J. Nonlinear Sci. Vol. 2: pp. 367–415 (1992)

Journal of Nonlinear Science

Gyroscopic Control and Stabilization

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Received May 9, 1991; accepted for publication May 5, 1992
Communicated by Jerrold Marsden

Summary. In this paper, we consider the geometry of gyroscopic systems with symmetry, starting from an intrinsic Lagrangian viewpoint. We note that natural mechanical systems with exogenous forces can be transformed into gyroscopic systems, when the forces are determined by a suitable class of feedback laws. To assess the stability of relative equilibria in the resultant feedback systems, we extend the energy-momentum block-diagonalization theorem of Simo, Lewis, Posbergh, and Marsden to gyroscopic systems with symmetry. We illustrate the main ideas by a key example of two coupled rigid bodies with internal rotors. The energy-momentum method yields computationally tractable stability criteria in this and other examples.

Key words. gyroscopic control, stabilization

1. Introduction

Geometric control theory has led to the development of a large body of results to analyze and design nonlinear feedback systems. There is a beautiful structure theory of nonlinear control systems that relates internal representations (state space models) to external representations (input-output models). Methods from the differential geometry of foliations play a key role in the solution of fundamental problems such as disturbance decoupling, noninteracting control, and so on, compare [21], [33]. Inspired in part by the success of the linear theory and partly by the search for suitable feedback invariants [10], the paradigm of feedback linearization has had diverse applications [38]. Much of this work is however concerned with generic dynamics and often does not specialize well to the context of natural mechanical systems. However, there is a growing body of literature devoted to development of a geometric control theory for Hamiltonian systems. See [12], [33], [22] for recent developments.

A parallel intellectual program to geometrize mechanics has been very successful and has had tremendous impact on many areas of mathematics (e.g., symplectic geometry and topology) and physics (e.g., gauge theory, geometric quantization). Some of the roots of this program may be traced to the problems of stability [4], symmetry and reduction, [44], [43], [13], and [32], and investigation of the topology of phase space [42]. Modern expositions of these developments may be found in [1], [5], and [6]. An exciting recent result is the *block-diagonalization theorem* for simple mechanical systems with symmetry, [39] and [41]. This theorem provides refined criteria for stability assessment for natural mechanical systems by careful exploitation of the underlying geometric structure.

One of the principal goals of the present paper is to demonstrate the applicability of geometric ideas to a large class of feedback systems derived from natural mechanical systems. Here the feedback controls are of the *gyroscopic type*. A key point is that simple mechanical systems with symmetry, when subject to exogenous forces determined by suitable classes of feedback laws, also admit Hamiltonian and Lagrangian structures. In [9] and [7], the Hamiltonian structures so derived are viewed as deformations (by feedback gains) of the Hamiltonian structure governing the open-loop unforced system. There, the methods of geometric mechanics such as reduction, reconstruction phases, and the energy-Casimir algorithm for stability analysis are brought to bear on a key example of rigid body control using external torques (as implemented by gas jets) and internal torques (via reaction wheels/rotors), and the relationships between these two methods of control. In the present paper, taking an *intrinsic Lagrangian* viewpoint, we develop a systematic theory of gyroscopic feedback systems with symmetry. Key examples of such systems include dual-spin satellites, and rigid-body satellites with magnetic torques, and so on.

The principal reason for taking a Lagrangian viewpoint is that it leads very naturally to the incorporation of exogenous forces/controls. Furthermore, in the setting of constrained nonholonomic systems, the Lagrange-D'Alembert principle is the basic principle of modeling. (See, however, related remarks about *vakonomic mechanics* and the role of variational principles for constrained systems in [6].) Good representations of higher-order tangent bundles together with the intrinsic/invariant formulation of Lagrangian mechanics lead to effective modeling of the systems of interest. Here we give variational principles for relative equilibria and their stability. One of the contributions of this paper is the extension of the (energy-momentum) block-diagonalization theorem to gyroscopic systems with symmetry. Thus, we are able to establish a set of refined stability criteria for a wide class of feedback systems by fully exploiting the underlying geometric and group-theoretic structures.

The outline of the paper is as follows. Section 2 lists relevant notations in geometric mechanics and group action. In Section 3, we give a brief treatment of Lagrangian mechanics in invariant form. We formulate the Lagrange-D'Alembert principle in geometric terms. Our exposition follows Vershik and Faddeev [45] in large part, except for certain conventions. We then display a useful representation of the second tangent and cotangent bundles associated to $SO(3)$. In Section 4, we axiomatize the notion of gyroscopic feedback system with symmetry. The dual-spin equations are derived as an example. In Section 5, we characterize relative equilibria by the Principle of Symmetric Criticality. We give a careful exposition of the concept of relative stability in the abstract setting. In Section 6, we prove the block-diagonalization theorem for gyroscopic systems with symmetry. This extends naturally the previous work of Simo, Lewis, Posbergh, and Marsden on block-diagonalization of simple mechanical systems with symmetry. A distinctive feature of the present work is that all computations are done on the velocity phase space TQ, or loosely, on the "Lagrangian side." We are aware that D. Lewis has carried out a similar program [27], but the present work was done independently and was primarily motivated by feedback stability problems.* The correct modification of the amended and augmented potentials to incorporate gyroscopic terms yields stability criteria that explicitly display said terms. This is further made clear in the detailed example of two coupled rigid bodies with internal rotors (the multibody dual-spin problem of [48]) studied in Section 7.

There are other aspects of gyroscopic feedback controls that we do not explore in this paper but that we think are quite promising. Control strategies based on bifurcation of relative equilibria may be effective in a variety of problems. We see instances in [54] and [8]. In the present setting it would be worthwhile to investigate bifurcations with respect to the gyroscopic feedback parameters. In the context of the dual-spin problem, this has been carried out by Krishnaprasad and Berenstein who gave a bifurcation diagram [25]. Also, in [7], the authors show in an example how the phenomenon of geometric phase shift is affected by gyroscopic parameters. We hope to discuss these aspects in a later paper. Some of the results in this paper appeared in the Ph.D. dissertation of Li-Sheng Wang [46].

We would like to thank Professor Jerrold Marsden for critical and helpful advice on an earlier version of this paper.

2. Notations and Preliminary

In this section, we introduce the notations used in later sections. First, we collect together some basic geometric objects of Lagrangian mechanics. We follow Vershik and Faddeev [45] closely, filling in details where needed.

Let Q be a smooth manifest with local coordinates x, TQ be the tangent bundle of Q with local coordinates (x, v), and $\pi : TQ \to Q$ be the canonical projection from TQ to Q. Let TTQ be the second tangent bundle with local coordinates (x, v, u, w). Let $T_{(x,v)}$ be the tangent space of TQ at (x, v), that is, $T_{(x,v)}TQ$, and denote the *vertical* tangent subspace of $T_{(x,v)}$ consisting of vectors tangent to the fiber of TQ by $T^V_{(x,v)}$. In local coordinates, each vector in $T^V_{(x,v)}$ can be written as $(0, w)$, for some $w \in T_xQ$. The tangent map $T\pi$ of the canonical project, $T\pi : TTQ \to TQ$, can be expressed in local coordinates as $T\pi_{(x,v)}(u, w) = u \in T_xQ$, which projects a vector in $T_{(x,v)}$ to its *horizontal* component. Define the map $\gamma_{(x,v)} : T_xQ \to T_{(x,v)}TQ$; $u \mapsto (0, u) \in T^V_{(x,v)}$, which establishes an isomorphism between T_xQ and $T^V_{(x,v)}$. Let $X : TQ \to TTQ$ be a vector field on TQ. The variable X is called a *vertical vector field* if $X(x, v) \in T^V_{(x,v)}$ or $X(x, v) = (0, w)$, for some $w \in T_xQ$. This is equivalent to saying $T\pi \cdot X = 0$. There is a unique vertical vector field X^{PV} called the *principal vertical field* defined by $X^{PV}(x, v) = \gamma_{(x,v)} \cdot v = (0, v)$.

* A feedback law is a rule that determines exogenous forces/controls as functions of the current state of a dynamical system. The feedback stabilization problem is to find feedback laws that achieve prescribed characteristics in the closed-loop system.

Now we consider the dual spaces. Let ω be a 1-form on TQ. It is said to be *horizontal* if for all vertical vector fields X, $\omega(X) = 0$, or in local coordinates, $\omega(x, v) = (\alpha, 0)$, for some $\alpha \in T_x^*Q$. The *dual* of the map $T\pi_{(x,v)} : T_{(x,v)}TQ \to T_xQ$ can be defined implicitly through

$$T\pi^*_{(x,v)} : T_x^*Q \to T^*_{(x,v)}TQ,$$
$$\langle T\pi^*_{(x,v)}\alpha, (u, w)\rangle = \langle \alpha, T\pi_{(x,v)}, (u, w)\rangle = \langle \alpha, u\rangle,$$

where $\alpha \in T_x^*Q$. Thus, in local coordinates, $T\pi^*_{(x,v)}\alpha = (\alpha, 0)$. Similarly, the dual of $\gamma_{(x,v)}$, denoted by $\gamma^*_{(x,v)} : T^*_{(x,v)}TQ \to T_x^*Q$, is defined by, in local coordinates,

$$\langle \gamma^*_{(x,v)}(\alpha, \beta), u\rangle = \langle (\alpha, \beta), \gamma_{(x,v)} \cdot u\rangle = \langle \beta, u\rangle.$$

Equivalently, $\gamma^*_{(x,v)}(\alpha, \beta) = \beta$.

With these dual mappings, we define the bundle map $\tau : T^*(TQ) \to T^*(TQ)$ as $\tau_{(x,v)} \triangleq T\pi^*_{(x,v)} \cdot \gamma^*_{(x,v)}$. In particular, for $(\alpha, \beta) \in T^*_{(x,v)}TQ$, we have

$$\tau_{(x,v)}(\alpha, \beta) = T\pi^*_{(x,v)} \cdot \gamma^*_{(x,v)}(\alpha, \beta) = (\beta, 0). \tag{2.1}$$

Thus $\tau_{(x,v)}$ maps any cotangent vector (*covector*) to a horizontal covector. Globally, τ maps any 1-form on TQ to a horizontal 1-form on TQ. On the other hand, we may define a bundle map from the second tangent bundle into itself, $\tau_* : TTQ \to TTQ$ as $\tau_{*(x,v)} \triangleq \gamma_{(x,v)} \cdot T\pi_{(x,v)}$. In local coordinates we can associate to each $(u, w) \in T_{(x,v)}TQ$, $\tau_{*(x,v)}(u, w) = (0, u)$. In other words, $\tau_{*(x,v)}$ maps any second tangent vector to a vertical tangent vector, and, globally, τ_* maps a vector field on TQ to a vertical vector field on TQ.

To treat second-order equations, we need the following concept. A vector field on TQ, $X \in \mathscr{X}(TQ)$, is a *special vector field* if and only if $\tau_*X = X^{PV}$. In local coordinates, assuming $X(x, v) = (u, w)$, it says $\tau_*X(x, v) = (0, v)$, which is equivalent to the condition $u = v$. It then follows that this definition of special vector field X is the same as saying X gives rise to a second-order equation on Q, compare p. 213 of Abraham and Marsden [1].

Let $T^{*H}_{(x,v)}$ denote the space of horizontal covectors at (x, v) in TQ. Define the map $\sigma : T^{*H}_{(x,v)} \to T_x^*Q$ to be, in local coordinates, $\sigma_{(x,v)}(\alpha, 0) \triangleq \alpha$, for $\alpha \in T_x^*Q$. This map will be used later in defining the *Legendre transformation*. The maps γ, τ, τ_*, and σ are intrinsic and do not depend on choices of local trivializations of the bundles involved.

Now we collect together basic notions of group actions on Riemannian manifolds necessary to discuss natural mechanical systems with symmetry. Let $(Q, \ll \cdot, \cdot \gg)$ be a manifold with Riemannian metric $\ll \cdot, \cdot \gg$. We sometimes write $K(x)(v_x, w_x) = \ll v_x, w_x \gg_x$, for $x \in Q$, and $v_x, w_x \in T_xQ$. The Riemannian metric induces a vector bundle isomorphism $K^b : TQ \to T^*Q$, defined by

$$\langle K^b(v_x), w_x\rangle_x = \ll v_x, w_x \gg_x, \quad \forall\, v_x, w_x \in T_xQ,$$

where $\langle \cdot, \cdot\rangle_x$ denotes the pairing between elements in T_x^*Q and T_xQ. Here, and in what follows, the notation $\langle \cdot, \cdot\rangle$ is used to denote the dual pairing between appropriate spaces. By the Riesz Representation Theorem, this isomorphism is well defined and we may write $K^{\#} = (K^{b})^{-1} : T^*Q \to TQ$, which is also a fiber-preserving mapping. By definition, for $\alpha_x \in T^*Q$ and $w_x \in TQ$, we have $\langle \alpha_x, w_x \rangle_x = \ll K^{\#} \cdot \alpha_x, w_x \gg_x$. Via the isomorphism $K^{\#}$, an induced inner product on T^*Q can be defined, for $\alpha_x, \beta_x \in T_x^*Q$,

$$< \alpha_x, \beta_x >_{T^*Q} \triangleq \ll K^{\#} \cdot \alpha_x, K^{\#} \cdot \beta_x \gg_x . \tag{2.2}$$

Let G be a Lie group, and $\Phi : G \times Q \to Q$ be a group action of G on the manifold Q. We shall use the notations $\Phi(g, x) \equiv \Phi_g(x) \equiv g \cdot x$ interchangeably to denote this action. The *tangent lift* Φ^T associated with Φ is defined as $\Phi_g^T \equiv T\Phi_g : TQ \to TQ$, or, in local coordinates, $\Phi_g^T(x, v) = (\Phi_g(x), T_x\Phi_g \cdot v)$. The *cotangent lift* Φ^{T*} associated to Φ on the cotangent bundle T^*Q, $\Phi^{T*} : G \times T^*Q \to T^*Q$, is $\Phi_g^{T*}(\alpha_x) \triangleq T^*\Phi_{g^{-1}} \cdot \alpha_x$, where $T^*\Phi_{g^{-1}}$ is the dual of $T\Phi^{g^{-1}}$. In local coordinates we have $\langle \phi_g^{T*}(x, \alpha), (g \cdot x, v) \rangle_{g \cdot x} = \langle \alpha, T_{g \cdot x}\Phi_{g^{-1}} \cdot v \rangle_x$. It is straightforward to verify that the tangent lift and cotangent lift are both well-defined actions on the spaces TQ and T^*Q respectively.

Let the Lie algebra of a Lie group G be denoted by $\mathcal{G}$, with its dual $\mathcal{G}^*$. Recall that the Lie algebra $\mathcal{G}$ is identified as the tangent space to G at the identity element e or, equivalently, the set of left invariant vector fields on G, compare also [34]. Given $\xi \in \mathcal{G}$, for a group action Φ on a manifold Q, we define

$$\xi_Q(x) \triangleq \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi_{\exp \epsilon\xi}(x) \quad \in T_xQ,$$

the *infinitesimal generator* of the action corresponding to ξ. The group G acts on $\mathcal{G}$ through the adjoint action

$$Ad : G \times \mathcal{G} \to \mathcal{G}; \quad (g, \xi) \mapsto T_e(R_{g^{-1}} \circ L_g)\xi = Ad_g\xi, \tag{2.3}$$

where L_g, R_g denote the left and right translation of a group element by $g \in G$, respectively. The map $g \mapsto Ad_g$ is also called the *adjoint representation* of G in $\mathcal{G}$. The infinitesimal generator of this adjoint action,

$$\xi_{\mathcal{G}}(\eta) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} Ad_{\exp \epsilon\xi}(\eta)$$

can be shown to be equal to the Lie bracket of ξ and η, namely,

$$\xi_{\mathcal{G}}(\eta) = [\xi, \eta] \triangleq ad_\xi \eta.$$

(We follow the sign convention for Lie brackets used in [1]). The group G also acts on the dual of the Lie algebra $\mathcal{G}^*$ through the coadjoint action

$$Ad^* : G \times \mathcal{G}^* \to \mathcal{G}^*; \quad (g, \mu) \mapsto Ad^*_{g^{-1}}\mu,$$

which is defined by, $\langle Ad_g^* \mu, \xi \rangle \triangleq \langle \mu, Ad_g\xi \rangle$, for all $\xi \in \mathcal{G}$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $\mathcal{G}^* \times \mathcal{G}$. The corresponding infinitesimal generator, $\xi_{\mathcal{G}*}$ can be shown to be determined by

$$\langle \xi_{\mathcal{G}*}(\mu), \eta \rangle = -\langle \mu, [\xi, \eta] \rangle \triangleq -\langle ad_\xi^* \mu, \eta \rangle,$$

for all $\eta \in \mathcal{G}$. From the previous definitions, it is easy to establish the identity, $\langle ad_\xi^* \mu, \eta \rangle = \langle \mu, ad_\xi \eta \rangle$. With the above structure, group G acts on Q and $\mathcal{G}^*$ through the actions Φ and Ad^* respectively. A map $J : Q \to \mathcal{G}^*$ is called *Ad^*-equivariant* if $J \circ \Phi_g = Ad^*_{g^{-1}} \circ J$. For $\mu \in \mathcal{G}^*$, we define the *isotropy subgroup* of μ by

$$G_\mu = \{g \in G : Ad_g^* \mu = \mu\}, \tag{2.4}$$

with isotropy Lie algebra

$$\mathcal{G}_\mu = \{\eta \in \mathcal{G} : ad_\eta^* \mu = 0\}, \tag{2.5}$$

which can be shown to be a subalgebra of $\mathcal{G}$.

The notion of invariance is now ready to be introduced. A Riemannian metric is *G-invariant* if it is invariant under the pull-back of the mapping Φ_g; that is, for all $g \in G$, $\Phi_g^* \cdot K = K$, or in local coordinates, $K(x)(v, w) = K(g \cdot x)(T_x\Phi_g \cdot v, T_x\Phi_g \cdot w)$, $\forall\ v, w \in T_xQ$. It then follows that the inner product on T^*Q defined in (2.2) is invariant under the cotangent lift, namely, $< \alpha_x,\ \beta_x >_{T^*Q} = < \Phi_g^{T^*} \alpha_x, \Phi_g^{T^*} \beta_x >_{T^*Q}$, for all $g \in G$. This can be shown by using the following identities,

$$K^\# \cdot \Phi_g^{T^*} \cdot \alpha_x = T_x\Phi_g \cdot K^\# \cdot \alpha_x, \quad \text{for } \alpha_x \in T^*Q, \tag{2.6a}$$

$$K^b \cdot T_x\Phi_g \cdot w_x = \Phi_g^{T^*} \cdot K^b \cdot w_x, \quad \text{for } w_x \in TQ. \tag{2.6b}$$

Similarly, a 1-form $\tilde{Y} \in T^*Q$ is called *G-invariant* if $\Phi_g^* \cdot \tilde{Y} = \tilde{Y}$, for all $g \in G$. A smooth function $V : Q \to \mathbb{R}$ is a *G-invariant function* on the manifold if, for all $g \in G$, $V(\Phi_g(x)) = V(x)$. A vector field Y on Q is a *G-invariant vector field* if for all $g \in G$, $(\Phi_g)_* \cdot Y = Y$, or $Y(x) = T\Phi_g \cdot Y(g^{-1} \cdot x)$, for $x \in Q$, $g \in G$.

Recall that a differential operator on the full tensor algebra can be defined from its restrictions on functions and vector fields, compare the theorem of Willmore [52]. Accordingly, the Lie derivative of a vector field on the tensor algebra can be found from its Lie derivative on functions (directional derivative) and Lie derivative on vector fields (Lie bracket). The following two lemmas are essential to the developments in Section 6. Their proofs can be found in, for example, [1] and [46].

Lemma 2.1. *Let Y, K be G-invariant vector field and Riemannian metric, respectively. Their Lie derivatives with respect to vector field η_Q, where $\eta \in \mathcal{G}$, vanish, namely, $L_{\eta_Q} Y = 0$, and $L_{\eta_Q} K = 0$.*

Lemma 2.2. *Let $X_1, X_2 \in \mathcal{X}(Q)$, the vector fields on Q. For $\eta \in \mathcal{G}$, we have*

$$L_{\eta_Q} \ll X_1, X_2 \gg_x = \ll L_{\eta_Q} X_1, X_2 \gg_x + \ll X_1, L_{\eta_Q} X_2 \gg_x .$$

3. Lagrangian Mechanics in Invariant Form

Lagrangian mechanics provides a systematic formulation of mechanical problems from a unified point of view. In contrast to working on the cotangent bundle as in most of Hamiltonian mechanics, Lagrangian mechanics formulates the problems on the tangent bundle, or the velocity phase space. As we shall see, it admits greater freedom in interpreting and formulating intuitive physical notions such as exogenous forces and the principle of virtual power. In this section, we present the invariant form of Lagrangian mechanics through local representations and show that the invariant form of the Lagrange-D'Alembert Principle gives rise to the Euler-Lagrange equations in local coordinates. Moreover, a similar equation can be applied to model mechanical systems on the special orthogonal group (rotation group) $SO(3)$ in a global sense.

Let Q be a smooth manifold viewed as the configuration space. Let TQ and TTQ denote its tangent bundle and second tangent bundle, respectively. Since Lagrangian mechanics is about second-order equations, we need to consider the corresponding elements in the *jet spaces* of Q, namely the second tangent vectors. Let $L : TQ \to \mathbb{R}$ be a smooth function (or *Lagrangian*). The corresponding differential 1-form $dL : TQ \to T^*(TQ)$ can be written in local coordinates as

$$\langle dL(x, v),\ (u, w)\rangle = TL_{(x,v)} \cdot (u, w) \tag{3.1}$$

or, in terms of Fréchét partial derivatives, $dL(x, v) = \big(D_1L(x, v), D_2L(x, v)\big)$. The horizontal 1-form Θ_L on TQ corresponding to L is defined to be, compare (2.1),

$$\Theta_L \triangleq \tau \cdot dL. \tag{3.2a}$$

In local coordinates, $\Theta_L(x, v) = \tau_{(x,v)} \cdot dL(x, v) = (D_2L(x, v), 0)$. Taking the exterior derivative of the 1-form Θ_L, we associate to L a 2-form on TQ, $\Omega_L : TTQ \times TTQ \to \mathbb{R}$, defined as

$$\Omega_L \triangleq -d\,\Theta_L. \tag{3.3a}$$

If we write $\Theta_L = D_{v^i}L\ dx^i$, then, by taking exterior derivatives on both sides, we get

$$\Omega_L = -dD_{v^i}L \wedge dx^i = D_{x^j}D_{v^i}L\, dx^i \wedge dx^j + D_{v^j}D_{v^i}L\, dx^i \wedge dv^j. \tag{3.3b}$$

On the other hand, let (u_1, w_1), $(u_2, w_2) \in T_{(x,v)}TQ$. From (3.3$a$), we derive the following formula in local coordinates,

$$\begin{aligned}\Omega_L(x, v)\big((u_1, w_1),\ (u_2, w_2)\big) &= \big(D_1D_2L(x, v) \cdot u_2\big) \cdot u_1 + \big(D_2D_2L(x, v) \cdot w_2\big) \cdot u_1 \\ &\quad - \big(D_1D_2L(x, v) \cdot u_1\big) \cdot u_2 - \big(D_2D_2L(x, v) \cdot w_1\big) \cdot u_3.\end{aligned} \tag{3.3c}$$

Next, we give the intrinsic form of *Legendre transformation*, which maps the velocity phase space to the momentum phase space. The Legendre transformation corresponding to the Lagrangian L can be defined as

$$\ell_L : TQ \to T^*Q, \qquad (x, v) \mapsto \big(x, \sigma_{(x,v)} \cdot \Theta_L(x, v)\big), \tag{3.4}$$

or, equivalently, $\ell_L(x, v) = \big(x, D_2L(x, v)\big)$, compare the definition through fiber derivatives in pp. 209, 219 of [1].

Assuming now that ℓ_L is a diffeomorphism (or L is *hyperregular*), we have $\ell_L^{-1} : T^*Q \to TQ$. (This condition implies that, in local coordinates, $D_2D_2L(x, v)$ is nonsingular.) Denote the space of k-forms on a manifold M as $\varpi^k(M)$. By the pull-back of ℓ_L^{-1}, $(\ell_L^{-1})^* : \varpi^2(TQ) \to \varpi^2(T^*Q)$, we can define a 2-form on T^*Q as $\omega_0 \triangleq (\ell_L^{-1})^*\Omega_L$. Although Ω_L is L-dependent, ω_0 defined above is invariant under the change of L. In fact, letting (x, p) be local coordinates of T^*Q, where $\langle p, w\rangle = \langle D_2L(x, v), w\rangle$, for $w \in T_xQ$, it can be shown that $\omega_0 = dx \wedge dp$, which is the canonical symplectic 2-form on the cotangent bundle.

Thus, when the Legendre transformation is diffeomorphic, the two approaches, either based on the cotangent bundle or directly on the tangent bundle, are equivalent. Moreover, (T^*Q, ω_0), (TQ, Ω_L) are both symplectic manifolds, carrying associated Poisson structures.

Remark 3.1. The closed 2-form Ω_L in (3.3*a*) is well defined for every Lagrangian L. It is, however, nondegenerate, and therefore a symplectic structure, only when L is *regular.* For a *singular* or irregular L, Ω_L becomes *presymplectic,* namely Ω_L is no longer of maximal rank. Discussions of this case may be found in, for example, [14] [15].

With the symplectic 2-form Ω_L, one constructs a correspondence between vector fields and 1-forms, $\Pi_L : \varpi^1(TQ) \to \mathscr{X}(TQ)$ through, for $\omega \in \varpi^1(TQ)$,

$$\Omega_L\big(\Pi_L(\omega), Z\big) = \omega(Z), \qquad \forall\, Z \in \mathscr{X}(TQ). \tag{3.5}$$

In terms of the inverse of Π_L, an alternative expression is $\Omega_L(X, Z) = \Pi_L^{-1}(X)(Z)$, for all $Z \in \mathscr{X}(TQ)$, or

$$\Pi_L^{-1}(X)(\cdot) = \Omega_L(X, \cdot). \tag{3.6}$$

It can be shown that Π_L maps horizontal 1-forms to vertical vector fields [45] [46]. Now define the *energy* function on TQ, $H_L : TQ \to \mathbb{R}$, as,

$$H_L \triangleq dL\big(X^{PV}\big) - L, \tag{3.7}$$

where X^{PV} is the principal vertical field defined in Section 2. In local coordinates, we have $H_L(x, v) = \langle \ell_L(x, v), v\rangle - L(x, v)$, which is exactly the same notion as the energy defined on p. 213 in [1]. In particular, the function $dL\big(X^{PV}\big)$ is sometimes called the *action* corresponding to L. From the energy function H_L on the velocity phase space, define the Hamiltonian on the momentum phase space as

$$H : T^*Q \to \mathbb{R} \qquad H = H_L \circ \ell_L^{-1}. \tag{3.8}$$

The Hamiltonian system (T^*Q, ω_0, H) is the customary object of study in Hamiltonian mechanics.

The *Lagrangian vector field* determined by L is defined as,

$$X_{H_L} \triangleq \Pi_L\big(dH_L\big), \tag{3.9a}$$

or, equivalently, $\Omega_L(X_{H_L}, Z) = dH_L(Z)$, for all $Z \in \mathscr{X}(TQ)$. In local coordinates, the matrix form is $X_{H_L}^T[\Omega_L]Z = \nabla H_L^T Z$. Thus we may write the Lagrangian vector field as

$$X_{H_L} = \big([\Omega_L]^{-1}\big)^T \nabla H_L. \tag{3.9b}$$

One could also think of X_{H_L} as the Hamiltonian vector field corresponding to the Hamiltonian H_L on the symplectic manifold (TQ, Ω_L), and thus H_L is a first integral (conserved quantity) along the vector field X_{H_L}. We say that we can *define consistent equations of motion* if such an X_{H_L} exists. It can be shown that X_{H_L} is a special vector field and thus gives rise to a second-order equation.

Now we introduce the important notion of Lagrangian force. Recall that in Lagrangian mechanics [45], virtual displacements can be thought as special vector fields on TQ, and forces can be modeled as horizontal 1-forms on TQ. For a Lagrangian L, the associated *Lagrangian force* on a virtual displacement X, $F_L(X)$, is defined through

$$F_L(X)(Z) \triangleq \Omega_L(X, Z) - dH_L(Z), \quad \forall\, Z \in \mathcal{X}(TQ). \tag{3.10a}$$

The Lagrangian force $F_L(X)$ is a 1-form on TQ. This 1-form can be shown to be a horizontal 1-form on TQ. In fact, in local coordinates, with $X(x, v) = (v, w)$, we have

$$F_L(X)(x, v)(u, w_2) = \big(-D_1D_2L(x, v)\cdot v - D_2D_2L(x, v)\cdot w + D_1L(x, v)\big)\cdot u. \tag{3.10b}$$

Thus it is a well-defined *force*.

Definition in (3.10a) holds even for L singular, compare Remark 3.1. If L is hyperregular, we may write, compare (3.5), $F_L(X) = \Pi_L^{-1}(X) - dH_L$. This is the definition used in [45] for the Lagrangian force. In the above setting, the Lagrange-D'Alembert Principle can be now stated in the following form.

Principle 3.2. *Lagrange d'Alembert Priniple. For a holonomic mechanical system, on the virtual displacement (special vector field) that determines the real trajectory of motion, the sum of the Lagrangian force and the exterior force vanishes.*

For natural systems, the Lagrangian force consists of resultant *force of inertia* and forces coming from the potential energy. Thus the principle here corresponds to the classical D'Alembert principle, see for example, [26]. As discussed in [26], a fundamental entity in analytical mechanics is virtual work, instead of the classical notion of force. Here we present a unified treatment in terms of horizontal 1-forms, where the *classical forces* are represented by the coordinates of this 1-form.

Let ω be an exterior force or a horizontal 1-form. The D'Alembert principle says that

$$F_L(X) + \omega = 0, \tag{3.11}$$

where X is a special vector field. The trajectories of motion of the mechanical system with Lagrangian L obey the flow of this vector field. In the absence of any exterior force and with L being regular, from (3.10a), we write $\Omega_L(X, Z) = dH_L(Z)$, for all $Z \in \mathcal{X}(TQ)$, which, by definition of X_{H_L}, implies that $X = X_{H_L}$, that is, the Lagrangian vector field gives the real trajectories of motion.

Now we express the D'Alembert principle in local coordinates where (3.11) reads

$$F_L(X)(x, v) + \omega(x, v) = 0.$$

Letting $\omega = (\alpha, 0)$, $X(x, v) = (v, w)$, we have, compare (3.10b),

$$\big(-D_1D_2L(x,v)\cdot v - D_2D_2L(x,v)\cdot w + D_1L(x,v)\big)\cdot u + \alpha\cdot u = 0, \quad \forall\, u \in T_xQ.$$

By including time derivatives as $v = \dot{x}$, $w = \dot{v}$, we get

$$\frac{d}{dt}D_2L(x,v)\cdot u = D_1L(x,v)\cdot u + \alpha\cdot u, \quad \forall\, u \in T_xQ. \tag{3.12}$$

Integrating both sides with respect to the variable t, this equation can be rewritten as

$$D_2L(x,v)\cdot u\,\Big|_0^T - \int_0^T D_2L(x,v)\cdot u_t\,dt = \int_0^T \big(D_1L(x,v)\cdot u + \alpha\cdot u\big)dt.$$

This corresponds to the *Principle of Virtual Power* in analytical mechanics, compare, for example, [51]. The tangent vector u is sometimes called *test function*. In the case that the pairing is nondegenerate, for example, in the finite dimensional case, we can write (3.12) as

$$\frac{d}{dt}D_2L(x,v) = D_1L(x,v) + \alpha, \tag{3.13}$$

which is the classical form of the Euler-Lagrange equation.

Example 3.3. (*On the group $SO(3)$*). Now we illustrate the Lagrange-D'Alembert principle in the setting of the special orthogonal group $SO(3)$ as configuration space. Recall that each element A in $SO(3)$ is an element in $GL(3)$, the group of all 3×3 nonsingular matrices, which satisfies the condition $A^TA = 1$ and $\det(A) = 1$. Let the operator $\hat{}$ denote the natural isomorphism from $\mathbb{R}^3$ to $so(3)$, the space of 3×3 skew-symmetric matrices, defined by

$$\widehat{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}. \tag{3.14}$$

Given $A \in SO(3)$, recall that $(A, A\hat{\Omega})$ is an element in $TSO(3)$. In rigid body mechanics, the variable Ω corresponds to the instantaneous angular velocity of the motion in body coordinates. With the representation of the elements in the second tangent bundle $TTSO(3)$ as

$$\big(A, A\hat{\Omega}, A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})\big), \tag{3.15}$$

and the *trace* pairing in $GL(3)$

$$\langle A, B\rangle = \frac{1}{2}tr\big(A^TB\big), \quad \text{for } A, B \in GL(3), \tag{3.16}$$

we can write elements in $T^*TSO(3)$ canonically as

$$(A, A\hat{\Omega}, A\big(\hat{b}\hat{\Omega} + \hat{a}\big), A\hat{b}). \tag{3.17}$$

Here $u,\ w,\ a,\ b$ are vectors in $\mathbb{R}^3$, and the pairing between $T^*TSO(3)$ and $TTSO(3)$ becomes

$$\left\langle \left(A, A\hat{\Omega}, A\left(\hat{b}\hat{\Omega} + \hat{a}\right), A\hat{b}\right),\ \left(A, A\hat{\Omega}, A\hat{u}, A\left(\hat{u}\hat{\Omega} + \hat{w}\right)\right)\right\rangle = a \cdot u + b \cdot w.$$

We remark here that these parametrizations of $TTSO(3)$ and $T^*TSO(3)$ are *globally defined* via the embedding of $SO(3)$ in $GL(3)$. Our goal has been to make the pairing analogous to that on Euclidean space. The global representations (3.15), (3.17) of the second tangent bundle and the dual of the second tangent bundle on $SO(3)$ also prove to be useful in computing the derivatives or variations of a function (Lagrangian) on $TSO(3)$ and in deriving the reduced Poisson bracket [49]. In the following, we state the Lagrange-D'Alembert principle in terms of these representations.

On $TSO(3)$, let a system be described by a Lagrangian L. The Lagrange-D'Alembert principle in the invariant form (3.11) applied to motions on $SO(3)$ gives rise to the Euler-Lagrange equation, namely for all $A\hat{u} \in T_ASO(3)$,

$$\left\langle \frac{d}{dt} D_2L\left(A, A\hat{\Omega}\right), A\hat{u} \right\rangle = \langle D_1L\left(A, A\hat{\Omega}\right), A\hat{u}\rangle + \langle \alpha, A\hat{u}\rangle, \tag{3.18}$$

where α is the exterior force. See [46] for detailed discussions.

4. Gyroscopic Control

In this section we demonstrate a class of feedback control laws that transform a simple mechanical system with symmetry *with exogenous forces* (controls) into a gyroscopic system with symmetry, compare Theorem 4.4 later in this section. Although the concept of a Lagrangian system with symmetry is by now well known (see [1], and [5]), in the interest of keeping our treatment self-contained, we give a rapid exposé of the basic ideas around the concept of a gyroscopic system with symmetry.

Definition 4.1. A *gyroscopic system with symmetry* is a 5-tuple, $(Q,\ K,\ Y,\ V,\ G)$, where

(1) (Q, K) is a Riemannian manifold.
(2) Y is a vector field on Q, called a *gyroscopic field*.
(3) V is a function on Q, a *potential*.
(4) G is a Lie group with an action $\Phi : G \times Q \to Q$, which leaves $K,\ Y,\ V$ invariant and is referred to as the *symmetry group*.
(5) The associated Lagrangian $L : TQ \to \mathbb{R}$ is given by

$$L(v_x) = \frac{1}{2}K(x)(v_x, v_x) + K(x)\left(v_x, Y(x)\right) - V(x). \tag{4.1}$$

On the other hand, in the framework of Hamiltonian mechanics, a gyroscopic system with symmetry is characterized by a Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$ of the following form,

$$H(\alpha_x) = \frac{1}{2} < \alpha_x - K^b\big(Y(x)\big), \alpha_x - K^b\big(Y(x)\big) >_{T^*Q} + V(x). \tag{4.2}$$

where $< \cdot, \cdot >_{T^*Q}$ is the induced metric on T^*Q defined in (2.2).

The word word "gyroscopic" comes from the second term in the Lagrangian (4.1), which includes the gyroscopic field Y. This term is *linear* in the velocity variables and is responsible for the paradoxical behavior of gyroscopes. The Coriolis force in a rotating reference system and the magnetic force due to electric currents are examples of the effect of gyroscopic terms in the Lagrangian function. To see how the gyroscopic term enters the dynamical equations, we restrict our attention for the moment to a gyroscopic system (without symmetry consideration) on $\mathbb{R}^n$ (or in local coordinates) described by the Lagrangian

$$L(x, v) = \frac{1}{2} < v, \quad M(x)v > + < \tilde{Y}(x), \quad v > - V(x), \tag{4.3}$$

where $M(x)$ is a symmetric positive-definite second-order tensor, vectors $x, v(= \dot{x})$ are in $\mathbb{R}^n$, $\tilde{Y}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and V is a real-valued function. The notation $< \cdot, \cdot >$ denotes the inner product on the Euclidean space $\mathbb{R}^n$. This is a gyroscopic system in the sense of Definition 4.1 with

$$K(x)(v, v) = v^T M(x) v \quad \text{and} \quad Y(x) = M(x)^{-1}\tilde{Y}(x).$$

Abstractly, $\tilde{Y}$ should be regarded as a 1-form in T^*Q.

To obtain the dynamical equations associated with the Lagrangian in (4.3), we invoke the classical Euler-Lagrange equations, compare (3.13). First, we find

$$\frac{\partial L}{\partial v} = M(x) \cdot v + \tilde{Y}(x).$$

By taking time derivatives, we get

$$\frac{d}{dt}\frac{\partial L}{\partial v} = M(x) \cdot \dot{v} + \left(\frac{\partial M}{\partial x}(x) \cdot v\right) \cdot v + \frac{\partial \tilde{Y}}{\partial x}(x) \cdot v,$$

where $\partial M / \partial x$ is a third-order tensor, and $\partial \tilde{Y} / \partial x$ is a second-order tensor. With standard notations in tensor algebra, compare, for example, [3], the above equation can be rewritten as

$$\frac{d}{dt}\frac{\partial L}{\partial v} = M(x) \cdot \dot{v} + \frac{\partial M}{\partial x}(x) : vv + \frac{\partial \tilde{Y}}{\partial x}(x) \cdot v. \tag{4.4}$$

Now we compute the partial derivative of L with respect to x. By definition,

$$\begin{aligned}\frac{\partial}{\partial x}\left(v^T M(x)v\right)\cdot \mathbf{w} &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} v^T M(x+\epsilon\mathbf{w})v \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} M(x+\epsilon\mathbf{w}) : (vv) \\ &= \left(\frac{\partial M}{\partial x}(x)\cdot \mathbf{w}\right) : (vv) = \left(\frac{\partial M}{\partial x}(x)^* : (vv)\right)\cdot \mathbf{w},\end{aligned}$$

where $(\partial M/\partial x)^*$ is the *cyclic transpose* of $\partial M/\partial x$ defined through

$$\frac{\partial M}{\partial x}(x)^*\cdot \mathbf{u}\cdot \mathbf{v}\cdot \mathbf{w} \triangleq \frac{\partial M}{\partial x}(x)\cdot \mathbf{w}\cdot \mathbf{u}\cdot \mathbf{v}, \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n.$$

Accordingly, we have

$$\frac{\partial}{\partial x}\left(v^T M(x)v\right) = \frac{\partial M}{\partial x}(x)^* : (vv). \tag{4.5}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial x}\left(v^T \tilde{Y}(x)\right)\cdot \mathbf{w} &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} v^T \tilde{Y}(x+\epsilon\mathbf{w}) \\ &= \left(\frac{\partial \tilde{Y}}{\partial x}(x)\cdot \mathbf{w}\right)\cdot v = \left(\frac{\partial \tilde{Y}}{\partial x}(x)^T\cdot v\right)\cdot \mathbf{w},\end{aligned}$$

where the superscript ^{T} denotes the transpose of the second-order tensor. Thus we have

$$\frac{\partial}{\partial x}\left(v^T \tilde{Y}(x)\right) = \frac{\partial Y}{\partial x}(x)^T\cdot v. \tag{4.6}$$

By substituting (4.4), (4.5), and (4.6) in the Euler-Lagrange equations, we obtain

$$M(x)\cdot \dot{v} + \frac{\partial M}{\partial x}(x) : (vv) + \frac{\partial \tilde{Y}}{\partial x}(x)\cdot v = \frac{1}{2}\frac{\partial M}{\partial x}(x)^* : (vv) + \frac{\partial \tilde{Y}}{\partial x}(x)^T\cdot v - \frac{\partial V}{\partial x}(x),$$

which can be further written as

$$M(x)\cdot \dot{v} = \left(\frac{1}{2}\frac{\partial M}{\partial x}(x)^* - \frac{\partial M}{\partial x}(x)\right) : (vv) - \left(\frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}}{\partial x}(x)^T\right)\cdot v - \frac{\partial V}{\partial x}(x).$$

Define

$$\mathcal{R} \triangleq \frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}}{\partial x}(x)^T, \tag{4.7a}$$

$$\mathcal{T} \triangleq \frac{1}{2}\frac{\partial M}{\partial x}(x)^* - \frac{\partial M}{\partial x}(x). \tag{4.7b}$$

The equations of motion for a gyroscopic system with Lagrangian of the form (4.1) can be then expressed as,

$$M(x) \cdot \ddot{x} = \mathcal{T} \cdot \dot{x} \cdot \dot{x} - \mathcal{R} \cdot \dot{x} - \frac{\partial V}{\partial x}(x). \tag{4.8}$$

Note that $\mathcal{R}$ is a skew-symmetric tensor, thus the second term on the right of (4.8) gives the gyroscopic force in the dynamical equations as discussed in [11]. We remark here that the component form of $M(x)^{-1}\mathcal{T}$ is nothing but the Christoffel symbol associated with the geodesic flow. Compare, for example, [1]. By multiplying both sides of (4.8) with M^{-1}, we get

$$\ddot{x} - M(x)^{-1}\mathcal{T} \cdot \dot{x} \cdot \dot{x} = M(x)^{-1}\mathcal{R} \cdot \dot{x} - M(x)^{-1}\frac{\partial V}{\partial x}(x). \tag{4.9}$$

In terms of covariant differentiation, we can write the left-hand side of (4.9) as $\nabla_{\dot{x}}\dot{x}$. Moreover, with $\tilde{Y}$ being a 1-form, by taking exterior derivative of $\tilde{Y}$, we obtain, in local coordinates,

$$\begin{aligned} d\tilde{Y}(x)(v, w) &= \big(D\tilde{Y}(x) \cdot v\big) \cdot w - \big(D\tilde{Y}(x) \cdot w\big) \cdot v \\ &= D\tilde{Y}(x) \cdot v \cdot w - D\tilde{Y}(x)^T \cdot v \cdot w = \mathcal{R} \cdot v \cdot w. \end{aligned}$$

Thus we may write

$$\mathcal{R} \cdot \dot{x} = d\tilde{Y}(x)(\dot{x}, \cdot),$$

which is actually a 1-form. On the other hand, the third term on the right of (4.8) also corresponds to a 1-form, namely, dV. Recall that $K^\#$ transforms a 1-form to a vector field. We define the following notations

$$\text{grad } V \triangleq K^\# \cdot dV,$$
$$(\text{curve}_{\dot{x}}\tilde{Y})^\# \triangleq K^\# \cdot d\tilde{Y}(x)(\dot{x}, \cdot).$$

Here grad V denotes the gradient of V. As a consequence, (4.9) can be expressed in its invariant form as follows.

Theorem 4.2. *The invariant form of the equations of motions of a gyroscopic system* $(Q, K, \tilde{Y}, V)$ *is*

$$\nabla_{\dot{x}}\dot{x} + \big(\text{curv}_{\dot{x}}\tilde{Y}\big)^\# = -\text{grad } V. \tag{4.10}$$

These are the equations of motion of a charged particle in a magnetic field $d\tilde{Y}$. They are thus a special (simplest, abelian) case of Wong's equation for the motion of a particle in a Yang-Mills field [53], [17]. The term $(\mathrm{curv}_{\dot{x}}\tilde{Y})^{\#}$ *is the corresponding force of interaction with the field.*

Example 4.3. We consider the dynamical system treated in [11] in the following form.

$$\ddot{x} = -\alpha x - g\dot{y}, \qquad \ddot{y}, = -\beta y + g\dot{x}. \tag{4.11}$$

The skew terms in velocities $-g\dot{y}$ and $g\dot{x}$ constitute the *gyroscopic forces* that do not network but affect the stability of the system. It is easily checked that this is a gyroscopic system with the Lagrangian in the form (4.3) with the following entities,

$$M(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Y}(x, y) = \begin{pmatrix} gy \\ 0 \end{pmatrix}, \quad V(x, y) = \frac{1}{2}(xy)\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

With the equations of motion of a gyroscopic system (4.8), or (4.10), we are ready to state the main ideas of *gyroscopic control.* This notion is isolated here to highlight the role of the gyroscopic term from the viewpoint of designing control algorithms. A simple mechanical system with symmetry with exterior forces can be transformed into a gyroscopic system with symmetry by using suitable feedback laws, that we refer to as *gyroscopic feedback*. This process is described in the following theorem.

Theorem 4.4. *Consider a simple mechanical system with symmetry,* (Q, K, V, G). *Let the exterior force exerted on this system be denoted by a horizontal 1-form* $(\alpha, 0)$. *Let* $\tilde{Y}$ *be any* G*-invariant 1-form on* Q. *Then, with the feedback law (gyroscopic feedback),*

$$\alpha(v_x) = -d\tilde{Y}(v_x, \cdot), \tag{4.12}$$

the closed-loop system becomes a gyroscopic system with symmetry (Q, K, Y, V, G) *where,*

$$Y = K^{\#} \cdot \tilde{Y}. \tag{4.13}$$

Proof. We prove this theorem in local coordinates where the Riemannian metric is expressed as

$$K(x)(v, w) = v^T M(x) w.$$

Also the feedback law (4.12) can be written in local coordinates as

$$\alpha(x, v) = -\left(\frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}^T}{\partial x}(x)\right)v. \tag{4.14}$$

Recalling the derivation of Equation (4.8), the dynamical equations for (Q, K, V, G) with exterior force can be found to be, in local coordinates,

$$M(x) \cdot \ddot{x} = \mathcal{T} \cdot \dot{x} \cdot \dot{x} - \frac{\partial V}{\partial x}(x) + \alpha, \tag{4.15}$$

where $\mathcal{T}$ is defined in (4.7b). With the feedback law (4.14), it is then easy to see that (4.15) becomes (4.8) which, in turn, corresponds to a system with Lagrangian in the form of (4.3). With the transformation rule (4.13) expressed in local coordinates,

$$Y(x) = M(x)^{-1}\tilde{Y}(x),$$

the system can be further identified as a gyroscopic system with symmetry, (Q, K, Y, V, G). The G-invariance property of Y follows from the G-invariance of the Riemannian metric and the 1-form $\tilde{Y}$.

Accordingly, we have a family of gyroscopic feedback laws induced by G-invariant 1-forms. The techniques used for analyzing gyroscopic systems with symmetry can then be applied to the study of the corresponding closed-loop system. In particular, the method for stability analysis based on the energy-momentum method that will be developed in the following is applicable. The gyroscopic term affects the dynamical behavior in many ways. For example, it changes the location of equilibria as well as their stability properties. As a consequence, suitable gyroscopic feedbacks may be chosen to fulfill design objectives. Much work remains to be done on general methods for selecting $\tilde{Y}$. The dual-spin problem illustrated below gives a simple example of gyroscopic systems with symmetry.

Example 4.5. (*Dual-Spin Problem*). Consider the system consisting of a rigid body (platform) with on-board rigid symmetric rotors moving in free space, compare Figure 4.1. With the rotors spinning at constant rates relative to the platform, the dynamical behavior can be captured by a gyroscopic system with symmetry.

First we consider the system dynamics with locked rotors. We assume that the center of mass of the system is fixed in some inertial frame. Let $B \in SO(3)$ denote the orthogonal transformation from the body frame to the spatial frame, and then describe the attitude of the body. We have $\dot{B} = B\hat{\Omega}$, where the operator $\hat{}$ is defined in (3.14) and Ω is the *instantaneous angular velocity* of the body relative to the body

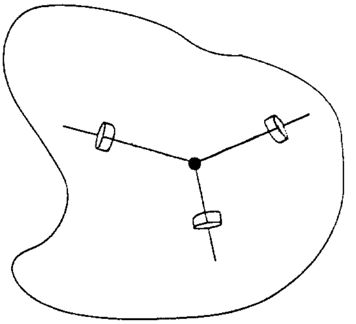

Fig. 4.1 Dual-spin configuration.

frame. Let $\mathbf{J}$ be the total moment of inertia of the platform with locked rotors. The Lagrangian for this locked system can be written as

$$L(B, B\hat{\Omega}) = \frac{1}{2}\langle \Omega, \mathbf{J}\Omega \rangle.$$

With this Lagrangian, we now invoke Lagrange-D'Alembert Principle specialized to the current situation, compare (3.18). Using the parametrizations introduced in Section 3, the dynamical equations are nothing but the Euler's equation for rigid body dynamics,

$$\mathbf{J}\dot{\Omega} = -\Omega \times \mathbf{J}\Omega.$$

Next we let the rotors spin at constant rates. Let θ_i, $i = 1, 2, 3$ denote the relative angles between the three rotors and platform, respectively, and $\Theta = (\theta_1, \theta_2, \theta_3)$. Let the corresponding moments of inertia of rotors relative to the spinning axis be denoted by $(\mathbf{I}_{S_i})_i$, $i = 1, 2, 3$, respectively. The reaction force exerted on the platform from the rotors can be derived from the following gyroscopic 1-form,

$$\tilde{Y}(B) = B\big(\widehat{\mathbf{I}^S\dot{\Theta}}\big), \tag{4.16}$$

where $\mathbf{I}^S = \mathrm{diag}\{(\mathbf{I}_{S_1})_1, (\mathbf{I}_{S_2})_2, (\mathbf{I}_{S_3})_3\}$. In fact, by using the formula (4.12), we can show that

$$\alpha(B, B\hat{\Omega}) = -B\big(\widehat{\Omega \times \mathbf{I}^S\dot{\Theta}}\big).$$

With this exogenous force, the dynamical equations for the closed-loop system become,

$$\mathbf{J}\dot{\Omega} = -\Omega \times \big(\mathbf{J}\Omega + \mathbf{I}^S\dot{\Theta}\big). \tag{4.17}$$

It is readily checked that this system is a gyroscopic system with symmetry with the following entities,

$$\begin{aligned} &Q = SO(3), \qquad K(B\hat{u}_1, B\hat{u}_2) = \langle u_1, \mathbf{J}u_2 \rangle, \\ &Y(B) = B\big(\widehat{\mathbf{J}^{-1}\mathbf{I}^S\dot{\Theta}}\big), \quad V(B) = 0, \quad G = SO(3) \end{aligned} \tag{4.18}$$

where $B\hat{u}_1, B\hat{u}_2 \in T_B SO(3)$. The group action here is $G \times Q \to Q$, $(R, B) \mapsto RB$, and the Lagrangian is

$$\tilde{L}(B, B\hat{\Omega}) = \frac{1}{2}\langle \Omega, \mathbf{J}\Omega \rangle + \langle \Omega, \mathbf{I}^S\dot{\Theta} \rangle. \tag{4.19}$$

Keeping in focus our program of understanding the closed-loop behavior of a system with gyroscopic feedback, we return to the abstract framework of gyroscopic systems with symmetry. We remark first that a simple mechanical system with symmetry in the sense of Smale [42] is a special case of a gyroscopic system with symmetry. We simply take $Y = 0$ and consider the quadruple (Q, K, V, G). Many key results in the category of simple mechanical systems with symmetry can be extended to gyroscopic systems [46]. First, for a gyroscopic system with symmetry, the Lagrangian (4.1) is invariant under the tangent lift Φ^T, which follows directly from the invariance of the metric K, the gyroscopic field Y, and the potential V. The Legendre transformation, compare (3.4), is given by

$$\ell_L(v_x) = K^b\big(v_x + Y(x)\big). \tag{4.20}$$

Its inverse can be then found as, for $\alpha_x \in T^*Q$,

$$\ell_L^{-1}(\alpha_x) = K^\#(\alpha_x) - Y(x).$$

It follows that ℓ_L is a diffeomorphism, and L is hyperregular. As a consequence, the space $(TQ,\ \Omega_L = -d\Theta_L)$ is a symplectic manifold, where the symplectic form Ω_L is defined as in (3.3) through the 1-form Θ_L, which in turn can be written as

$$\Theta_L(v_x)\cdot(u, w) = K(x)\big(v + Y(x), u\big).$$

From Lemma 4.6, the group G acts on TQ through the tangelt lift Φ^T as a symmetry group. It can be further verified that this action is symplectic, namely, $(\Phi^T)^*\Omega_L = \Omega_L$. Within this framework, a momentum mapping $J : TQ \to \mathcal{G}^*$ can be constructed such that the infinitesimal generator of the action Φ^T corresponding to $\xi \in \mathcal{G}$ is the vector field induced by the function $\langle J, \xi\rangle : TQ \to \mathbb{R}$, through the symplectic structure, compare (3.9). Consequently, we have the following theorem.

Theorem 4.6. *The gyroscopic system with symmetry (Q, K, Y, V, G) has the following properties:*

(i) The 1-form corresponding to L defined in (3.2) is invariant under the tangent lift, that is, $(\Phi_g^T)^*\Theta_L = \Theta_L$.

(ii) There is an associated Ad^*-equivariant momentum mapping $J : TQ \to \mathcal{G}^*$,

$$J(v_x)(\xi) = \langle \ell_L(v_x),\ \xi_Q(x)\rangle_x = \ll v_x + Y(x),\ \xi_Q(x) \gg_x, \tag{4.21}$$

where $\xi \in \mathcal{G}$ is an element in the Lie algebra of G and $\xi_Q(x)$ denotes the infinitesimal generator of ξ on Q. Here $\mathcal{G}^*$ denotes the dual of the Lie algebra $\mathcal{G}$.

(iii) The momentum mapping defined in (4.21) is a vector-valued integral of any vector field induced by a G-invariant function on TQ through an analogous formula in (3.9). In particular, it is an integral of the Lagrangian vector field X_{H_L}.

Proof. For (i), we note that $(\Phi_g^T)^*L = L$, by Lemma 4.6. Since the exterior differentiation commutes with the pull-back operator, (i) follows immediately. Statements (ii), and (iii) can be shown by directly applying Theorem 4.2.2 and Corollary 4.2.14. in [1].

The quadruple $(TQ, \Omega_L, \Phi^T, J)$ is an example of a *Hamiltonian G-space*. The energy function for the gyroscopic system can be derived as, compare (3.7),

$$\begin{aligned} H_L(v_x) &= \langle \ell_L(v_x), v_x \rangle_x - L(v_x) \\ &= \ll v_x + Y(x), v_x \gg_x - \frac{1}{2} \ll v_x, v_x \gg_x - \ll v_x, Y(x) \gg_x + V(x) \\ &= \frac{1}{2} \ll v_x, v_x \gg_x + V(x). \end{aligned} \tag{4.22}$$

It is easy to see that the energy function is not affected by the presence of the gyroscopic field Y. However, the dynamics are different from what one would see if $Y = 0$. The differences in the dynamical behavior arise from the Y-dependent symplectic 2-form Ω_L. In particular, the gyroscopic term in the Lagrangian gives rise to the *magnetic terms* in the symplectic 2-form. On the other hand, on the momentum phase space T^*Q, the Hamiltonian associated to the system is, compare (4.2),

$$\begin{aligned} H(\alpha_x) &= H_L \circ \ell_L^{-1}(\alpha_x) \\ &= \frac{1}{2} \ll K^{\#}(\alpha_x) - Y(x), K^{\#}(\alpha_x) - Y(x) \gg_x + V(x) \\ &= \frac{1}{2} < \alpha_x - K^b\big(Y(x)\big), \alpha_x - K^b\big(Y(x)\big) >_{T^*Q} + V(x). \end{aligned}$$

Accordingly, on the momentum phase space, the Hamiltonian is affected by the gyroscopic term through a momentum shift, while the canonical 2-form ω_0 is unchanged. This subtlety is best explained by the following example.

Example 4.7. We consider again the system in Example 4.3. The energy associated with (4.11) on TQ is

$$H_L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2 + \alpha x^2 + \beta y^2\big),$$

with the symplectic 2-form in matrix representation

$$\big[\Omega_L\big] = \begin{pmatrix} 0 & g & 1 & 0 \\ -g & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This can be checked from the differential equation (4.11), compare (3.9b),

$$\dot{\mathbf{x}} = X_{H_L}(\mathbf{x}) = \Big(\big[\Omega_L\big]^{-1}\Big)^T \nabla H_L,$$

where $\mathbf{x} = (x, y, \dot{x}, \dot{y})$. The left-upper 2×2 block in $\big[\Omega_L\big]$ is called the *magnetic part*. On the other hand, on T^*Q, we have the conjugate momentum variables defined by

$$p_1 = \dot{x} + gy, \qquad p_2 = \dot{y}.$$

The dynamical equation (4.11) can be written as

$$\begin{aligned} \dot{x} &= p_1 - gy, & \dot{y} &= p_2, \\ \dot{p}_1 &= -\alpha x, & \dot{p}_2 &= -\beta y + g(p_1 - gy), \end{aligned}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(x, y, p_1, p_2) = \frac{1}{2}\left((p_1 - gy)^2 + p_2^2 + \alpha x^2 + \beta y^2\right).$$

The symplectic structure is the canonical symplectic 2-form ω_0, that is, in matrix representation,

$$[\omega_0] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

In summary, the gyroscopic term affects the symplectic 2-form on the TQ side, whereas, on the T^*Q side, it affects the Hamiltonian function. To gain more insight on how the gyroscopic field enters the symplectic structure Ω_L, we consider an even simpler case than (4.3). We assume that the second-order tensor M is independent of x in (4.3). It can be easily seen that the symplectic 2-form is now, compare (3.3),

$$\begin{aligned} &\Omega_L(q, v)\big((u_1, w_1), (u_2, w_2)\big) \\ &\quad = \frac{\partial \tilde{Y}}{\partial x} \cdot u_2 \cdot u_1 + M \cdot w_2 \cdot u_1 - \frac{\partial \tilde{Y}}{\partial x} \cdot u_1 \cdot u_2 - M \cdot w_1 \cdot u_2 \\ &\quad = (u_1 \; w_1) \cdot \begin{pmatrix} \dfrac{\partial \tilde{Y}}{\partial x} - \dfrac{\partial \tilde{Y}}{\partial x}^T & M \\ -M^T & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ w_2 \end{pmatrix}. \end{aligned}$$

The block $\partial \tilde{Y}/\partial x - \partial \tilde{Y}^T/\partial x$ is the so-called magnetic term.

5. Reduction, Relative Equilibria, and Stability

By recognizing the symmetry, under suitable regularity hypotheses, it is possible to reduce a gyroscopic system with symmetry (Q, K, Y, V, G) to a lower-order dynamical system. The reduction process has a long history. For Jacobi and Liouville [1] [5], this meant reduction of the Hamilton's equation via first integrals in involution. For Routh [37], this meant a process of eliminating ignorable variables. In the following, we shall discuss the reduction from two modern points of view, namely, symplectic reduction and Poisson reduction.

First, we consider symplectic reduction in the sense of [32]. As discussed in Section 3, if L is regular, $(TQ, \ \Omega_L)$ is a well-defined symplectic manifold. By the Property (i) in Theorem 4.6, the Lie group G acts symplectically on $(TQ, \ \Omega_L)$. Also, from Property (ii) in Theorem 4.6, there is an Ad^*-equivariant momentum mapping J

for this action. Thus all the conditions in the Symplectic Reduction Theorem, see Theorems 4.3.1, 4.3.5, pp. 299, 304 in [1] are satisfied, and we can state the following reduction theorem specialized to gyroscopic systems with symmetry.

Theorem 5.1. *(Marsden-Weinstein). Consider the gyroscopic system with symmetry* (Q, K, Y, V, G). *Assume that* $\mu \in \mathcal{G}^*$ *is a regular value of the momentum mapping* J, *as defined in (4.21), and that the isotropy subgroup* G_μ, *defined by* $G_\mu = \{g \in G : Ad^*_{g^{-1}}\mu = \mu\}$, *under the* Ad^* *action on* $\mathcal{G}^*$ *acts freely and properly on* $J^{-1}(\mu)$, *then* $(TQ)_\mu \triangleq J^{-1}(\mu)/G_\mu$, *has a unique symplectic form* Ω_μ *with the property* $\pi^*_\mu\Omega_\mu = i^*_\mu\Omega_L$, *where* $\pi_\mu : J^{-1}(\mu) \to (TQ)_\mu$ *is the canonical projection and* $i_\mu : J^{-1}(\mu) \hookrightarrow TQ$ *is the inclusion map. Letting* H_L *be as in (4.22), the flow* F_t *of* X_{H_L} *induces a flow* F^μ_t *on* $(TQ)_\mu$ *satisfying* $\pi_\mu \cdot F_t = F^\mu_t \cdot \pi_\mu$. *This flow is a Hamiltonian flow on* $(TQ)_\mu$ *with a Hamiltonian function* H^μ_L *satisyfing* $H^\mu_L \cdot \pi_\mu = H_L \cdot i_\mu$, *with respect to the symplectic structure* Ω_m.

The function H^μ_L on the reduced space is called the *reduced energy*. The corresponding vector field $H_{H^\mu_L}$ on the reduced space $(TQ)_\mu$ is called the *reduced dynamics*. Thus in symplectic reduction, we first restrict the dynamics to a level set of the momentum mapping, and then factor out the isotropy subgroup.

Next, we consider Poisson reduction [31]. We first recall the basic setup of Poisson manifolds. A Poisson manifold P is a smooth manifold equipped with an $\mathbb{R}$-bilinear map (Poisson structure) on the space of smooth functions, $\{\cdot, \cdot\}_P : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ satisfying the axioms, for $f, g \in C^\infty(P)$,

(i) $\{f, g\}_P = -\{g, f\}_P$
(ii) $\{fg, h\}_P = g\{f, h\}_P + f\{g, h\}_P$
(iii) $\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0.$

Associated to a Poisson structure, there is a unique twice contravariant skew-symmetric, smooth tensor field Λ on P such that $\{f, g\}_P = \Lambda(df, dg)$, where df, dg are differentials of f, g, respectively. The tensor field Λ defines a vector-bundle morphism, $\Lambda^\# : T^*P \to TP$; $\alpha_x \mapsto \Lambda^\#(\alpha_x) \in T_xP$, satisfying,

$$\beta_x\big(\Lambda^\#(\alpha_x)\big) = \Lambda(x)(\alpha_x, \beta_x) \quad \forall\ \beta_x \in T^*_xP\,.$$

Let G be a Lie group and let $\Psi : G \times P \to P$, $(g, x) \mapsto \Psi_g(x)$, be a group action such that $\Psi_g(\cdot)$ is a Poisson morphism for every $g \in G$, that is, $\Psi_g : P \to P$ is an isomorphism and preserves the Poisson structure. Suppose that the action is proper and free. Then the quotient space P/G is a manifold that carries a Poisson structure $\{\cdot, \cdot\}_{P/G}$ induced from the one on P satisfying, for $f, g \in C^\infty(P/G)$,

$$\{f, g\}_{P/G} \circ \pi = \{f \circ \pi, g \circ \pi\}_P. \tag{5.1}$$

Here $\pi : P \to P/G$ is the canonical projection. By construction, it is a Poisson morphism.

G-equivariant dynamics on P induce dynamics on P/G. Suppose $h : P \to \mathbb{R}$ is a G-invariant Hamiltonian function on P, that is, $h\big(\Phi_g(x)\big) = h(x), \forall\ g \in G$. Define a vector field X_h through

$$X_h[f] = \{f, h\}_P \qquad \forall\, f \epsilon C^\infty(P) \tag{5.2}$$

where $X_h[f]$ denotes the Lie-derivative of the vector field X_h on the function f. The Hamiltonian h descends to $\tilde{h} : P/G \to \mathbb{R}$ and determines a *Poisson-reduced* dynamics $x_{\tilde{h}}$ on P/G by

$$X_{\tilde{h}}[\tilde{f}] = \{\tilde{f}, \tilde{h}\}_{P/G} \qquad \forall\, \tilde{f} \epsilon C^\infty(P/G). \tag{5.3}$$

Here $\tilde{h}([\![x]\!]) = h(x)$ for an equivalence class $[\![x]\!]$ in P.

Recall that the symplectic manifold (TQ, Ω_L) has a Poisson structure induced from the symplectic structure, namely, for $f, g \in C^\infty(TQ)$,

$$\{f, g\}_L(v_x) \triangleq df(v_x) \cdot X_g(v_x) \equiv \Omega_L(v_x)(\Pi_L(df), \Pi_L(dg)), \tag{5.4}$$

compare (3.5). Since the energy function H_L is G-invariant, we carry out the Poisson reduction as follows. Assume G acts on TQ freely and properly. Let $\tilde{\tau}$ be the projection from TQ to TQ/G, $\tilde{f}, \tilde{g} \in C^\infty(TQ/G)$, the induced Poisson bracket of $\tilde{f}$ and $\tilde{g}$ is defined analogous to (5.1) as

$$\{\tilde{f}, \tilde{g}\}_i \circ \tilde{\tau} = \{\tilde{f} \circ \tilde{\tau}, \tilde{g} \circ \tilde{\tau}\}_L. \tag{5.5}$$

Referring to the framework of Poisson reduction, we can identify the induced Hamiltonian $\tilde{H}_L$ and associated dynamics $X_{\tilde{H}_L}$ as:

$$\tilde{H}_L \circ \tilde{\tau}(v_x) = H_L(v_x), \tag{5.6}$$

$$X_{\tilde{H}_L}[\tilde{f}] = \{\tilde{f}, \tilde{H}_L\}_i \qquad \forall\, \tilde{f} \in C^\infty(TQ/G). \tag{5.7}$$

Here the vector field $X_{\tilde{H}_L}$ is called the *project Hamiltonian vector field* on TQ/G.

The reductions discussed here are on the Lagrangian side, or TQ side. We could perform a similar reduction process on T^*Q side, or Hamiltonian side, by noting that the Hamiltonian function on T^*Q, namely H in (4.2), is invariant under the cotangent lift Φ^{T^*} (this follows from (2.6)). The underlying symplectic manifold is (T^*Q, ω_0), with the corresponding momentum mapping,

$$J : T^*Q \to \mathcal{G} \quad \langle J(\alpha_x), \xi\rangle = \langle \alpha_x, \xi_Q(x)\rangle_x \quad \forall\, \xi \in \mathcal{G} \tag{5.8}$$

Since L is hyperregular, the reductions on TQ and T^*Q are equivalent, but we shall use the one on TQ side in the following development, bearing in mind that the Lagrange-D'Alembert Principle is formulated there.

We proceed to discuss a characterization of relative equilibria. The concept of relative equilibrium goes back to Poincaré. In the context of symplectic reduction, we define the notion of relative equilibrium as follows, compare Theorem 5.1.

Definition 5.2. A point v_x in TQ is called a *relative equilibrium* if $\pi_\mu(v_x) \in (TQ)_\mu$ is a fixed point for the symplectic-reduced vector field $X_{H_L^\mu}$, where $\mu = J(v_x)$.

In the context of Poisson reduction, we can define a similar notion, compare (5.6), (5.7). A point v_x in TQ is called a *relative equilibrium* for X_{H_L} with respect to the Poisson reduction if $X_{\tilde{H}_L}(\tilde{\tau}(v_x)) = 0$. It turns out that the two notions of relative equilibrium are equivalent. In fact, it can be shown, compare [1], that, for both cases, v_x is a relative equilibrium *iff* there exists a $\xi \in \mathcal{G}$ such that the flow of X_{H_L},

$$F^t_{X_{H_L}}(v_x) = \Phi_{exp(t\xi)}(v_x), \tag{5.9}$$

namely, the dynamical orbit is simply a group orbit. Thus if the observer were to be set in uniform motion according to the one-parameter group $exp(t\xi)$, then for such a moving observer, a relative equilibrium will appear to be stationary. For instance, if $G = SO(3)$, then $F^t_{X_{H_L}}(v_x)$ corresponds to a uniform rotation about a fixed axis ξ in space with the rotational speed $|\xi|$. In a central force field, a relative equilibrium for the motion of a point mass corresponds to a circular orbit, compare [49].

Relative equilibria can be characterized by the following result of Souriau-Smale-Robbin.

Theorem 5.3. *$v_x \in TQ$ is a relative equilibrium for X_H iff there exists a $\xi \in \mathcal{G}$ such that v_x is a critical point of $H_\xi \triangleq H_L - \langle J\, , \xi\rangle$, where $\langle J\, , \xi\rangle : TQ \to R$ is the real-valued function given by $v_x \mapsto \langle J(v_x), \xi\rangle$, associated to the momentum mapping J.*

In particular, for gyroscopic systems with symmetry, we have, compare (5.8), (4.22),

$$\begin{aligned} H_\xi(v_x) &= \frac{1}{2} \ll v_x, v_x \gg_x + V(x) - \ll v_x + Y(x), \xi_Q(x) \gg_x, \\ &= \frac{1}{2} \ll v_x - \xi_Q(x), v_x - \xi_Q(x) \gg_x \\ &+ V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x . \end{aligned} \tag{5.10}$$

From Theorem 5.3, it is then easy to check that the necessary and sufficient conditions for v_x to be a relative equilibrium are

$$v_x = \xi_Q(x), \tag{5.11}$$

and

$$d_x \left[V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x \right] = 0.$$

We thus have the following algorithm (*principle of symmetric criticality* [35]) to find relative equilibria.

Algorithm 5.4.

0. Pick $\xi \in \mathcal{G}$.
1. Search for the critical points x_e of the function

$$V_\xi : Q \to R \qquad V_\xi(x) \triangleq V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x . \tag{5.12}$$

2. Substitute x_e in (5.11) to find the corresponding $v_e = \xi_Q(x_e)$.

We note that the computation in step 1 is fully on the configuration space. Thus the process of searching for a relative equilibrium is greatly simplified. We remark that, for simple mechanical systems with symmetry, the principle of symmetric criticality stated above appears as Theorem 1.1 in Part II of Smale[42]. Smale also notes that special versions have been known earlier, for example, in the study of symmetric geodesics. See also p. 355 of [1], Theorem 16.7 in Hermann [19], Arnold [6], and Palais [35]. Here the *augmented potential* function V_ξ has one additional term to accommodate the gyroscopic effects. Through this term, we can change the number and locations of the critical points. This provides us an effective tool in *controlling* the phase portrait. Compare [46].

There is an additional symmetry in the augmented potential V_ξ. First, we define the *stabilizer* of ξ to be $G_\xi = \{g \in G | Ad_g(\xi) = \xi\} \subset G$, where Ad is the adjoint action of G on $\mathcal{G}$ defined in (2.3). The stabilizer G_ξ is actually a subgroup of G, and thus defines an action on Q. By an argument similar to the one in the proof of Lemma 4.6, it can be shown that V_ξ is invariant under the action of G_ξ on Q, that is,

$$V_\xi(\Phi_g(x)) = V_\xi(x) \quad \forall\, g \in G_\xi. \tag{5.13}$$

We assume that the quotient space Q/G_ξ is well defined. Denote the projection from Q to Q/G_ξ by π_ξ. By (5.13), we can define an induced function $\tilde{V}_\xi$ on Q/G_ξ from the augmented potential such that the diagram in Figure 5.1 commutes, namely,

$$\tilde{V}_\xi \circ \pi_\xi = V_\xi. \tag{5.14}$$

This symmetry of V_ξ will be used later in establishing a stability result associated with V_ξ.

As mentioned earlier, reductions could be worked out on the Hamiltonian side as well. Thus there is a similar algorithm corresponding to Algorithm 5.4 on the T^*Q side. We only need to find the corresponding conjugate momentum variable p_e, by substituting x_e obtained in Step 1 of Algorithm 5.4 in the formula, $p_e = K^b(Y(x_e) - \xi_Q(x_e))$. The point (x_e, p_e) in the momentum phase space T^*Q is then a relative equilibrium corresponding to the reduction on T^*Q with respect to the cotangent lift action.

We now address the stability of relative equilibria. Although both symplectic reduction and Poisson reduction lead to equivalent notions of relative equilibria, the

$$\begin{array}{ccc} Q & \xrightarrow{\quad V_\xi \quad} & \mathbb{R} \\ & \searrow^{\pi_\xi} \quad \nearrow_{\tilde{V}_\xi} & \\ & Q/G_\xi & \end{array}$$

Fig. 5.1 Symmetry of V_ξ.

associated stability issues are quite different. In the following, we shall state the main ideas in a somewhat more general setting than needed for this paper. However, we think that this added generality keeps the treatment free of confusing details. In general, let B, P be differentiable manifolds, and G be a Lie group. Consider a *principal G-bundle,* (P, G, B), namely, G acts differentiably on P freely and properly, $B = P/G$ is the quotient space of P with the canonical projection $\pi : P \to B$ being differentiable. Moreover, P is locally trivial, that is, every point $u \in B$ has a neighborhood U such that there is a mapping from $\pi^{-1}(U)$ to $U \times G$, $z \mapsto (\pi(z), \phi(z))$ that is a diffeomorphism and $\phi(g \cdot z) = g \cdot \phi(z)$, for all $g \in G$. See Figure 5.2 for an illustration of the geometric structure of such an object. For more details, see, for example, [34].

A vector field X on P is said to be *projectable* if for each $\tilde{f} \in \mathcal{F}(B)$, there exists a $\bar{f} \in \mathcal{F}(B)$ such that $X[\tilde{f} \circ \pi] = \bar{f} \circ \pi$, compare, for example, [30], [18]. Now, given a projectable vector field X on P, the corresponding *projected vector field* $\tilde{X}$ on B is defined in the following way. Given a smooth function $\bar{f}$ on B, the Lie derivative of $\tilde{X}$ on $\tilde{f}$ is defined as

$$\tilde{X}[\tilde{f}] \triangleq \bar{f} \quad \text{or} \quad \tilde{X}[\tilde{f}] \circ \pi = X[\tilde{f} \circ \pi]. \tag{5.15}$$

It is easy to verify that the vector field X_h defined in (5.2) is projectable with the projected vector field $X_{\tilde{h}}$ defined in (5.3) in the above sense.

Definition 5.5. For the principal G-bundle, (P, G, B), a point $z \in P$ is called a *relative equilibrium* of a projectable vector field $X \in \mathcal{X}(P)$ if $\pi(z)$ is an equilibrium of the associated projected vector field $\tilde{X} \in \mathcal{X}(B)$. Moreover, a relative equilibrium $z \in P$ is *relatively stable modulo G* if the equilibrium $\pi(z)$ is Lyapunov stable with respect to the projected vector field $\tilde{X}$.

Remark 5.6. In [28], the smooth manifold structure of the quotient space P/G is not explicitly invoked in defining the notion of *stationary motion* and *relative stability modulo G*. However, when the group action is free and proper, P/G is a manifold.

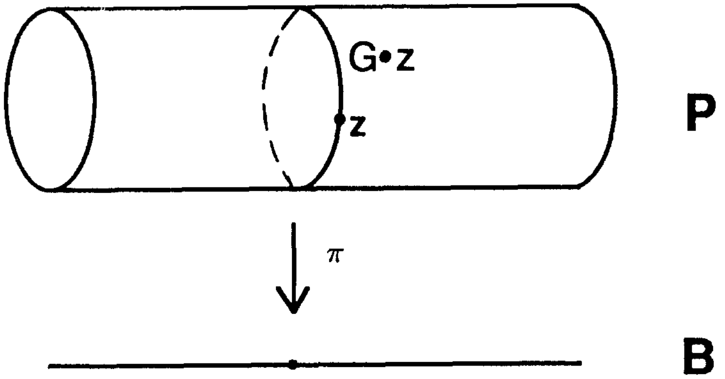

Fig. 5.2 Principle G-bundle.

This is the case considered in this paper, and hence Definition 8.13, p. 242 in [28] is equivalent to Definition 5.5.

For a gyroscopic system with symmetry, the definition of relative equilibrium $v_x \in TQ$ in Definition 5.4 matches with the Definition 5.5 by noting that the principal G-bundle is now $(TQ, G, TQ/G)$. Accordingly, the relative equilibrium v_x is *relatively stable modulo G in TQ* if $\tilde{\tau}(v_x)$ is a stable equilibrium with respect to the projected Hamiltonian vector field $X_{\tilde{H}_L}$. On the other hand, in the symplectic reduction process, we have the bundle structure $\big(J^{-1}(\mu), G_\mu, (TQ)_\mu\big)$. The relative equilibrium defined in Definition 5.2 can be regarded as a relative equilibrium with respect to this principle G-bundle. Correspondingly, we may define *relative stability modulo G_μ in $J^{-1}(\mu)$* with respect to the reduced dynamics $X_{H_L^\mu}$. Since the space $(TQ)_\mu$ is diffeomorphic to a symplectic leaf in TQ/G, relative stability modulo G in TQ implies relative stability modulo G_μ in $J^{-1}(\mu)$. The converse is illustrated by the following theorem from [28], Theorem 8.17, p. 244, see also [50] [24].

Theorem 5.7. *Let v_x^e be a relative equilibrium, compare Definition 5.2. Definiteness of the Hessian $D^2H_L^\mu$ at $\pi_\mu(v_x^e) \in (TQ)_\mu$ implies the relative stability modulo G in TQ of v_x^e, if there exists a neighborhood W of $\tilde{\tau}(v_x^e) \in TQ/G$ such that the rank of the Poisson structure $\{\cdot, \cdot\}_i$, defined in (5.5), is constant in W.*

Those points v_x in TQ satisfying the constant-rank condition stated in the above theorem will be referred to as *generic points*. The following example demonstrates that the sufficient condition in Theorem 5.7 is essential. This example is from [28]. A detailed discussion can be also found in [24].

Example 5.8. Consider a symplectic manifold (P, ω), where

$$P = \mathbb{R}^4 = \big\{(q_1, q_2, p_1, p_2)\big\} \quad \omega = dg_1 \wedge dp_1 + dq_2 \wedge dp_2. \tag{5.16}$$

Let

$$\begin{aligned} G = Aff_+(\mathbb{R}) \triangleq \{(a, b) \in \mathbb{R}^2\} \quad &\text{with the group structure} \\ (a, b) \cdot (c, d) &= (a + c, b + e^a d)\,. \end{aligned} \tag{5.17}$$

It can be shown that G defined in (5.17) is a Lie group. We define the action of G on P as

$$G \times P \to P \qquad \big((a, b), (q_1, q_2, p_1, p_2)\big) \mapsto (a + q_1, b + e^a q_2, p_1, e^{-a} p_2).$$

It is easy to check that this is a symplectic action on P. This action is also free and proper. It follows that P/G is a manifold ($\approx \mathbb{R}^2$). The symplectic structure ω in (5.16) defines a Poisson bracket on $\mathcal{F}(P)$, which, in turn, induces a Poisson structure on P/G. Let a Hamiltonian function H be defined at $H(q_1, q_2, p_1, p_2) = p_2 e^{q_1}$, which is a G-invariant function. It can be checked that $(0, t, 0, 0) \in P$ is a relative equilibrium corresponding to the vector field X_H. Moreover, this relative equilibrium is relatively stable modulo G_μ in $J^{-1}(\mu)$, since the quotient space $J^{-1}(\mu)/G_\mu$ degenerates to a point. However, it has been shown in [24] that this relative equilibrium is not relatively stable modulo G in P. Note also that the induced Poisson structure does not have a constant rank at $(0, t, 0, 0)$ and hence the condition in Theorem 5.7 does not hold.

There are several methods for determining relative stability in the appropriate sense. For example, the energy-Casimir method [20] [46], or the Lagrange multiplier method [29] [49] [46] can be used to determine relative stability modulo G in TQ. On the other hand, the energy-momentum method [39] [41] is useful in determining relative stability modulo G_μ in $J^{-1}(\mu)$. For simplicity, we will drop the underlying spaces in the definition of relative stability, for example, we say merely *relative stability modulo* G_μ. The underlying space is clear from the context. In the next section, the energy-momentum method will be adopted to study stability properties of relative equilibria for gyroscopic systems with symmetry.

6. Energy-Momentum Method for Gyroscopic Systems

As pointed out in Section 4, the use of gyroscopic feedback laws can affect the location of relative equilibria and their stability properties. In the work of Bloch, Krishnaprasad, Marsden, and Alvarez [7], an example of rigid body stabilization using such a control law is considered. Here we give a general method to explore stability under gyroscopic feedback laws. A key requirement is to obtain stability criteria that are explicit in the parameters of the feedback law, for example, the gyroscopic field.

In this section, the relative stability modulo G_μ will be examined via the energy-momentum mapping. Here we apply the energy-momentum method to the general framework of gyroscopic systems with symmetry. The block-diagonalization technique for simple mechanical systems with symmetry is extended here to account for gyroscopic terms. The decomposition of the symplectic structure is also presented. Key references for this section are [39] [41].

Let (P, ω) be a symplectic manifold on which the Lie group G acts symplectically, and let $J : P \to \mathcal{G}^*$ be an Ad^*-equivariant momentum mapping for this action (see Section 2 for definitions). Assume we could perform symplectic reduction on P in the sense of Marsden and Weinstein [32]. The reduced phase space is denoted by $P_\mu = J^{-1}(\mu)/G_\mu$. Let $H : P \to \mathbb{R}$ be invariant under the action of G. It induces a Hamiltonian function H^μ on P_μ satisfying $H^\mu \circ \pi_\mu = H \circ i_\mu$, where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \hookrightarrow P$ is the inclusion map. We are interested in the stability property of a relative equilibrium associated with the reduced dynamics X_{H^μ} on the reduced space P_μ, or the relative stability modulo G_μ in $J^{-1}(\mu)$. By construction, H^μ is a first integral of the reduced dynamics. Thus if H^μ has a strict local minimum at $\pi_\mu(z_e)$ where z_e is a relative equilibrium, then H^μ serves as a Lyapunov function. Standard Lyapunov stability analysis can be applied to conclude stability. Since, for $z \in J^{-1}(\mu) \subset P$,

$$H^\mu\big(\pi_\mu(z)\big) = H\big(i_\mu(z)\big) = H\,\big|_{J^{-1}(\mu)}(z)\,,$$

the condition for $\pi_\mu(z_e)$ to be a strict local minimum of H^μ is equivalent to the condition for z_e to be a strict local minimum of $H\,\big|_{J^{-1}(\mu)}$ modulo the tangent directions of the group orbit, $G_\mu \cdot z$. This in turn corresponds to checking that the relative equilibrium z_e solves the constrained minimization problem,

$$\textit{minimize} \quad H(z) \quad \textit{subject to} \quad J(z) = \mu_e = J(z_e).$$

This problem could be further formulated as checking z_e to be a strict local minimum of $H - \langle J, \xi \rangle$ in all directions on $J^{-1}(\mu_e)$ except along the tangent directions to the group orbit generated by G_μ, where ξ serves as the Lagrange multipler. These heuristic remarks are formalized in the following, giving rise to the *energy-momentum method,* compare [40], [41], [36], [39].

Define the energy-momentum functional

$$H_\xi(z) = H(z) - \langle J(z), \xi \rangle. \tag{6.1}$$

From the relative equilibrium theorem, compare Theorem 5.3, each relative equilibrium of the system is a critical point of H_ξ, for some $\xi \in \mathcal{G}$, namely,

$$DH_\xi(z_e) \cdot \delta z = 0 \quad \forall\, \delta z \in T_{z_e}P.$$

From previous discussions, the definiteness of the second variation of H_ξ on a subspace $\mathcal{S}$ of $T_{z_e}P$ satisfying

$$\mathcal{S} \cong T_{z_e}J^{-1}(\mu_e)/T_{z_e}(G_\mu \cdot z_e), \tag{6.2}$$

implies the relative stability modulo G_μ of the relative equilibrium z_e. One way to find such a space $\mathcal{S}$ is to construct a complement of $T_{z_e}(G_\mu \cdot z_e)$ in $T_{z_e}J^{-1}(\mu_e)$ such that $T_{z_e}J^{-1}(\mu_e) = \mathcal{S} \oplus T_{z_e}(G_\mu \cdot z_e)$. Since $T_{z_e}J^{-1}(\mu_e) = \operatorname{Ker} DJ(z_e)$, which is the kernel of the operator $DJ(z_e)$, we summarize the energy-momentum method for relative stability as follows.

Algorithm 6.1. (Energy-Momentum Method).

0. Pick $\xi \in \mathcal{G}$.
1. Solve the problem $DH_\xi(z) \cdot \delta z = 0$, $\forall\, \delta z \in T_zP$, for a relative equilibrium z_e.
2. Compute $\mu_e = J(z_e)$, and determine the space $\operatorname{Ker} DJ(z_e)$.
3. Find $\mathcal{S} \subset \operatorname{Ker} DJ(z_e)$ such that $\operatorname{Ker} DJ(z_e) = \mathcal{S} \oplus T_{z_e}(G_\mu \cdot z_e)$.
4. Check the second variation of H_ξ on $\mathcal{S}$. Definiteness of the second variation implies stability.

For visualizing the geometric picture, see Figure 6.1.

Now we restrict our consideration to gyroscopic systems with symmetry introduced in Section 4. The underlying space is $P = TQ$ with the symplectic structure Ω_L. In this setting, the momentum mapping is given by, compare (4.21),

$$J(v_x)(\xi) = \ll v_x + Y(x), \xi_Q(x) \gg_x, \tag{6.3}$$

and the energy-momentum functional is $H_\xi(v_x) = K_\xi(v_x) + V_\xi(x)$, where, compare (5.12),

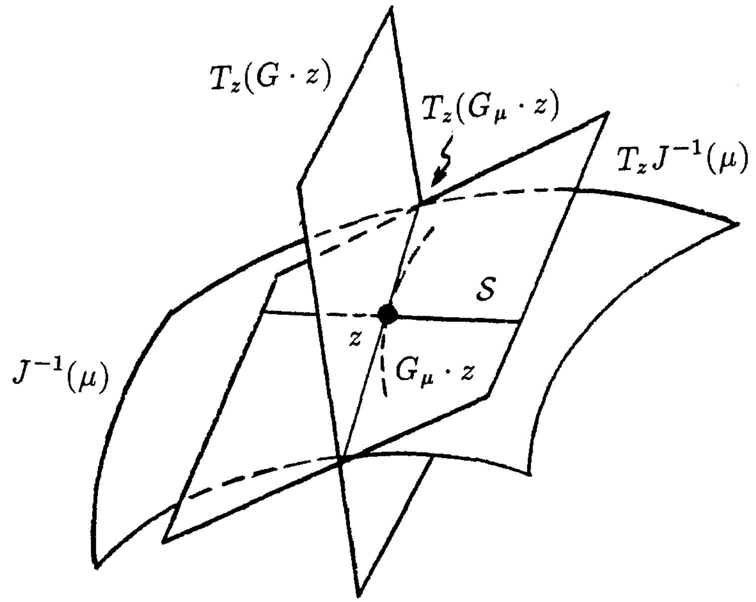

Fig. 6.1 Energy-momentum method.

$$K_\xi(v_x) = \frac{1}{2} \ll v_x - \xi_Q(x), v_x - \xi_Q(x) \gg_x,$$

$$V_\xi(x) = V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x.$$

We define the space

$$\mathcal{N}_x \triangleq \{\eta_Q(x) : \eta \in \mathcal{G}\}, \tag{6.4}$$

which is a subspace of $T_x Q$, and thus $\mathcal{N} \triangleq \cup_{x \in Q} \mathcal{N}_x$, is a subbundle of TQ. We then decompose $T_x Q = \mathcal{N}_x \oplus \mathcal{N}_x^\perp$ where $\mathcal{N}_x^\perp$ is the orthogonal complement of $\mathcal{N}_x$ with respect to the inner product associated with the Riemannian metric. Every element $v \in T_x Q$ can be thus written uniquely as $v = \eta_Q(x) + \overline{v}$, for $\eta \in \mathcal{G}$, $\overline{v} \in \mathcal{N}_x^\perp$. With this decomposition, the function K_ξ could be further written as

$$K_\xi(x, v) = \frac{1}{2} \left\| \eta_Q(x) - \xi_Q(x) \right\|^2 + \frac{1}{2} \|\overline{v}\|^2.$$

Note that at relative equilibrium $(x_e, \xi_Q(x_e))$, we have $v_e = \xi_Q(x_e)$, and $\overline{v} = 0$. Thus the second term in K_ξ is nonnegative and vanishes at a relative equilibrium with a positive semidefinite second variation. Define

$$\tilde{H}_\xi(x, \eta) = \frac{1}{2} \left\| \eta_Q(x) - \xi_Q(x) \right\|^2 + V_\xi(x). \tag{6.5}$$

From the above observations, relative stability modulo G_μ can be determined from this modified function on the space of $Q \times \mathcal{G}$. Moreover, taking variations in the space $Q \times \mathcal{G}$ corresponds to taking variations in the subbundle $\mathcal{N} \subset TQ$. We define the following embedding:

$$\mathfrak{X} : Q \times \mathcal{G} \to TQ, \qquad (x, \eta) \mapsto \big(x, \eta_Q(x)\big).$$

Hence $\tilde{H}_\xi = H_\xi \circ \mathfrak{X}$. A G-action on the space $Q \times \mathcal{G}$ can be constructed as

$$\begin{aligned} \Psi : G \times (Q \times \mathcal{G}) &\to Q \times \mathcal{G} \\ \big(g, (x, \eta)\big) &\mapsto \big(g \cdot x, Ad_g \eta\big), \end{aligned} \tag{6.6}$$

where *Ad* is the adjoint action defined in (2.3). It can be checked that

$$\Psi_g^T \circ \mathfrak{X} = \mathfrak{X} \circ \Psi_g. \tag{6.7}$$

Define the *premomentum mapping*

$$\tilde{J}(x, \eta) = J \circ \mathfrak{X}(x, \eta) = J\big(x, \eta_Q(x)\big) \tag{6.8}$$

A straightforward argument shows that the premomentum mapping $\tilde{J} : Q \times \mathcal{G} \to \mathcal{G}^*$ is Ad^*-equivariant. Thus the level set $\tilde{J}^{-1}(\mu)$ is invariant under the action of the isotropy subgroup G_μ. Furthermore, let $\tilde{H} : Q \times \mathcal{G} \to \mathbb{R}$ be defined as $\tilde{H} = H \circ \mathfrak{X}$. From (6.7), the function $\tilde{H}$ is invariant under the group action Ψ. The functional $\tilde{H}_\xi$ can be now written as $\tilde{H}_\xi = \tilde{H} - \langle \tilde{J}, \xi \rangle$. By the invariance properties of $\tilde{H}$ and $\tilde{J}$, the restriction of $\tilde{H}_\xi$ on $\tilde{J}^{-1}(\mu)$,

$$\tilde{H}_\xi \Big|_{\tilde{J}^{-1}(\mu)} = \tilde{H}\Big|_{\tilde{J}^{-1}(\mu)} - \langle \mu, \xi \rangle.$$

is invariant under the group action of G_μ. As a consequence, the geometric picture is the same as in Figure 6.1. An algorithm analogous to Algorithm 6.1 can then be applied to check if (x_e, ξ) is a local minimizer of $\tilde{H}_\xi$ restricted to $\tilde{J}^{-1}(\mu)$. Before doing so, we introduce a few notations. The Riemannian metric restricted to the subspace $\mathcal{N}_x$ provides an x-dependent bilinear form on the Lie algebra $\mathcal{G}$. This, in turn, induces a pairing (locked inertia tensor associated to $x \in Q$), $\mathbf{I}_{lock}(x) : \mathcal{G} \to \mathcal{G}^*$, defined through

$$\langle \xi, \mathbf{I}_{lock}(x)\eta \rangle \triangleq \ll \xi_Q(x), \eta_Q(x) \gg_x, \tag{6.9}$$

for $\xi, \eta \in \mathcal{G}$. From the symmetry property of the Riemannian metric, we have

$$\langle \xi, \mathbf{I}_{lock}(x)\eta \rangle = \langle \mathbf{I}_{lock}(x)\xi, \eta \rangle,$$

namely, $\mathbf{I}_{lock}(x)$ is symmetric. Also, we assume that, at x, the locked inertia tensor has an inverse, $\mathbf{I}_{lock}(x)^{-1} : \mathcal{G}^* \to \mathcal{G}$. On the other hand, the gyroscopic field also induces for each $x \in Q$ an element $\mathbf{I}_Y(x)$ in $\mathcal{G}^*$ defined by

$$\langle \mathbf{I}_Y(x), \eta \rangle \triangleq \ll Y(x), \eta_Q(x) \gg_x \quad \forall\, \eta \in \mathcal{G}. \tag{6.10}$$

We refer to $\mathbf{I}_Y(x)$ as the (x-dependent) *gyromomentum*. The function $\tilde{H}_\xi$ can now be expressed as

$$\begin{aligned}\tilde{H}_\xi(x,\eta) &= \frac{1}{2}\langle \eta-\xi, \mathbf{I}_{lock}(x)(\eta-\xi)\rangle \\ &\quad + V(x) - \frac{1}{2}\langle \xi, \mathbf{I}_{lock}(x)\xi\rangle - \langle \mathbf{I}_Y(x), \xi\rangle \qquad (6.11) \\ &= \frac{1}{2}\langle \eta-\xi, \mathbf{I}_{lock}(x)(\eta-\xi)\rangle + V_\xi(x)\end{aligned}$$

with the premomentum mapping, from (6.3), (6.8), for $\eta \in \mathcal{G}$,

$$\begin{aligned}\langle \tilde{J}(x,\eta), \zeta\rangle &= \langle J\big(x, \eta_Q(x)\big), \zeta\rangle \\ &= \ll \eta_Q(x), \zeta_Q(x) \gg_x + \ll Y(x), \zeta_Q(x) \gg_x \\ &= \langle \mathbf{I}_{lock}(x)\eta, \zeta\rangle + \langle \mathbf{I}_Y(x), \zeta\rangle,\end{aligned}$$

or we may write

$$\tilde{J}(x,\eta) = \mathbf{I}_{lock}(x)\eta + \mathbf{I}_Y(x). \qquad (6.12)$$

For $\mu \in \mathcal{G}^*$, the associated isotropy subalgebra $\mathcal{G}_{\mu_e}$ is defined in (2.4). With the inner product induced on $\mathcal{G}$ by the locked inertia tensor at x_e, we define the orthogonal complement of $\mathcal{G}_{\mu_e}$ to be

$$\mathcal{G}^{\perp}_{\mu_e} \triangleq \left\{\zeta \in \mathcal{G} : \langle \zeta, \mathbf{I}_{lock}(x_e)\eta\rangle = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\right\}. \qquad (6.13)$$

Following the notations used in [39], we define the maps $\tilde{\mathcal{A}} : \mathcal{G} \to \mathcal{G}^*$, and $\mathcal{A} : \mathcal{G} \to \mathcal{G}$, by

$$\tilde{\mathcal{A}}(\eta) \triangleq ad^*_\mu \mu_e, \quad \mathcal{A}(\eta) \triangleq \mathbf{I}_{lock}(x_e)^{-1}\tilde{\mathcal{A}}(\eta), \qquad (6.14)$$

respectively. As proved in [39], we have the following lemma.

Lemma 6.2. *Provided that $\mathcal{G}^{\perp}_{\mu_e}$ is finite dimensional or $\mathcal{A}$ is elliptic with respect to the inner product induced by $\mathbf{I}_{lock}(x_e)$, we have*

(i) $\mathcal{A}$ maps $\mathcal{G}$ onto $\mathcal{G}^{\perp}_{\mu_e}$.
(ii) $\tilde{\mathcal{A}}$ maps $\mathcal{G}$ onto $\mathcal{G}^a_{\mu_e} \subset \mathcal{G}^*$, where $\mathcal{G}^a_{\mu_e} = \left\{\mu \in \mathcal{G}^* : \langle \mu, \eta\rangle = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\right\}$, is the annihilator of $\mathcal{G}_{\mu_e}$.

With these notations, we are ready to apply Algorithm 6.1 to check if (x_e, ξ) is a local minimizer of $\tilde{H}_\xi$ restricted to $\tilde{J}^{-1}(\mu)$. We proceed as follows.

Step 0. Fix $\xi \in \mathcal{G}$.

Step 1. It is straightforward to derive

$$D\tilde{H}_\xi(x,\eta)(\delta x,\delta\eta) = DV_\xi(x)\delta x + \langle \delta\eta, \mathbf{I}_{lock}(x)(\eta-\xi)\rangle + \frac{1}{2}\langle \eta-\xi,\ \big(D\mathbf{I}_{lock}(x)\delta x\big)(\eta-\xi)\rangle.$$

The relative equilibrium is given by the conditions $DV_\xi(x_e) = 0$, $\eta_e = \xi$, which match with the conditions we obtained in Algorithm 5.4.

Step 2. For the relative equilibrium determined by the pair (x_e,ξ), we have

$$\mu_e = \tilde{J}(x_e,\xi) = \mathbf{I}_{lock}(x_e)\xi + \mathbf{I}_Y(x_e). \tag{6.15}$$

Now we find the space Ker $D\tilde{J}(x_e,\xi)$. From (6.12),

$$\begin{aligned} D\tilde{J}(x,\eta)(\delta x,\delta\eta) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \tilde{J}(x+\epsilon\delta x,\eta+\epsilon\delta\eta) \\ &= \big(D\mathbf{I}_{lock}(x)\delta x\big)\eta + \mathbf{I}_{lock}(x)\delta\eta + D\mathbf{I}_Y(x)\delta x. \end{aligned} \tag{6.16}$$

Here again $x+\epsilon\delta x$ denotes the integral curve corresponding to the tangent vector δx at x. For $(\delta x,\delta\eta)$ to be in Ker $D\tilde{J}(x_e,\xi)$, we must have, from (6.16),

$$\begin{aligned} \delta\eta &= -\mathbf{I}_{lock}(x_e)^{-1}\Big(\big(D\mathbf{I}_{lock}(x_e)\,\delta x\big)\xi + D\mathbf{I}_Y(x_e)\delta x\Big) \\ &= \mathbf{I}_{lock}(x_e)^{-1}\mathrm{ident}_\xi^Y(x_e)\delta x. \end{aligned}$$

where the map $\mathrm{ident}^Y : \mathcal{G}\times TQ \to \mathcal{G}^*$ is defined by, for $(x,\delta x)\in TQ$,

$$\mathrm{ident}_\xi^Y(x)\delta x \triangleq -\Big(\big(D\mathbf{I}_{lock}(x)\delta x\big)\xi + D\mathbf{I}_Y(x)\delta x\Big). \tag{6.17}$$

This map specializes to the map ident_ξ defined in [39] when $Y=0$, that is, for simple mechanical systems with symmetry. The properties of this map play an important role in our subsequent development. We need the following lemma.

Lemma 6.3. *For $x\in Q$ and $\zeta,\nu,\eta\in\mathcal{G}$, we have the following identities,*

$$\langle \zeta, \big(D\mathbf{I}_{lock}(x)\eta_Q(x)\big)\nu\rangle = \langle [\zeta,\eta], \mathbf{I}_{lock}(x)\nu\rangle + \langle [\nu,\eta], \mathbf{I}_{lock}(x)\zeta\rangle, \tag{6.18}$$

$$\langle \zeta, D\mathbf{I}_Y(x)\eta_Q(x)\rangle = \langle \mathbf{I}_Y(x), [\zeta,\eta]\rangle. \tag{6.19}$$

Proof. The proof of (6.18) can be found in [39]. Here we only verify (6.19). By definition (6.10),

$$\begin{aligned}\langle \zeta, D\mathbf{I}_Y(x)\eta_Q(x)\rangle &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \langle \mathbf{I}_Y(\exp \epsilon\eta \cdot x)\zeta\rangle \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \ll Y(\exp\epsilon\eta \cdot x), \zeta_Q(\exp\epsilon\eta \cdot x) \gg_{\exp\epsilon\eta\cdot x} \\ &= L_{\eta_Q} \ll Y, \zeta_Q \gg (x) \\ &= \ll L_{\eta_Q} Y(x),\ \zeta_Q(x) \gg_x + \ll Y(x),\ L_{\eta_Q}\zeta_Q(x) \gg_x,\end{aligned}$$

by using Lemma 2.2. Also we have the identity $L_{\eta_Q}\zeta_Q = [\zeta, \eta]_Q$. With Lemma 2.1, it follows that

$$\langle \zeta, D\mathbf{I}_Y(x)\eta_Q(x)\rangle = \ll Y(x),\ [\zeta, \eta]_Q(x) \gg_x = \langle \mathbf{I}_Y(x), [\zeta, \eta]\rangle.$$

We now evaluate the map $\text{ident}_\xi^Y(x_e)$ restricted to the space $\mathcal{N}_{x_e}$.

Lemma 6.4. *For $\eta \in \mathcal{G}$, at relative equilibrium (x_e, ξ),*

$$\text{ident}_\xi^Y(x_e)\eta_Q(x_e) = ad_\eta^*\mu_e + \mathbf{I}_{lock}(x_e)[\eta, \xi]. \tag{6.20}$$

Proof. From the definition (6.17), for arbitrary $\nu \in \mathcal{G}$,

$$\langle \text{ident}_\xi^Y(x_e)\eta_Q(x_e), \nu\rangle = -\langle \nu, \big(D\mathbf{I}_{lock}(x_e)\eta_Q(x_e)\big)\xi\rangle - \langle D\mathbf{I}_Y(x_e)\eta_Q(x_e), \nu\rangle.$$

From Lemma 6.3, this could be further written as

$$\begin{aligned}&-\langle[\nu, \eta], \mathbf{I}_{lock}(x_e)\xi\rangle - \langle[\xi, \eta], \mathbf{I}_{lock}(x_e)\nu\rangle - \langle \mathbf{I}_Y(x_e), [\nu, \eta]\rangle \\ &\quad= \langle \mathbf{I}_{lock}(x_e)\xi + \mathbf{I}_Y(x_e), [\eta, \nu]\rangle + \langle \mathbf{I}_{lock}(x_e)[\xi, \eta], \nu\rangle, \\ &\quad= \langle ad_\eta^*\mu_e, \nu\rangle + \langle \mathbf{I}_{lock}(x_e)[\xi, \eta], \nu\rangle,\end{aligned}$$

where the formula for μ_e in (6.15) has been used. We thus established (6.20).
The discussions in Step 2. can be summarized by writing

$$\text{Ker } D\tilde{J}(x_e, \xi) = \left\{(\delta x, \eta) \in T_{(x_e,\xi)}(Q \times \mathcal{G}) : \eta = \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x\right\}. \tag{6.21}$$

Step 3. As seen in (6.21), the component of $\mathcal{G}$ in Ker $D\tilde{J}\big(x_e, \xi\big)$ is determined from the variation δx in $T_{x_e}Q$. We thus only need to decompose the kernel space with respect to $T_{x_e}(G_{\mu_e} \cdot x_e)$. Since

$$\mathcal{N}_x^{\mu_e} \triangleq T_{x_e}\big(G_{\mu_e} \cdot x_e\big) = \left\{\eta_Q(x_e) \in T_{x_e}Q : \eta \in \mathcal{G}_{\mu_e}\right\}, \tag{6.22}$$

we find the orthogonal complement of $\mathcal{N}_x^{\mu_e}$ with respect to the Riemannian metric as

$$\mathcal{V} = \left\{\delta x \in T_{x_e}Q : \ll \delta x, \eta_Q(x_e) \gg_{x_e} = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\right\} \tag{6.23}$$

Consequently, the space $\tilde{\mathcal{S}}$ can be written as

$$\tilde{\mathcal{S}} = \left\{(\delta x, \eta) \in \mathcal{V} \times \mathcal{G} : \eta = \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x\right\}. \tag{6.24}$$

and we obtain Ker $D\tilde{J}(x_e, \xi) = \tilde{\mathcal{S}} \oplus T_{(x_e,\xi)}\big(G_{\mu_e} \cdot (x_e, \xi)\big)$, where, with respect to the action Ψ defined in (6.6),

$$T_{(x_e,\xi)}\big(G_{\mu_e} \cdot (x_e, \xi)\big) = \{(\zeta_Q(x_e), ad_\zeta \xi) : \zeta \in \mathcal{G}_{\mu_e}\},$$

can be shown to be a subspace of Ker $D\tilde{J}(x_e, \xi)$.

Step 4. Now we check the definiteness of the second variation of $\tilde{H}_\xi$ on the space $\tilde{\mathcal{S}}$. The block diagonalization techniques prove to be useful in this context. First, we note that, under conditions specified in Lemma 6.6 (see below), the space $\mathcal{V}$ can be decomposed as

$$\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}, \tag{6.25}$$

where

$$\mathcal{V}_{RIG} \triangleq \left\{\zeta_Q(x_e) : \zeta \in \mathcal{G}_{\mu_e}^{\perp}\right\}, \tag{6.26}$$

$$\mathcal{V}_{INT} \triangleq \left\{\delta x \in \mathcal{V} : \langle \zeta, \mathrm{ident}_\xi^Y(x_e)\delta_x \rangle = 0,\ \forall\, \zeta \in \mathcal{G}_{\mu_e}^{\perp}\right\}. \tag{6.27}$$

It is this decomposition that the block diagonalization is based on. On the other hand, by definitions (6.14), (6.22), and (6.26), we have $\mathcal{N}_x = \mathcal{N}_x^{\mu_e} \oplus \mathcal{V}_{RIG}$. The relationship between these spaces is geometrically depicted in Figure 6.2.

Next we check the second variation of $\tilde{H}_\xi$ given by

$$\begin{aligned} D^2\tilde{H}_\xi(x_e, \xi) \cdot (\Delta x_1 \eta_1) \cdot (\delta x_2, \eta_2) \\ &= \langle \eta_1, \mathbf{I}_{lock}(x_e)\eta_2 \rangle + D^2 V_\xi(x_e) \cdot \delta x_1 \cdot \delta x_2 \\ &= \langle \mathrm{ident}_\xi^Y(x_e)\delta x_1, \mathbf{I}_{lock}(x_e)^{-1}\mathrm{ident}_\xi^Y(x_e)\delta x_2 \rangle + D^2 V_\xi(x_e) \cdot \delta x_1 \cdot \delta x_2, \end{aligned}$$

for $(\delta x_1, \eta_1), (\delta x_2, \eta_2) \in$ Ker $D\tilde{J}(x_e, \xi)$. For convenience, a bilinear form on $T_{x_e}Q \times T_{x_e}Q$ is defined as

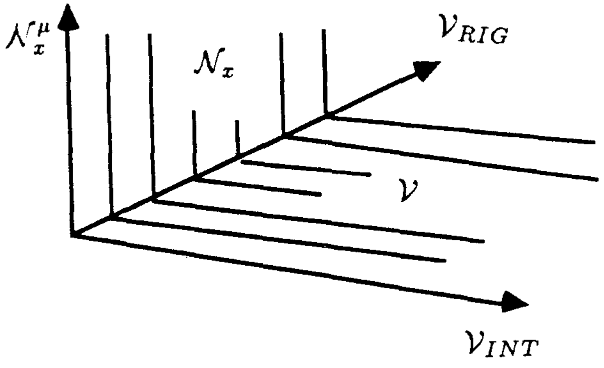

Fig. 6.2 Decomposition of T_xQ.

$$\begin{aligned}\mathbf{B}_\xi(\delta x_1, \delta x_2) \triangleq \langle \text{ident}_\xi^Y(x_e)\delta x_1, \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x_2\rangle \\ + D^2 V_\xi(x_e)\cdot \delta x_1 \cdot \delta x_2.\end{aligned} \tag{6.28}$$

Accordingly,

$$D^2\tilde{H}_\xi(x_e, \xi)\cdot(\delta x_1, \eta_1)\cdot(\delta x_2, \eta_2) = \mathbf{B}_\xi(\delta x_1, \delta x_2).$$

We have the following key proposition.

Proposition 6.5. For $\eta_Q(x_e) \in \mathcal{V}_{RIG}$, and $\delta x \in \mathcal{V}_{INT}$, $\mathbf{B}_\xi\big(\eta_Q(x_e), \delta x\big) = 0$.

Proof. We first find the second variation of V_ξ. By the property that V is G-invariant and Lemma 6.3, we have

$$\begin{aligned}DV_\xi(x)\cdot\eta_Q(x) &= DV(x)\cdot\eta_Q(x) - \frac{1}{2}\langle\xi, \big(D\mathbf{I}_{lock}(x)\cdot\eta_Q(x)\big)\xi\rangle - \langle D\mathbf{I}_Y(x)\cdot\eta_Q(x), \xi \\ &= -\langle[\xi,\eta], \mathbf{I}_{lock}(x)\xi + \mathbf{I}_Y(x)\rangle.\end{aligned}$$

It is then easy to see that, compare (6.17),

$$\begin{aligned}D^2V_\xi(x)\cdot\eta_Q(x)\cdot\delta x &= -\langle[\xi,\eta], \big(D\mathbf{I}_{lock}(x)\cdot\delta x\big)\xi + D\mathbf{I}_Y(x)\cdot\delta x\rangle. \\ &= \langle[\xi,\eta], \text{ident}_\xi^Y(x)\delta x\rangle.\end{aligned} \tag{6.29}$$

Next we evaluate the bilinear form on $\mathcal{V}_{RIG}\times\mathcal{V}_{INT}$. Combining (6.28), (6.29) and using Lemma 6.4, we obtain

$$\begin{aligned}\mathbf{B}_\xi\big(\eta_Q(x_e), \delta x\big) &= \langle\text{ident}_\xi^Y(x_e)\eta_Q(x_e), \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x\rangle + \langle[\xi,\eta], \text{ident}_\xi^Y(x_e)\delta x \\ &= \langle ad_\eta^*\mu_e + \mathbf{I}_{lock}(x_e)[\eta,\xi], \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x\rangle \\ &\quad + \langle[\xi,\eta], \text{ident}_\xi^Y(x_e)\delta x\rangle \\ &= \langle ad_\eta^*\mu_e, \mathbf{I}_{lock}(x_e)^{-1}\text{ident}_\xi^Y(x_e)\delta x\rangle \\ &= \langle\mathcal{A}(\eta), \text{ident}_\xi^Y(x_e)\delta x\rangle,\end{aligned} \tag{6.30}$$

where $\mathcal{A}$ is defined in (6.14). From Lemma 6.2, $\mathcal{A}(\eta)\in\mathcal{G}_{\mu_e}^\perp$. For $\delta x\in\mathcal{V}_{INT}$, by the definition of $\mathcal{V}_{INT}$, compare (6.27), the desired property follows.

With this proposition, the second variation of $\tilde{H}_\xi$ on $\mathcal{S}$ at relative equilibrium is diagonalized into two blocks. Checking the definiteness of $PD^2\tilde{H}_\xi$ on $\tilde{\mathcal{S}}$ is thus equivalent to checking the definiteness of $\mathbf{B}_\xi$ on the spaces of $\mathcal{V}_{RIG}\times\mathcal{V}_{RIG}$ and $\mathcal{V}_{INT}\times\mathcal{V}_{INT}$ independently, under the assumption that (6.25) holds. These techniques often simplify the computations quite significantly. In particular, the form of $\mathbf{B}_\xi$ on $\mathcal{V}_{RIG}\times\mathcal{V}_{RIG}$ can be worked out explicitly. From (6.30),

$$\begin{aligned}\mathbf{B}_\xi\big(\eta_Q(x_e), \eta_Q(x_e)\big) &= \langle\mathcal{A}(\eta), \text{ident}_\xi^Y(x_e)\eta_Q(x_e)\rangle \\ &= \langle\mathcal{A}(\eta), ad_\eta^*\mu_e + \mathbf{I}_{lock}(x_e)[\eta,\xi]\rangle \\ &= \langle ad_\eta^*\mu_e, \mathbf{I}_{lock}(x_e)^{-1}ad_\eta^*\mu_e\rangle + \langle ad_\eta^*\mu_e, ad_\eta\xi\rangle.\end{aligned} \tag{6.31}$$

This is the *Arnold block* analogous to the one in simple mechanical systems with symmetry [41]. The gyro-momentum is buried in μ_e and can affect definitiveness of this block. Definiteness of this block ensures the decomposition (6.25) of the space $\mathcal{V}$, which is proved in the following lemma.

Lemma 6.6. *Positive definiteness of* $\mathbf{B}_\xi$ *on* $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ *implies that* $\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}$.

Proof. The proof is analogous to the one in [39]. We only consider here the finite dimensional case. Letting $\zeta_Q(x_e) \in \mathcal{V}_{RIG} \cap \mathcal{V}_{INT}$, we have $\zeta \in \mathcal{G}^{\perp}_{\mu_e}$, and

$$\langle \nu, \mathrm{ident}^Y_\xi(x_e)\zeta_Q(x_e)\rangle = 0 \quad \forall\ \nu \in \mathcal{G}^{\perp}_{\mu_e}. \tag{6.32}$$

We choose, in (6.32), $\nu = \mathcal{A}(\zeta) \in \mathcal{G}^{\perp}_{\mu_e}$, which is ensured by Lemma 6.2. By comparing with (6.31), we get

$$\mathbf{B}_\xi\big(\zeta_Q(x_e), \zeta_Q(x_e)\big) = 0.$$

Since, by assumption, $\mathbf{B}_\xi$ is positive definite, this implies $\zeta = 0$. Namely, $\mathcal{V}_{RIG} \cap \mathcal{V}_{INT} = \{0\}$. On the other hand, $\dim \mathcal{V}_{RIG} + \dim \mathcal{V}_{INT} = \dim \mathcal{V}$. Thus the decomposition (6.25) holds.

With this Lemma, we do not need to verify the decomposition (6.25) explicitly. It is guaranteed by checking the definiteness of the Arnold block. We summarize the discussion in this step in the following theorem.

Theorem 6.7. *If the bilinear form* $\mathbf{B}_\xi$ *is positive definite on both* $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ *and* $\mathcal{V}_{INT} \times \mathcal{V}_{INT}$, *then the relative equilibrium* $\big(x_e, \xi_Q(x_e)\big) \in TQ$ *is relatively stable modulo* G_μ.

Now we have completed the process of Algorithm 6.1 of determining the relative stability for a gyroscopic system with symmetry. The block diagonalization of the second variation of $\tilde{H}_\xi$ is achieved on the constrained subspace $\tilde{\mathcal{S}}$. A few explanatory remarks follow. First, we note a necessary condition for relative equilibrium. Namely, at relative equilibrium (x_e, ξ),

$$ad^*_\xi \mu_e = 0. \tag{6.33}$$

This result holds in the general setting of Hamiltonian systems with symmetry, see Proposition 1.2 of [39] for the proof. Next we consider the amended potential introduced for simple mechanical systems with symmetry. From (6.12), we may construct a mapping from $Q \times \mathcal{G}^*$ to $Q \times \mathcal{G}$ as

$$(x, \mu) \mapsto \Big(x, \mathbf{I}_{lock}(x)^{-1}\big(\mu - \mathbf{I}_Y(x)\big)\Big).$$

With this transformation, the functional $\tilde{H}_\xi$ on the space $Q \times \mathcal{G}^*$ can be expressed as, from (6.11),

$$\tilde{\tilde{H}}_\xi(x, \mu) = V_\mu(x) - \langle \mu, \xi \rangle,$$

where the function

$$V_\mu(x) \triangleq V(x) + \frac{1}{2}\langle \mu - \mathbf{I}_Y(x), \mathbf{I}_{lock}(x)^{-1}\big(\mu - \mathbf{I}_Y(x)\big)\rangle, \tag{6.34}$$

is called the *amended potential*. It can be shown that, at relative equilibrium (x_e, ξ), we have

$$\begin{aligned} DV_\mu(x_e)\delta x &= DV_\xi(x_e)\delta x, \\ D^2V_\mu(x_e)\cdot\delta x_1\cdot\delta x_2 &= \mathbf{B}_\xi(\delta x_1, \delta x_2). \end{aligned}$$

Thus the stability conditions in Theorem 6.7 are equivalent to the conditions for the relative equilibrium to be a constrained strict local minimizer of the function V_μ. This conclusion is analogous in spirit to the Lagrange-Dirichlet theorem [6]. We phrase it as a theorem.

Theorem 6.8. *For gyroscopic systems with symmetry, the components of relative equilibria in the configuration space are the critical points of the function V_μ. If the configuration component x_e of a relative equilibrium is a constrained strict local minimizer of the function V_μ (i.e., by taking out the neutral directions tangent to $G_{\mu_e}\cdot x_e$), then the associated relative equilibrium is relatively stable modulo G_{μ_e}.*

Remark 6.9. In most practical problems, the augmented potential V_ξ is easier to compute than the amended potential V_μ. From Theorem 6.7 and (6.28), it is clear that positive-definiteness of the second variation of V_ξ on $\mathcal{V}$ is sufficient for stability. Following arguments similar to the discussion regarding V_μ, we get an analogous statement as in Theorem 6.8 with V_μ replaced by the augmented potential V_ξ. However, sufficient conditions obtained from V_μ are clearly weaker than the conditions from V_ξ.

Remark 6.10. Theorem 4.4 and Theorem 6.8 can be combined to assess the stability of closed-loop relative equilibria obtained by the application of a gyroscopic feedback law, compare 4.12, to a simple mechanical system with symmetry and exterior force.

In the following, we consider two special cases. First, it is easy to see that for the case of $Q = G$, compare Figure 6.2, $\mathcal{V}_{INT} = \{0\}$. Consequently, we only need to consider the Arnold block for stability. Secondly, for the case of $G = SO(3)$, we have

$$ad_{\hat{\xi}}\hat{\eta} = \widehat{\xi\times\eta} \qquad ad^*_{\hat{\xi}}\hat{\mu} = \widehat{\mu\times\xi}, \tag{6.35}$$

where $\hat{\xi}$, $\hat{\eta} \in so(3)$, and $\hat{\mu} \in so^*(3)$. Thus, condition (6.33) implies

$$\mu_e \times \xi = 0 \quad \text{or} \quad \mu_e = \lambda\xi, \tag{6.36}$$

where $\lambda \in \mathbb{R}$ is a scalar. It follows that $\mathcal{G}_{\hat{\mu}_e}$ is the subspace spanned by the vector ξ, which, in turn, implies that $\mathcal{G}_{\hat{\mu}_e} = \mathcal{G}_{\hat{\xi}}$. Recall that from (5.13), V_ξ is invariant along the group orbit, $\mathcal{G}_{\hat{\xi}}\cdot x_e$. From Remark 6.9, we conclude that for this case, the function $\tilde{V}_\xi$ defined in (5.14) is sufficient for determining stability. We summarize the discussion in the following corollary.

Corollary 6.11. *We consider a gyroscopic system with symmetry* (Q, K, Y, V, G).

(i) For the case that $Q = G$, positive definiteness of $\mathbf{B}_\xi$ at relative equilibrium, defined in (6.28), on $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ implies relative stability modulo G_μ.
(ii) For the case that $G = SO(3)$, a strict local minimizer of the function $\tilde{V}_\xi$, defined in (5.14) on the space Q/G_ξ induced by the augmented potential V_ξ gives rise to a stable relative equilibrium.

These observations are very useful in applying the energy-momentum method to specific problems.

After discussing the block-diagonalization of the bilinear form, or the second variation of $\tilde{H}_\xi$, we consider the decomposition of the symplectic structure on T^*Q side for gyroscopic systems with symmetry. We follow closely the derivations in [39]. Recall that for gyroscopic systems with symmetry, the symplectic structure on T^*Q side is unaffected by the presence of the gyroscopic term, and is hence the canonical one. The associated momentum mapping is given by $J : T^*Q \to \mathcal{G}^*$ with $\langle J(x, p), \xi \rangle = \langle p, \xi_Q(x) \rangle$. We define the fundamental mechanical connection $\alpha : TQ \to \mathcal{G}$ as

$$\alpha(v_x) \triangleq \mathbf{I}_{lock}^{-1}(x) J\big(x, K^b(v_x)\big). \tag{6.37}$$

It can be shown that α is a connection on the G-bundle $Q \mapsto Q/G$ and $\alpha(\xi_Q(x)) = \xi$. Given $\mu \in \mathcal{G}^*$, a 1-form $\mathfrak{X}_\mu : Q \to T^*Q$ is induced through the connection α,

$$\langle \mathfrak{X}_\mu(x), v_x \rangle = \langle \mu, \alpha(v_x) \rangle. \tag{6.38}$$

Regarding $\mathfrak{X}_\mu$ as a map from Q to T^*Q, we can find the corresponding tangent map $T\mathfrak{X}_\mu : TQ \to TT^*Q$. Recall that the space $\mathcal{V} \subset T_xQ$ is decomposed into $\mathcal{V}_{RIG}$ and $\mathcal{V}_{INT}$ as defined in (6.26), (6.27), respectively, by assuming that the Arnold's block is definite. We define

$$\begin{aligned} \mathcal{S}_{RIG} &\triangleq \big\{ T_{x_e}\mathfrak{X}_{\mu_e} \cdot \zeta_Q(x_e) : \zeta \in \mathcal{G}_{\mu_e}^{\perp} \big\}, \\ \mathcal{W}_{INT} &\triangleq \big\{ T_{x_e}\mathfrak{X}_{\mu_e} \cdot \delta x : \delta x \in \mathcal{V}_{INT} \big\}, \\ \mathcal{W}_{INT}^* &\triangleq \big\{ (0, \delta p) : \langle \delta p, \eta_Q(x_e) \rangle = 0, \forall\ \eta \in \mathcal{G} \big\}. \end{aligned} \tag{6.39}$$

It can be verified that $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{W}_{INT} \oplus \mathcal{W}_{INT}^*$. Also, we have the following proposition.

Proposition 6.12. At the relative equilibrium, $z_e = (x_e, p_e) \in T^*Q$, we have

(1) for $\zeta,\ \nu \in \mathcal{G}_{\mu_e}^{\perp}$,

$$\omega_0(z_e)\big(T_{x_e}\mathfrak{X}_{\mu_e} \cdot \zeta_Q(x_e), T_{x_e}\mathfrak{X}_{\mu_e} \cdot \nu_Q(x_e)\big) = -\langle \mu_e, [\zeta, \nu] \rangle,$$

(2) for $\zeta \in \mathcal{G}_{\mu_e}^{\perp}$, $(\delta x, \delta p) \in \mathcal{W}_{INT} \oplus \mathcal{W}_{INT}^*$,

$$\omega_0(z_e)\big(T_{x_e}\mathfrak{X}_{\mu_e} \cdot \zeta_Q(x_e), (\delta x, \delta p)\big) = -\langle \mu_e, [\zeta, \alpha(\delta x)] \rangle,$$

(3) for $\delta x_1, \delta x_2 \in \mathcal{V}_{INT}$,

$$\omega_0(z_e)\big(T_{x_e}\mathfrak{X}_{\mu_e}\cdot\delta x_1, T_{x_e}\mathfrak{X}_{\mu_e}\cdot\delta x_2\big) = d\mathfrak{X}_{\mu_e}(\delta x_2,\delta x_1) - \langle \mathfrak{X}_{\mu_e}, [\delta x_2, \delta x_1]\rangle,$$

(4) for $\eta \in \mathcal{G}$, $(0,\delta p)\in \mathcal{W}^*_{INT}$, $\omega_0(z_e)\big(T_{x_e}\mathfrak{X}_{\mu_e}\cdot\eta_Q(x_e),\ (0,\delta p)\big) = 0$,

(5) for $(0,\delta p_1)$, $(0,\delta p_2)\in\mathcal{W}^*_{INT}$, $\omega_0(z_e)\big((0,\delta p_1),\ (0,\delta p_2)\big) = 0$.

The proof is omitted here. It is very similar to the proof of Proposition IV.4, pp. 61–62 in [39]. Define the map

$$\rho : \mathcal{G}^{\perp}_{\mu_e} \times \mathcal{V}_{INT} \times \big[\mathcal{G}\cdot x_e\big]^A \to \mathcal{S},$$

where $\big[\mathcal{G}\cdot x_e\big]^A$ is the annihilator of $\mathcal{G}\cdot x_e$ in T^*Q, as

$$\rho(\zeta,\delta x,\delta p) = T_{x_e}\mathfrak{X}_{\mu_e}\cdot\zeta_Q(x_e) + T_{x_e}\mathfrak{X}_{\mu_e}\cdot\delta x + (0,\delta p).$$

This map induces a bilinear form

$$\omega_\rho\big((\zeta_1,\delta x_1,\delta p_1),\ (\zeta_2,\delta x_2,\delta p_2)\big) = \omega_0(z_e)\big(\rho(\zeta_1,\delta x_1,\delta p_1), \rho(\zeta_2,\delta x_2,\delta p_2)\big).$$

From Proposition 6.12, this bilinear form can be written as

$$\begin{aligned}\omega_\rho&\big((\zeta_1,\delta x_1,\delta p_1),\ (\zeta_2,\delta x_2,\delta p_2)\big)\\ &= -\langle\mu_e,\ [\zeta_1,\zeta_2]\rangle - \langle\mu_e,\ [\zeta_1,\ \alpha(\delta x_2)]\rangle + \langle\mu_e,\ [\zeta_2,\ \alpha(\delta x_1)]\rangle\\ &\quad + d\mathfrak{X}_{\mu_e}(\delta x_2,\delta x_1) - \langle\mathfrak{X}_{\mu_e},\ [\delta x_2,\delta x_1]\rangle + \langle\delta p_2,\delta x_1\rangle - \langle\delta p_1,\delta x_2\rangle.\end{aligned}$$

This shows that the restriction of the symplectic structure can be block-diagonalized as in Figure 6.3. This is analogous to the case of simple mechanical systems with symmetry [39]. However, the gyroscopic effects enters through the definition of μ_e, compare (6.15).

Upon the completion of the discussion of the abstract framework, now we implement the energy-momentum method in more detail for the special case of $G = SO(3)$. This physically corresponds to the study of stability properties of rotating structures. Through the isomorphism between $\mathbb{R}^3$ and skew symmetric matrices defined in (3.14), we define the *locked inertia dyadic* $\mathbf{I}^o_{lock}(x)$ as

$$\langle\hat{\xi}, \mathbf{I}_{lock}(x)\hat{\eta}\rangle = \xi\cdot\mathbf{I}^o_{lock}(x)\eta, \tag{6.40}$$

where we have used the *trace* pairing, compare (3.16). The matrices $\mathbf{I}_{lock}(x)$, $\mathbf{I}^o_{lock}(x)$ are related by the following formula. For

	$\mathcal{G}^{\perp}_{\mu_e}$	$\mathcal{V}_{INT}$	$\big[\mathcal{G}\cdot x_e\big]^A$
$\mathcal{G}^{\perp}_{\mu_e}$	Lie-Poisson Bracket	Rigid-Internal Coupling	0
$\mathcal{V}_{INT}$	−Rigid-Internal Coupling	Canonical Symplectic Structure	
$\big[\mathcal{G}\cdot x_e\big]^A$	0	plus a Magnetic Term	

Fig. 6.3 Block-diagonalization of ω_ρ

$$\mathbf{I}_{lock}(x) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}, \mathbf{I}^o_{lock}(x) = \frac{1}{2}\begin{pmatrix} I_{22}+I_{33} & -I_{12} & -I_{13} \\ -I_{12} & I_{11}+I_{33} & -I_{23} \\ -I_{13} & -I_{23} & I_{11}+I_{22} \end{pmatrix}.$$

Also, we define

$$\mathbf{I}_Y(x) = \widehat{\mathbf{I}^o_Y(x)}, \tag{6.41}$$

where $\mathbf{I}^o_Y(x) \in \mathbb{R}^3$. Namely,

$$\langle \mathbf{I}_Y(x), \hat{\eta} \rangle = \mathbf{I}^o_Y(x) \cdot \eta. \tag{6.42}$$

With these two objects, we have the following new representations,

$$\tilde{J}^o\big(x, \hat{\zeta}\big) = \mathbf{I}^o_{lock}(x)\zeta + \mathbf{I}^o_Y(x), \tag{6.43a}$$

$$V_{\hat{\xi}} = V(x) - \frac{1}{2}\xi \cdot \mathbf{I}^o_{lock}(x)\xi - \mathbf{I}^o_Y(x) \cdot \xi, \tag{6.43b}$$

$$\text{ident}^{Y,o}_{\hat{\xi}}(x)\delta x = -\big(D\mathbf{I}^o_{lock}(x)\delta x\big)\xi - D\mathbf{I}^o_Y(x)\delta x, \tag{6.43c}$$

The bilinear form defined in (6.28) is now

$$\begin{aligned} \mathbf{B}_{\hat{\xi}}(\delta x_1, \delta x_2) &= \text{ident}^{Y,o}_{\xi}(x_e)\delta x_1 \cdot \mathbf{I}^o_{lock}(x_e)^{-1}\text{ident}^{Y,o}_{\xi}(x_e)\delta x_2 \\ &\quad + D^2 V_{\hat{\xi}}(x_e) \cdot \delta x_1 \cdot \delta x_2. \end{aligned} \tag{6.44}$$

The Arnold block in (6.31) can be then written as, compare (6.35),

$$\begin{aligned} &\langle ad^*_{\hat{\eta}}\hat{\mu}_e, \ \mathbf{I}_{lock}(x_e)^{-1} ad^*_{\hat{\eta}}\hat{\mu}_e \rangle + \langle ad^*_{\hat{\eta}}\hat{\eta}_e, ad_{\hat{\eta}}\hat{\xi} \rangle \\ &\quad = \big(\mu_e \times \eta\big) \cdot \mathbf{I}^o_{lock}(x_e)^{-1}(\mu_e \times \eta) + (\mu_e \times \eta) \cdot (\eta \times \xi) \\ &\quad = \lambda^2(\xi \times \eta) \cdot \left(\mathbf{I}^o_{lock}(x_e)^{-1} - \frac{1}{\lambda}1\right)(\xi \times \eta) \end{aligned} \tag{6.45}$$

It is thus clear that for the Arnold block, we need to check the definiteness of the matrix $\mathbf{I}^o_{lock}(x_e)^{-1} - (1/\lambda)1$ along all directions except ξ. Note that here λ is *not* an eigenvalue of the locked inertia dyadic in contrast with the case of simple mechanical systems with symmetry. The gyroscopic field affects λ through the gyro-momentum term, compare (6.36). The above formulae will be used in the following section for the example of two rigid bodies with rotors coupled via a ball-in-socket joint.

7. Two Coupled rigid bodies with internal rotors

We now apply the energy-momentum method developed in Section 6 to a multibody analog of the dual-spin problem. In [48], we show that, with an appropriate an damping mechanism, the system depicted in Figure 7.1 asymptotically approaches one of the stable relative equilibria corresponding to an associated gyroscopic system with symmetry. Here we will compute certain stable relative equilibria.

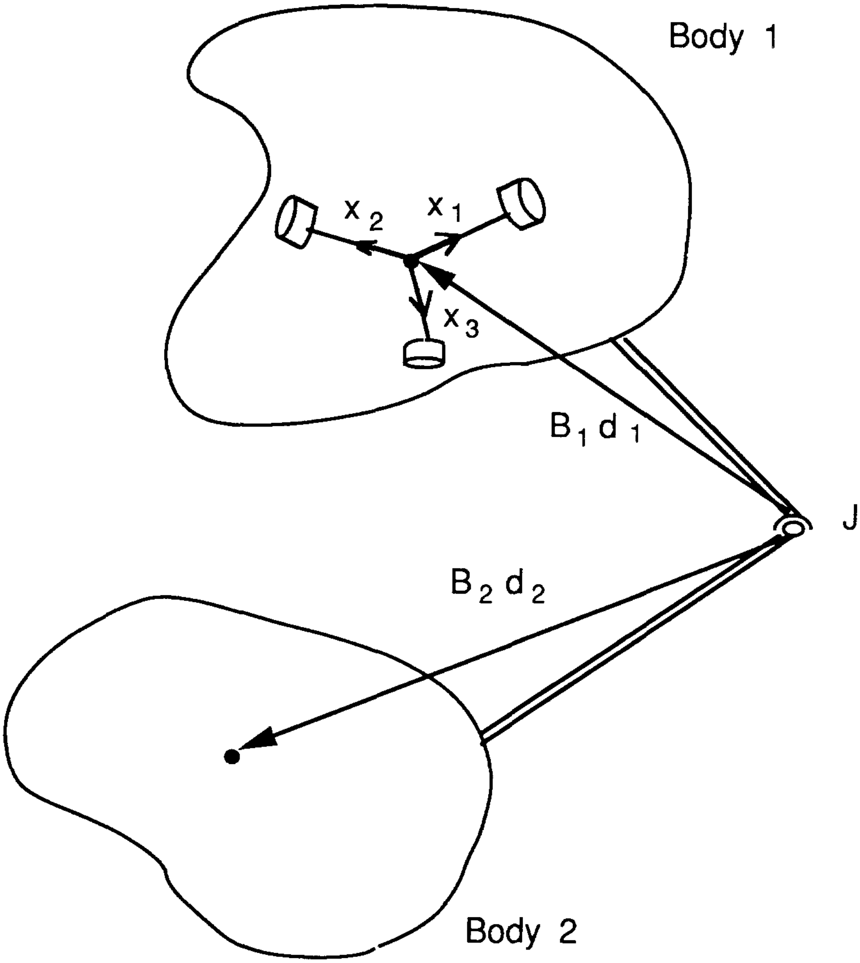

Fig. 7.1 Two rigid bodies with rotors.

The system under consideration consists of two rigid bodies connected by a three-degree-of-freedom spherical (ball-in-socket) joint and three symmetric rotors mounted on the center of mass of one body along its three principal axes, see Figure 7.1. These rotors, called *driven rotors,* are set in constant motion relative to the carrier body. We assume that the assembly is moving in a free space. For simplicity, the inertial reference frame is placed at the center of mass of the assembly. This corresponds to the reduction by the translational invariance of the system as discussed in [16], [47]. Let m_1, m_2, m_{S_i}, $i = 1, 2, 3$, and $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_{S_i}$, $i = 1, 2, 3$ denote the masses and the moments of inertia of body 1, body 2, and driven rotors, respectively. Let $\varepsilon = m_2(m_1 + m_{s_1} + m_{s_2} + m_{s_3})/(m_1 + m_{s_1} + m_{s_2} + m_{s_3} + m_2)$, the reduced mass. As in the classical dual-spin example of Section 4, this system can be put in the category of gyroscopic systems with symmetry with the following entities, compare [46],

$$Q = SO(3) \times SO(3),$$

$$K\Big((B_1\hat{u}_1, B_2\hat{u}_2), (B_1\hat{w}_1, B_2\hat{w}_2)\Big) = \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \begin{pmatrix} \mathbf{J}_1 & \mathbf{J}_{12} \\ \mathbf{J}_{12}^T & \mathbf{J}_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$Y(B_1, B_2) = (B_1\hat{y}_1, B_2\hat{y}_2), \quad V(B_1, B_2) = 0, \quad G = SO(3),$$

where,

$$\mathbf{J}_1 = \mathbf{I}_1 + \varepsilon \hat{d}_1^T \hat{d}_1 + \sum_{i=1}^{3} \mathbf{I}_{s_i},$$

$$\mathbf{J}_2 = \mathbf{I}_2 + \varepsilon \hat{d}_2^T \hat{d}_2 + \sum_{i=1}^{3} \mathbf{I}_{D_i},$$

$$\mathbf{J}_{12} = \varepsilon \hat{d}_1 B_1^T B_2 \hat{d}_2,$$

and the components y_1, y_2 of the gyroscopic field are given by solving the equations

$$\mathbf{J}_1\, y_1 + \mathbf{J}_{12} y_2 = l = \mathbf{I}^S \dot{\Theta}, \qquad \mathbf{J}_{12}^T y_1 + \mathbf{J}_2\, y_2 = 0.$$

The associated Lagrangian is

$$L(B_1, \Omega_1, B_2, \Omega_2) = \frac{1}{2} < \Omega_1, \mathbf{J}_1\Omega_1 > + \frac{1}{2} < \Omega_2, \mathbf{J}_2\Omega_2 >$$
$$+ \varepsilon < \Omega_1, \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2 > + < \Omega_1, l > .$$

This system can be also viewed as a closed-loop system of a simple mechanical system with gyroscopic feedback as described in Example 4.5.

Now we find the quantities introduced in Section 6. For the system under consideration, the locked inertia dyadic, compare (6.40), can be determined from

$$< \xi, \mathbf{I}^o_{lock}(B_1, B_2)\eta >$$
$$= \ll \hat{\xi}_Q(B_1, B_2), \hat{\eta}_Q(B_1, B_2) \gg$$
$$= < \xi, \big(B_1\mathbf{J}_1B_1^T + B_2\mathbf{J}_2B_2^T + B_1\mathbf{J}_{12}B_2^T + B_2\mathbf{J}_{12}^TB_1^T\big)\eta > .$$

Thus we have

$$\mathbf{I}^o_{lock}(B_1, B_2) = B_1\mathbf{J}_1B_1^T + B_2\mathbf{J}_2B_2^T + B_1\mathbf{J}_{12}B_2^T + B_2\mathbf{J}_{12}^TB_1^T.$$

The gyro-momentum in $\mathcal{G}^*$ induced by the gyroscopic field Y can be shown, compare definition (6.42), to be

$$\mathbf{I}^o_Y = B_1\, l.$$

Accordingly, the momentum mapping is

$$\begin{aligned} \mu &= \mathbf{I}^o_{lock}(x)\xi + \mathbf{I}^o_Y(x) \\ &= \big(B_1\mathbf{J}_1B_1^T + B_2\mathbf{J}_2B_2^T + B_1\mathbf{J}_{12}B_2^T + B_2\mathbf{J}_{12}^TB_1^T\big)\xi + B_1 l. \end{aligned} \tag{7.1}$$

The augmented potential function V_ξ is

$$V_\xi(B_1, B_2) = -\frac{1}{2} < \xi, \mathbf{I}^o_{lock}\xi > - < B_1 l, \xi >$$
$$= -\frac{1}{2} < \xi, (B_1\mathbf{J}_1B_1^T + B_2\mathbf{J}_2B_2^T + B_1\mathbf{J}_{12}B_2^T + B_2\mathbf{J}_{12}^TB_1^T)\xi >$$
$$- < \xi, B_1 l >$$
$$= -\frac{1}{2} < \xi, (B_1\mathbf{J}_1B_1^T + B_2\mathbf{J}_2B_2^T + \varepsilon\widehat{B_2 d}_1\widehat{B_2 d}_2 + \varepsilon\widehat{B_2 d}_2\widehat{B_1 d}_1)\xi >$$
$$- < \xi, B_1 l > .$$

Now we apply the principle of symmetric criticality to find the conditions for relative equilibria. The first variation of the augmented potential is derived as follows,

$$DV_\xi(B_1, B_2)\cdot(\hat{u}_1B_1, \hat{u}_2B_2) = < \hat{\xi}B_1\mathbf{J}_1B_1^T\xi + \hat{\xi}B_1 l + \varepsilon\widehat{B_1 d}_1\hat{\xi}\widehat{B_2 d}_2\xi, u_1 >$$
$$+ < \hat{\xi}B_2\mathbf{J}_2B_2^T\xi + \varepsilon\widehat{B_2 d}_2\hat{\xi}\widehat{B_1 d}_1\xi, u_2 > .$$

From the above formula, we immediately read out the conditions for the configuration components of the relative equilibrium $(B_{1e},\ B_{2e})$ as satisfying

$$\xi \times (B_1\mathbf{J}_1B_1^T\xi + B_1 l) + \varepsilon B_1 d_1 \times (\xi \times (B_2 d_2 \times \xi)) = 0, \tag{7.2a}$$

$$\xi \times (B_2\mathbf{J}_2B_2^T\xi) + \varepsilon B_2 d_2 \times (\xi \times (B_1 d_1 \times \xi)) = 0. \tag{7.2b}$$

These are very similar to the conditions derived in [47], except that a gyroscopic term enters. By taking dot product with ξ on both side of (7.2a), and letting $s_1 = B_1 d_1$, $s_2 = B_2 d_2$, we obtain the *coplanarity condition,* compare [47],

$$\xi \cdot (s_1 \times s_2) = 0. \tag{7.3}$$

Accordingly, the gyroscopic term does not affect the coplanarity condition for the relative equilibrium for this problem. With this condition (7.3), equations (7.2) may be re-expressed as

$$\xi \times (B_1\mathbf{J}_1B_1^T\xi + B_1 l) - \varepsilon(B_1 d_1 \cdot \xi)(B_2 d_2 \times \xi) = 0, \tag{7.4a}$$

$$\xi \times (B_2\mathbf{J}_2B_2^T\xi) - \varepsilon(B_2 d_2 \cdot \xi)(B_1 d_1 \times \xi) = 0. \tag{7.4b}$$

Now we find a particular relative equilibrium for this problem. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be the coordinate frames corresponding to body 1, body 2, respectively, such that

$$\mathbf{J}_1\mathbf{e}_i = \mathbf{J}_{1i}\mathbf{e}_i, \quad \mathbf{J}_2\mathbf{f}_i = \mathbf{J}_{2i}\mathbf{f}_i, \quad i = 1, 2, 3.$$

It can be checked that if the following conditions hold,

$$\xi = |\xi| B_1 \, \mathbf{e}_1 = |\xi| B_2 \, \mathbf{f}_1, \tag{7.5a}$$

$$l = l_1 \, \mathbf{e}_1, \tag{7.5b}$$

$$d_1 = a_1 \, \mathbf{e}_2, \qquad d_2 = a_2 \, \mathbf{f}_2, \tag{7.5c}$$

conditions (7.4) are satisfied. Thus the conditions (7.5) are associated to a relative equilibrium (B_{1e}, B_{2e}). From (7.5a), we know that

$$B_{1e} \, \mathbf{e}_1 = B_{2e} \, \mathbf{f}_1. \tag{7.6}$$

By substituting (7.5a), (7.5c) in the coplanarity condition (7.3), we get

$$\mathbf{e}_1 \cdot \left(\mathbf{e}_2 \times B_e \mathbf{f}_2 \right) = 0,$$

where $B_e = B_{1e}^T B_{2e}$. With (7.6), this only happens when $B_3 \, \mathbf{f}_2 = \pm \mathbf{e}_2$. Thus, we have two sets of relative equilibria expressed in terms of the relative shape variable B_e,

$$B_e \, \mathbf{f}_1 = \mathbf{e}_1, \quad B_e \, \mathbf{f}_2 = \mathbf{e}_2, \quad B_e \, \mathbf{f}_3 = \mathbf{e}_3, \tag{7.7a}$$

$$B_e \, \mathbf{f}_1 = \mathbf{e}_1, \quad B_e \, \mathbf{f}_2 = -\mathbf{e}_2, \quad B_e \, \mathbf{f}_3 = -\mathbf{e}_3. \tag{7.7b}$$

In the following, we will study the stability property of the relative equilibrium corresponding to (7.7b) with (7.5). This configuration is depicted in Figure 7.2.

The energy-momentum method is adopted here to determine the stability. We first need to compute the second variation of the augmented potential. It can be found as follows,

$$\begin{aligned}
& D^2 V_\xi(B_1, B_2) \cdot (\hat{u}_1 B_1, \hat{u}_2 B_2) \cdot (\hat{u}_1 B_1, \hat{u}_2 B_2) \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} D V_\xi \left(e^{\epsilon \hat{u}_1} B_1, e^{\epsilon \hat{u}_2} B_2 \right) \cdot \left(\hat{u}_1 e^{\epsilon \hat{u}_1} B_1, \hat{u}_2 e^{\epsilon \hat{u}_2} B_2 \right) \\
&= < \hat{u}_1 B_1 \mathbf{J}_1 B_1^T \xi - B_1 \mathbf{J}_1 B_1^T \hat{u}_1 \xi + \hat{u}_1 B_1 l, \hat{u}_1 \xi > \qquad (7.8) \\
&\quad + < \hat{u}_2 B_2 \mathbf{J}_2 B_2^T \xi - B_2 \mathbf{J}_2 B_2^T \hat{u}_2 \xi, \hat{u}_2 \xi > \\
&\quad + \varepsilon < \left(\hat{u}_1 \widehat{B_1 d_1} \right) \hat{\xi} \widehat{B_2 d_2} \xi, u_1 > + \varepsilon < \left(\hat{u}_2 \widehat{B_2 d_2} \right) \hat{\xi} \widehat{B_1 d_1} \xi, u_2 > \\
&\quad + 2\varepsilon < \left(\hat{u}_1 \widehat{B_1 d_1} \right) \xi, \left(\hat{u}_2 \widehat{B_2 d_2} \right) \xi > .
\end{aligned}$$

Define

$$\mathbf{u}_1 \triangleq B_{1e}^T u_1, \qquad \mathbf{u}_2 \triangleq B_{2e}^T u_2.$$

The components of $\mathbf{u}_1$, $\mathbf{u}_2$ will be denoted by (u_{11}, u_{12}, u_{13}) and (u_{21}, u_{22}, u_{23}), respectively. Also, we will use the notations,

$$\mathbf{J}_1 = \mathrm{diag}\{ \mathrm{J}_{11}, \mathrm{J}_{12}, \mathrm{J}_{13} \}, \qquad \mathbf{J}_2 = \mathrm{diag}\{ \mathrm{J}_{21}, \mathrm{J}_{22}, \mathrm{J}_{23} \}.$$

At relative equilibrium (B_{1e}, B_{2e}) such that (7.7b), (7.5) hold, we can further write the second variation of the augmented potential as

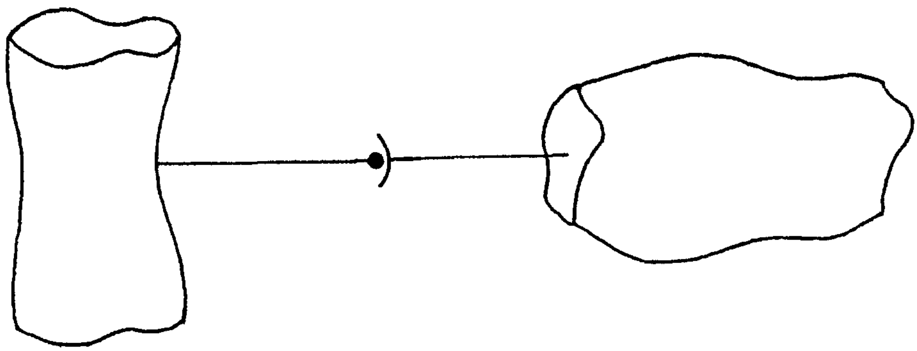

Fig. 7.2 Relative equilibrium configuration.

$$D^2 V_\xi(B_{1e}, B_{2e}) \cdot (B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}}) \cdot (B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}})$$
$$= \varepsilon|\xi|^2 a_1 a_2 (\mathrm{u}_{11} - \mathrm{u}_{21})^2 + \Big((\mathrm{J}_{11} - \mathrm{J}_{13})|\xi| + l_1\Big)|\xi| u_{12}^2$$
$$+ \Big((\mathrm{J}_{11} - \mathrm{J}_{12} + \varepsilon a_1 a_2)|\xi| + l_1\Big)|\xi|\mathrm{u}_{13}^2 + (\mathrm{J}_{21} - \mathrm{J}_{23})|\xi|^2 \mathrm{u}_{22}^2 \qquad (7.9)$$
$$+ (\mathrm{J}_{21} - \mathrm{J}_{22} + \varepsilon a_1 a_2)|\xi|^2 \mathrm{u}_{23}^2.$$

From Remark 6.9, we check the positive definiteness of the second variation of the augmented potential on the space of $\mathcal{V}$. For the relative equilibrium under investigation, we have the momentum mapping, compare (7.1),

$$\mu_e = \Big((\mathrm{J}_{11} + \mathrm{J}_{21} + 2\varepsilon a_1 a_2)|\xi| + l_1\Big) B_{1e}\, \mathbf{e}_1.$$

Thus the Lie algebra corresponding to the isotropy group is $\mathcal{G}_{\mu_e} = \mathrm{Span}\{B_{1e}\, \mathbf{e}_1\}$, with the orthogonal complement with respect to the locked inertia tensor,

$$\mathcal{G}_{\mu_e}^{\perp} = \mathrm{Span}\{B_{1e}\, \mathbf{e}_2,\ B_{1e}\, \mathbf{e}_3\}.$$

The space $\mathcal{V}$ is given by

$$\mathcal{V} = \{(B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}}) : \ll (B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}}),\ (\hat{\eta} B_{1e}, \hat{\eta} B_{2e}) \gg = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\}$$
$$= \{(B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}}) : (\mathrm{J}_{11} + \varepsilon a_1 a_2)\mathrm{u}_{11} + (\mathrm{J}_{21} + \varepsilon a_1 a_2)\mathrm{u}_{21} = 0\}.$$

The second variation of the augmented potential restricted to $\mathcal{V}$ is now

$$D^2 V_\xi(B_{1e}, B_{2e})\,|_{\mathcal{V}\times\mathcal{V}} \cdot (B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}}) \cdot (B_{1e}\widehat{\mathrm{u_1}}, B_{2e}\widehat{\mathrm{u_2}})$$
$$= \varepsilon|\xi|^2 a_1 a_2 \left(\frac{\mathrm{J}_{21} + \varepsilon a_1 a_2}{\mathrm{J}_{11} + \varepsilon a_1 a_2} + 1\right)^2 \mathrm{u}_{21}^2$$
$$+ \Big((\mathrm{J}_{11} - \mathrm{J}_{13})|\xi| + l_1\Big)|\xi|\mathrm{u}_{12}^2 + \Big((\mathrm{J}_{11} - \mathrm{J}_{12} + \varepsilon a_1 a_2)|\xi| + l_1\Big)|\xi|\mathrm{u}_{13}^2$$
$$+ (\mathrm{J}_{21} - \mathrm{J}_{23})|\xi|^2 \mathrm{u}_{22}^2 + (\mathrm{J}_{21} - \mathrm{J}_{22} + \varepsilon a_1 a_2)|\xi|^2 \mathrm{u}_{23}^2. \qquad (7.10)$$

Consequently, we can read off the sufficient conditions for stability from (7.10) as

$$\begin{aligned}
\big(J_{11} - J_{13}\big)|\xi| + l_1 &> 0,\\
\big(J_{11} - J_{12} + \varepsilon a_1 a_2\big)|\xi| + l_1 &> 0,\\
J_{21} - J_{23} &> 0,\\
J_{21} - J_{22} + \varepsilon a_1 a_2 &> 0.
\end{aligned}$$

The above discussions are summarized in the following theorem.

Theorem 7.1. *For the multibody dual-spin problem, conditions (7.5), (7.7b) give rise to a relative equilibrium* (B_{1e}, B_{2e}). *Furthermore, assuming that*

$$\frac{l_1}{|\xi|} > J_{13} - J_{11} \quad \textit{and} \quad \frac{l_1}{|\xi|} > J_{12} - J_{11} - \varepsilon a_1 a_2,$$

the relative equilibrium (B_{1e}, B_{2e}) *is stable if*

$$J_{21} - J_{23} > 0 \quad \textit{and} \quad J_{21} - J_{22} + \varepsilon a_1 a_2 > 0.$$

Remark 7.2. It may be checked that the positive definiteness conditions for the Arnold block are

$$\begin{aligned}
\big(J_{11} - J_{13} + J_{21} - J_{23}\big)|\xi| + l_1 &> 0,\\
\big(J_{11} - J_{12} + J_{21} - J_{22} + 2\varepsilon a_1 a_2\big)|\xi| + l_1 &> 0.
\end{aligned}$$

These conditions ensure the decomposition of the space $\mathcal{V}$, compare Lemma 6.6. It is easy to see that these conditions are implied by the conditions in Theorem 7.1. However, this is not sufficient for stability. There are additional conditions coming from the other block. Thus, for such a coupled system, we could never regard the system as a whole rigid body. The coupling effects should be suitably accommodated.

Now we consider the other relative equilibrium coming from (7.7a). The second variation of the augmented potential corresponding to the case that the relative shape is identity [i.e., that the two bodies are *folded* can be found from (7.8) to be, compare (7.9)],

$$\begin{aligned}
D^2 V_\xi\big(B_{1e}, B_{2e}\big) &\cdot \big(B_{1e}\widehat{\mathrm{u}_1}, B_{2e}\widehat{\mathrm{u}_2}\big) \cdot \big(B_{1e}\widehat{\mathrm{u}_1}, B_{2e}\widehat{\mathrm{u}_2}\big)\\
&= -\varepsilon|\xi|^2 a_1 a_2\big(\mathrm{u}_{11} - \mathrm{u}_{21}\big)^2 + \Big(\big(J_{11} - J_{13}\big)|\xi| + l_1\Big)|\xi|\mathrm{u}_{12}^2\\
&\quad + \Big(\big(J_{11} - J_{12} - \varepsilon a_1 a_2\big)|\xi| + l_1\Big)|\xi|\mathrm{u}_{13}^2 + \big(J_{21} - J_{23}\big)|\xi|^2\mathrm{u}_{22}^2\\
&\quad + \big(J_{21} - J_{22} - \varepsilon a_1 a_2\big)|\xi|^2\mathrm{u}_{23}^2.
\end{aligned}$$

Even restricted to the space $\mathcal{V}$, there is always one negative term. This fact suggests that this relative equilibrium may be unstable, irrespective of the rotor speed. Further analysis is needed to justify this statement.

8. Conclusions

The Lagrange-D'Alembert principle is a starting point for modeling natural mechanical systems subject to exogenous forces. If the forces are determined through feedback laws, then the structure of the closed-loop system can be used to assess stability properties of the system. In the present paper, using an intrinsic formulation of the Lagrange-D'Alembert principle, we have identified a class of feedback laws that lead to gyroscopic systems with symmetry. The (energy-momentum) block-diagonalization theorem for simple mechanical systems with symmetry has been extended to gyroscopic systems with symmetry, or the closed-loop system. Working consistently on the tangent bundle side, we establish the splitting that block-diagonalizes the second variation of the energy-momentum function at a relative equilibrium. The splitting depends on a quantity that we refer to as the gyro-momentum that can be computed in terms of the given gyroscopic vector field.

The gyro-momentum also enters the stability criteria. From the viewpoint of this paper, the gyro-momentum is the key control parameter and thus it is possible, using the methods of this paper, to determine whether a specific gyroscopic feedback law is a stabilizing feedback law. This is illustrated in the example of Section 7 on two coupled rigid bodies with internal rotors. This example is a natural generalization of the single rigid body dual-spin problem studied by P. S. Krishnaprasad [23], Sanchez de Alvarez [2], and more recently in the collaborative work with Bloch and Marsden [7]. Other more complicated examples, including dual-spin satellites in central gravitational fields and with flexible attachments, appear in the dissertation of Wang [46].

In future work, we plan to investigate bifurcations of relative equilibria with respect to the gyro-momentum. Examples of this appear in the work of Krishnaprasad and Berenstein [25]. Control strategies based on bifurcation of relative equilibria may be effective in a variety of problems. We hope to discuss these and other aspects in a later paper.

Acknowledgment

This work was supported in part by the AFOSR University Research Initiative Program under grant AFOSR-90-0105, and by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012.

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