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Cournot–Walras equilibria in markets with a continuum of traders*

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Summary. In a model with a nonatomic continuum of traders some of which behave strategically while the others remain price-takers, the set of Cournot–Walras equilibria coincides with the set of Walras equilibria. This property is no longer valid when the strategic agents are represented by atoms.

1. Introduction

In a recent paper (Codognato and Gabszewicz (1991)), – published in french, but available in english (Codognato and Gabszewicz (1990)), – we have analyzed the oligopolistic behaviour of economic agents in the framework of a *finite* exchange economy. In particular, we propose a noncooperative concept of equilibrium – the Cournot–Walras equilibrium –, which is the natural counterpart of the same concept defined in Gabszewicz and Vial (1972) for an economy with production. Furthermore, we study an example revealing that, in the finite framework, the Cournot–Warlas equilibrium generally differs from the Walrasian outcome: The oligopolists "exploit" the "small" traders at the noncooperative equilibrium. Nevertheless, using the same example, we prove that the Cournot–Walras equilibrium converges by replication to the Walrasian equilibrium.

The above analysis invites spontaneously to examine whether, in a model "à la Aumann" with a *continuum* of traders (Aumann (1964)), some of which behave strategically (the "oligopolists") while the others (the "small" traders) remain price-takers, the strategic agents can still exploit the small traders at a Cournot–Walras equilibrium. The main result of the present paper shows that this cannot be the case in a model with an *atomless* continuum of oligopolists and small traders: in this model, the set of Cournot–Walras equilibria coincides with the set of Walras equilibria. Nonetheless this result rests crucially on the fact that the actions of the oligopolists are negligible in an atomless continuum, even if they act consciously in a strategic way. One may wonder whether in a *mixed* exchange economy "à la

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Shitovitz" consisting of both an atomless sector (the small traders) and atoms (the oligopolists), a similar result would still hold. This possibility should indeed be considered since this equivalence property remains valid for the core of such a mixed exchange economy whenever the atoms are of the same type (Shitovitz (1973)). Here we provide a negative answer to this question: We show by an example that, if the oligopolists are represented by atoms, the Cournot–Walras equilibrium may differ from the competitive outcome.

The equilibrium concept used in our analysis is of the Cournot type. The oligopolists use quantity strategies, manipulating the fraction of their initial holdings they send to the market for trade: Each oligopolist can a priori exert a partial control over the equilibrium prices, via his individual supply. By contrast the small traders behave as price takers without trying to influence the market clearing mechanism. A Cournot–Walras equilibrium is a noncooperative equilibrium of the game whose players are the oligopolists, strategies are their individual supplies, and whose payoffs are the utility levels which can be reached through the exchange mechanism operating under the market clearing conditions.

In section 2, we present the continuous version of the exchange model and state our main result, while section 3 is devoted to the proof of this result. Section 4 provides the example referred to above, through which we show that in a market with atoms, the equivalence result fails. Finally in our last section, we discuss the literature related to the approach considered in this paper.

2. The model

We shall be working in the space R^l_+. The dimension l represents the number of different commodities traded in the market. We shall denote by $x = (x^1, \dots, x^h, \dots, x^l)$ a vector of R^l_+. Let $(T, \mathscr{T}, \mu)$ be a measure space of economic agents, where T denotes the set of traders. We take $T = [0, 2]$, $\mathscr{T}$ the σ-field of Lebesgue measurable subsets of T and μ the Lebesgue measure on $\mathscr{T}$. A trader $t \in [0,1] \underset{\text{Def}}{=} T_1$ is called an *oligopolist*, while a trader t in $[1,2] \underset{\text{Def}}{=} T_2$ denotes a *small trader*.

A *commodity bundle* is a point in R^l_+. An *assignment* (of commodity bundles to traders) is an integrable function $\boldsymbol{x}$ from T to R^l_+. All integrals are with respect to $t, t \in T$. There is a fixed *initial assignment* $\boldsymbol{w}$. We assume that

$$\begin{aligned} \boldsymbol{w}(t) &= (w^1(t), 0, \dots, 0), \quad t \in T_1 \\ &= (0, w^2(t), \dots, w^h(t), \dots, w^l(t)), \quad t \in T_2, \end{aligned}$$

and $\int_T \boldsymbol{w}(t) d\mu \gg 0$. Accordingly, the oligopolists are assumed to have a corner on commodity 1, while the ownership of commodities $2, \dots, l$ is spread among the small traders.

For each trader t, a continuous *utility function* $U_t(x)$ is defined on R^l_+ satisfying the following assumptions:

(A.1) $$x > y \Rightarrow U_t(x) > U_t(y);$$

(A.2) $$U_t(x) \geq U_t(y) \Rightarrow \forall \alpha \in (0,1),\ U_t(\alpha x + (1-\alpha) y) > U_t(y).$$

An *allocation* is an assignment $\boldsymbol{x}$ for which

$$\int_T \boldsymbol{x}(t)d\mu = \int_T \boldsymbol{w}(t)d\mu.$$

A *price system* is an l-tuple of nonnegative real numbers, not all of which vanish. Given a price system p we define the *budget set* $B_t(p)$ of a trader t by

$$B_t(p) = \{x \in R^l_+ \mid p\cdot x \leq p\cdot \boldsymbol{w}(t)\}.$$

A *strategy* for oligopolist t, t in T_1, is a real number in the interval $S_t \underset{\text{Def}}{=} [0, \boldsymbol{w}^1(t)]$. A *strategy profile* is a real valued integrable function $\boldsymbol{e}$ defined on T_1 such that, for all $t\in T_1$, $\boldsymbol{e}(t)\in S_t$. Accordingly, given a strategy profile $\boldsymbol{e}$, $\boldsymbol{e}(t)$ represents the amount of commodity 1 that oligopolist t chooses to sell on the market for commodity 1. Consider a small trader $t\in T_2$ and a fixed price system p. By assumption, this trader behaves competitively on all markets so that his demand is the solution to the problem

$$\max_{x\in B_t(p)} U_t(x).$$

Under assumptions (A.1) and (A.2), there exists a unique solution to this problem which we denote by $x(t,p)$. Now consider an oligopolist $t\in T_1$, a price system p and a strategy profile $\boldsymbol{e}$. Given the choice $\boldsymbol{e}(t)$, the income of oligopolist t (i.e. the value of resources brought to the market) is equal to $p^1\cdot \boldsymbol{e}(t)$. If the oligopolist t, endowed with income $p^1\cdot \boldsymbol{e}(t)$ chooses to buy a commodity bundle $(x^2,\dots,x^l)$ of the commodities owned initially by the small traders, he reaches a utility level equal to $U_t(\boldsymbol{w}^1(t)-\boldsymbol{e}(t), x^2,\dots,x^l)$. Accordingly, given p and $\boldsymbol{e}(t)$, each oligopolist in T_1 solves the problem

$$\max_{(x^2,\dots,x^l)} U_t(\boldsymbol{w}^1(t)-\boldsymbol{e}(t), x^2,\dots,x^l) \quad \text{s.t.} \sum_{h=2}^{l} p^h x^h = p^1\cdot \boldsymbol{e}(t).$$

Under assumptions (A.1) and (A.2), there exists also a unique solution to this problem, and, for $t\in T_1$, we represent by $x(t,p)$ the vector $(\boldsymbol{w}^1(t)-\boldsymbol{e}(t),\ x^2(t,p),\dots,x^h(t,p),\dots,x^l(t,p))$, where $(x^2(t,p),\dots,x^h(t,p),\dots,x^l(t,p))$ denotes this unique solution. Let $\boldsymbol{x}(\cdot,p)$ be the function on T with values in R^l_+ defined by $\boldsymbol{x}(t,p)=x(t,p)$. We shall assume that, for all $p\in R^l_+$, $\boldsymbol{x}(\cdot,p)$ is an assignment. Given a strategy profile $\boldsymbol{e}$ we denote by $p(\boldsymbol{e})$ a price system such that

(i) $\int_0^1 \boldsymbol{e}(t)d\mu = \int_1^2 \boldsymbol{x}^1(t,p(\boldsymbol{e}))d\mu;$

(ii) $\int_0^2 \boldsymbol{x}^h(t,p(\boldsymbol{e}))d\mu = \int_1^2 \boldsymbol{w}^h(t)d\mu, \quad h=2,\dots,l:$

$p(\boldsymbol{e})$ is a price system which clears all markets in the exchange economy where traders t in T_1 are endowed with initial holdings $(\boldsymbol{e}(t),0,\dots,0)$ while traders t in T_2 have their initial assignment vectors $\boldsymbol{w}(t)$.

We assume that, for all strategy profiles $\boldsymbol{e}$, $p(\boldsymbol{e})$ exists and is unique. We denote by $\boldsymbol{e}\backslash e(\tau)$ the strategy profile which coincides with $\boldsymbol{e}$ for all $t\in T_1$, except for $t=\tau$ with $e(\tau)\in S_\tau$, $e(\tau)\neq \boldsymbol{e}(\tau)$.

Given a strategy profile e, consider the assignment $x(\cdot, p(e))$. Clearly, for $h = 1$, we obtain

$$\int_0^2 x^1(t, p(e))d\mu = \int_0^1 w^1(t)d\mu - \int_0^1 e(t)d\mu + \int_1^2 x^1(t, p(e))d\mu$$

$$= \int_0^1 w^1(t)d\mu, \quad \text{by (i)}$$

and, for $h = 2, \ldots, l$

$$\int_0^2 x^h(t, p(e))d\mu = \int_1^2 w^h(t)d\mu, \quad \text{by (ii)}$$

so that the assignment $x(\cdot, p(e))$ is an allocation. A *Cournot–Walras equilibrium* is a pair $(\tilde{e}, \tilde{x})$ consisting of a strategy profile $\tilde{e}$ and an allocation $\tilde{x}$ such that,

$$\tilde{x}(t) = x(t, p(\tilde{e})), \quad \forall t \in T$$

and, $\forall t \in T_1$, $\forall e(t) \in S_t$,

$$U_t(\tilde{x}(t)) = U_t(w^1(t) - \tilde{e}(t), x^2(t, p(\tilde{e})), \ldots, x^l(t, p(\tilde{e})))$$

$$\geq U_t(w^1(t) - e(t), x^2(t, p(\tilde{e} \backslash e(t))), \ldots, x^l(t, p(\tilde{e} \backslash e(t)))).$$

At a Cournot–Walras equilibrium, each oligopolist chooses his supply of commodity 1 in such a way that, given the supplies chosen by the other oligopolists on the same market and the resulting equilibrium prices, no deviation from this choice can increase his utility. A *Walras equilibrium* is a pair (p^*, x^*) consisting of a price system p^* and an allocation x^* such that, for all $t \in T$, $x^*(t) \in B_t(p^*)$ and $B_t(p^*) \cap \{x \mid U_t(x) > U_t(x^*(t))\} = \phi$.

Now we may state:

Theorem: If $(\tilde{e}, \tilde{x})$ is a Cournot–Walras Equilibrium, there exists a price system $\tilde{p}$ such that $(\tilde{p}, \tilde{x})$ is a Walras equilibrium. Conversely, if (p^*, x^*) is a Walras equilibrium, there exists a strategy profile e^* such that (e^*, x^*) is a Cournot–Walras equilibrium.

3. Proof of the Theorem

As a first step towards proving the theorem, we first demonstrate

Lemma: For all strategy profiles $\hat{e}$ and all $\tau \in T_1, e(\tau) \in S_\tau$,

$$p(\hat{e}) = p(\hat{e} \backslash e(\tau)).$$

Proof. By definition of $p(\hat{e})$, we have

$$\int_0^1 \hat{e}(t)d\mu = \int_1^2 x^1(t, p(\hat{e}))d\mu$$

and

$$\int_0^2 x^h(t, p(\hat{e}))d\mu = \int_0^1 w^h(t)d\mu, \quad h = 2, \ldots, l.$$

We notice that $p(\hat{\mathbf{e}})$ only depends on the integral $\int_0^1 \hat{\mathbf{e}}(t)d\mu$ so that we may write $p(\hat{\mathbf{e}})$ as $p(\hat{\mathbf{e}}) = p(\int_0^1 \hat{\mathbf{e}}(t)d\mu)$. Consider the strategy profile $\hat{\mathbf{e}}\backslash e(\tau)$. Since μ is non atomic, $\{\tau\}$ is a set of measure 0 so that

$$\int_0^1 \hat{\mathbf{e}}(t)d\mu = \int_0^1 \hat{\mathbf{e}}(t)\backslash e(\tau)d\mu.$$

Thus

$$p(\hat{\mathbf{e}}) = p\left(\int_0^1 \hat{\mathbf{e}}(t)d\mu\right) = p\left(\int_0^1 \hat{\mathbf{e}}(t)\backslash e(\tau)d\mu\right) = p(\hat{\mathbf{e}}\backslash e(\tau)). \quad \blacksquare$$

Now we prove the first part of the theorem.

Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot–Walras equilibrium and $\tilde{p} \underset{\text{Def}}{=} p(\tilde{\mathbf{e}})$ the price system for which (i) and (ii) hold, given $\tilde{\mathbf{e}}$. We shall show that $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. First consider a trader t in T_1 and his budget set $B_t(\tilde{p})$. We show that, for all $t \in T_1$, $\tilde{\mathbf{x}}(t) = x(t, p(\tilde{\mathbf{e}})) \in B_t(\tilde{p})$. By definition of a Cournot–Walras equilibrium, we must have, for all $t \in T_1$,

$$\sum_{h=2}^{l} \tilde{p}^h x^h(t, \tilde{p}) = \tilde{p}^1 \cdot \tilde{\mathbf{e}}(t). \tag{3.1}$$

Consider the commodity bundle $\tilde{\mathbf{x}}(t) = (w^1(t) - \tilde{\mathbf{e}}(t), \mathbf{x}^2(t, \tilde{p}), \ldots, x^l(t, \tilde{p}))$. We evaluate $\tilde{p} \cdot \tilde{\mathbf{x}}(t)$, i.e.

$$\begin{aligned} \tilde{p} \cdot \tilde{\mathbf{x}}(t) &= \tilde{p}^1(w^1(t) - \tilde{\mathbf{e}}(t))) + \sum_{h=2}^{l} \tilde{p}^h x^h(t, \tilde{p}) \\ &= \tilde{p}^1 w^1(t) - \tilde{p}^1 \tilde{\mathbf{e}}(t) + \sum_{h=2}^{l} \tilde{p}^h x^h(t, \tilde{p}) \\ &= \tilde{p}^1 w^1(t), \end{aligned}$$

where the last equality follows from (3.1). By definition of a Cournot–Walras equilibrium, it is immediate that $\tilde{\mathbf{x}}(t) \in B_t(\tilde{p})$ for all $t \in T_2$. Now let us show that, for all $t \in T$, $\tilde{\mathbf{x}}(t)$ is a maximal element for U_t in $B_t(\tilde{p})$. That this is so is an immediate consequence of the definition of a Cournot–Walras equilibrium whenever $t \in T_2$. Consider now a trader t in T_1 and for each price system p and strategy $e \in S_t$, define $\varphi_t(e, p)$ as the commodity bundle which solves the problem

$$\max_{x^2, \ldots, x^l} U_t(w^1(t) - e, x^2, \ldots, x^l) \quad \text{s.t.} \sum_{h=2}^{l} p^h x^h = p^1 \cdot e.$$

Clearly, according to assumptions (A.1) and (A.2), $\varphi_t(e, p)$ exists and is unique for all p and $e \in S_t$. Notice that $\tilde{\mathbf{x}}(t) = \varphi_t(\tilde{\mathbf{e}}(t), \tilde{p})$. Also, by the definition of a Cournot–Walras equilibrium, $\tilde{\mathbf{e}}(t)$ must solve

$$\max_{e \in S_t} U_t(\varphi_t(e, p(\tilde{\mathbf{e}}, \backslash e))) = \max_{e \in S_t} U_t(\varphi_t(e, \tilde{p})),$$

where the last equality follows from the lemma. Accordingly, for all $e \in S_t$,

$$U_t(\varphi_t(e, p(\tilde{\mathbf{e}}\backslash e))) \leq U_t(\varphi_t(\tilde{\mathbf{e}}(t), \tilde{p})) = U_t(\tilde{\mathbf{x}}(t)). \tag{3.2}$$

Now suppose that there exists a trader t in T_1 for which $\tilde{\mathbf{x}}(t)$ is not a maximal element in $B_t(\tilde{p})$. Then there exists a commodity bundle $\bar{x} \in B_t(\tilde{p})$ such that

$$U_t(\bar{x}) > U_t(\tilde{\mathbf{x}}(t)). \tag{3.3}$$

Define $\bar{e}$ by

$$\bar{x}^1 = w^1(t) - \bar{e}.$$

Notice that we must have $\sum_{h=2}^{l} \tilde{p}^h \bar{x}^h \leq \tilde{p}^1 \bar{e}$. Indeed we obtain

$$\begin{aligned} \tilde{p} \cdot \bar{x} &= \sum_{h=2}^{l} \tilde{p}^h \bar{x}^h + \tilde{p}^1 w^1(t) - \tilde{p}^1 \bar{e} \\ &\leq \tilde{p}^1 w^1(t), \end{aligned}$$

where the last equality follows from the fact that $\bar{x} \in B_t(\bar{p})$. Consequently,

$$\sum_{h=2}^{l} \tilde{p}^h \bar{x}^h \leq \tilde{p}^1 \cdot \bar{e}.$$

Thus it follows from the definition of $\varphi_t(\bar{e}, p(\tilde{\mathbf{e}} \backslash \bar{e}))$ that

$$U_t(\bar{x}) \leq U_t(\varphi_t(\bar{e}, p(\tilde{\mathbf{e}} \backslash \bar{e}))),$$

which, by (3.2), implies

$$U_t(\bar{x}) \leq U_t(\varphi_t(\tilde{\mathbf{e}}(t), \tilde{p})) = U_t(\tilde{\mathbf{x}}(t)). \tag{3.4}$$

But (3.4) contradicts (3.3). Consequently, for all $t \in T$, $\tilde{\mathbf{x}}(t)$ is a maximal element for U_t in the budget set $B_t(\tilde{p})$ so that $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.

Now we proceed to the proof of the second part of the theorem. Let $(p^*, \boldsymbol{x}^*)$ be a Walras equilibrium. We shall exhibit a strategy profile $\mathbf{e}^*$ and show that the pair $(e^*, \boldsymbol{x}^*)$ is a Cournot–Walras equilibrium. To this end, consider, for each $t \in T_1$, the following problem:

$$\max_{(x^2, \ldots, x^l)} U_t(\boldsymbol{x}^{*1}(t), x^2, \ldots, x^l) \quad \text{subject to} \quad \sum_{h=2}^{l} p^{*h} x^h = p^{*1}(w^1(t) - \boldsymbol{x}^{*1}(t)). \tag{3.5}$$

First, it is clear that $(\boldsymbol{x}^{*2}(t), \ldots, \boldsymbol{x}^{*l}(t))$ is a solution to this problem for, otherwise, there would exist a point $(\bar{x}^2, \ldots, \bar{x}^l)$ satisfying (3.5) such that $U_t(\boldsymbol{x}^{*1}(t), \bar{x}^2, \ldots, \bar{x}^l) > U_t(\boldsymbol{x}^{*1}(t), \ldots, \boldsymbol{x}^{*l}(t))$. But then, $\boldsymbol{x}^*(t)$ would not be the maximal element in the budget set $B_t(p^*)$, contrary to the fact that $(p^*, \boldsymbol{x}^*)$ is a Walras equilibrium. Define the strategy profile $\boldsymbol{e}^*$ by

$$\boldsymbol{e}^*(t) = w^1(t) - \boldsymbol{x}^{*1}(t).$$

We have just shown that, for all $t \in T_1$, the bundle $(w^1(t) - \boldsymbol{e}^*(t), \boldsymbol{x}^{*2}(t), \ldots, \boldsymbol{x}^{*l}(t))$ solves

$$\max_{(x^2, \ldots, x^l)} U_t(w^1(t) - \boldsymbol{e}^*(t), x^2, \ldots, x^l) \quad \text{s.t.} \sum_{h=2}^{l} p^{*h} x^h = p^{*1} \cdot \boldsymbol{e}^*(t),$$

so that this bundle is equal, for all $t \in T_1$, to $\boldsymbol{x}(t, p^*)$. Clearly, for all $t \in T_2$, we have also that $x^*(t) = x(t, p^*)$. Now we verify that $p^* = p(e^*)$. To this end, we notice that

$$\int_0^1 e^*(t)d\mu = \int_0^1 w^1(t)d\mu - \int_0^1 x^{*1}(t, p^*)d\mu.$$

Since x^* is an allocation, we have that

$$\int_0^1 w^1(t)d\mu = \int_0^1 x^1(t, p^*)d\mu + \int_1^2 x^1(t, p^*)d\mu,$$

so that

$$\int_0^1 e^*(t)d\mu = \int_1^2 x^1(t, p^*)d\mu,$$

which is (i). On the other hand, since $\int_0^1 w^h(t)d\mu = 0$ for all $h = 2, \ldots, l$, it follows from the fact that x^* is an allocation that

$$\int_0^2 x^h(t, p^*)d\mu = \int_1^2 w^h(t), d\mu, \quad h = 2, \ldots, l$$

which is (ii).

To complete the proof that (e^*, x^*) is a Cournot–Walras equilibrium, it remains to show that no trader t in T_1 has an advantageous deviation from $e^*(t)$. Suppose on the contrary that there exists a trader t in T_1 and a strategy $e(t) \in S_t$ such that

$$\begin{aligned} U_t(x^*(t)) &= U_t(w^1(t) - e^*(t), x^2(t, p^*), \ldots, x^l(t, p^*)) \\ &< U_t(w^1(t) - e(t), x^2(t, p(e^*\backslash e(t))), \ldots, x^l(t, p(e^*\backslash e(t)))). \quad (3.6) \end{aligned}$$

By the lemma, $p(e^*\backslash e(t)) = p(e^*) = p^*$. Thus, it follows that the bundle $(w^1(t) - e(t), \ldots, x^l(t, p(e^*\backslash e(t))))$ solves

$$\max_{x^2, \ldots, x^l} U_t(w^1(t) - e(t), x^2, \ldots, x^l) \quad \text{s.t.} \sum_{h=2}^{l} p^{*h}x^h = p^{*1} \cdot e(t),$$

so that

$$\sum_{h=2}^{l} p^{*h} \cdot x^h(t, p(e^*\backslash e(t))) = p^{*1} \cdot e(t).$$

Accordingly we have

$$p^{*1}w^1(t) - p^{*1}e(t) + \sum_{h=2}^{l} p^{*h}x^h(t, p(e^*\backslash e(t))) = p^{*1}w^1(t),$$

so that the bundle $(w^1(t) - e(t), \ldots, x^l(t, p(e^*\backslash e(t)))) \in B_t(p^*)$. But then, according to (3.6), the bundle $x^*(t) = (w^1(t) - e(t), x^2(t, p^*), \ldots, x^l(t, p^*))$ would not be the maximal element in the budget set $B_t(p^*)$, contrary to the fact that (p^*, x^*) is a Walras equilibrium. ■

4. An example with a continuum of traders and atoms

The theorem we have just demonstrated shows that noncooperative strategic behaviour is totally ineffective when the strategic agents form an atomless ocean facing a continuum of small traders. The question arises whether a similar result would hold if oligopolists are represented in the model by *atoms*. Surprisingly enough, Shitovitz (1973) has shown that this is indeed the case if the outcome of the exchange has to be in the *core*: If the oligopolists (atoms) are of the same type (same endowments and preferences), any allocation in the core is Walrasian. Now we propose an example in the framework of a mixed model with a continuum of traders and atoms, and show that the corresponding Cournot–Walras equilibrium is *not* Walrasian: On the contrary, the atoms exploit the small traders at the Cournot–Walras equilibrium. Consider an exchange economy embodying two atoms a_1 and a_2, each of measure $\mu(\{a_1\}) = \mu(\{a_2\}) = 1$ and an atomless continuum of traders represented by the unit interval $[0, 1]$ with the Lebesgue measure μ. We define the initial holdings of the traders by

$$\boldsymbol{w}(\{a_1\}) = \boldsymbol{w}(\{a_2\}) = (1, 0)$$

and, for $t \in [0, 1]$,

$$\boldsymbol{w}(t) = (0, 1).$$

Furthermore, we set $U_{a_1}(x) = U_{a_2}(x) = U_t(x) = x^1 \cdot x^2$. Thus, the total holdings of commodity 1 are in the hands of duopolists $\{a_1\}$ and $\{a_2\}$, while the ownership of commodity 2 is spread among the small traders t in $[0, 1]$. A strategy of duopolist $\{a_i\}, i = 1, 2,$ is any number e_i, $0 \leq e_i \leq 1$. Denote by $x(t, p)$ the solution to the problem

$$\max_{x_1, x_2} x^1 \cdot x^2 \quad \text{s.t. } px^1 + x^2 = p \cdot \boldsymbol{w}(t) = 1, \quad t \in [0, 1]:$$

We normalize the price vector by letting the price of commodity 2 be equal to one. Thus the vector $\left(\frac{1}{2p}, \frac{1}{2}\right)$ is the aggregate demand vector of the small traders at the price system $(p, 1)$. Given $(p, 1)$ the income of the atom $\{a_i\}$ is equal to $p \cdot e_i$. The problem of the atom $\{a_i\}$ is

$$\max_{x^2} (1 - e_i) \cdot x^2 \quad \text{s.t.} \quad x^2 = p \cdot e_i, \quad i = 1, 2$$

which reduces to

$$x^2(\{a_i\}, p) = p \cdot e_i.$$

Now we compute $p(e_1, e_2)$ as the solution of the equation

$$\int_0^1 \boldsymbol{x}^1(t, p) d\mu = e_1 + e_2,$$

or $p(e_1, e_2) = \dfrac{1}{2(e_1 + e_2)}$. Thus we obtain

$$\boldsymbol{x}(\{a_i\}, p(e_1, e_2)) = \left(1 - e_i, \frac{e_i}{2(e_1 + e_2)}\right), \quad i = 1, 2$$

and

$$\boldsymbol{x}(t, p(e_1, e_2)) = \left(e_1 + e_2, \frac{1}{2}\right), \quad t \in [0, 1].$$

The corresponding payoff of the atom $\{a_i\}$ is equal to $(1-e_i)\left(\frac{e_i}{2(e_1+e_2)}\right)$, $i=1,2$. The Cournot–Walras equilibrium obtains as the simultaneous solution to the problems

$$\max_{e_1}(1-e_1)\left(\frac{e_1}{2(e_1+e_2)}\right)$$

and

$$\max_{e_2}(1-e_2)\left(\frac{e_2}{2(e_1+e_2)}\right).$$

The first order conditions give us the following system:

$$e_2-e_1^2-2e_1e_2=0;$$
$$e_1-e_2^2-2e_1e_2=0.$$

Noting that payoffs are symmetric in e_1 and e_2, we may solve this system by imposing $e_1=e_2$, from which we get $\tilde{e}(\{a_1\})=\tilde{e}(\{a_2\})=\frac{1}{3}$ and $p(\tilde{e}(\{a_1\},\ \tilde{e}(\{a_2\}))=\frac{3}{4}$. Accordingly, we obtain

$$\tilde{x}(\{a_1\})=\tilde{x}(\{a_2\})=\left(\frac{2}{3},\frac{1}{4}\right)$$

and, for $t\in[0,1]$,

$$\tilde{x}(t)=\left(\frac{2}{3},\frac{1}{2}\right).$$

Thus we conclude that the pair of strategies $(\tilde{e}(\{a_1\}),\ \tilde{e}(\{a_2\}))$ and the allocation $\tilde{x}$ is a Cournot–Walras equilibrium.

The Walras equilibrium of the economy is clearly the pair (p^*, x^*) defined by

$$x^*(\{a_1\})=x^*(\{a_2\})=\left(\frac{1}{2},\frac{1}{4}\right)$$

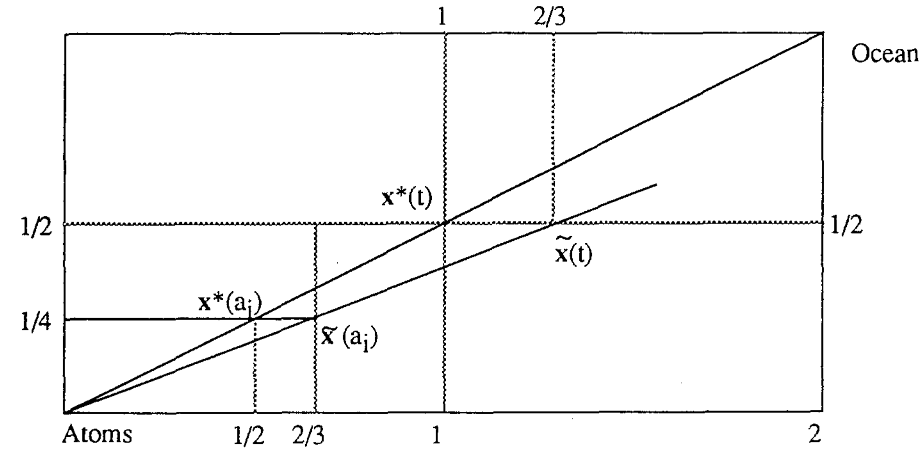
$$x^*(t)=\left(1,\frac{1}{2}\right),\quad \text{for all } t\in[0,1]$$

and

$$p^*=\left(\frac{1}{2},1\right).$$

From direct comparison between the outcomes at the Cournot–Walras and at the Walras equilibrium, we observe that the duopolists exploit the small traders when they move from a price-taking to a strategic behaviour.

The following Edgeworth box describes the Cournot–Walras and Walras equilibrium corresponding to this example.



5. Discussion of the literature

The noncooperative approach to general equilibrium has developed along two major lines. The first can be defined as an essay to "throw some light into the shadowy transition zone between 'many' and 'few'" (Shapley (1976)). This calls for building a model which treats all traders symmetrically and allows them to individually manipulate the price structure. Then it is shown that this possibility vanishes when the number of traders grows indefinitely: The price constellation at the noncooperative equilibrium approaches the competitive prices because they result from vanishing responsiveness to the traders buy/sell decisions. It is along this line of research that numerous recent developments in the noncooperative theory of exchange are articulated (see, for instance, Shapley and Shubik (1977), Dubey and Shubik (1978), Dubey, Mas-Colell and Shubik (1980), Mas-Colell (1982), Sahi and Yao (1989) and Amir *et al.* (1990)).

The second development of noncooperative theory aims at explaining the outcome to be expected in an economy in which there is an inherent asymmetry in the "market power" of the agents: Even if markets embody an "ocean" of small anonymous traders, individual merchants may also be present which are *not* anonymous, either because their endowment of some commodity is large compared with the endowments of the entire market, or because they represent market participants who have combined into a unique decision center. This approach calls for building a model in which the behaviour of the agents is no longer symmetric: "Significant" merchants do manipulate the price structure, while the anonymous traders behave as price-takers. The present paper is within this line of research and is inspired by the work of Gabszewicz and Vial (1972) (henceforth G–V); other papers in the same vein are Fitzroy (1974), Laffont and Laroque (1976) and Gary-Bobo (1989). However there is a significant difference between the G–V approach and the present one. The paper by G–V considers a production economy, and the asymmetry among the agents comes from the fact that *firms* are the strategic agents while consumers behave competitively. As a consequence of this assumption, the decision criterion of the "oligopolists" is profit-maximization in terms of a price system that firms can manipulate via their shareholders on the markets where the goods produced are exchanged. Two difficulties arise from this representation. First, there may be situations in which the maximization of monetary profits cannot be regarded as a rational decision criterion for the firms which are able to manipulate prices. By choosing an output vector so as to maximize the wealth of the firm's owners, the firm may neglect alternative strategies which would yield higher utility levels for its shareholders; an example of this situation is considered in Gabszewicz and Vial (1972), p. 395. On the other hand, it turns out that the Cournot–Walras equilibrium in G–V depends on the rule which is chosen to normalize prices. An example is provided in G–V, in which the Cournot–Walras equilibrium of the production economy is different when two different normalization rules are chosen. None of these difficulties appear in the present paper: The decision criterion of oligopolists is the usual criterion of utility maximization, and the Cournot–Walras equilibrium is invariant with respect to the normalization rule which has been chosen. Another difference between the two models is that, in G–V, the analog of our main theorem in the present paper is an approximation theorem showing that, when the number of oligopolists increases, the Cournot–Walras equilibrium tends to the Walras equilibrium (on this problem, see also Roberts (1980)). Here we have worked with a model "at the limit", which induces an equivalence result. Of course, it would be interesting to study an asymptotic version of the finite model and show that the associated sequence of Cournot–Walras equilibria converges to the set of Walras equilibria of the basic economy.

Two difficulties which appeared in the G–V paper also appear in the present work. First, we have assumed that, for any strategy profile, there exists a unique Walras equilibrium at which all markets clear. This assumption is the counterpart of assumption A_1 in G–V, and all the comments provided there (cfr. Gabszewicz and Vial (1972), p. 396) about this assumption apply, *mutatis mutandis*, to the present version of the model. On the other hand, even if an existence theorem for a Cournot–Walras equilibrium is provided in G–V, it rests crucially on the assumption of quasi-concavity of the oligopolists' payoff functions. This hypothesis is clearly unsatisfactory since one cannot ascertain under which conditions it is satisfied, starting from the basic data of the model. Roberts and Sonnenschein (1977) have shown that it is possible to construct standard general equilibrium models where the assumption of quasi-concavity is violated. Dierker and Grodal (1986) have shown that there are similar models where not even mixed strategy equilibria exist. A similar existence problem of a Cournot–Walras equilibrium appears for the finite version studied in Codognato and Gabszewicz (1991). Nevertheless, thanks to our equivalence theorem in section 2, we know that Cournot–Walras equilibria exist in a model with a continuum of oligopolists and small traders since, in that case, Cournot–Walras equilibria coincide with Walras equilibria. Thus we may apply the existence result obtained by Aumann (1966) for economies with a continuum of traders.

Finally, much remains to be done in view of building a bridge between the two lines of research referred to in the beginning of this conclusion. To this end, it would be of great interest to analyse in depth the noncooperative equilibria of a mixed exchange model of the type considered in our example of section 4.

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