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On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game*

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Summary. We provide elementary proofs of Scarf's theorem on the non-emptiness of the core and of the K-K-M-S thoerem, based on Kakutani's fixed point theorem. We also show how these proofs can be modified to apply a coincidence theorem of Fan instead of Kakutani's fixed point theorem, for some additional simplicity.

1 Introduction

Scarf (1967) showed that every NTU game whose characteristic function is 'balanced' has a non-empty core, i.e., a set of feasible utility vectors that no subset of players can improve upon. The central idea in his algorithmic proof was to approximate the general case by finitely generated balanced games. For the finitely generated case, Scarf used an ingenious ordinal analog of the simplex method of linear programming to construct points in the core.

Shapley (1973) provided an alternative proof of this result, including Billera's (1970) extension, which suggested an intimate connection to classical fixed point theory. In fact, he first extended Sperner's (1928) lemma on completely labelled cells in a triangulated simplex to the case of set-valued labels. Whereas Sperner's lemma leads directly to the theorem of Knaster, Kuratowski and Mazurkiewicz (1929) on labelled simplices (the 'K-K-M theorem'), which is virtually equivalent to Brouwer's (1912) fixed point theorem, the generalized Sperner's lemma leads to an analogous generalization of the K-K-M theorem, now called the 'K-K-M-S theorem', which in turn leads in a fairly straightforward way[1] to Scarf's theorem (or, in its π-balanced' form, to Billera's extension).

* The results presented here were first reported in Shapley (1987) and Vohra (1987). A version of our proof of Theorem 1 has also been presented in a recent book by C.D. Aliprantis, D.J. Brown and O. Burkinshaw, *Existence and Optimality of Competitive Equilibria* (1989) Springer-Verlag. We are grateful to Ky Fan, Wanda Gorgol, Tatsuro Ichiishi and Ali Khan for comments on earlier drafts. Vohra's research has been supported in part by NSF grant SES-8605630.

[1] Hildenbrand and Kirman (1976, 1988) and Ichiishi (1983) have given an especially lucid account of this construction. See also Kannai (1988), where a generalized Sperner's lemma is proved through Brouwer's fixed point theorem.

The importance of Scarf's theorem in mathematical economics has led several authors to further explore the logical connections between it and fixed-point theory, either by proving the K-K-M-S theorem from one of the standard fixed point theorems or by going directly to Scarf's core theorem by a different route. Thus, Ichiishi (1981) used Fan's (1969) coincidence theorem to obtain the K-K-M-S theorem. (Previously, in unpublished lecture notes Todd (1978) had used Kakutani's fixed point theorem (1941) to prove a special case of the K-K-M-S theorem, sufficient to prove the core theorem.) Keiding and Thorlund-Peterson (1985) prove the core theorem through the K-K-M theorem. And Ichiishi (1988) proves a dual version of the K-K-M-S theorem, again using a coincoidence theorem of Fan, and then applies it to the core existence problem.

In this paper we provide elementary proofs of both Scarf's theorem and the K-K-M-S theorem, based on Kakutani's fixed point theorem. This is of particular interest given the importance of the core and Walrasian equilibria in economics, and the fact that most results on the existence of Walrasian equilibria are based on an application of Kakutani's fixed point theorem. Indeed, even the fixed point mapping we use is not unfamiliar in general equilibrium theory; similar mappings were used by Nash (1951) and Negishi (1960).

2 Non-emptiness of the core

In this section we shall consider cooperative games in characteristic function form. Let $N=\{1,\dots,n\}$ denote the set of players and let $\mathcal{N}=2^N\setminus\{\emptyset\}$ denote the set of all non-empty subsets of N. An element of $\mathcal{N}$ is referred to as a coalition. For any coalition $S\in\mathcal{N}$, let R^S denote the $|S|$ dimensional Euclidean space with coordinates indexed by the elements of S. For $u\in R^N$, u_S will denote its restriction on R^S. We shall use the convention $\gg, >, \geq$ to order vectors in R^N. R^N_+ refers to the positive orthant of R^N and for any set $Y\subseteq R^N$, $\mathrm{Co}(Y)$, ∂Y and $\mathring{Y}$ will denote its convex hull, boundary and interior respectively. Each coalition S has a feasible set of payoffs or utilities denoted $V_S\subseteq R^S$. It is convenient to describe the feasible utilities of a coalition as a set in R^N. For $S\in\mathcal{N}$ let $V(S)=\{u\in R^N\mid u_S\in V_S\}$; i.e., $V(S)$ is a cylinder in R^N and can be alternatively defined as $V(S)=V_S\times R^{N\setminus S}$. With this interpretation in mind, we can now define an NTU game.

Definition 1. *A (normalized) non-transferable utility (NTU) game is a pair* (N, V) *where the correspondence* $V:\mathcal{N}\mapsto R^N$ *satisfies the following:*
(1.1) *$V(S)$ is non-empty, non-full and closed for all $S\in\mathcal{N}$,*
(1.2) *$V(S)$ is comprehensive for all $S\in\mathcal{N}$ in the sense that $V(S)=V(S)-R^N_+$,*
(1.3) *$V(S)$ is cylindrical for all $S\in\mathcal{N}$ in the sense that if $x\in V(S)$ and $y\in R^N$ such that $y_S=x_S$, then $y\in V(S)$,*
(1.4) *there exists $v^0\in R^N$ such that $v^0\gg 0$ and for every $j\in N$, $V(\{j\})=\{x\in R^N\mid x_j\leq v^0_j\}$,*
(1.5) *$V(S)$ is 'bounded' for all[2] $S\in\mathcal{N}$ in the sense that there exists a real number $q>0$, such that if $x\in V(S)$ and $x_S\geqq 0$, then $x_i<q$ for all $i\in S$.*

Since the core concept is translation invariant, the partial normalization we chosen, namely $v^0\gg 0$ in condition (1.4), rather the more common condition $v^0=0$ is simply a matter of convenience. It does bear mentioning that the 'boundedness'

[2] It is easy to check that in balanced games (see definition 4 below) this condition will hold for all $S\in\mathcal{N}$ if it holds for N.

condition is usually imposed on $x_S \geqq v_S^0$ rather than on $x_S \geqq 0$, as specified in (1.5). But, with a little effort, the reader can verify that given comprehensiveness there is no loss of generality in using condition (1.5).

Definition 2. The *core* of a game (N, V) is defined as

$$C(N, V) = \{u \in V(N) \mid \nexists S \in \mathcal{N} \quad \text{and } \bar{u} \in V(S) \quad \text{such that} \quad \bar{u}_S \gg u_S\}.$$

It is the set of all outcomes feasible for the grand coalition which cannot be improved upon by any coalition and can be alternatively defined as

$$C(N, V) = V(N) \setminus \bigcup_{S \in \mathcal{N}} \mathring{V}(S).$$

For any $S \in \mathcal{N}$ let e^S denote the vector in R^N whose ith coordinate is 1 if $i \in S$ and 0 otherwise. We shall also use the notation e for e^N and e^i for $e^{\{i\}}$.

Let A be the unit simplex in R^N. For every $S \in \mathcal{N}$ define

$$A^S = \mathrm{Co}\{e^i \mid i \in S\}.$$

Thus $A^N = A$, and the other A^S are its closed $(|S| - 1)$-dimensional faces. Finaly, for each $S \in \mathcal{N}$, define

$$m^S = \frac{e^S}{|S|};$$

these are the centers of gravity of the respective sets A^S as well as of the sets $\{e^i \mid i \in S\}$.

Definition 3. A set $\mathcal{B} \subseteq \mathcal{N}$ is said to be *balanced* if there exist non-negative weights λ^S, $S \in \mathcal{B}$, such that

$$\sum_{S \in \mathcal{B}} \lambda^S e^S = e^N.$$

It is easily verified that $\mathcal{B}$ is balanced if and only if

$$m^N \in \mathrm{Co}\{m^S \mid S \in \mathcal{B}\}. \tag{1}$$

Definition 4. A game (N, V) is said to be *balanced* if $\bigcap_{S \in \mathcal{B}} V(S) \subseteq V(N)$ for any balanced collection $\mathcal{B}$.

Theorem 1 (Scarf). *A balanced game has a non-empty core.*

To prove the non-emptiness of the core we shall make use of a mapping from A to a suitable modification of the boundary of the 'utility set' (defined as the union over all S of the sets $V(S)$). Towards this end, given q as in (1.5), let $Q = \{x \in R^N \mid x \leqq qe\}$ and define

$$W = \left(\bigcup_{S \in \mathcal{N}} V(S)\right) \cap Q,$$

It will be useful to consider some elementary properties of ∂W. Since every $V(S)$ is comprehensive and Q is comprehensive, so is W, i.e.

$$\text{If } u \in \partial W \quad \text{and} \quad v \gg u, \quad \text{then } v \notin W. \tag{2}$$

By (1.3) and (1.4) there exist $v^0 \gg 0$ such that, for any $j \in N$, $(q,\ldots,v_j^0,\ldots,q) \in V(\{j\})$. Clearly, $(q,\ldots,v_j^0,\ldots,q)$ also belongs to W. This, along with (2) yields

$$\text{if } u \in \partial W \text{ and } u_j = 0 \text{ for some } j \in N, \quad \text{then } u_i = q \text{ for some } i \in N. \tag{3}$$

We shall prove Scarf's theorem by showing the existence of a balanced collection $\mathscr{T}$ and $\bar{u} \in \partial W$ such that $\bar{u} \in \bigcap_{S \in \mathscr{T}} V(S)$.

Proof of Theorem 1. Consider W as constructed above and define $f: A \mapsto \partial W$ by

$$f(x) = \{y \in \partial W \mid y = tx \text{ for some } t \geq 0\}.$$

We now show that f is a well defined, continuous function. For any $x \in A$, $n(q+1)x \notin W$, while by (1.4), $0 \in \mathring{W}$. This implies that for any $x \in A$, there exists $t \in [0, n(q+1)]$ such that $tx \in \partial W$, i.e. $f(x)$ is non-empty. It is easy to verify that f is upper hemicontinuous given that for any $x \in A$, the set $\{t \in R_+ \mid tx \in \partial W\} \subseteq [0, n(q+1]$ is bounded. We now show that f is single valued. Suppose not. Then there exist $x \in A$, $y \in f(x)$, and $\hat{y} \in f(x)$ such that $y = tx$, $\hat{y} = \hat{t}x$ and $\hat{t} > t$. If $x \gg 0$, $\hat{y} \gg y$, which contradicts (2). If $I = \{i \in N \mid x_i > 0\}$ and $K = \{k \in N \mid x_k = 0\}$ are both non-empty, $tx_i < \hat{t}x_i$ for all $i \in I$. We also know, from the construction of W, that $\hat{t}x_i \leq q$ for all $i \in N$. We now have $tx_i < q$ for all $i \in I$ and $tx_k = 0$ for all $k \in K$, which contradicts (3) and establishes that f is a continuous function.

Let $G: A \mapsto A$ be defined by

$$G(x) = \{m^S \mid S \in \mathscr{N} \text{ and } f(x) \in V(S)\}.$$

Since, for all $x \in A$, $f(x) \in \bigcup_{S \in \mathscr{N}} V(S)$ and f is non-empty, so is G. We now show that G is upper hemicontinuous. Suppose $x^q \to x$, $x^q \in A$ for all q, $y^q \in G(x^q)$ for all q and $y^q \to y$. We need to establish that $y \in G(x)$. Since $G(A)$ is a finite set, there exists $\hat{q}$ such that for all $q > \hat{q}$, $y^q = y$. This implies that for all $q > \hat{q}$, $y \in G(x^q)$, i.e. there exists $S \in \mathscr{N}$ such that $y = m^S$ and $f(x^q) \in V(S)$. Since $x^q \to x$, f is continuous and, by (1.1), $V(S)$ is closed, $f(x) \in V(S)$, which means that $y \in G(x)$ and completes the proof that G is upper hemicontinuous.

Finally, for any x, $g \in A$, let $h: A \times A \mapsto A$ be defined by

$$h_i(x, g) = \frac{x_i + \max(g_i - 1/n, 0)}{1 + \sum_{j \in N} \max(g_j - 1/n, 0)} \quad \text{for all } i \in N.$$

Certainly, h is a well defined, continuous function. Now consider the mapping $h \times \mathrm{Co}(G): A \times A \mapsto A \times A$. Clearly, this satisfies the conditions of Kakutani's fixed point theorem. We can therefore assert that there exists $(\bar{x}, \bar{g}) \in A \times A$ such that $\bar{x} = h(\bar{x}, \bar{g})$ and $\bar{g} \in \mathrm{Co}(G(\bar{x}))$. This also means that

$$\bar{x}_i = \frac{\bar{x}_i + \max(\bar{g}_i - 1/n, 0)}{1 + \sum_{j \in N} \max(\bar{g}_j - 1/n, 0)} \quad \text{for all } i \in N.$$

Or,

$$\bar{x}_i\left(\sum_{j\in N}\max(\bar{g}_j-1/n,0)\right)=\max(\bar{g}_i-1/n,0)\qquad\text{for all } i\in N. \tag{4}$$

We shall now establish that $m^N\in \mathrm{Co}(G(\bar{x}))$ by showing that $\bar{g}=m^N$. Suppose not. Then, $\sum_{j\in N}\max(\bar{g}_j-1/n,0)>0$. Let $I=\{i\in N\mid \bar{x}_i>0\}$ and $K=\{k\in N\mid \bar{x}_k=0\}$. It follows from (4) that for every $i\in I$, $\bar{g}_i>1/n$. If K is empty this is impossible since $\bar{g}\in A$. Let us therefore assume that K is also non-empty. As we have already seen, for every $i\in I$, $\bar{g}_i>1/n>0$. Since $\bar{g}\in \mathrm{Co}(G(\bar{x}))$, given the construction of G, this means that for every $i\in I$, there exists S such that $i\in S$ and $f(\bar{x})\in V(S)$. Since $f(\bar{x})\geqq 0$, (1.5) implies that for every $i\in I$, $f_i(\bar{x})<q$. But $f_k(\bar{x})=0$ for all $k\in K$ which contradicts (3) and completes the proof that $\bar{g}=m^N$.

Let $\mathscr{T}=\{S\in\mathscr{N}\mid f(\bar{x})\in V(S)\}$. Since $G(\bar{x})=\{m^S\mid S\in\mathscr{T}\}$ and $m^N\in\mathrm{Co}(G(\bar{x}))$, it follows from (1) that $\mathscr{T}$ is balanced. Let $\bar{u}=f(\bar{x})$. From the definition of G it follows that $\bar{u}\in\bigcap_{S\in\mathscr{T}}V(S)$. Since the game is balanced, this implies that $\bar{u}\in V(N)$. Given (1.5) and the fact that $\bar{u}\geqq 0$, this in turn implies that $\bar{u}\ll qe$. We now claim that $\bar{u}\in C(N,V)$. Suppose not. Since $\bar{u}\in V(N)$, this must mean that there exists $S\in\mathscr{N}$ and $v\in V(S)$ such that $v_S\gg\bar{u}_S$. Since $\bar{u}\ll qe$, by (1.3) we can find $\bar{v}\gg\bar{u}$such that $\bar{v}\ll qe$ and $\bar{v}\in V(S)$; i.e. $\bar{v}\gg\bar{u}$ and $\bar{v}\in W$. Since $\bar{u}=f(\bar{x})\in\partial W$, this contradicts (2). □

3 A proof of the K-K-M-S theorem

Shapley (1973) proved the following generalization of the K-K-M theorem.

Theorem 2 (K-K-M-S). *Let $\{C^S\mid S\in\mathscr{N}\}$ be a family of closed subsets of A such that*

$$\bigcup_{S\subseteq T}C^S\supseteqq A^T\quad\textit{for each } T\in\mathscr{N}. \tag{5}$$

Then there is a balanced set $\mathscr{B}$such that

$$\bigcap_{S\in\mathscr{B}}C^S\neq\{\emptyset\}.$$

Definition 5. A *proper set-labelling* of A is a correspondence $L: A\mapsto\mathscr{N}$ of the form

$$L(x)=\{S\mid x\in C^S\},$$

where the C^S are closed subsets of A satisfying (5).

Thus, if $x\in A^T$ then $L(x)$ contains at least one subset of T. Theorem 2 is then equivalent to the statement that in any properly set labelled simplex at least one point has a balanced label.

Proof of Theorem 2. The idea of the proof is to embed A in a larger simplex A' and construct a 'Kakutani' mapping on A' with the property that its fixed points all lie in A and have balanced labels.

First, we define $G: A \mapsto A$ by

$$G(x) = \mathrm{Co}\,\{m^S \mid S \in L(x)\}\,.$$

By construction, G is convex valued. By the same argument as in the proof of Theorem 1 it can be shown that it is upper hemicontinuous. Also by (5) with $S = N$, it follows that $G(x)$ is non-empty.

Next, define $F: A \mapsto \bar{A}$ by

$$F(x) = x + m^N - G(x)\,.$$

This too is convex valued, non-empty and upper hemicontinuous. But note that its range may extend beyond A into $\bar{A}$, the hyperplane in R^N which contains A.

We see from (1) that F has a desirable property, namely, that $L(x)$ is balanced if and only if x is a fixed point of F. The domain and range of F do not agree, however. Consider the larger simplex

$$A' = \left\{ x \in R^N \,\middle|\, \sum_{i \in N} x_i = 1 \text{ and } x_i \geqq -1 \text{ for all } i \right\}.$$

We observe that $A' \supseteq A + A - A$, so A' easily contains $F(A)$—with room to spare.

Next we define a mapping which shrinks A' down to A. Let $h: A' \mapsto A$ be defined as,

$$h_i(y) = \frac{\max\,(y_i, 0)}{\sum_{j \in N} \max\,(y_j, 0)} \quad \text{for all } i \in N.$$

Next we extend the labelling function to A':

$$L'(y) = \{S \in L(h(y)) \mid y_i \geqq 0, \text{ for all } i \in S\}\,. \tag{6}$$

Since L is assumed to be proper, $L'(y)$ is non-empty; moreover, L' is upper hemicontinuous[3] and agrees with L on A. Continuing, let

$$G'(y) = \mathrm{Co}\,\{m^S \mid S \in L'(y)\}\,.$$

G' too is non-empty, convex valued and upper hemicontinuous.

Let $F': A' \mapsto A'$ be defined as

$$F'(y) = h(y) + m^N - G'(y)\,.$$

Clearly, F' too is non-empty, upper hemicontinuous and convex valued. Since $h(A') = A$, we have $F'(y) \subseteq A + A - A \subseteq A'$ for all $y \in A'$. We can, therefore, appeal to Kakutani's fixed point theorem to assert that there exists $\bar{x} \in F'(\bar{x})$.

Notice that $F'(x) = F(x)$ for all $x \in A$, so that to complete the proof it only remains to be shown that $\bar{x} \in A$. If $\bar{x} \in F'(\bar{x})$,

$$\bar{x} = h(\bar{x}) + m^N - g \quad \text{for some } g \in G'(\bar{x})\,.$$

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[3] It may be instructive to compare our mappings with those of Todd (1978). He defined $h(y)$ in a different way. More importantly, he constructed L' as

$$L'(y) = \{S \in L(h(y)) \mid h_i(y) > 0 \quad \text{for all} \quad i \in S\}\;,$$

a mapping that is not necessarily upper hemicontinuous.

Suppose $\bar{x} \in A' \setminus A$. Then $\bar{x}_i < 0$ for some i. This implies that $h_i(\bar{x}) = 0$. By (6) it also follows that $g_i = 0$. But we now have

$$\bar{x}_i = 0 + 1/n - 0 > 0\,,$$

a contradiction. Thus, $\bar{x} \in A$ and $L(\bar{x})$ is balanced. □

4 Fan's coincidence theorem

Theorems 1 and 2, which we proved by using Kakutani's fixed point theorem can both be proved, more simply, by applying a coincidence theorem of Fan (Theorem 6, (1969)).

Theorem 3 (Fan). *Let X be a non-empty compact, convex set in a real locally convex, Hausdorff topological vector space E. Let $G, H: X \mapsto E$ be two non-empty, convex-valued, upper demicontinuous correspondences and, for each $x \in X$, let at least one of $G(x)$, $H(x)$ be compact. Suppose*

$$\text{for each } x \text{ in the algebraic boundary of } X \text{ there exists } g \in G(x),\ h \in H(x) \text{ and a real number } \lambda > 0 \text{ such that } x + \lambda(g - h) \in X. \tag{7}$$

Then there exists $\bar{x} \in X$ such that $G(\bar{x}) \cap H(\bar{x}) \neq \emptyset$.

The existence issues dealt with in Theorems 1 and 2 can be viewed as problems of obtaining a 'coincidence' of the type considered in Fan's coincidence theorem. Moreover, we are interested in the special case where G, H map from A to A. Then for any $x \in A$, $g \in G(x)$ and $h \in H(x)$, we have $\sum_{j \in N} (g_j - h_j) = 0$. This means that we can choose λ small enough to ensure that (7) is satisfied, provided there exist $g \in G(x)$ and $h \in H(x)$ such that $g_j \geqq h_j$ for all j such that $x_j = 0$. Notice also that Theorem 3 remains valid for $\lambda < 0$. This provides us with the following corollary of Theorem 3.

Corollary. *Let $G, H: A \mapsto A$ be two non-empty, convex-valued, upper hemicontinuous correspondences. Suppose*

$$\text{for all } x \in A \text{ there exists } g \in G(x) \text{ and } h \in H(x) \text{ such that either } g_{\{j \in N \mid x_j = 0\}} \geqq h_{\{j \in N \mid x_j = 0\}} \text{ or } g_{\{j \in N \mid x_j = 0\}} \leqq h_{\{j \in N \mid x_j = 0\}}\,. \tag{8}$$

Then there exists $\bar{x} \in A$ such that $G(\bar{x}) \cap H(\bar{x}) \neq \emptyset$.

Recall that the basic argument in our proof of Theorem 1 is to find $x \in A$ for which $m^N \in \mathrm{Co}(G(x))$. The corollary to Theorem 3 can be applied to the mappings $\mathrm{Co}(G)$ and $H: A \mapsto A$, where $H(x) = m^N$ for all $x \in A$, to show the existence of $\bar{x}$ such that $m^N \in \mathrm{Co}(G(\bar{x}))$. All we need to verify is that (8) is satisfied. Suppose $x \in A$. Let $K = \{k \in N \mid x_k = 0\}$. From (1.2), (1.3) and (1.4) we know that $f(x) \in V(\{k\})$ for all $k \in K$. Thus $e^k \in G(x)$ for all $k \in K$. Clearly then, $m^K \in \mathrm{Co}(G(x))$ and, since $m^K_K \geqq m^N_K$, this establishes condition (8).

To prove Theorem 2 using the above corollary, define, as in Sect. 3, $G: A \mapsto A$ by

$$G(x) = \mathrm{Co}(m^S \mid S \in L(x)\}\,.$$

It is clear that G is convex valued. As observed in Sect. 3, it is also upper hemicontinuous and non-empty. Let $H(x) = m^N$ for all $x \in A$. Again, we are interested in finding $\bar{x} \in A$ such that $G(\bar{x}) \cap H(x) \neq \emptyset$. Suppose $x \in A$ and $K = \{k \in N \mid x_k = 0\}$. To verify condition (8), notice that from (5) it follows directly that there exists $g \in G(x)$ such that $g_K = 0$, i.e., $g_K \leqq m_K^N$.

Remark. Notice that in applying Kakutani's fixed point theorem to prove Theorem 1, our proof involves verifying a certain 'boundary condition'. It is in this step that we make crucial use of our normalization (1.4). An application of Fan's coincidence theorem requires a weaker 'boundary condition', for which it is possible to choose $v^0 = 0$ in (1.4); see the proof of Lemma 2 in Vohra (1990) for details. It is for a similar reason that in proving Theorem 2 through Kakutani's fixed point theorem we need to extend F from A to A'. Again, as we have seen, this is not necessary if we use Fan's coincidence theorem to prove Theorem 2.

5 π-balancedness

From our proofs of Theorems 1 and 2 it should be clear that our argument does not depend on the particular location of any of the points m^S within the relative interiors of their respective faces A^S, $S \in \mathcal{N}$. Our results, therefore, continue to hold for a wide class of generalized concepts of balancedness, as was first demonstrated by Billera (1970), and again in Shapley (1973).

Suppose there is a given array of positive numbers

$$\pi = \{\pi_{S,i} \mid S \in \mathcal{N}, i \in S\}.$$

Definition 5. A set $\mathcal{B} \subseteq \mathcal{N}$ is said to be *π-balanced* if there exist non-negative weights λ^S, $S \in \mathcal{B}$, such that

$$\sum_{\{S \in \mathcal{B} \mid i \in S\}} \lambda^S \pi_{S,i} = 1, \qquad \text{for all } i \in N.$$

It is clear that a set is π-balanced if and only if it is $\bar{\pi}$-balanced, where $\bar{\pi}$ is the 'normalization' of π given by

$$\bar{\pi}_{S,i} = \frac{\pi_{S,i}}{\sum_{j \in S} \pi_{S,j}}, \qquad \text{for all } S \in \mathcal{N}.$$

For each $S \in \mathcal{N}$ let

$$m^S(\pi) = \sum_{i \in S} \bar{\pi}_{S,i} e^i,$$

be the π-weighted center of gravity of the set $\{e^i \mid i \in S\}$. Since π is strictly positive,[4] $m^S(\pi)$ lies in the relative interior of A^S. Moreover, any collection of points in the relative interiors of A^S, $S \in \mathcal{N}$ can be expressed as the π-weighted centers of gravity of the sets $\{e^i \mid i \in S\}$, for some strictly positive π. It is easy to see that a set $\mathcal{B} \subseteq \mathcal{N}$ is π-balanced if and only if

$$m^N \in \mathrm{Co}(m^S(\pi) \mid S \in \mathcal{B}\}.$$

Definition 6. A game (N, V) is said to be *π-balanced* if $\bigcap_{S \in \mathcal{B}} V(S) \subseteq V(N)$ for any π-balanced set $\mathcal{B}$.

If we replace the mapping $G(x)$ in the proof of Theorem 1 by

$$G_\pi(x) = \mathrm{Co}(m^S(\pi) \mid S \in \mathcal{N} \text{ and } f(x) \in V(S)\},$$

we obtain, without any further changes in the proof, a result which asserts that every π-balanced game has a non-empty core. Similarly, in the proof of Theorem 2, replacing $G(x)$ by

$$G_\pi(x) = \mathrm{Co}(m^S(\pi) \mid S \in L(x)\} \ ,$$

yields the π-balanced version of the K-K-M-S theorem. It is also possible to apply Fan's coincidence to prove π-balanced versions of Theorems 1 and 2 with similar modifications to the argument presented in Sect. 4.

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