $a^3 \sum_p \phi_a(p) a^{-2} d\theta(p)$ converges as $a \to 0$ to the right side of (1.2) for any test 2-form ϕ if $\beta^{-1} = ag^2$. We are interested however in the nonlinear, periodic field variables

$$F_a(\phi_a) = a^3 \sum_p \phi_a(p) a^{-2} \sin d\theta(p), \tag{8.4}$$

where ϕ is a test 2-form on R^3. We saw in Sect. 7 that the only possible source of monopoles in the continuum limit for the Wilson action is the cubic term in the expansion of $\sin d\theta(p)$. Our objective here is to show that there are no monopoles in the continuum limit of the nonlinear fields (8.4).

Theorem 8.1. *Let ψ be a test 3-form on R^3. Then*

$$\lim_{a \to 0} \langle F_a(\delta_a \psi)^2 \rangle_{(ag^2)^{-1}} = 0. \tag{8.5}$$

Remark. 8.2. The same result holds if $\sin x$ is replaced by any odd polynomial. Only the third order Hermite polynomial requires some care and this proceeds in much the same way as we shall show for $\sin x$. The higher order Hermite polynomials however contribute zero in the limit for the same reasons as in Sect. 7.

Lemma 8.3. *For plaquettes p and q define $S_\beta(p,q) = \langle \sin d\theta(p) \sin d\theta(q) \rangle_\beta$. Then*

$$S_\beta(p,q) = \exp[-\beta^{-1} G] \sinh[\beta^{-1} G(p,q)], \tag{8.6}$$

where $G = G(p,q)$ is a constant.

Proof. Write $2 \sin x \sin y = \cos(x-y) - \cos(x+y) = \mathrm{Re}\,[e^{i(x-y)} - e^{i(x+y)}]$, and use (8.3) to get

$$\begin{aligned} 2\langle \sin d\theta(p) \sin d\theta(q) \rangle_\beta &= \mathrm{Re}\, \langle e^{id\theta \cdot (C_p - C_q)} - e^{id\theta \cdot (C_p + C_q)} \rangle_\beta \\ &= \exp[(2\beta)^{-1}(\Delta^{-1}(C_p - C_q), C_p - C_q)] \\ &\quad - \exp[(2\beta)^{-1}(\Delta^{-1}(C_p + C_q), C_p + C_q)] \\ &= e^{-\beta^{-1}G}(e^{\beta^{-1}((-\Delta)^{-1}C_p, C_q)} \\ &\quad - e^{-\beta^{-1}((-\Delta)^{-1}C_p, C_q)}). \end{aligned}$$

Lemma 8.4. *Fix q. Then $p \to G(p,q)$ is a 2-cochain and its exterior derivative is zero. That is, $dG(\cdot,q)(c) = 0$ for all cubes c.*

Proof. $\sum_{p \in \partial c} G(p,q) = \beta \sum_{p \in \partial c} \langle d\theta(p) d\theta(q) \rangle_\beta = \beta \langle d^2\theta(c) d\theta(q) \rangle_\beta = 0.$

Lemma 8.5. *Write $|p-q|_L$ for the distance of the center of the plaquette p in $(Z^3)^{(2)}$ to the center of the plaquette q measured in units in which adjacent points in Z^3* have distance one. *There is a constant A and a number $\gamma > 1/2$ such that*

$$|G(p,q)| \leqq A|p-q|_L^{-\gamma}.$$

Proof. $G(p,q) = ((-\Delta)^{-1} \delta C_p, \delta C_q)$. The kernel $(-\Delta)^{-1}(x,y)$ acting on scalar functions is known to fall off like $|x-y|_L^{-1}$ for large separation between x and y, see e.g. [20, § 26] or [16, page 387]. The inner product defining $G(p, q)$ is a finite sum of differences of this kernel and therefore falls off at least as fast. One can therefore take $\gamma = 1$.

Remark 8.6. An explicit computation for $G(p, q)$ for the two cases, p parallel to q or p perpendicular to q, shows that $G(p,q) = O(|p-q|_L^{-3} \log|p-q|_L)$. With more care the log can probably be eliminated. It is illuminating, however, to see how fast a fall off is actually required to eliminate monopoles. $\gamma > 1/2$ will be shown to be sufficient.

Lemma 8.7. *There is a constant M such that*

$$|S_\beta(p,q) - e^{-\beta^{-1}G_\beta^{-1}} G(p,q)| \leqq M\beta^{-3}|G(p,q)^3|$$

if $\beta \geqq 1$.

Proof. Use Lemma 8.3 and the fact that $\beta^{-1}G(p,q)$ is bounded for $\beta \geqq 1$.

Proof of Theorem 8.1.

$$\begin{aligned}\langle F_a(\delta_a\psi)^2\rangle_\beta &= a^6 \sum_{p,q} (\delta_a\psi)(p)(\delta_a\psi)(q) a^{-4} \langle \sin d\theta(p) \sin d\theta(q)\rangle_\beta \\ &= a^2 \sum_{p,q} (\delta_a\psi)(p)(\delta_a\psi)(q) S_\beta(p,q).\end{aligned}$$

Now $\sum_p (\delta_a\psi)(p) G(p,q) = a^{-1} \sum_c \psi_a(c)\, dG(\cdot, q)(c) = 0$ by Lemma 8.4. Hence

$$\begin{aligned}\langle F_a(\delta_a\psi)^2\rangle_\beta &= a^2 \sum_{p,q} (\delta_a\psi)(p)(\delta_a\psi)(q)[S_\beta(p,q) - e^{-\beta^{-1}G_\beta^{-1}} G(p,q)] \\ &\leqq M\beta^{-3}a^2 \sum_{p,q} |(\delta_a\psi)(p)||(\delta_a\psi)(q)||G(p,q)|^3.\end{aligned}$$

Choose a number $\gamma > \frac{1}{2}$ as in Lemma 8.5 and a number $t < \frac{2}{3}$ such that $3\gamma t > 1$. Then, writing M' for a bound on $M|G(p,q)|^3$, we have

$$\begin{aligned}\langle F_a(\delta_a\psi)^2\rangle_\beta \leqq{}& M'\beta^{-3}a^2 \sum_{|p-q|_L \leqq a^{-t}} |(\delta_a\psi)(p)||(\delta_a\psi)(q)| \\ &+ M\beta^{-3}a^2 \sum_{|p-q|_L > a^{-t}} |(\delta_a\psi)(p)||(\delta_a\psi)(q)| A^3 a^{3\gamma t}. \qquad (8.7)\end{aligned}$$

Now ψ has compact support in R^3. Let r be the radius of a ball centered at zero containing the support of ψ. Then $\delta_a\psi$ is supported in Z^d on a ball of radius r/a. For each plaquette p in this ball the number of plaquettes q for which $|p-q|_L \leqq a^{-t}$ is on the order $(a^{-t})^3$. Thus the first sum on the right of (8.7) is $O((r/a)^3(a^{-t})^3)$ as $a \downarrow 0$, since $\delta_a\psi(p)$ is uniformly bounded in a and p by the mean value theorem. Thus the first term on the right of (8.7), with $\beta^{-1} = ag^2$ is $O(a^5 a^{-3-3t}) = O(a^{2-3t})$. Since $t < \frac{2}{3}$ this goes to zero as $a \to 0$. The second term on the right of (8.7) is at most $Mg^6 a^{5+3\gamma t} \sum_{p,q} |(\delta_a\psi)(p)(\delta_a\psi)(q)|$. Since $5 + 3\gamma t > 6$ this also converges to zero as a approaches zero.

Acknowledgement. I would like to thank the Institute for Mathematics and its Applications at the University of Minnesota and the Institute for Advanced Study in Princeton for support during part of this work. It is a pleasure also to thank D. Amit, D. Brydges, H. Hamber, A. Knapp, E. lieb. F. Spitzer and A. Wightman for useful discussions and especially O. Alvarez, G. Gallavotti, A. Guth, M. Peskin and B. Svetitsky.

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Communicated by A. Jaffe
Received April 19, 1983