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On the regularity of the pressure of weak solutions of Navier-Stokes equations

By

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1. Introduction. Let Ω be a bounded domain or an exterior domain of $\mathbb{R}^n$ with $n \geqq 3$. We assume that Ω has a smooth boundary $\partial\Omega$, i.e. $\partial\Omega$ is of class C^∞. On $\Omega^T := \Omega \times (0, T)$ with some $T > 0$, we consider the equations of Navier-Stokes

(1.1) $$u' - \Delta u + u \cdot \nabla u + \nabla \pi = f, \quad \nabla \cdot u = 0, \quad u(0) = u_0, \quad u|_{\partial\Omega} = 0$$

and a weak solution $u \in L^\infty(0, T, L^2(\Omega)^n) \cap L^2(0, T, \mathring{H}^{1,2}(\Omega)^n)$ of these equations. For some applications it is important to know regularity properties of the pressure π (see Caffarelli-Kohn-Nirenberg [2]). In case $\Omega = \mathbb{R}^3$ it has been shown in [2] that $\pi \in L^{5/3}(\Omega^T)$ holds (under some assumptions on f, u_0). For a bounded domain, v. Wahl [12] has shown the property $\pi \in L^{5/4}(\Omega^T)$. A conjecture in [2, p. 780] states that $\pi \in L^{5/3}(\Omega^T)$ holds in the general case; this property would have some important consequences for the partial regularity theory of weak solutions of (1.1). It is the aim of the present paper to prove this conjecture for a bounded domain Ω. However, combining the method of the proof with the method given in [10], the result follows for an exterior domain too.

For exterior domains, the result $\pi \in L^{5/3}(\Omega^T)$ is a global one; it follows that π has a certain decay for $|x| \to \infty$. Another consequence is the existence of a weak solution u in an exterior domain $\Omega \subset \mathbb{R}^3$ which is smooth for large $|x|$. In the case $\Omega = \mathbb{R}^3$, this has been proved by Caffarelli-Kohn-Nirenberg [2].

Our method rests on a potential theoretical estimate for the linear part of (1.1) with different integration exponents in space and time (given in [13] for the parabolic case) and on the method to regularize weak solutions by the Yosida approximation (given in [8] and [9]). The main idea can be briefly explained as follows:

In case $\Omega = \mathbb{R}^3$ the result $\pi \in L^{5/3}(\Omega^T)$ follows by a careful investigation of the equation $\Delta\pi = \nabla \cdot (f - \nabla \cdot (u u))$ in the sense of distributions; it follows that π has essentially the same regularity as $u u$ which belongs to $L^{5/3}(\Omega^T)$ (see [2]). Of course, this method fails for arbitrary domains $\Omega \subseteqq \mathbb{R}^3$ because π does not fulfill any boundary conditions on $\partial\Omega$. A similar argument yields only that $\nabla\pi$ has the same regularity as $u \cdot \nabla u$ which belongs to $L^{5/4}(\Omega^T)$ (see [12]); therefore the integration exponent of π in space is better than $\frac{5}{4}$ by Sobolev's embedding theorem; however it is not possible to improve the exponent in time by this argument. The splitting of the integration exponents in space and time overcomes this difficulty. It holds more generally $u \cdot \nabla u \in L^s(0, T; L^p)$ with $4 = \dfrac{2}{s} + \dfrac{3}{p}$ in $\Omega \subseteqq \mathbb{R}^3$

and we obtain $u \cdot \nabla u \in L^{5/3}(0, T; L^{15/14})$ for $s = 5/3$; from this we get $\nabla \pi \in L^{5/3}(0, T; L^{15/14})$, and Sobolev's embedding theorem yields $\pi \in L^{5/3}(0, T; L^{5/3}) = L^{5/3}(\Omega^T)$ because of $\frac{1}{3} + \frac{3}{5} = \frac{14}{15}$. Thus we arrive at desired result $\pi \in L^{5/3}(\Omega^T)$.

We introduce some notations. In the following we need the usual spaces $L^p(\Omega)$, $H^{k,p}(\Omega)$, $\mathring{H}^{k,p}(\Omega)$, $C^k(\Omega)$, $C^k(\bar{\Omega})$, $\mathring{C}^k(\Omega)$, $\mathring{C}^k(\bar{\Omega})$ with $k = 0, 1, 2, \ldots$ and $1 < p < \infty$. The norm in $L^p(\Omega)$ is denoted by $\|v\|_{L^p(\Omega)} = \|v\|_p = \|v\|_{1/p}$; $\bar{\Omega}$ is the closure of Ω. We set $D_i := \frac{\partial}{\partial x_i}$ with $i = 1, 2, \ldots, n$, $x = (x_1, \ldots, x_n) \in \Omega$. All spaces are real. The corresponding spaces for vector functions $v = (v_1, v_2, \ldots, v_n)$ are denoted by $L^p(\Omega)^n$, $H^{k,p}(\Omega)^n, \ldots$. For a Banach space H, $L^p(0, T; H)$ is the usual space with the norm $\|v\|_{L^p(0,T;H)} = \left(\int_0^T \|v\|_H^p \, dt\right)^{\frac{1}{p}}$; we use the notation $\|v\|_{L^s(0,T;L^p(\Omega))} = \|v\|_{p,s} = \|v\|_{\frac{1}{p},\frac{1}{s}}$ $(1 < p < \infty,\ 1 < s < \infty)$.

Let $H_p(\Omega)$ be the closure of $\{v \in \mathring{C}^\infty(\Omega)^n \mid \operatorname{div} v = 0\}$ with respect to the $L^p(\Omega)^n$-norm. There exists a bounded linear operator $P_p: L^p(\Omega)^n \to H_p(\Omega)$ with $P_p^2 = P_p$ such that we have for every $v \in L^p(\Omega)^n$ a decomposition $v = P_p v + \nabla \pi$ with $\pi \in L^p_{\text{loc}}(\Omega)$ and $\nabla \pi \in L^p(\Omega)^n$ ([4], [7], [11]). Similarly, for every $v \in L^s(0, T; L^p(\Omega)^n)$ we get a decomposition $v = P_p v + \nabla \pi$ with $P_p v \in L^s(0, T; H_p(\Omega))$, $\pi \in L^s(0, T; L^p(K))$ for every compact set $K \subseteqq \Omega$, and $\nabla \pi \in L^s(0, T; L^p(\Omega)^n)$.

We use the notation $u' = \frac{\partial}{\partial t} u$, $(u, v) := \int_\Omega u(x) \cdot v(x) \, dx$ with $u(x) \cdot v(x) = u_1(x) v_1(x) + \cdots + u_n(x) v_n(x)$, $(\nabla u, \nabla v) := (D_1 u, D_1 v) + \cdots + (D_n u, D_n v)$, $\operatorname{div} u = \nabla \cdot u = D_1 u_1 + \cdots + D_n u_n$, $\nabla := (D_1, \ldots, D_n)$, $u \cdot \nabla u := (u \cdot \nabla u_1, \ldots, u \cdot \nabla u_n)$, $u u := (u_i u_j)_{i,j=1,\ldots,n}$, $\nabla \cdot (u u) := (\nabla \cdot (u u_1), \ldots, \nabla \cdot (u u_n))$, $(u u, \nabla v) := (u u_1, \nabla v_1) + \cdots + (u u_n, \nabla v_n)$.

In the following definition of weak solutions u we use testing functions of the form $v(x, t) = w(x) h(t)$ only. Instead of the usual condition $f \in L^2(0, T; H^{-1,2}(\Omega)^n)$, we require $f \in L^s(0, T; L^p(\Omega)^n)$ in our assumptions; this is possible because we do not need the energy inequality here; $v \in \mathring{C}^\infty(\bar{\Omega})^n$ means that $v \in C^\infty(\Omega)^n$ and that the support of v is compact and contained in $\bar{\Omega}$.

1.2 Definition. Let $u_0 \in H_2(\Omega)$ and $f \in L^s(0, T; L^p(\Omega)^n)$ with $s, p \in (1, \infty)$. A weak solution of (1.1) with data u_0, f is a weakly continuous function u: $[0, T] \to H_2(\Omega)$ with the following properties:

a) $u \in L^\infty(0, T; H_2(\Omega)^n) \cap L^2(0, T; \mathring{H}^{1,2}(\Omega)^n)$, $u(0) = u_0$.

b) $-\int_0^T (u, v')\, dt + \int_0^T (\nabla u, \nabla v)\, dt - \int_0^T (u u, \nabla v)\, dt = (u_0, v(0)) + \int_0^T (f, v)\, dt$ for all $v = w h$ with $w \in \mathring{C}^\infty(\bar{\Omega})^n$, $\operatorname{div} w = 0$, $w|_{\partial\Omega} = 0$, $h \in C^1([0, T])$, $h(T) = 0$.

Remark. The weak continuity of u follows also from $u \in L^\infty(0, T; H_2(\Omega)^n) \cap L^2(0, T; \mathring{H}^{1,2}(\Omega)^n)$ and b); $u(0) = u_0$ follows also from b).

In what follows, $c, c_1, c_2, \ldots$ denote positive constants which may change from line to line.

2. A potential theoretical estimate for the linearized equation. Let $\Delta_p: D(\Delta_p) \to L^p(\Omega)^n$ be the usual Laplace operator in $L^p(\Omega)^n$ with $D(\Delta_p) := H^{2,p}(\Omega)^n \cap \mathring{H}^{1,p}(\Omega)^n$. The Stokes operator $P_p \Delta_p: D(P_p \Delta_p) \to H_p(\Omega)$ is defined by $D(P_p \Delta_p) := D(\Delta_p) \cap H_p(\Omega)$

and $P_p \Delta_p v := P_p(\Delta_p v)$. We suppose always $1 < p < \infty$. I denotes the identity operator. We set

$$A_p := -P_p \Delta_p + I, \quad B_p := -\Delta_p + I.$$

In case of a bounded domain $\Omega \subseteqq \mathbb{R}^n$, it is possible to set $A_p := -P_p \Delta^p$ and $B_p := -\Delta_p$. It is well known that $-A_p$ and $-B_p$ generate analytic semigroups in (the complexification of) $H_p(\Omega)$ and $L_p(\Omega)^n$ respectively ([5], [14], [11]). The fractional powers A_p^γ and B_p^γ $(0 \leqq \gamma \leqq 1)$ are well defined ([6]) and we get $D(A_p^\gamma) = D(B_p^\gamma) \cap H_p(\Omega)$ with equivalent graph norms; $D(A_p^\gamma)$ and $D(B_p^\gamma)$ are always equipped with the graph norms. We obtain

$$\|e^{-tA_p}\|_{H_p(\Omega)} \leqq c\, e^{-\delta t}, \quad t \geqq 0, \tag{2.1}$$

for some $\delta > 0$. A well known embedding property yields

$$\|A_q^\alpha v\|_q \leqq c \, \|A_p^\beta v\|_p \quad \text{for} \quad \frac{2\beta}{n} - \frac{1}{p} \geqq \frac{2\alpha}{n} - \frac{1}{q}, \quad \infty > q \geqq p > 1$$

where the constant $c = c(p, q, \alpha, \beta, \Omega) > 0$ is independent of $v \in D(A_p^\beta)$.

The following theorems are well known in the special case $s = p$. However the splitting of these integration exponents is important for our method. First we consider the case $u_0 = 0$.

2.2 Theorem. *Let* $1 < p < \infty$, $1 < s < \infty$ *and* $f \in L^s(0, T; H_p(\Omega))$. *Then* $u(t) := \int_0^t e^{-(t-\tau)A_p} f(\tau)\, d\tau \in D(\Delta_p)$ *for a.e.* $t \in [0, T]$, $u', A_p u \in L^s(0, T; H_p(\Omega))$, $u' + A_p u = f$, *and*

$$\int_0^T (\|u'\|_p^s + \|A_p u\|_p^s)\, dt \leqq c \int_0^T \|f\|_p^s\, dt,$$

where the constant $c = c(T, p, s, \Omega) > 0$ *is independent of* f.

Proof. According to [11], [12], [14], the conclusion of the theorem holds in the case $s = p$. There exists a unique $u \in L^p(0, T; H^{2,p}(\Omega)^n \cap \mathring{H}^{1,p}(\Omega)^n \cap H_p(\Omega))$ with $u' \in L^p(0, T; H_p(\Omega))$, $u(0) = 0$, such that

$$u' + A_p u = f \text{ a.e. on } [0, T]. \tag{2.3}$$

The following estimate holds:

$$\int_0^T \|u'\|_p^p\, dt + \sum_{|\alpha| \leqq 2} \int_0^T \|D^\alpha u\|_p^p\, dt \leqq c \int_0^T (\|f\|_p^p + \|u\|_p^p)\, dt. \tag{2.4}$$

Approximating f by functions $f_\nu \in C^1([0, T] H_p(\Omega))$ in $L^p(0, T; H_p(\Omega))$ it is easily seen that

$$u(t) = \int_0^t e^{-(t-\tau)A_p} f(\tau)\, d\tau \quad \text{on} \quad [0, T]. \tag{2.5}$$

(2.1) furnishes the estimate

$$\int_0^T \|u\|_p^p \,\mathrm{d}t \leqq c \int_0^T \|f\|_p^p \,\mathrm{d}t. \tag{2.6}$$

The norm equivalence

$$\frac{1}{c}\|u\|_{H^{2,p}(\Omega)^n} \leqq \|A_p u\|_{H_p(\Omega)} \leqq c\,\|u\|_{H^{2,p}(\Omega)^n}$$

together with (2.6), (2.4) and (2.5) yields

$$\int_0^T \|A_p u\|_p^p \,\mathrm{d}t \leqq c \int_0^T \|f\|_p^p \,\mathrm{d}t.$$

[13, p. 492–493] with $H_p(\Omega)$ instead $L^p(\Omega)$ completes the proof. □

The next theorems concern the case $u_0 \neq 0$.

2.7 Theorem. *Let* $1 < p < \infty$, $1 < s < \infty$, $f \in L^s(0, T; H_p(\Omega))$ *and* $u_0 \in D\left(A_p^{1-\frac{1}{s}+\varepsilon}\right)$ *for some* $\varepsilon > 0$ *with* $1 - \dfrac{1}{s} + \varepsilon \leqq 1$. *Then there is a unique u with*

$$\begin{aligned} &u' \in L^s(0, T; H_p(\Omega)),\\ &u \in L^s(0, T; H^{2,p}(\Omega)^n \cap \mathring{H}^{1,p}(\Omega)^n \cap H_p(\Omega)),\\ &u' - P_p \Delta_p u = f \quad \text{a.e. on} \quad [0, T],\\ &u(0) = u_0; \end{aligned} \tag{2.8}$$

moreover the following estimate holds:

$$\int_0^T \|u'\|_p^s \,\mathrm{d}t + \int_0^T \|u\|_{H^{2,p}(\Omega)^n}^s \,\mathrm{d}t \leqq c\left(\|A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s + \int_0^T \|f\|_p^s \,\mathrm{d}t\right), \tag{2.9}$$

where $c = c(T, p, s, \Omega) > 0$ *is independent of* f, u_0.

Proof. We set $v(t) := e^{-tA_p} u_0$ and consider the equation

$$(e^{-t} u - v)' + A_p(e^{-t} u - v) = e^{-t} f.$$

Setting $w(t) := e^{-t} u(t) - v(t)$ we solve the problem

$$w' + A_p w = e^{-t} f, \quad w(0) = 0. \tag{2.10}$$

Arguing as in the proof of Theorem 2.2 we get that

$$w(t) = \int_0^t e^{-(t-\tau)A_p} e^{-\tau} f(\tau)\,\mathrm{d}\tau$$

gives the unique solution of (2.10) with $w' \in L^s(0, T; H_p(\Omega))$, $w \in L^s(0, T; H^{2,p}(\Omega)^n \cap \mathring{H}^{1,p}(\Omega)^n \cap H_p(\Omega))$, $w(0) = 0$. From 2.2 we get the following estimate

$$\int_0^T \|w'\|_p^s \,\mathrm{d}t + \int_0^T \|w\|_{H^{2,p}(\Omega)^n}^s \,\mathrm{d}t \leqq c \int_0^T \|f\|_p^s \,\mathrm{d}t. \tag{2.11}$$

We observe that

$$\begin{aligned}\int_0^T \|v\|^s_{H^{2,p}(\Omega)^n}\,dt &\leqq c_1 \int_0^T \|A_p v\|_p^s\,dt\\ &= c_1 \int_0^T \|A_p^{\frac{1}{s}-\varepsilon}\, e^{-tA_p}\, A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s\,dt,\\ &\leqq c_2\left(\int_0^T \left(\frac{1}{t^{\frac{1}{s}-\varepsilon}}\right)^s dt\right)\|A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s\\ &\leqq c_3 \|A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s.\end{aligned}$$

From this estimate and (2.11) we obtain for $u(t) = e^t(w(t) + v(t))$ the estimate (2.9), and u solves the equation $u' - P_p\Delta_p u = f$ with $u(0) = u_0$. This completes the proof. □

Next we turn to the instationary Stokes problem, the linear part of (1.1).

2.12 Theorem. *Let* $1 < p < \infty$, $1 < s < \infty$, $f \in L^s(0, T; L^p(\Omega)^n)$ *and* $u_0 \in D(B_p^{1-\frac{1}{s}+\varepsilon}) \cap H_p(\Omega)$ *for some* $\varepsilon > 0$ *with* $1 - \frac{1}{s} + \varepsilon \leqq 1$. *Then there exist a unique* u *with*

$$u \in L^s(0, T; H^{2,p}(\Omega)^n \cap \mathring{H}^{1,p}(\Omega)^n \cap H_p(\Omega)),$$

$$u' \in L^s(0, T; H_p(\Omega)), \quad u(0) = u_0,$$

and a π *such that*

$$\pi \in L^s(0, T; L^p(K)) \quad \textit{for every compact set} \quad K \subseteqq \bar{\Omega},$$

$$\nabla\pi \in L^s(0, T; L^p(\Omega)^n),$$

$$u' - \Delta u + \nabla\pi = f \quad \textit{in} \quad L^s(0, T; L^p(\Omega)^n), \quad \nabla \cdot u = 0,$$

and

$$\int_0^T \|u'\|_p^s\,dt + \int_0^T \|u\|^s_{H^{2,p}(\Omega)^n}\,dt + \int_0^T \|\nabla\pi\|_p^s \leqq c\left(\|B_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s + \int_0^T \|f\|_p^s\,dt\right), \tag{2.13}$$

where $c = c(T, s, p, \Omega) > 0$ *is independent of* f, u_0.

Proof. From Giga's result [6] it follows $D(A_p^{1-\frac{1}{s}+\varepsilon}) = D(B_p^{1-\frac{1}{s}+\varepsilon}) \cap H_p(\Omega)$ and therefore $u_0 \in D(A_p^{1-\frac{1}{s}+\varepsilon})$. We may apply Theorem 2.7 to u_0 and $P_p f \in L^s(0, T; H_p(\Omega))$ and get a solution u of the equation $u' - P_p\Delta_p u = P_p f$, $u(0) = u_0$.

Therefore it follows $P_p(f - u' + \Delta_p u) = 0$ and the element $f = u' + \Delta_p u \in L^s(0, T; L^p(\Omega)^n)$ has a decomposition of the form $f - u' + \Delta_p u = P_p(f - u' + \Delta_p u) + \nabla\pi = \nabla\pi$, where π has the desired properties. Therefore we get

$$u' - \Delta_p u + \nabla\pi = f$$

in $L^s(0, T; L^p(\Omega)^n)$. Using (2.9) with $P_p f$ instead f and the equivalence of the norms $\|u\|_{H^{2,p}(\Omega)^n}$ and $\|\Delta_p u\|_p + \|u\|_p$, we get

$$\begin{aligned}\|\nabla\pi\|_{p,s} &\leqq \|f\|_{p,s} + \|u'\|_{p,s} + \|\Delta_p u\|_{p,s} + \|u\|_{p,s}\\ &\leqq \|f\|_{p,s} + c_1\left(\int_0^T \|u'\|_p^s \,dt + \int_0^T \|u\|_{H^{2,p}(\Omega)^n}^s \,dt\right)^{\frac{1}{s}}\\ &\leqq \|f\|_{p,s} + c_2\left(\|A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s + \int_0^T \|P_p f\|_p^s \,dt\right)^{\frac{1}{s}}\\ &\leqq c_3\left(\|A_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s + \int_0^T \|f\|_p^s \,dt\right)^{\frac{1}{s}}.\end{aligned}$$

Using (2.9) again we obtain immediately the estimate (2.13). This completes the proof. □

Remark. For some applications it is of interest to have an estimate instead of (2.13) which holds uniformly for all $T > 0$. This can be done as in [15, pp. 139–141] and yields the following result: Under the assumption of Theorem 2.12 we have

$$\begin{aligned}&\int_0^T \|u'\|_p^s \,dt + \int_0^T \|u\|_{H^{2,p}(\Omega)^n}^s \,dt + \int_0^T \|\nabla\pi\|_p^s \,dt\\ &\leqq c\left(\|B_p^{1-\frac{1}{s}+\varepsilon} u_0\|_p^s + \int_0^T \|f\|_p^s \,dt + \int_0^T \|u\|_p^s \,dt\right)\end{aligned}$$

where $c = c(s, p, \Omega) > 0$ is independent of f, u_0 and T.

3. Properties of these pressure π of weak solutions. We try to get information on π by writing the Navier-Stokes equation in the form

$$u' - \Delta u + \nabla\pi - \tilde{f} := f - u \cdot \nabla u \tag{3.1}$$

and using potential theoretical estimates of the linear equation. In principle this procedure is well known; however, we will use here the following new idea.

It can be shown by Sobolev's embedding theorem that $u \cdot \nabla u \in L^s(0, T; L^p(\Omega)^n)$ holds with $n + 1 \leqq \frac{2}{s} + \frac{n}{p}$. In case $p = s$ we obtain $p \leqq \frac{n+2}{n+1}$ and therefore $\nabla\pi \in L^{(n+2)/(n+1)}(0, T; L^{(n+2)/(n+1)}(\Omega)^n)$. However, the integration exponent of π with respect to space can be improved by using Sobolev's embedding theorem; of course this is not possible with respect to the time t. Therefore we have introduced the splitting in space and time, in order to get an improvement of the exponent for the time: We may take s larger and p smaller than $\frac{n+2}{n+1}$ and later on we improve p by Sobolev's theorem to get the same value s again. By this method we get $\pi \in L^s(0, T, L^p(\Omega))$ with $n \leqq \frac{2}{s} + \frac{n}{p}$ for a bounded domain; in particular we get $\pi \in L^{(n+2)/n}(0, T; L^{(n+2)/n}(\Omega))$ and $\pi \in L^{5/3}(\Omega^T)$ for $n = 3$.

In order to apply potential theoretical estimates to a weak solution u of (3.1), we will regularize u by the Yosida approximation as in [8], [9]. We set $J_k := \left(I + \frac{1}{k} A_2\right)^{-\delta}$ with $\delta = \frac{n}{4}$ and $k = 1, 2, \ldots$ and get the properties $\|J_k\| \leqq c$ and $J_k v \to v$ for $k \to \infty$. The regularized functions $J_k u$ possess better regularity properties and are strong solutions of a modified equation. First we will explain the procedure (see [9] for details).

Let u_0, f and u be as in the Definition 1.2. In case of an exterior domain we suppose in addition $p \leqq 2$. In 1.2 we can take testing functions of the form $v = wh$ with $w \in \mathring{C}^\infty(\bar{\Omega})^n)$, $\nabla \cdot w = 0$, $w|_{\partial\Omega} = 0$, $w \in D(A_2^\delta)$ for $\delta = \frac{n}{4}$, $h \in C^1([0, T])$, $h(T) = 0$. Then we write $w = A_2^{-\delta} A_2^\delta w = A_2^{-\delta} \tilde{w}$ with $\tilde{w} := A_2^\delta w$ and the set of all $\tilde{w}$ of this form is dense in $H_2(\Omega)$. Thus from 1.2 we obtain the equation

$$- \int_0^T (u, A_2^{-\delta} \tilde{w}) h' \, dt + \int_0^T (\nabla u, \nabla A_2^{-\delta} \tilde{w}) h \, dt - \int_0^T (u u, \nabla A_2^{-\delta} \tilde{w}) h \, dt$$

$$= (u_0, A_2^{-\delta} \tilde{w}) h(0) + \int_0^T (f, A_2^{-\delta} \tilde{w}) h \, dt.$$

Using the embedding property $D(A_\sigma^\alpha) \subseteqq D(A_\varrho^\beta)$, $\frac{2\alpha}{n} - \frac{1}{\sigma} \geqq \frac{2\beta}{n} - \frac{1}{\varrho}$, $\infty > \varrho \geqq \sigma > 1$, and a well known interpolation theorem we get

$$\begin{aligned}
\|\nabla \cdot (u u)\|_r &= \|u \cdot \nabla u\|_{\frac{1}{r}} \leqq c_1 \|u\|_{\frac{1}{r} - \frac{1}{2}} \|\nabla u\|_{\frac{1}{2}} \\
&= c_1 \|u\|_{-\left[\frac{2}{n} \cdot \frac{n}{2}\left(1 - \frac{1}{r}\right) - \frac{1}{2}\right]} \|\nabla u\|_{\frac{1}{2}}, \\
&\leqq c_2 \left\| A_2^{\frac{n}{2}\left(1 - \frac{1}{r}\right)} u \right\|_{\frac{1}{2}} \|\nabla u\|_{\frac{1}{2}} \\
&\leqq c_3 \left\| A_2^{\frac{1}{2}} u \right\|_{\frac{1}{2}}^{n\left(1 - \frac{1}{r}\right)} \|u\|_{\frac{1}{2}}^{1 - n\left(1 - \frac{1}{r}\right)} \|\nabla u\|_{\frac{1}{2}}, \\
&\leqq c_4 \left\| A_2^{\frac{1}{2}} u \right\|_2^{1 + n\left(1 - \frac{1}{r}\right)} \|u\|_2^{1 - n\left(1 - \frac{1}{r}\right)} \\
&\leqq c_5 (\|\nabla u\|_2 + \|u\|_2)^{1 + n\left(1 - \frac{1}{r}\right)} \cdot \|u\|_2^{1 - n\left(1 - \frac{1}{r}\right)},
\end{aligned}$$

and with $1 < r \leqq \frac{n+2}{n+1}$ we obtain $r\left(1 - n\left(1 - \frac{1}{r}\right)\right) \leqq 2$,

$$\|u \cdot \nabla u\|_{r,r}^r = \int_0^T \|u \cdot \nabla u\|_{\frac{1}{r}}^r \, dt \leqq c_6 \|u\|_{2,\infty}^{1 - n\left(1 - \frac{1}{r}\right)} \int_0^T (\|\nabla u\|_2 + \|u\|_2)^2 \, dt < \infty.$$

Therefore it holds $u \cdot \nabla u \in L^r(0, T, L^r(\Omega)^n)$ for $1 < r \leqq \frac{n+2}{n+1}$. From this we conclude that $A_2^{-\delta} P_r u \cdot \nabla u \in L^2(0, T, L^2(\Omega)^n)$ for $\delta = \frac{n}{4}$. This follows from the estimates

$$\begin{aligned}
\| A_2^{-\delta} P_r u \cdot \nabla u \|_2 &= \| A_2^{-\left(\delta - \frac{1}{2}\right)} A_2^{-\frac{1}{2}} P_r \nabla \cdot (u u) \|_{\frac{1}{2}} \\
&\leqq c_1 \| A_2^{-\frac{1}{2}} P_r \nabla \cdot (u u) \|_{\frac{1}{2} + \frac{2}{n}\left(\delta - \frac{1}{2}\right)} \leqq c_2 \| u u \|_{\frac{1}{2} + \frac{2}{n}\left(\delta - \frac{1}{2}\right)} \\
&= c_2 \| u u \|_{1 - \frac{1}{n}} \leqq c_3 \| u \|^2_{\frac{1}{2}\left(1 - \frac{1}{n}\right)} = c_3 \| u \|^2_{-\left(\frac{2}{n} \cdot \frac{1}{4} - \frac{1}{n}\right)} \\
&\leqq c_4 \| A_2^{\frac{1}{4}} u \|_2^2 \leqq c_5 \| A_2^{\frac{1}{2}} u \|_2 \| u \|_2 \\
&\leqq c_6 (\| \nabla u \|_2 + \| u \|_2) \| u \|_2,
\end{aligned}$$

$$\| A_2^{-\delta} P_r u \cdot \nabla u \|_{2,2}^2 \leqq c_6 \| u \|_{2,\infty}^2 \int_0^T (\| \nabla u \|_2 + \| u \|_2)^2 \, dt < \infty .$$

In the same way we conclude $P_2 \Delta A_2^{-\delta} u \in L^2(0, T; L^2(\Omega)^n)$ and $A_2^{-\delta} P_p f \in L^s(0, T; L^2(\Omega)^n)$. Here we use $p \leqq 2$ for exterior domains. Thus we obtain

$$\begin{aligned}
& - \int_0^T (A_2^{-\delta} u, w) h' \, dt \\
& = \int_0^T [(A_2^{-\delta} P_2 \Delta u, \tilde{w}) - (A_2^{-\delta} P_r u \cdot \nabla u, \tilde{w}) + (A_2^{-\delta} P_p f, \tilde{w})] h \, dt \\
& \quad + (A_2^{-\delta} u_0, \tilde{w}) h(0),
\end{aligned}$$

$$(A_2^{-\delta} u, \tilde{w})' = (P_2 \Delta A_2^{-\delta} u, \tilde{w}) - (A_2^{-\delta} P_r u \cdot \nabla u, \tilde{w}) + (A_2^{-\delta} P_p f, \tilde{w}) \quad \text{a.e. on } [0, T],$$

$$(A_2^{-\delta} u(0), \tilde{w}) = (A_2^{-\delta} u_0, \tilde{w}),$$

$$(A_2^{-\delta} u)' \in L^\gamma(0, T, L^2(\Omega)^n) \quad \text{with} \quad \gamma := \min(s, 2),$$

and

(3.1) $$(A_2^{-\delta} u)' - P_2 \Delta A_2^{-\delta} u + A_2^{-\delta} P_r u \cdot \nabla u = A_2^{-\delta} P_p f.$$

This condition is equivalent with 1.2, b); it is easy to apply potential theoretical methods to (3.1).

Of course we can replace $A_2^{-\delta}$ by $J_k = \left(I + \frac{1}{k} A_2\right)^{-\delta}$ for $k = 1, 2, \ldots$. Thus we obtain the equations

(3.2) $$(J_k u)' - P_2 \Delta J_k u + J_k P_r u \cdot \nabla u = J_k P_q f$$

which are regularizations of (1.1) in a certain sense.

The next theorem is our main result for bounded domains. It yields the desired regularity result for π and some regularity properties of the weak solution u.

3.3. Theorem. *Let $\Omega \subseteqq \mathbb{R}^n$ be a smooth bounded domain with $n \geqq 3$, let $s, p \in (1, \infty)$ with $n \leqq \frac{2}{s} + \frac{n}{p}$, $\frac{1}{p} + \frac{1}{n} < 1$, $\frac{1}{q} := \frac{1}{p} + \frac{1}{n}$, $f \in L^s(0, T; L^q(\Omega)^n)$ for some $T > 0$ and $u_0 \in D(A_q^{1-\frac{1}{s}+\varepsilon}) \cap H_2(\Omega)$ for some $\varepsilon > 0$, $1 - \frac{1}{s} + \varepsilon \leqq 1$. Let u be a weak solution of the Navier-Stokes equation (1.1) with data u_0 and f. Then we have*

$$u' \in L^s(0, T; L^q(\Omega)^n), \quad \Delta u \in L^s(0, T; L^q(\Omega)^n), \quad u \cdot \nabla u \in L^s(0, T; L^q(\Omega)^n),$$

and there exists a pressure $\pi \in L^s(0, T; L^p(\Omega))$ with $\nabla \pi \in L^s(0, T; L^q(\Omega)^n)$ and

$$u' - \Delta u + u \cdot \nabla u + \nabla \pi = f \quad \text{in} \quad L^s(0, T; L^q(\Omega)^n).$$

Remarks. We are mainly interested in the result $\pi \in L^s(0, T; L^p(\Omega))$ with $\frac{1}{p} + \frac{1}{n} < 1$, $n \leqq \frac{2}{s} + \frac{n}{p}$. In case $n = 3$ we may take $s = p = \frac{5}{3}$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{3}$, and we get the result $\pi \in L^{5/3}(\Omega^T)$. Of course, π is only determined up to a function $\pi_0 \colon [0, T] \to \mathbb{R}$ with $\pi_0 \in L^s(0, T)$ (for a bounded domain). If $u_0 \in D(A_2) \cap H_2(\Omega)$ $(= D(A_2))$, the assumption on u_0 is satisfied for all possible s and p. For all $n \geqq 3$, we may take $p = s = \frac{n+2}{n}$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$ in the hypothesis of 3.3.

The theorem proves the conjecture in [2; p. 780] for bounded domains.

Proof. The property $u \cdot \nabla u \in L^s(0, T; L^q(\Omega)^n)$ follows from the following estimates:

$$\begin{aligned}
\| u \cdot \nabla u \|_{\frac{1}{q}} &= \| u \cdot \nabla u \|_{\frac{1}{q} - \frac{1}{2} + \frac{1}{2}} \leqq c_1 \| u \|_{\frac{1}{q} - \frac{1}{2}} \| \nabla u \|_{\frac{1}{2}} \\
&= c_1 \| u \|_{-\left[\frac{2}{n} \cdot \frac{n}{2}\left(1 - \frac{1}{q}\right) - \frac{1}{2}\right]} \| \nabla u \|_{\frac{1}{2}}, \\
&\leqq c_2 \left\| A_2^{\frac{n}{2}\left(1 - \frac{1}{q}\right)} u \right\|_{\frac{1}{2}} \| \nabla u \|_{\frac{1}{2}} \\
&\leqq c_3 \left\| A_2^{\frac{1}{2}} u \right\|_2^{n\left(1 - \frac{1}{q}\right)} \| u \|_2^{1 - n\left(1 - \frac{1}{q}\right)} \| \nabla u \|_2 \\
&\leqq c_4 \left\| A_2^{\frac{1}{2}} u \right\|_2^{1 + n\left(1 - \frac{1}{q}\right)} \| u \|_2^{1 - n\left(1 - \frac{1}{q}\right)} \\
&\leqq c_5 (\| \nabla u \|_2 + \| u \|_2)^{1 + n\left(1 - \frac{1}{q}\right)} \cdot \| u \|_2^{1 - n\left(1 - \frac{1}{q}\right)},
\end{aligned}$$

$$\begin{aligned}
\| u \cdot \nabla u \|_{q,s}^s &= \int_0^T \| u \cdot \nabla u \|_q^s \, \mathrm{d}t \\
&\leqq c_6 \| u \|_{2,\infty}^{s\left(1 - n\left(1 - \frac{1}{q}\right)\right)} \int_0^T (\| \nabla u \|_2 + \| u \|_2)^{s\left(1 + n\left(1 - \frac{1}{q}\right)\right)} \, \mathrm{d}t.
\end{aligned}$$

Observe that $q < 2$ because of $n + 1 \leqq \frac{2}{s} + \frac{n}{q}$, $1 - \frac{1}{n}\left(\frac{2}{s} - 1\right) \leqq \frac{1}{q}$, $\frac{1}{n}\left(\frac{2}{s} - 1\right) < \frac{1}{2}$, $\frac{1}{s} < 1$.

From $n \leqq \frac{2}{s} + \frac{n}{p}$ we get $n + 1 \leqq \frac{2}{s} + n\left(\frac{1}{p} + \frac{1}{n}\right) = \frac{2}{s} + \frac{n}{q}$ and therefore it holds $s\left(1 + n\left(1 - \frac{1}{q}\right)\right) \leqq 2$. Thus we get

$$\| u \cdot \nabla u \|_{q,s}^{s} \leqq c_7 \, \| u \|_{2,\infty}^{s\left(1 - n\left(1 - \frac{1}{q}\right)\right)} \int_0^T (\| \nabla u \|_2 + \| u \|_2)^2 \, dt < \infty .$$

In the next step we use the equations (3.2), write them in the form

$$(J_k u)' - P_q \Delta (J_k u) = J_k P_q f - J_k P_q u \cdot \nabla u, \quad k = 1, 2, \ldots,$$

and apply the estimate (2.9) to obtain

$$\int_0^T \| (J_k u)' \|_q^s \, dt + \int_0^T \| J_k u \|_{H^{2,p}(\Omega)^n}^s \, dt \leqq c\left(\| A_q^{1 - \frac{1}{s} + \varepsilon} u_0 \|_q^s + \int_0^T \| f \|_q^s + \| u \cdot \nabla u \|_q^s) \, dt \right).$$

Since the right hand side does not depend on k, we get immediately $u' \in L^s(0, T; L^q(\Omega)^n)$, $u \in L^s(0, T; H^{2,q}(\Omega)^n)$ and $\Delta u \in L^s(0, T; L^q(\Omega)^n)$. Letting $k \to \infty$ we get the equation

$$u' - P_q \Delta u + P_q u \cdot \nabla u = P_q f$$

in $L^s(0, T; L^q(\Omega)^n)$. In this space we have a decomposition of the form $f - u' + \Delta u - u \cdot \nabla u = P_q(f - u' + \Delta u - u \cdot \nabla u) + \nabla \pi = \nabla \pi$, where $\nabla \pi$ is uniquely determined. Thus we obtain the equation $u' - \Delta u + u \cdot \nabla u + \nabla \pi = f$ in $L^s(0, T; L^q(\Omega)^n)$ and the property $\nabla \pi \in L^s(0, T; L^q(\Omega)^n)$. From $\frac{1}{n} - \frac{1}{q} = -\frac{1}{p}$, $p \geqq q$ we get $\| \pi \|_p \leqq c \, \| \nabla \pi \|_q$ and $\| \pi \|_{p,s} \leqq c \, \| \nabla \pi \|_{q,s}$ using the boundedness of Ω. It follows $\pi \in L^s(0, T; L^p(\Omega))$. This completes the proof. □

Remark. This proof leads to an estimate on the form

$$\int_0^T \| u' \|_q^s \, dt + \int_0^T \| u \|_{H^{2,q}(\Omega)^n}^s \, dt + \int_0^T \| \nabla \pi \|_q^s \, dt \leqq c\left(\| A_q^{1 - \frac{1}{s} + \varepsilon} u_0 \|_q^s + \int_0^T \| f \|_q^s \, dt + \| u \|_{2,\infty}^{s\left(1 - n\left(1 - \frac{1}{q}\right)\right)} \cdot \int_0^T (\| \nabla u \|_2 + \| u \|_2)^2 \, dt \right)$$

where $c = c(a, s, \varepsilon, T, \Omega) > 0$ is independent of u_0 and f. In particular we get

$$\int_0^T \| \pi \|_p^s \, dt \leqq c\left(\| A_q^{1 - \frac{1}{s} + \varepsilon} u_0 \|_q^s + \int_0^T \| f \|_q^s \, dt + \| u \|_{2,\infty}^{s\left(1 - n\left(1 - \frac{1}{q}\right)\right)} \cdot \int_0^T (\| \nabla u \|_2 + \| u \|_2)^2 \, dt \right).$$

The conclusion of Theorem 3.3 holds essentially for an exterior domain Ω too. This can be shown by combining the method given above with the method in [10] which is a localization procedure; an essential step there is the use of the formula of Bogovski-Erig ([1], [3]) concerning the equation div $w = f$, $\int_\Omega f\,dx = 0$, $w \in \mathring{H}^{1,q}(\Omega)^n$. To localize the Navier-Stokes equation we have to multiply (1.1) by a certain function $\xi \in \mathring{C}^\infty(\bar{\Omega})$ and with the help of the formula of Bogovski-Erig it is possible to make ξu divergence-free.

For an exterior domain we define $\hat{\mathring{H}}^{1,p}(\Omega)^n$ to be the closure of $\mathring{C}^\infty(\Omega)^n$ with respect to the Dirichlet norm $\|\nabla u\|_p$; let $\hat{H}^{-1,p'}(\Omega)^n$ with $\frac{1}{p'} + \frac{1}{p} = 1$ be the dual space of $\hat{\mathring{H}}^{1,p}(\Omega)^n$. Then it holds $\hat{H}^{-1,p'}(\Omega)^n \subset H^{-1,p'}(\Omega)^n$ for an exterior domain, where $H^{-1,p'}(\Omega)^n := (\mathring{H}^{1,p}(\Omega)^n)^*$ is the usual space.

We have the following result.

3.4 Theorem. *Let $\Omega \subseteqq \mathbb{R}^n$ be a smooth exterior domain with $n \geqq 3$ and $s, p, q \in (1, \infty)$ with $n \leqq \frac{2}{s} + \frac{n}{p}$, $\frac{1}{p} \leqq \frac{1}{q} \leqq \frac{1}{p} + \frac{1}{n}$, $n + 1 \leqq \frac{2}{s} + \frac{n}{q}$. We suppose $f \in L^s(0, T; \hat{H}^{-1,p}(\Omega)^n) \cap L^s(0, T; L^q(\Omega)^n)$ and $u_0 \in D(A_q^{1-\frac{1}{s}+\varepsilon}) \cap D(A_p^{1-\frac{1}{s}+\varepsilon}) \cap H_2(\Omega)$ for some $\varepsilon > 0$, $1 - \frac{1}{s} + \varepsilon \leqq 1$.*

Let u be a weak solution of the Navier-Stokes equation (1.1) *with data u_0 and f. Then we have*

$$u' \in L^s(0, T; L^q(\Omega)^n), \quad \Delta u \in L^s(0, T; L^q(\Omega)^n), \quad u \cdot \nabla u \in L^s(0, T; L^q(\Omega)^n),$$

$$u \in L^s(0, T; L^p(\Omega)^n), \quad \nabla u \in L^s(0, T; L^p(\Omega)^{n^2}), \quad u\,u \in L^s(0, T; L^p(\Omega)^{n^2}),$$

and there exists a pressure $\pi \in L^s(0, T; L^p(\Omega))$ with $\nabla \pi \in L^s(0, T; L^q(\Omega)^n)$ and

$$u' - \Delta u + u \cdot \nabla u + \nabla \pi = f \quad in \quad L^s(0, T; L^q(\Omega)^n).$$

R e m a r k s. The conditions on the data u_0 and f are always satisfied if the following holds: $u_0 \in \mathring{C}^2(\bar{\Omega})$, $f \in \mathring{C}(\bar{\Omega} \times [0, T])$. The conditions on s, p, q are satisfied in the following case: $s = p = \frac{n+2}{n}$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$. In the case $n = 3$ we get the result $\pi \in L^{5/3}(\Omega^T)$ as for a bounded domain. However, the properties $\pi \in L^s(0, T; L^p(\Omega))$, $u \in L^s(0, T; L^p(\Omega)^n)$, ... are global results for the exterior domain Ω; π and u must have a certain decay for $|x| \to \infty$.

An important consequence of the result $\pi \in L^{5/3}(\Omega^T)$, $n = 3$, is the smoothness of u for sufficiently large $|x|$, provided u fulfills additionally a generalized form of the energy inequality as in [2]. This result is known in the case $\Omega = \mathbb{R}^3$ by Caffarelli-Kohn-Nirenberg [2]; we have extended this result to arbitrary smooth exterior domains [10]. Theorem 3.4 proves the conjecture in [2; p. 780] for exterior domains.

Using the method given in [10] and Prop. 1 in [2, p. 775], we can prove the following result.

3.5 Theorem. *Let Ω be a smooth exterior domain of $\mathbb{R}^3$, $T > 0$, $p = \frac{5}{3}$, $r > \frac{5}{2}$, $\frac{1}{q} = \frac{3}{5} + \frac{1}{3}$, $f \in L^p(0, T; \hat{H}^{-1,p}(\Omega)^3) \cap L^2(0, T; H^{-1,2}(\Omega)^3) \cap L^p(0, T; L^q(\Omega)^3) \cap L^r(\Omega^T)$ and $u_0 \in D(A_q^{3/5}) \cap D(A_r^{1-\frac{1}{r}}) \cap H_2(\Omega)$.*

Then there exists a weak solution u of (1.1) which is smooth for sufficiently large $|x|$. This means that there exist constants $M > 0$, $K > 0$ such that $|u(x, t)| \leqq M$ holds for almost all x, t with $|x| \geqq K$, $t \geqq 0$.

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