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On the continuity of expected utility*

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Summary. We provide necessary and sufficient conditions for weak (semi)continuity of the expected utility. Such conditions are also given for the weak compactness of the domain of the expected utility. Our results have useful applications in cooperative solution concepts in economies and games with differential information, in noncooperative games with differential information and in principal-agent problems.

1 Introduction

Recent work on cooperative solution concepts in economies and games with differential information (e.g. Yannelis [25], Krasa–Yannelis [16], Allen [2, 3], Koutsougeras–Yannelis [17], Page [22]) has necessitated the consideration of conditions that guarantee the (semi)continuity of an agent's expected utility.[1]

Specifically, in this paper $(\Omega, \mathscr{F}, P)$ is a probability space, representing the *states of the world* and their governing distribution, $(V, \|\cdot\|)$ a separable Banach space of *commodities*, and $X: \Omega \to 2^V$ a set-valued function, prescribing for each state ω of the world the set $X(\omega)$ of possible *consumptions*. We define the set $\mathscr{L}^1_X$ of *feasible state contingent consumption plans* to consist of all Bochner integrable a.e. selections of X, that is, the set of all $x \in \mathscr{L}^1_V$ such that

$$x(\omega) \in X(\omega) \quad \text{a.e. in } \Omega.$$

As usual, $\mathscr{L}^1_V$ stands for the (prequotient) set of all Bochner-integrable V-valued functions on $(\Omega, \mathscr{F}, P)$; the $\mathscr{L}^1$*-seminorm* on this space is defined by

$$\|x\|_1 := \int_\Omega \|x(\omega)\| P(d\omega).^2$$

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[1] This problem also arises naturally in principal-agent problems (see for example Page [19, 21]) and Kahn [14], as well as in noncooperative games with differential information (see for example, Yannelis-Rustichini [27]).

[2] Since all principal results hold modulo sets of measure zero, one could alternatively work with the usual equivalence class structure. One consequence of choosing for the prequotient setup is, of course, that the L_1-norm is traded in for its seminorm analogue.

Let $U:\Omega \times V \to [-\infty, +\infty)$ be a given *utility* function. Then the *expected utility* $I_U(x)$ of a consumption plan $x \in \mathscr{L}_X^1$ is given by

$$I_U(x) := \int_\Omega U(\omega, x(\omega))P(d\omega),$$

assuming that this integral exists. Clearly, if for each $\omega \in \Omega$ the function $U(\omega, \cdot)$ is norm-continuous and if U is integrably bounded, then the $\mathscr{L}^1$-seminorm-continuity of I_U would follow directly from Lebesgue's dominated convergence theorem [11]. However, the corresponding $\mathscr{L}^1$-compactness of $\mathscr{L}_X^1$, on which I_U is defined, is only found under quite heavy conditions, even when X has only *finite* sets as its values:

Example 1.1. *Consider for $(\Omega, \mathscr{F}, P)$ the unit interval cum Lebesgue measure. Let the consumption set $X(\omega)$ be $\{-1, +1\}$ for all ω. Then the sequence (x_k) of* Rademacher *functions $x_k:[0,1] \to \{-1, +1\}$, defined by*

$$x_k(\omega) := \operatorname{sgn}(\sin(2\pi k\omega)),$$

forms a sequence of consumption plans that does not contain any subsequence which converges in $\mathscr{L}^1$-seminorm; obviously, this implies that the set $\mathscr{L}_X^1$ cannot be compact for the $\mathscr{L}^1$-seminorm. Indeed, if such a subsequence did exist, the corresponding limit consumption plan would have to a.e. equal to zero (note that $\int_B x_k \to 0$ for every interval $B := [\alpha, \beta]$; start by observing that when $\alpha, \beta \in [0, 1]$ have finite binary expansions this is trivial). But since $\|x_k\|_1 = \int_{[0,1]} |x_k(\omega)| d\omega = 1$ for all k, the $\mathscr{L}^1$-norm of the limit consumption plan would have to be equal to 1 at the same time.

Thus, in such situations the attainment of a maximum of the expected utility is not guaranteed. To this end stronger continuity conditions (viz. *weak* continuity in the second variable) must be imposed on U. The corresponding continuity found for I_U in this way is weak continuity. At the same time, imposing weak compactness upon the values of X yields weak compactness of the set $\mathscr{L}_X^1$ (Diestel's theorem [26]). Hence, in this situation attainment of the maximum of I_U is guaranteed.

The purpose of this paper is to investigate the necessary and sufficient conditions for the following properties:

- weak and strong (semi)continuity of I_U on $\mathscr{L}_X^1$,
- weak and strong closedness and weak compactness of $\mathscr{L}_X^1$.

In view of recent work on cooperative and noncooperative solution concepts in economies and games with differential information, as well as in principal-agent problem, an answer to the above question is of fundamental importance. For this enables us to prove – via the usual forms of analysis – the existence of value and core allocations in economies with differential information, as well as the existence of a correlated equilibrium in games with differential information. The techniques employed in this paper are mostly based on classical developments in the calculus of variations and optimal control theory.

This paper is organized as follows: First, we state our principal results (section 2), and their economic applications (section 3). Our mathematical tools, their proofs, as well as all other proofs have been collected in section 4. Some notation to be used below is as follows: V^* stands for the topological dual space of $(V, \|\cdot\|)$. As usual

$\|\cdot\|^*$ stands for the dual norm on V^* [i.e., $\|x^*\|^* := \sup\{\langle x, x^*\rangle : x \in V, \|x\| \le 1\}$, where $\langle x, x^*\rangle := x^*(x)$.

2 Main results

Let us observe that the probability space $(\Omega, \mathscr{F}, P)$ can always be decomposed into an atomless part Ω_1 and a countable union Ω_2 of atoms. Let $U : \Omega \times V \to [-\infty, +\infty)$ be a given *utility function*, which we suppose to be $\mathscr{F} \times \mathscr{B}(V)$-measurable; here $\mathscr{B}(V)$ stands for the Borel σ-algebra on $(V, \|\cdot\|)$. The *expected utility* functional I_U on $\mathscr{L}_V^1$ is given by

$$I_U(x) := \int_\Omega U(\omega, x(\omega)) P(d\omega),$$

where we use the following *convention* regarding the integration of any $\mathscr{F}$-measurable function $\phi : \Omega \to [-\infty, +\infty]$: $\int \phi := \int \phi^+ - \int \phi^-$, with $+\infty - +\infty := -\infty$. Let $X : \Omega \to 2^V$ be a given set-valued function; we imagine the *consumption* set $X(\omega)$ to comprise all feasible (e.g., budgetary) consumption plans under the state of nature ω. The graph of X is supposed to be $\mathscr{F} \times \mathscr{B}(V)$-measurable. We define the set $\mathscr{L}_X^1$ of all integrable state contingent *consumption plans* by

$$\mathscr{L}_X^1 := \{x \in \mathscr{L}_V^1 : x(\omega) \in X(\omega) \ P\text{-a.e. in } \Omega\}.$$

We distinguish between *strong* and *weak* (semi)continuity of the expected utility functional I_U on $\mathscr{L}_X^1$. The first kind of continuity is with respect to the seminorm $\|\cdot\|_1$ (see section 1), and the second kind of continuity is with respect to the weak topology $\sigma(\mathscr{L}_V^1, \mathscr{L}_{V^*}^\infty[V])$, restricted to $\mathscr{L}_X^1$. Here $\mathscr{L}_{V^*}^\infty[V]$ stands for the set of all functions $p : \Omega \to V^*$ that are bounded [i.e., $\sup_{\omega \in \Omega} \|p(\omega)\|^* < +\infty$] and V-scalarly measurable [i.e., $\omega \mapsto \langle x, p(\omega)\rangle$ is $\mathscr{F}$-measurable for every $x \in V$]. It is well-known that $\mathscr{L}_{V^*}^\infty[V]$ is the dual of $(\mathscr{L}_V^1, \|\cdot\|_1)$ [12, VI]). Recall also that $\sigma(\mathscr{L}_V^1, \mathscr{L}_{V^*}^\infty[V])$ is defined as the weakest topology on $\mathscr{L}_V^1$ for which all functionals

$$x \mapsto \int_\Omega \langle x(\omega), p(\omega)\rangle P(d\omega), \quad p \in \mathscr{L}_{V^*}^\infty[V],$$

are continuous. In other words, this is the weakest topology that one could define for the consumption plans so that at least all the very simple utility functions of the type $U_p(\omega, x) := \langle x, p(\omega)\rangle$, $p \in \mathscr{L}_{V^*}^\infty[V]$, one would have the corresponding expected utility functionals $I_{U_p}(x)$ depend continuously upon the consumption plan variable x. With the same topologies in mind, we can also distinguish between strong and weak closedness of the set $\mathscr{L}_X^1$ of consumption plans. Similarly, on the commodity space V we make a distinction between the weak topology $\sigma(V, V^*)$ and the strong norm-topology (however, the corresponding σ-algebras on V coincide). Thus, we shall be considering two weak topologies and two strong topologies, respectively on the space $\mathscr{L}_V^1$ (and/or its subsets) and on the space V (and/or its subsets); from the context the reader can always deduce which space is intended.

The following *nontriviality hypothesis* will be adopted in this entire section:

$$\text{there exists at least one } \bar{x} \in \mathscr{L}_X^1 \text{ with } -\infty < I_U(\bar{x}).$$

Of course, this hypothesis is extremely mild: it only prevents a completely trivial situation. On some occasions we shall require the only slightly more restrictive *strict nontriviality hypothesis*

$$\text{there exists at least one } \bar{x}\in\mathscr{L}^1_X \text{ with } -\infty < I_U(\bar{x}) < +\infty,$$

but when this reinforcement is needed, it will always be stated explicitly.

Our first result concerns a necessary and sufficient condition for the weak closedness of the set $\mathscr{L}^1_X$ of integrable consumption plans:

Theorem 2.1. *The following statements are equivalent.*

i. $X(\omega)$ *is convex and closed a.e. in* Ω_1, *and weakly closed a.e. in* Ω_2.[3]

ii. $\mathscr{L}^1_X$ *is weakly closed.*

By Mazur's theorem the adjective "closed" for a *convex* subset of V can be interpreted equivalently as weakly closed and as strongly closed; hence "convex and closed" above needs no further specification.

Our second result is similar in nature, but now the strong closedness of the set of integrable consumption plans is addressed:

Theorem 2.2. *The following statements are equivalent.*

i. $X(\omega)$ *is strongly closed a.e. in* Ω,

ii. $\mathscr{L}^1_X$ *is strongly closed.*

In this connection it is useful to recall the following related result which has to do with weak compactness of the set of integrable consumption plans. The necessity part comes from [15, Thm. 3.6]; the sufficiency part in the above result – frequently referred to as Diestel's theorem – is better known (see for instance [26]). It has been refined in [8], using *K-convergence*, a Cesaro-type of pointwise convergence (for arithmetic averages).

Theorem 2.3 (Klei). *Suppose that the set* $\mathscr{L}^1_X$ *of integrable consumption plans is relatively weakly compact. Then*

$$X(\omega) \text{ is relatively weakly compact a.e. in } \Omega.$$

The converse implication holds also, provided that X *is integrably bounded.*

Recall here that the multifunction X is said to be *integrably bounded* if for some $\psi\in\mathscr{L}^1_{\mathbf{R}}$

$$\sup_{x\in X(\omega)} \|x\| \leq \psi(\omega) \text{ a.e. in } \Omega.$$

Note that this additional condition is essential for the sufficiency part, as is shown by the following counterexample.

Example 2.4. *Consider* $\Omega := (0,1)$, *equipped with the Borel* σ*-algebra and the Lebesgue measure* P. *Define* $X(\omega) := [0, 1/\omega]$. *Then the the sequence* $(x_k)\in\mathscr{L}^1_X$, *defined by* $x_k(\omega) := 1/\omega$ *if* $1/k \leq \omega < 1$, *and* $x_k(\omega) := 0$ *otherwise, does not have a convergent subsequence, since it is not even uniformly integrable.*

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[3] Such condensed formulations are used throughout: we mean to say that *for P-almost every* $\omega\in\Omega_1$ the set $X(\omega)\subset V$ is convex and closed, etc.

Corollary 2.5. *Suppose that the set* $\mathscr{L}_X^1$ *of integrable consumption plans is weakly compact. Then*

$$X(\omega) \text{ is convex and weakly compact a.e. in } \Omega_1,$$

$$X(\omega) \text{ is weakly compact a.e. in } \Omega_2.$$

The converse implication holds also, provided that X is integrably bounded.

Proof. Combine Theorems 2.1 and 2.3. QED

It is interesting to observe that for the strong topologies the counterpart to the above result fails as far as the sufficiency part is concerned [15, p. 316], even if $V = \mathbf{R}$ (the necessity part has an analogue [15, Prop. 3.12]). Next, we occupy ourselves with necessary conditions for weak upper semicontinuity and weak continuity of the expected utility.

Theorem 2.6. *Suppose that the expected utility* I_U *is weakly upper semicontinuous and that the set* $\mathscr{L}_X^1$ *of all integrable consumption plans is weakly closed. Suppose also that for each of the countably many atoms* $A \subset \Omega_2$ *there exist constants* $M_A, K_A > 0$ *such that*

$$U(\omega,\cdot) \leq K_A + M_A \|\cdot\| \text{ on } X(\omega) \text{ a.e. in } A.$$

Then

i. $U(\omega,\cdot)$ *is concave and upper semicontinuous on the convex closed set* $X(\omega)$ *a.e. in* Ω_1,

ii. $U(\omega,\cdot)$ *is weakly upper semicontinuous on the weakly closed set* $X(\omega)$ *a.e. in* Ω_2.

Corollary 2.7. *Suppose that the expected utility* I_U *is weakly continuous and that the set* $\mathscr{L}_X^1$ *of all integrable consumption plans is weakly closed. Suppose also that for each of the countably many atoms* $A \subset \Omega_2$ *there exists contains* $M_A, K_A > 0$ *such that*

$$|U(\omega,\cdot)| \leq K_A + M_A \|\cdot\| \text{ on } X(\omega) \text{ a.e. in } A.$$

Then, under the strict nontriviality hypothesis,

i. $U(\omega,\cdot)$ *is affine and continuous on the convex closed set* $X(\omega)$ *a.e. in* Ω_1,

ii. $U(\omega,\cdot)$ *is weakly continuous on the weakly closed set* $X(\omega)$ *a.e. in* Ω_2.

The corresponding sufficient conditions for weak upper semicontinuity and weak continuity of the expected utility are as follows:

Theorem 2.8. *Suppose that a.e. in* Ω_1

$$X(\omega) \text{ is convex and closed,}$$
$$U(\omega,\cdot) \text{ is concave and upper semicontinuous on } X(\omega),$$

and

$$U(\omega,\cdot) \leq \psi(\omega) + M\|\cdot\|$$

for some $M > 0$ *and* $\psi \in \mathscr{L}_{\mathbf{R}}^1$. *Suppose further that a.e. in* Ω_2

$$X(\omega) \text{ is weakly closed,}$$

$$U(\omega,\cdot) \text{ is weakly upper semicontinuous on } X(\omega).$$

Then I_U *is weakly upper semicontinuous on the weakly closed set* $\mathscr{L}_X^1$.

Corollary 2.9. *Suppose that a.e. in* Ω_1

$$X(\omega) \text{ is convex and closed,}$$

$$U(\omega,\cdot) \text{ is affine and continuous on } X(\omega),$$

and

$$|U(\omega,\cdot)| \leq \psi(\omega) + M\|\cdot\|$$

for some $M > 0$ *and* $\psi \in \mathscr{L}^1_{\mathbf{R}}$. *Suppose further that a.e. in* Ω_2

$$X(\omega) \text{ is weakly closed,}$$

$$U(\omega,\cdot) \text{ is weakly continuous on } X(\omega).$$

Then I_U *is weakly continuous on the weakly closed set* $\mathscr{L}^1_X$.

For strong continuity of the expected utility we have the following characterization:

Theorem 2.10. *Suppose that there exists a constant* $M > 0$ *and* $\psi \in \mathscr{L}^1_{\mathbf{R}}$ *such that*

$$U(\omega,\cdot) \leq \psi(\omega) + M\|\cdot\| \text{ on } X(\omega) \text{ a.e. in } \Omega.$$

Then the following statements are equivalent:

i. $U(\omega,\cdot)$ *is strongly upper semicontinuous on the strongly closed set* $X(\omega)$ *a.e. in* Ω,

ii. $\mathscr{L}^1_X$ *is strongly closed and* I_U *is strongly upper semicontinuous on* $\mathscr{L}^1_X$.

Corollary 2.11. *Suppose that there exist a constant* $M > 0$ *and* $\psi \in \mathscr{L}^1_{\mathbf{R}}$ *such that*

$$|U(\omega,\cdot)| \leq \psi(\omega) + M\|\cdot\| \text{ on } X(\omega) \text{ a.e. in } \Omega.$$

Then, under the strict nontriviality hypothesis, the following statements are equivalent:

i. $U(\omega,\cdot)$ *is strongly continuous on the strongly closed set* $X(\omega)$ *a.e. in* Ω,

ii. $\mathscr{L}^1_X$ *is strongly closed and* I_U *is strongly continuous on* $\mathscr{L}^1_X$.

3 Applications

3.1 Market games with differential information

Consider an *exchange economy with differential information* $\mathscr{E} = \{(X_i, U_i, \mathscr{F}_i, e_i, P): i \in I\}$, $I := \{1, \ldots, n\}$, where

i. $X_i: \Omega \to 2^V$ is a multifunction prescribing agent i's potential *consumption sets* [i.e., $X_i(\omega)$ is i's potential consumption set in state $\omega \in \Omega$],

ii. $U_i: \Omega \times V \to \mathbf{R}$ is the state dependent *utility function* of agent i,

iii. $\mathscr{F}_i$ is a sub σ-algebra of $(\Omega, \mathscr{F})$) denoting the *private information* of agent i about the state of nature,

iv. $e_i: \Omega \to V$ is the *initial endowment* of agent i, where e_i is $\mathscr{F}_i$-measurable and $e_i(\omega) \in X_i(\omega)$ P-a.e.,

v. P is a probability measure on Ω representing the *common probability beliefs* of the players concerning states of nature.

Suppose that for the economy $\mathscr{E}$ the following assumptions hold for each $i \in I$:

$$X_i(\omega) \text{ is convex, nonempty and weakly compact a.e. in } \Omega, \tag{3.1}$$

$$X_i \text{ is integrably bounded,} \tag{3.2}$$

$$U_i(\omega,\cdot) \text{ is concave and upper semicontinuous on } X_i(\omega) \text{ a.e. in } \Omega, \tag{3.3}$$

$$U_i \text{ is integrably bounded from above.} \tag{3.4}$$

Note that if the commodity space V is assumed to be a Banach lattice with an order continuous norm (which implies that the order intervals are weakly compact [1]), then it is reasonable to assume that the state contingent consumption set $X_i(\omega)$ of each agent i is contained in the order interval $[0, e(\omega)]$, where $e(\omega) := \sum_{i\in I} e_i(\omega)$.

In this case we may replace (3.1)–(3.2) with simple integrable boundedness of e_i for each agent i.

We will now indicate how our results can be used to prove the existence of a Shapley value allocation for an exchange economy with differential information (see for example [16]). For this one associates with the economy $\mathscr{E}$ the following game with side-payments: for each collection $\lambda := \{\lambda_1, \ldots, \lambda_n\}$ of nonnegative weights $\lambda_i, \sum_{i=1}^n \lambda_i = 1$, define the *side payment game* (I, V_λ) according to the following rule: for each *coalition* $S \in 2^I$, let

$$V_\lambda(S) := \sup_x \sum_{i\in S} \lambda_i \int_\Omega U_i(\omega, x_i(\omega)) P(d\omega),$$

where the supremum is taken over all $x := (x_i)_{i\in S}$, $x_i \in \mathscr{L}^1_{X_i}$, subject to

$$\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega) \text{ a.e. in } \Omega.$$

Here $\mathscr{L}^1_{X_i}$ stands for the collection of all $x_i \in \mathscr{L}^1_V(\Omega, \mathscr{F}_i, P)$ such that x_i is $\mathscr{F}_i$-measurable and $x_i(\omega) \in X_i(\omega)$ a.e. in Ω.

First, let us verify that the supremum above is actually attained, by the Weierstrass theorem. By Theorem 2.3 each $\mathscr{L}^1_{X_i}$ is weakly compact, $i \in S$; hence, so is their product. Since $x \mapsto \sum_{i\in S} x_i$ is obviously weakly continuous, we conclude that the above supremum is taken over a weakly compact set. Since each U_i satisfies the conditions in Theorem 2.8, each I_{U_i} is weakly upper semicontinuous, $i \in S$; hence, so is their sum. This proves the attainment of the supremum in the definition of the Shapley value of the game (I, V_λ). The above existence problem arises naturally if one wants either to prove the existence of a Shapley value allocation in an exchange economy with differential information or to show that a TU market game in characteristic function form is well-defined for such an economy (see for instance [25] or [2, 3]).

We now examine an application to the core of an exchange economy with differential information. Following Yannelis [25], the *private core* of $\mathscr{E}$ is defined as follows. The vector $x \in \prod_{i=1}^n \mathscr{L}^1_{X_i}$ is said to be a *private core allocation* for $\mathscr{E}$ if

i. $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$,

ii. there does not exist $S \subset I$ and $(y_i)_{i\in S} \in \prod_{i\in S} \mathscr{L}^1_{X_i}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $I_{U_i}(y_i) > I_{U_i}(x_i)$ for all $i \in S$.

Following Shapley–Shubik [24], we may convert the economy $\mathscr{E}$ to a *market game* (V, I) as follows: Define $V: 2^I \to \mathbf{R}^n$ by

$$V(S) = \left\{ z \in \mathbf{R}^{|S|} : z_i \le I_{U_i}(x_i), x_i \in \mathscr{L}^1_{X_i}, i \in S, \sum_{i\in S} x_i = \sum_{i\in S} e_i \right\};$$

here $|S|$ stands for the number of elements in S. For $S \in I$, clearly the set $V(S)$ is convex, nonempty and bounded from above. In view of Theorem 2.8 the function I_{U_i} is weakly upper semicontinuous; hence, $V(S)$ must be closed. Hence, the market game (V, I) is balanced, and has therefore a nonempty core (Scarf's theorem [23]). Standard arguments can now be applied to show that nonemptiness of the core of the game (V, I) implies nonemptiness of the core of the economy $\mathscr{E}$. Related arguments have been employed by Allen [3] to show nonemptiness of the private core of an economy with a finite-dimensional commodity space. Using the K-compactness of the $\mathscr{L}^1_{X_i}$ (as introduced in [8, Corollary 4.2]) and the sequential weak upper semicontinuity of expected utilities I_U, Page [22] has shown that the market game (V, I) corresponding to an exchange economy with an infinite dimensional commodity space is well-defined and balanced, and hence has a nonempty core.

3.2 *Principal-agent contracting games with adverse selection*

Consider a principal-agent contracting game $\mathscr{G} = \{T, X, U_1, U_2, P, Q\}$, where

i. $(T, \mathscr{T})$ is a measurable space of *agent types*,
ii. $X: \Omega \to 2^V$ prescribes the *potential payoffs* in each state of nature (i.e., $X(\omega)$ is the set of potential contract payoffs in state $\omega \in \Omega$),
iii. $U_1: T \times \Omega \times V \to \mathbf{R}$ is the *principal's utility function*, type and state dependent,
iv. $U_2: T \times \Omega \times V \to \mathbf{R}$ is the *agent's utility function*, again type and state dependent,
v. P is a probability measure on Ω, representing the principal's and the agent's common beliefs concerning states of nature,
vi. Q is a probability measure on T, representing the principal's probability beliefs concerning agent types.

Suppose that for the game $\mathscr{G}$ the following assumptions hold:

$$\text{the } \sigma\text{-algebra } \mathscr{F} \text{ is countably generated,} \tag{3.5}$$

$$X(\omega) \text{ is convex, nonempty and weakly compact a.e. in } \Omega, \tag{3.6}$$

$$X \text{ is lower measurable and integrably bounded.} \tag{3.7}$$

As a consequence, $\mathscr{L}^1_X$ forms the set of all (measurable) state contingent contracts. Also, we require:

$$\text{for each } t \in T,\ U_1(t, \omega, \cdot) \text{ is concave and upper semicontinuous on } X(\omega) \text{ a.e. in } \Omega, \tag{3.8}$$

$$\text{for each } t \in T,\ U_2(t, \omega, \cdot) \text{ is affine and continuous on } X(\omega) \text{ a.e. in } \Omega, \tag{3.9}$$

$$U_1 \text{ is product measurable and integrably bounded from above with respect to } P \times Q. \tag{3.10}$$

$$U_2 \text{ is product measurable and integrably bounded with respect to } P \times Q. \tag{3.11}$$

Note that (3.10)–(3.11) must be understood as follows: there exist $P \times Q$-integrable functions $\gamma_1, \gamma_2 : T \times \Omega \to \mathbf{R}$ with

$$\sup_{x \in V} U_1(t, \omega, x) \leq \gamma(t, \omega) \quad \text{in } T \times \Omega$$

and

$$\sup_{x \in V} |U_2(t, \omega, x)| \leq \gamma(t, \omega) \quad \text{in } T \times \Omega.$$

If the agent is of type $t \in T$ and the principal and agent enter into the contract $x \in \mathscr{L}^1_X$, then

$$I_{U_1}(t, x) := \int_\Omega U_1(t, \omega, x(\omega)) P(d\omega)$$

is the principal's expected utility, while the type t agent's expected utility is given by

$$I_{U_2}(t, x) := \int_\Omega U_2(t, \omega, x(\omega)) P(d\omega).$$

By Corollary 2.9, $I_{U_2}(t, \cdot)$ is weakly continuous on $\mathscr{L}^1_X$ for each $t \in T$, and by assumption (3.8) above, $I_{U_2}(t, \cdot)$ is also affine on $\mathscr{L}^1_X$ for each $t \in T$. Finally, I_{U_2} is $\mathscr{T} \times \mathscr{B}_w$-measurable on $T \times \mathscr{L}^1_X$, where $\mathscr{B}_w$ denotes the Borel σ-algebra for the weak topology on $\mathscr{L}^1_V$.

A *contract mechanism* is a mapping $\xi : T \to \mathscr{L}^1_X$ from agent types into the set of contracts. Let Ξ denote the set of all $(\mathscr{T}, \mathscr{B}_w)$-measurable contract mechanisms. The principal's contracting problem, with adverse selection, is now given by

$$\sup_{\xi \in \Xi} J(\xi) := \int_T I_{U_1}(t, \xi(t)) Q(dt) \tag{3.12}$$

subject to

$$I_{U_2}(t, \xi(t)) \geq I_{U_2}(t, \xi(t')) \quad \text{for all } t, t' \text{ in } T, \tag{3.13}$$

$$I_{U_2}(t, \xi(t)) \geq 0 \text{ for all } t \text{ in } T. \tag{3.14}$$

Verbally, this contracting problem can be described as follows: The principal chooses a mechanism $\xi \in \Xi$. Given the mechanism ξ chosen by the principal, the agent responds by making a report to the principal concerning his/her type. If a type t agent reports his/her type as t' (i.e., the agent lies about his/her type), then the principal and agent enter into contract $\xi(t') \in \mathscr{L}^1_X$. Constraints (3.13) are *incentive compatibility* constraints; they guarantee that the mechanism chosen by the principal induces truthful reporting by the agent, and constraints (3.14), the *individual rationality* constraints, guarantee that the mechanism chosen by the principal is such that, given truthful reporting by the agent, it is rational for the agent – no matter what his/her type – to enter into a contract with the principal. Let Ξ_0 denote the set of all $\xi \in \Xi$ satisfying (3.13)–(3.14); it is trivial to verify that Ξ_0 is convex.

In order to guarantee that there exists at least one mechanism in Ξ_0, the following nontriviality hypothesis is sufficient:

$$\text{there exists an } \bar{x} \in \mathscr{L}^1_X \text{ such that } I_{U_2}(t, \bar{x}) \geq 0 \quad \text{for all } t \in T.$$

Indeed, then the corresponding constant mechanism belongs to Ξ_0. Using the general existence result of [9], to which the above properties precisely apply, one can then conclude the existence of an optimal contract mechanism for the principal (the proof in [9] still depends heavily on the K-convergence results of [8], and thus follows essentially the same line of proofs as [21, 20], but uses the equivalence result in [10, III.2] for compact-valued multifunctions to obtain a slightly better result).

4 Mathematical preliminaries and proofs of the main results

In this section we develop the tools to be used in deriving the main results of this paper. Let $f: \Omega \times V \to [-\infty, +\infty]$ be a given function, which we suppose to be $\mathscr{F} \times \mathscr{B}(V)$-measurable. We define the *integral functional* $I_f: \mathscr{L}^1_V \to [-\infty, +\infty]$ by

$$I_f(v) := \int_\Omega f(\omega, v(\omega)) P(d\omega),$$

using the opposite of the integration convention introduced in section 2: for any $\mathscr{F}$-measurable function $\phi: \Omega \to [-\infty, +\infty]$ we still set $\int \phi := \int \phi^+ - \int \phi^-$, but this time with $+\infty - +\infty := +\infty$. Sometimes we shall wish to restrict considerations to a particular integration domain $B \subset \Omega$. We then define $I^B_f: \mathscr{L}^1_V(B) \to [-\infty, +\infty]$ by obvious restriction:

$$I^B_f(v) := \int_B f(\omega, v(\omega)) P(d\omega).$$

Throughout this section the following truly minimal *nontriviality hypothesis* will be in force:

there exists at least one $\bar{v} \in \mathscr{L}^1_V$ with $I_f(\bar{v}) < +\infty$.

We start out by giving necessary conditions for weak lower semicontinuity of I_f in the presence of atomlessness. Of course, any necessary condition for strong lower semicontinuity automatically qualifies as a necessary condition for weak lower semicontinuity (but not conversely). The following result, as well as its proof, can be found in [18] (as shown here, the fact that V is finite-dimensional in [18], does not affect the validity of the result in our present context).

Lemma 4.1. *Assume that* $(\Omega, \mathscr{F}, P)$ *is atomless*[4] *Suppose that* I_f *is strongly lower semicontinuous on* $\mathscr{L}^1_V$. *Then there exist a constant* $M > 0$ *and a function* $\psi \in \mathscr{L}^1_{\mathbf{R}}$ *such that*

$$f(\omega, \cdot) \geq \psi(\omega) - M \|\cdot\| \text{ on } V \text{ a.e. in } \Omega. \tag{4.1}$$

Proof. Suppose that (4.1) does not hold. Then for arbitrary $n \in \mathbf{N}$ the function $\psi_n: \Omega \to [-\infty, +\infty]$, defined by

$$\psi_n(\omega) := \inf_{x \in V} [f(\omega, x) + n \|x\|],$$

[4] I.e., assume that the purely atomic part Ω_2 is a null set.

and measurable by [10, III.39], satisfies

$$\int_\Omega \psi_n dP = -\infty.$$

Note here that $\psi_n(\omega) \leq f^+(\omega, \bar{v}(\omega) + n\|\bar{v}(\omega)\|$, and by virtue of the nontriviality hypothesis the right side forms a P-integrable function. By the fact that $(\Omega, \mathscr{F}, P)$ is atomless, we can find a measurable partition of Ω, all whose n components have P-measure $P(\Omega)/n$. Now for at least one such component, which we denote by $A_n \in \mathscr{F}$, it must be true that $\int_{A_n} \psi_n dP = -\infty$, by the above. Hence also $\int_{A_n} (\psi_n + 1) dP = -\infty$, and this implies in turn that $\int_{B_n} (\psi_n + 1)^- dP = +\infty$, where B_n is defined as the set of those $\omega \in A_n$ for which $\psi_n < -1$. By definition of the latter integral, there exists and integrable function $s_n: B_n \to \mathbf{R}$, $0 \leq s_n \leq (\psi_n + 1)^-$ on B_n (e.g., a step function), such that $i_n := \int_{B_n} s_n dP \geq 1$. Setting $\phi_n := -i_n^{-1} s_n$ now gives

$$P(B_n) \leq P(A_n) = P(\Omega)/n, \quad \int_{B_n} \phi_n dP = -1,$$

$$0 \geq \phi_n \geq -i_n^{-1}(\psi_n + 1)^- \geq -(\psi_n + 1)^- = \psi_n + 1 \text{ on } B_n.$$

The last inequality guarantees that for every $\omega \in B_n$ the set

$$\{x \in V : f(\omega, x) + n\|x\| \leq \phi_n(\omega)\}$$

is nonempty. So by the Von Neumann–Aumann measurable selection theorem [10, III.22] there exists a $\mathscr{F}$-measurable function $v_n: B_n \to V$ such that for a.e. ω in B_n

$$f(\omega, v_n(\omega)) + n\|v_n(\omega)\| \leq \phi_n(\omega).$$

Now either (i) $\int_{B_n} \|v_n\| dP \leq n^{-1}$ or (ii) $\int_{B_n} \|v_n\| dP > n^{-1}$. In case (i) we set $C_n := B_n$, and in case (ii) atomlessness guarantees the existence of a measurable subset C_n of B_n with $\int_{C_n} \|v_n\| dP = n^{-1}$. Outside C_n we set $v_n := \bar{v}$. In this way we end up with

$$\|v_n - \bar{v}\|_1 \leq \int_{C_n} (\|v_n\| + \|\bar{v}\|) dP \leq \frac{1}{n} + \int_{B_n} \|\bar{v}\| dP.$$

In view of $P(C_n) \leq P(B_n) \leq n^{-1}$, this shows that the sequence (v_n) converges in $\|\cdot\|_1$ to $\bar{v}$. But by the above

$$I_f(v_n) \leq \int_{\Omega \backslash C_n} f(\cdot, \bar{v}(\cdot)) dP + \int_{C_n} (\phi_n - n\|v_n\|) dP.$$

By (i)–(ii) above it is easy to see that, either way, the second integral on the right is at most -1. This means $\liminf_n I_f(v_n) \leq I_f(\bar{v}) - 1$, so that a contradiction with the lower semicontinuity hypothesis has been reached. QED

Thus, we see that for atomless $(\Omega, \mathscr{F}, P)$ the most obvious condition for the integral functional I_f to be nowhere $-\infty$, is, at the same time, a necessary condition for its strong semicontinuity. In Example 4.4 below we show that atomlessness is essential for this finding.

We shall now discuss some results which specifically address *weak* lower semicontinuity. Let us denote the duality between $\mathscr{L}^1_V$ and its dual $\mathscr{L}^\infty_{V^*}[V]$ (cf. section 2) by

$$\prec v,p \succ := \int_\Omega \langle v(\omega), p(\omega)\rangle P(d\omega).$$

The following result is well-known; for generalizations, see [13, 18, 7]. It shows that weak lower semicontinuity of I_f forces the integrand not only to be lower semicotinuous in the second variable, but convex as well. Here atomlessness is again an essential ingredient, as borne out by Example 4.4 below.

Proposition 4.2. *Assume that $(\Omega, \mathscr{F}, P)$ is atomless. Suppose that I_f is weakly lower semicontinuous on $\mathscr{L}^1_V$. Then*
i. I_f is convex on $\mathscr{L}^1_V$.
ii. $f(\omega,\cdot)$ is convex and lower semicontinuous on V a.e. in Ω.

Proof. i. Consider the *epigraph E of I_f* [5, p. 11], defined by

$$E := \{(v,\alpha)\in\mathscr{L}^1_V \times \mathbf{R} : \alpha \geq I_f(v)\}.$$

Clearly, E is a closed set for the product of the weak topology (on $\mathscr{L}^1_V$) and the ordinary topology (on $\mathbf{R}$), as a consequence of the hypothesis. We must prove that E is a convex set. To do so, we first establish the following convexity criterion: E is convex if and only if for every finite subset $\{p_1,\ldots,p_N\}$, $N\in\mathbf{N}$, of the dual space $\mathscr{L}^\infty_{V^*}[V]$ one has that

$$C := \{(\prec v,p_1\succ,\ldots,\prec v,p_N\succ,\alpha) : (v,\alpha)\in E\} \tag{4.2}$$

is a convex subset of $\mathbf{R}^{N+1}$. Indeed, for arbitrary $0<\lambda<1$, (v,α), $(v',\alpha')\in E$, we have to check that $(w,\gamma) := \lambda(v,\alpha) + (1-\lambda)(v',\alpha')$ belongs to E, viz., that $I_f(w)\leq\gamma$. If this were not true, then, by closedness of E, there would be a weakly open subset W of $\mathscr{L}^1_V$, containing w, and a $\delta>0$ such that $(v,\alpha)\notin E$ whenever $v\in W$ and $\gamma-\delta<\alpha<\gamma+\delta$. By definition of the basis of the weak topology, there exists a finite collection $\{p_1,\ldots,p_N\}\subset\mathscr{L}^\infty_{V^*}[V]$ for some $N\in\mathbf{N}$, such that for every $v\in\mathscr{L}^1_V$

$$|\prec v-w,p_i\succ| < 1,\ i=1,\ldots,N, \text{ implies } v\in W.$$

Let C be the convex set of (4.2). Evidently, by convexity, the $N+1$-vector with coordinates $\prec w,p_i\succ$, $i=1,\ldots,N$, and last coordinate γ, belongs to C. By definition of C, this means that there exists $(\tilde{v},\tilde{\alpha})\in E$ such that $\prec\tilde{v},p_i\succ = \prec w,p_i\succ$, $i=1,\ldots,N$ and $\tilde{\alpha}=\gamma$. But then the above implies $\tilde{v}\in W$ and $(\tilde{v},\tilde{\alpha})\notin E$. This contradiction proves the validity of the convexity criterion. Next, it is easy to establish that all sets C of the form (4.2) are indeed convex: Let $0<\lambda<1$ and $(v,\alpha),(v',\alpha')\in E$ be arbitrary. Then for $(w,\gamma) := \lambda(v,\alpha)+(1-\lambda)(v',\alpha')$ to belong to C it is enough to verify the existence of some $\tilde{v}\in\mathscr{L}^1_V$ with $\prec\tilde{v},p_i\succ = \prec w,p_i\succ$, $i=1,\ldots,N$ and $I_f(\tilde{v})\leq\gamma$. By Liapunov's theorem (which we may invoke because $(\Omega,\mathscr{F},P)$ is assumed to be atomless) there exists a measurable subset B of Ω such that

$$\int_B (\langle v,p_1\rangle,\ldots,\langle v,p_N\rangle, f(\cdot,v(\cdot)), \langle v',p_1\rangle,\ldots,\langle v',p_N\rangle, f(\cdot,v'(\cdot))dP$$

$$= \lambda(\prec v,p_1\succ,\ldots,\prec v,p_N\succ, I_f(v), \prec v',p_1\succ,\ldots,\prec v',p_N\succ, I_f(v')).$$

(Note that $I_f(v), I_f(v') \in \mathbf{R}$ by Lemma 4.1 and by $I_f(v) \leq \alpha$, $I_f(v') \leq \alpha'$.) Then setting $\tilde{v} := v$ on B and $\tilde{v} := v'$ on the complement of B gives the desired integrable function. This establishes the convexity of E, which immediately implies convexity of I_f.

ii. By i, I_f is a convex semicontinuous function on $\mathscr{L}^1_V$. Moreover, we find $I_f > -\infty$ (by Lemma 4.1) and $I_f(\bar{v}) < +\infty$ (by nontriviality). By a well-known result from convex analysis this implies

$$I_f^{**}(v) = I_f(v) \quad \text{for all } v \in \mathscr{L}^1_V,$$

where it should be recalled that

$$I_f^{**}(v) := \sup_{p \in \mathscr{L}^\infty_{V^*}[V]} [\prec v, p \succ - I_f^*(p)]$$

with

$$I_f^*(p) := \sup_{v \in \mathscr{L}^1_V} [\prec v, p \succ - I_f(v)]$$

define two successive instances of *Fenchel conjugation.* Now as a consequence of *decomposability* [10, p. 197] of $\mathscr{L}^1_V$ and $\mathscr{L}^\infty_{V^*}[V]$ – a formalization of the fact that these spaces are both rich in measurable functions – and the Von Neumann–Aumann measurable selection theorem one has the following integral functional representation [10, VII.7]:

$$I_f^{**}(v) = I_{f^{**}}(v) := \int_\Omega f^{**}(\omega, v(\omega)) P(d\omega).$$

Here

$$f^{**}(\omega, x) := \sup_{x^* \in V^*} [\langle x, x^* \rangle - f^*(\omega, x^*)]$$

with

$$f^*(\omega, x^*) := \sup_{x \in V} [\langle x, x^* \rangle - f(\omega, x)]$$

denote two successive Fenchel-conjugations with respect to the second argument. It should be kept in mind that for every $\omega \in \Omega$

$$f^{**}(\omega, \cdot) \text{ is the convex lower semicontinuous hull of } f(\omega, \cdot).$$

It follows therefore that $I_f(v) = I_{f^{**}}(v)$ for all $v \in \mathscr{L}^1_V$. By decomposability of $\mathscr{L}^1_V$, the nontriviality hypothesis and Lemma 4.1 we may apply [6, Thm. B.2]. This implies that

$$f(\omega, \cdot) = f^{**}(\omega, \cdot) \text{ a.e. in } \Omega.$$

This finishes the proofs. QED

We shall now obtain a characterization of *strong* lower semicontinuity of I_f, which will play an essential role in our study of the necessary conditions for *weak* lower semicontinuity on atoms; this result is valid for a general finite measure space.

Proposition 4.3. *Suppose that there exist a constant $M > 0$ and $\psi \in \mathscr{L}^1_{\mathbf{R}}$ such that*

$$f(\omega, \cdot) \geq \psi(\omega) - M \|\cdot\| \text{ on } V \text{ a.e. in } \Omega.$$

Then the following statements are equivalent:

i. $f(\omega,\cdot)$ *is strongly lower semicontinuous on* V *a.e. in* Ω,

ii. I_f *is strongly lower semicontinuous on* $\mathscr{L}^1_V$.

Proof. ii ⇒ i: From the given inequality for f it follows that

$$I_f(v) \geq \int_\Omega \psi\, dP - M\|v\|_1 \text{ for all } v \in \mathscr{L}^1_V.$$

Hence, it follows by lower semicontinuity of I_f that for every $v \in \mathscr{L}^1_V$

$$I_f(v) = \sup_{n\in\mathbf{N}} \inf_{w\in\mathscr{L}^1_V} [n\|v-w\|_1 + I_f(w)].$$

This follows by [4, p. 391]. In view of the nontriviality hypothesis and the decomposability of $\mathscr{L}^1_V$ (already used in the proof of Proposition 4.2) it follows by [6, Thm. B.1] (or by mimicking the proof of [10, Theorem VII.7]) that

$$I_f(v) = \sup_{n\in\mathbf{N}} \int_\Omega \inf_{y\in V} [n\|v(\omega)-y\| + f(\omega,y)]P(d\omega).$$

Note that, by our given inequality for f, the monotone convergence theorem can be invoked, giving

$$I_f(v) = \int_\Omega \bar{f}(\omega, v(\omega))P(d\omega), \tag{4.3}$$

where we define

$$\bar{f}(\omega,x) := \sup_{n\in\mathbf{N}} \inf_{y\in V} [n\|x-y\| + f(\omega,y)].$$

By the given inequality for f and easy ad hoc inspection (cf. [4, p. 391]) it follows from this definition that for a.e. ω

$$\bar{f}(\omega,\cdot) \text{ is the strongly lower semicontinuous hull of } f(\omega,\cdot).$$

By the nontriviality hypothesis and (4.3) it follows from [6, Thm. B.2] that

$$f(\omega,\cdot) = \bar{f}(\omega,\cdot) \text{ a.e. in } \Omega,$$

giving i.

i ⇒ ii: Let (v_k) be an arbitrary sequence in $\mathscr{L}^1_V$ such that $\|v_k - v_0\|_1 \to 0$. Let $\gamma := \liminf_k I_f(v_k)$. Then for some subsequence (v_{k_i}) we shall actually have $\gamma = \lim_i I_f(v_{k_i})$. By [4, 2.5.3] there exists a further subsequence of (v_{k_i}), say (v_{k_j}), such that for a.e. ω

$$\lim_{j\to\infty} \|v_{k_j}(\omega) - v_0(\omega)\| = 0.$$

Therefore, Fatou's lemma gives

$$\gamma + M\|v_0\|_1 = \lim_j \int_\Omega [f(\omega, v_{k_j}(\omega)) + M\|v_{k_j}(\omega)\|]P(d\omega) \geq I_f(v_0) + M\|v_0\|_1$$

(the integrand in the middle expression is minorized by the integrable function $\psi(\omega)$). This shows the validity of ii. QED

Note the similarity of our proofs of Proposition 4.2 and of the necessity part of the above result. A much more complicated, hybrid version of both results was given in [6], in connection with certain classical notions in the calculus of variations.

Even though Proposition 4.3 captures the semicontinuity aspect of its counterpart Proposition 4.2, there can be no question of emulating the convexity aspect of Proposition 4.2 or the boundedness feature of Lemma 4.1 if atomlessness is no longer satisfied:

Example 4.4. *Let $(\Omega, \mathscr{F}, P)$ be the purely atomic measure space consisting of the singleton $\{\tilde{\omega}\}$ with $P(\{\tilde{\omega}\})=1$. Consider as V the separable Banach space formed by all continuous real-valued functions on the unit interval* $[0,1]$; *the norm on V is the usual supremum norm. Define $f(\tilde{\omega}, x) := -[x(0)]^2$; this is evidently a nonconvex function. However, if $v_\iota \to v_0$ weakly, then (equivalently) $v_\iota(\tilde{\omega}) \to v_0(\tilde{\omega})$ weakly in V. Now V^* is known to be identifiable with the set of all bounded signed Borel measures on* $[0,1]$; *in particular, V^* contains the point probability concentrated at* 0. *This immediately implies the convergence of $I_f(v_\iota) = f(\tilde{\omega}, v_\iota(\tilde{\omega}))$ to $I_f(v_0) = f(\tilde{\omega}, v_0(\tilde{\omega}))$. Thus, I_f is weakly continuous, but $f(\omega, \cdot)$ is neither convex – let alone affine – nor does it obey the lower bound in Lemma* 4.1.

Necessary conditions for weak lower semicontinuity of I_f take on a particularly easy form on atoms. We shall see how Proposition 4.3 plays an auxiliary role in connection with the following lemma:

Lemma 4.5. *Let A be an atom of $(\Omega, \mathscr{F}, P)$. Then every function $v: \Omega \to V$ which is measurable with respect to $\mathscr{F}$ and $\mathscr{B}(V)$ is constant a.e. on A. More generally, every multifunction $\Gamma: \Omega \to 2^V$ which has strongly closed values and for which* gph $\Gamma :=$ $\{(\omega, x) \in \Omega \times V : x \in \Gamma(\omega)\}$ *is $\mathscr{F} \times \mathscr{B}(V)$-measurable, is equal to a constant set a.e. on A.*

Proof. Let (x_j) be a sequence in V which is strongly dense. For arbitrary $j \in \mathbf{N}$, the function

$$\phi_j : \omega \mapsto \operatorname{dist}(x_j, \Gamma(\omega)) := \inf_{x \in \Gamma(\omega)} \|x - x_j\|$$

is measurable by [10, III.30]. By an elementary property of measurable, real-valued functions on atoms, ϕ_j must be a.e. constant on A for every j. It remains to observe that when two strongly closed subsets C, D of V satisfy $\operatorname{dist}(x_j, C) = \operatorname{dist}(x_j, D)$ for all j, then $C = D$. QED

Proposition 4.6. *Let A be an atom of $(\Omega, \mathscr{F}, P)$. Suppose that I_f^A is weakly lower semicontinuous on $\mathscr{L}_V^1(A)$ and that there exist constants M, $K > 0$ such that*

$$f(\omega, \cdot) \geq K - M \|\cdot\| \text{ on } V \text{ a.e. in } A.$$

Then

$$f(\omega, \cdot) \text{ is weakly lower semicontinuous on } V \text{ a.e. in } A.$$

Proof. A *fortiori*, I_f^A is strongly lower semicontinuous on $\mathscr{L}_V^1(A)$, so by Proposition 4.3.

$$f(\omega, \cdot) \text{ is strongly lower semicontinuous on } V \text{ a.e. in } A. \tag{4.4}$$

Therefore, the multifunction $\Gamma:\Omega\to 2^{V\times\mathbf{R}}$, defined by

$$\Gamma(\omega):=\{(x,\lambda)\in V\times\mathbf{R}:\lambda\geq f(\omega,x)\},$$

satisfies all conditions of Lemma 4.5. It follows that there exist a null set N and a closed set $C\subset V\times\mathbf{R}$ such that $\Gamma(\omega)=C$ for all $\omega\in A\setminus N$. It thus follows that there exists a strongly lower semicontinuous function $g:V\to(-\infty,+\infty]$ such that

$$f(\omega,\cdot)=g \text{ for all } \omega\in A\setminus N. \tag{4.5}$$

It remains to show that g is also *weakly* lower semicontinuous. To this end, let (x_ι) be a generalized sequence weakly converging to x_0 in V. Define, correspondingly, $v_\iota\in\mathscr{L}^1_V(A)$ by $v_\iota(\omega):=x_\iota$; then (v_ι) converges weakly in $\mathscr{L}^1_V(A)$ to v_0, so we get

$$P(A)g(x_0)=\int_A f(\omega,v_0(\omega))P(d\omega)\leq\liminf_\iota\int_A f(\omega,v_\iota(\omega))P(d\omega)=P(A)\liminf_\iota g(x_\iota),$$

thanks to lower semicontinuity of I^A_f. QED

The pattern emerging from the aforegoing results is as follows: (a) in the presence of atomlessness, weak lower semicontinuity of the integral functional is associated with lower semicontinuity and convexity of the integrand (in the second variable); (b) on atoms this is associated with weak lower semicontinuity of the integrand (*without* convexity). This impression is confirmed by the following result.

Proposition 4.7. *Assume that $(\Omega,\mathscr{F},P)$ is atomless. Suppose that a.e. in Ω*

$$f(\omega,\cdot) \text{ is convex and lower semicontinuous on } V,$$

and

$$f(\omega,\cdot)\geq\psi(\omega)-M\|\cdot\| \text{ on } V$$

for some constant $M>0$ and $\psi\in\mathscr{L}^1_{\mathbf{R}}$. Then I_f is weakly lower semicontinuous on $\mathscr{L}^1_V$.

Proof. The integral functional I_f is strongly semicontinuous (by Proposition 4.3) and convex (obvious). Therefore, it must also be weakly lower semicontinuous (Mazur's theorem [5, I.3.5]). QED

Remark 4.8. *Combining Lemma 4.1 and Proposition 4.2.ii, we observe that the converse of the implication in Proposition 4.7 is also valid.*

On atoms, on the other hand, the situation is even simpler:

Proposition 4.9. *Let A be an atom of $(\Omega,\mathscr{F},P)$. Suppose that*

$$f(\omega,\cdot) \text{ is weakly lower semicontinuous on } V \text{ a.e. in } A.$$

Then the integral functional I^A_f is weakly lower semicontinuous on $\mathscr{L}^1_V(A)$.

Proof. Note first that *a fortiori*

$$f(\omega,\cdot) \text{ is strongly lower semicontinuous on } V \text{ a.e. in } A.$$

So we can repeat the part of the proof of Proposition 4.6 leading from (4.4) to (4.5). Using the notation introduced there, we get for every $v\in\mathscr{L}^1_V(A)$

$$I^A_f(v)=\int_A f(\omega,v(\omega))P(d\omega)=P(A)g(x),$$

where x stands for the a.e. constant value taken by v on the atom A (Lemma 4.5), and where g is weakly lower semicontinuous. The proof is now easily finished.

QED

Proof of Theorem 2.1. ii⇒i: Define the $\mathscr{F} \times \mathscr{B}(V)$-measurable function $f_X: \Omega \times V \to \{0, +\infty\}$ by setting $f_X(\omega, x) := 0$ if $x \in X(\omega)$ and $f_X(\omega, x) := +\infty$ if not. Clearly, the integral functional $I := I_{f_X}$ is as follows: $I(v) = 0$ if $v \in \mathscr{L}^1_X$ and $I(v) = +\infty$ if not (note in particular that $I(\bar{x}) < +\infty$ by the nontriviality hypothesis). Therefore, weak closedness of $\mathscr{L}^1_X$ is *equivalent* to I being weakly lower semicontinuous on $\mathscr{L}^1_V$. Because of the obvious identity

$$I(v) = \int_{\Omega_1} f_X(\omega, v(\omega))P(d\omega) + \int_{\Omega_2} f_X(\omega, v(\omega))P(d\omega) = : I_1(v) + I_2(v),$$

we see that this is equivalent to having I_1 weakly lower semicontinuous on $\mathscr{L}^1_V(\Omega_1)$ and I_2 on $\mathscr{L}^1_V(\Omega_2)$ separately. By Proposition 4.2 (note that $f_X \geq 0$) the semicontinuity of I_1 implies

$$f_X(\omega, \cdot) \text{ is convex and lower semicontinuous for a.e. } \omega \in \Omega_1,$$

which in turn is precisely equivalent to the first part of i. Also, semicontinuity of I_2 on $\mathscr{L}^1_V(\Omega_2)$ implies that on every atom A which is part of Ω_2 (note that $f_X \geq 0$)

$$f_X(\omega, \cdot) \text{ is weakly lower semicontinuous on } A,$$

by virtue of Proposition 4.6. Since Ω_2 is the countable union of such atoms, this finishes the proof of *i*.

i⇒ii: f_X now clearly satisfies the conditions of Proposition 4.7 on Ω_1 and Proposition 4.9 on Ω_2. Therefore, I is weakly lower semicontinuous; in view of what was said about $I := I_{f_X}$ above, this implies ii. QED

Proof of Theorem 2.2. Define f_X and $I := I_{f_X}$ as in the proof of the previous theorem. Then the result follows directly from Proposition 4.3. QED

Proof of Theorem 2.6. By Theorem 2.1 we already know the stated facts about the values $X(\omega)$. Define the $\mathscr{F} \times \mathscr{B}(V)$-measurable function $f: \Omega \times V \to [-\infty, +\infty]$ as follows: set $f(\omega, x) := -U(\omega, x)$ if $x \in X(\omega)$ and $f(\omega, x) := +\infty$ if not. Then I_f equals $-I_U$ on $\mathscr{L}^1_X$ (by the integration conventions) and $+\infty$ on $\mathscr{L}^1_V \backslash \mathscr{L}^1_X$ [note how the switch in sign precisely explains the difference in integration conventions and nontriviality hypotheses between sections 2 and the present one!]. It follows directly from the hypotheses that I_f is weakly lower semicontinuous on $\mathscr{L}^1_V$, so by splitting I_f over the atomless part Ω_1 and its complement Ω_2, as done in the proof of Theorem 2.1, and successively applying Propositions 4.2 and 4.6, we find that for a.e. $\omega \in \Omega_1$

$$f(\omega, \cdot) \text{ is convex and lower semicontinuous on } V, \tag{4.6}$$

and for a.e. $\omega \in \Omega_2$

$$f(\omega, \cdot) \text{ is weakly lower semicontinuous on } V.$$

In view of the already established properties of $X(\omega)$, the former is equivalent to

$$U(\omega, \cdot) \text{ is convex and lower semicontinuous on } X(\omega),$$

and the latter to

$U(\omega, \cdot)$ is weakly lower semicontinuous on $X(\omega)$. QED

Proof of Theorem 2.8. Define f as in the previous proof. Then our conditions guarantee that Propositions 4.7 and 4.9 may be applied. In view of the already established weak closedness of $\mathscr{L}^1_X$ (Theorem 2.1), the desired weak continuity of $-I_U$ follows from the nature of I_f, established in the previous proof. QED

Proof of Theorem 2.10. The proof essentially consists of an application of Proposition 4.3 to the function f used in the previous two proofs. Details are left to the reader. QED

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