On The Spectrum of Schrödinger Operators with a Random Potential

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Abstract. We investigate the spectrum of Schrödinger operators H_ω of the type: $H_\omega = -\Delta + \Sigma q_i(\omega) f(x - x_i + \xi_i(\omega))$ ($q_i(\omega)$ and $\xi_i(\omega)$ independent identically distributed random variables, $i \in \mathbb{Z}^d$). We establish a strong connection between the spectrum of H_ω and the spectra of deterministic periodic Schrödinger operators. From this we derive a condition for the existence of "forbidden zones" in the spectrum of H_ω. For random one- and three-dimensional Kronig–Penney potentials the spectrum is given explicitly.

Introduction

In this paper we study the spectra of random Schrödinger operators H_ω of the form:

$$H_\omega = -\Delta + \sum q_i(\omega) f(x - x_i + \xi_i(\omega)),$$

where $\{x_i\}_{i \in \mathbb{Z}^d}$ is a Bravais Lattice and $\{q_i\}_{i \in \mathbb{Z}^d}$ and $\{\xi_i\}_{i \in \mathbb{Z}^d}$ are independent, identically distributed random variables. Physically speaking H_ω corresponds to a random "charge"-configuration $\{q_i(\omega)\}$, each $q_i(\omega)$ being located at the random position $x_i - \xi_i(\omega)$ and producing a potential $q_i(\omega) f(x - x_i + \xi_i(\omega))$. Thus H_ω can be used as the Hamiltonian of a model for a "mixed" crystal with centers of strength $q_i(\omega)$ at perturbed lattice positions $x_i - \xi_i(\omega)$ or of a model of a liquid.

Models of this kind were considered by many authors, see for example: Halperin [10], Frisch and Lloyd [7], Luttinger [15], Borland [4], Lieb and Mattis [14] and references therein. Random operators of a more or less different kind are studied e.g. in Pastur [18] and [19], Kunz and Souillard [13], Fukushima, Nagai and Nakao [8], Nakao [17] and references given there.

In [11] the present authors showed that the spectrum of a wide class of random operators, containing the H_ω given above, is a nonrandom set Σ. In the present paper we determine the spectrum of the above operator more precisely.

In the first section we give conditions under which the operator H_ω is well defined and moreover essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, the infinitely differentiable

functions with compact support. This turns out to be the case if the moments of $\{q_i\}$ up to a sufficiently high order are finite and the function f is not too singular.

In Sect. 2 we investigate the relation between the spectrum of H_ω and the spectra of well ordered "charge" configurations, i.e. of Schrödinger operators of the form

$$H_{\lambda,u} = -\Delta + \sum \lambda_i f(x - x_i + u_i),$$

where λ_i as well as u_i are periodic and nonrandom. We get that the set Σ, the spectrum of H_ω, is completely determined by the spectra of operators of the form $H_{\lambda,u}$. Thus Σ has a band structure, in the sense that $\Sigma = \cup [a_i, b_i]$ but the intervals $[a_i, b_i]$ may overlap.

In Sect. 3 for $\xi_i \equiv 0$ we give a sufficient condition for the existence of forbidden zones ("gaps") in Σ (Theorem 5). Under a very mild condition (Assumption A) on f and $\{q_i\}$ it is shown that (α, β) is a gap for H_ω if it is a gap for all "pure" Hamiltonians $H_\lambda = -\Delta + \lambda \sum f(x - x_i)$, where λ runs through the (connected) component of supp p'_{q_0} the support of the probability distribution of $q_0(\omega)$. Theorem 5 can be looked upon as a generalisation of a famous conjecture by Saxon and Hutner [21].

These authors conjectured that a common gap for two pure solids is also a gap for an alloy of these solids, at least in a one-dimensional model with point interactions. In the latter case the conjecture was proved by Luttinger [15], but it was shown to be wrong for other potentials (see e.g.: Lieb and Mattis [14], Halperin [10]).

In Sect. 4 we study three examples. First we choose f to be a square-well potential in one dimension without overlapping of the wells. For this the existence of infinitely many gaps is shown. The second example is a random point interaction in one dimension, a generalisation of the model considered by Luttinger [15], the nonrandom version of which goes back to Kronig and Penney [12]. Specifically we give the spectrum of H_ω, as in the nonrandom case. This result contains Luttinger's result mentioned above. Our last example shows the existence of a gap for a random point interaction in three dimensions.

Section 1

Let $\{q_i(\omega)\}_{i \in \mathbb{Z}^d}$ be independent, identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$. Let $\{x_i\}_{i \in \mathbb{Z}^d}$ be a Bravais Lattice, i.e. $i \to x_i$ is a representation of the group $\mathbb{Z}^d$ into $\mathbb{R}^d$ such that the $\{x_i\}_{i \in \mathbb{Z}^d}$ span the space $\mathbb{R}^d$. By introducing a new norm on $\mathbb{R}^d$, if necessary, we can assume the lattice $\{x_i\}_{i \in \mathbb{Z}^d}$ to be $\mathbb{Z}^d$. Furthermore let f be a real $L^p_{\mathrm{loc}}(\mathbb{R}^d)$-function, for some $p \geq d/2$ for $d \geq 4$, and $p = 2$ for $d \leq 3$, such that:

$$\sum_{\substack{i \in \mathbb{Z}^d \\ |i| \geq k_0}} \sup_{x \in C_0 + x_i} |f(x)| < \infty \tag{1}$$

for some constant K_0; where C_0 is the unit cube around the point $x_0 = 0$.

Then we define the potential $V_\omega(x)$ by:

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega) f(x - x_i). \tag{2}$$

Clearly we have to put some restrictions on the random variables $\{q_i\}_{i\in\mathbb{Z}^d}$ in order that $V_\omega(x)$ is well defined.

Lemma 1. *If the first two moments of $q_0(\omega)$ are finite, i.e. $E|q_0(\omega)| < +\infty, E|q_0(\omega)|^2 < +\infty$, where E denotes the expectation with respect to the probability measure P, then $\sum_{i\in\mathbb{Z}^d} |q_i(\omega)||f(x-x_i)|$ is a locally $L^2(\mathbb{R}^d)$-function with probability one.*

The proof is omitted since it is standard. From the above Lemma we get that $V_\omega(x)$ is a well-defined function locally square integrable for almost all $\omega\in\Omega$. Hence the operator: $H_\omega = -\Delta + V_\omega$ on $L^2(\mathbb{R}^d)$, where Δ denotes the Laplacian, is a densely defined (e.g. on $C_0^\infty(\mathbb{R}^d)$) symmetric operator on $L^2(\mathbb{R}^d)$ and by V. Neumann's theorem (see e.g. Reed–Simon II [20]) has selfadjoint extensions.

Actually with mild additional assumptions on f and on $\{q_i\}_{i\in\mathbb{Z}^d}$ we can prove the following result:

Theorem 1. *Let the real function $f(x)$ on $\mathbb{R}^d$ be as before (i.e. f satisfies assumption (1) and $f\in L^p(\mathbb{R}^d)$ for some $p > 2$ for $d \leq 3$ and $p > d/2$ for $d > 3$) and assume that $E|q_0|^k < +\infty$ with $k > pd/2(p-2)$ for $d \leq 3$ and $k > dp/(2p-d)$ for $d > 3$. Then the Hamiltonian $-\Delta + \sum_{i\in\mathbb{Z}^d} q_i(\omega)f(x-x_i)$ on $L^2(\mathbb{R}^d)$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ with probability one.*

Proof. For simplicity we assume the constant k_0 appearing in (1) equal to 1. The idea of the proof is to show that with probability one it is possible to split V_ω into: $V_\omega = V_\omega^{(1)} + V_\omega^{(2)}$, in such a way that $V_\omega^{(1)}(x) > -c(\omega)x^2$ for some positive constant $c(\omega)$ and $V_\omega^{(2)}\in L^q_{\mathrm{un,loc}}(\mathbb{R}^d)$, i.e. $\int_{C_0+y} |V_\omega^{(2)}(x)|^q dx < c_1$ for any $y\in\mathbb{R}^d$ and a constant $c_1 = c_1(\omega)$, independent of y, for some $q > d/2$ if $d > 3$, $q = 2$ if $d \leq 3$. Then the essential selfadjointness follows from the Faris–Lavine theorem (see e.g. Reed–Simon II [20]) combined with Theorem XIII 96 of Reed–Simon IV [20].

We will treat only the case $d \geq 3$; the other cases, i.e. $d = 2$ or $d = 1$, can be handled exactly in the same way.

By definition, $V_\omega(x)$ can be written as:

$$V_\omega(x) = \sum_{j \neq i(x)} q_j(\omega)f(x-x_j) + q_{i(x)}f(x-x_{i(x)})$$
$$\equiv \bar{V}_\omega(x) + q_{i(x)}f(x-x_{i(x)}), \tag{3}$$

where $i(x)$ is such that $x\in C_0 + x_{i(x)}$. $\bar{V}_\omega(x)$ is the non-singular part of $V_\omega(x)$ and by assumption (1) is finite for almost every $\omega\in\Omega$.

We will show that:

$$P\left(\sum_{j \neq i}|q_j(\omega)| \sup_{x\in C_0+x_i} |f(x-x_j)| > \inf_{x\in C_0+x_i} |x|^2\right.$$
$$\left. \text{for infinitely many } x_i\right) = 0. \tag{4}$$

From (4) we can conclude that:

$$\sup_{x\in C_0+x_i} |\bar{V}_\omega(x)| > \inf_{x\in C_0+x_i} |x|^2 \text{ only for finitely many } x_i$$

almost surely; hence, since $\bar{V}_\omega(x)$ is finite almost surely we can find with probability one a positive constant $C(\omega)$ such that

$$\bar{V}_\omega(x) > -C(\omega)|x|^2. \tag{5}$$

By the Borel–Cantelli Lemma in order to prove (4) it suffices to show for a suitable chosen M:

$$\sum_{\substack{i\in\mathbb{Z}^d \\ |i|>M}} P\left(\sum_{j\neq i}|q_j|\sup_{x\in C_0+x_i}|f(x-x_j)| > \inf_{x\in C_0+xi}|x|^2\right) < +\infty; \tag{6}$$

by the Chebyshev inequality we have:

$$P\left(\sum_{j\neq i}|q_j|\sup_{x\in C_0+x_i}|f(x-x_j)| > \inf_{x\in C_0+x_i}|x|^2\right)$$

$$\leqq \frac{E\left(\sum_{j\neq i}|q_j|\sup_{x\in C_0+x_i}|f(x-x_j)|\right)^k}{\left(\inf_{x\in C_0+x_i}|x|^2\right)^k}, \tag{7}$$

where k is as in the statement of the theorem.
The right hand side of (7) can be bounded from above by

$$E|q_0|^k D^k |x_i'|^{-2k}, \tag{8}$$

where $D = \sum_{j\neq 0}\sup_{x\in C_0}|f(x-x_j)| < +\infty$ by assumption (1), and $x_i'^2$ means $\inf_{x\in C_0+x^i}|x|^2$.

Inserting estimate (8) in (6) we get:

$$\sum_{\substack{i\in\mathbb{Z}^d \\ |i|>M}} P\left(\sum_{j\neq i}|q_j|\sup_{x\in C_0+x_i}|f(x-x_i)| > \inf_{x\in C_0+x_i}|x|^2\right)$$

$$\leqq E|q_0|^k D^k \sum_{\substack{i\in\mathbb{Z}^d \\ |i|>M}}|x_i'|^{-2k} < +\infty, \tag{9}$$

since $k > dp/(2p-d) > d/2$ in the case $d > 3$, and $k > dp/2(p-2) > d/2$ in the case $d = 3$.

We now consider the singular part of $V_\omega(x)$: $\tilde{V}_\omega(x) \equiv q_{i(x)}f(x-x_{i(x)})$, where as before $i(x)$ is such that $x\in C_0+x_{i(x)}$. $\tilde{V}_\omega(x)$ in turn can be decomposed as follows:

$$\tilde{V}_\omega(x) = q_{i(x)}^{(\omega)}f(x-x_{i(x)})\chi(q_{i(x)}^{(\omega)}f(x-x_{i(x)}) \geqq -x^2)$$
$$+ q_{i(x)}^{(\omega)}f(x-x_{i(x)})\chi(q_{i(x)}^{(\omega)}f(x-x_{i(x)}) < -x^2). \tag{10}$$

Here $\chi(\cdot)$ denotes the characteristic function on $\mathbb{R}^d$.

The proof of the theorem is now complete if we are able to prove that $q_{i(x)}^{(\omega)}f(x-x_{i(x)})\chi(q_{i(x)}^{(\omega)}f(x-x_{i(x)}) < -x^2)$ belongs to $L_{\text{un,loc}}^q(\mathbb{R}^d)$ almost surely, for some $q > d/2$ for $d > 3$, $q = 2$, $d = 3$. For this it is sufficient to show:

$$\int_{C_0+x_i}|q_i(\omega)|^q|f(x-x_i)|^q\chi(|q_i||f(x-x_i)| > x^2)dx < M_1(\omega) \tag{11}$$

for some $q > d/2$ for $d > 3$, $q = 2$ for $d = 3$ and some constant $M_1(\omega)$ independent of $i\in\mathbb{Z}^d$ with probability one. Since $f\in L_{\text{loc}}^p(\mathbb{R}^d)$ with $p > d/2$ for $d > 3$ and $p > 2$ for

$d = 3$, we can choose q in such a way that $\max(2, d/2) < q < p$ for $d \geqq 3$. As before it is enough to check that:

$$P\left(\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i)| > x^2) dx > 1 \right. \tag{12}$$

for infinitely many $i \in \mathbb{Z}^d\Big) = 0$.

In fact if (12) is true then $\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i)| > x^2) dx > 1$ only for finitely many $i \in Z^d$, and since $f(x) \in L_{loc}^p(\mathbb{R}^d)$ for $d/2 < q < p$ in the case $d > 3$, and $f \in L_{loc}^2(\mathbb{R}^d)$ for $d = 3$, it is always possible to find a constant $M_1(\omega)$ such that (11) is satisfied for any $i \in \mathbb{Z}^d$.

Again, by the Borel–Cantelli lemma, the problem is reduced to give an estimate of:

$$P\left(\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i)| > x^2) dx > 1\right) \tag{13}$$

such that the sum:

$$\sum_{\substack{i \in \mathbb{Z} \\ |i| > M_0}} P\left(\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i) > x^2)|dx > 1\right) \tag{14}$$

is finite. By the Hölder inequality the integral appearing in formula (13) can be estimated from above by:

$$|q_i|^q \left[\int_{C_0 + x_i} |f(x - x_i)|^p dx\right]^{q/p} \left[\int_{C_0 + x_i} \chi(|q_i||f(x - x_i)| > x^2) dx\right]^{1 - q/p}$$

$$\leqq |q_i|^p \left[\int_{C_0 + x_i} |f(x - x_i)|^p dx\right]^{q/p} b_1 x_i'^{-2p(1 - q/p)} \tag{15}$$

for some constant $b_1 > 0$, where we have used the fact that $f \in L_w^p(C_0)$ (see e.g. Reed–Simon II page 30, [20]). Using now the Chebyshev inequality and estimate (15) we obtain:

$$P\left(\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i)| > x^2) dx > 1\right)$$

$$\leqq b_2 E|q_0|^k |x_i'|^{-2k(1 - q/p)} \tag{16}$$

for some constant $b_2 > 0$, where K is as in the statement of the theorem.

Inserting (16) in (14) we get that:

$$\sum_{\substack{i \in \mathbb{Z}^d \\ |i| > M_0}} P\left(\int_{C_0 + x_i} |q_i|^q |f(x - x_i)|^q \chi(|q_i||f(x - x_i)| > x^2) dx > 1\right)$$

$$\leqq \sum_{\substack{i \in \mathbb{Z}^d \\ |i| > M_0}} b_2 E|q_0|^k |x_i'|^{-2k(1 - q/p)}$$

is finite if $2k(1 - q/p) > d$, i.e. if $q < p - dp/2k$. Since $k > dp/(2p - d)$ for $d > 3$, $k > dp/2(p - 2)$ for $d = 3$, $p - dp/2k > d/2$ for $d > 3$ and $p - dp/2k > 2$ for $d = 3$, we can always find a p such that: $d/2 < p < p - dp/2k$ for $d > 3$ and $2 < p < p - dp/2k$ for $d = 3$. The convergence of (14) is thus assured. $\qquad \square$

Remark. The results of this section, namely Lemma 1 and Theorem 1 can be extended to the following case:

$$V_\omega(x) = \sum q_i(\omega) f(x - x_i + \xi_i(\omega)), \tag{17}$$

where $\{\xi_i\}_{i\in\mathbb{Z}^d}$ are new independent, identically distributed random variables satisfying $|\xi_i(\omega)| < N_0$ for any ω.

Section 2

In this section we investigate the spectra of Schrödinger operators of the form $H_\omega = -\Delta + V_\omega$ with V_ω as in (17) with the stronger assumption that the $\{q_i(\omega)\}_{i\in\mathbb{Z}^d}$ are independent identical distributed random variables. The following theorem was proved in Kirsch–Martinelli [11]:

For almost each $\omega\in\Omega$, let H_ω be a selfadjoint operator on a separable Hilbert space H, such that for each $z\in\mathbb{R}$ the function $\omega \to (z - H_\omega)^{-1}$ is weakly measurable. Furthermore assume that there exist measure preserving transformations $\{T_i\}_{i\in I}$ (I an arbitrary index set), which are ergodic in the sense that every $A\in\mathscr{F}$ which is invariant under all $\{T_i\}_{i\in\mathbb{Z}^d}$ has either probability one or probability zero. Suppose furthermore that there are unitary operators. $\{U_i\}_{i\in I}$ on H such that

$$H_{T_i\omega} = U_i H_\omega U_i^*.$$

Then we have

Theorem 2 (Theorem 1 and 2 of [11]).

1) *The spectrum $\sigma(H_\omega)$ of H_ω is a nonrandom set.*

2) *The pure point part, the singular continuous part and the absolutely continuous part of $\sigma(H_\omega)$ are nonrandom sets.*

This theorem can be applied to our situation, since the $\{q_i\}_{i\in\mathbb{Z}^d}$ (as well as the $\{\xi_i\}_{i\in\mathbb{Z}^d}$) as independent identically distributed random variables form a stationary, metrically transitive random field, i.e. there are measure preserving, ergodic transformations $\{T_i\}_{i\in\mathbb{Z}^d}$ such that

$$q_j(T_i\omega) = q_{j-i}(\omega).$$

Hence with: $U_i f(x) := f(x - x_i)$ one has immediately:

$$H_{T_i\omega} = -\Delta + V_{T_i\omega} = -\Delta + U_i V_\omega U_i^* = U_i(-\Delta + V_\omega)U_i^* = U_i H_\omega U_i^*.$$

The measurability of $(z - H_\omega)^{-1}$ can be obtained by Corollary 3 in [11]. There it was proved that $(z - (-\Delta + V_\omega))^{-1}$ is weakly measurable if V as a function of ω and x is measurable and $-\Delta + V_\omega$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ almost surely. By Theorem 1 we know that in our case $H_\omega = -\Delta + V_\omega$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, hence Theorem 2 holds for this operators. Moreover an application of Corollary 1 in [11] shows that the discrete part of the spectrum of H_ω is almost surely empty.

Applications of Theorem 2 to more general random operators were discussed in [11]. In our special case however it is possible to investigate the spectrum of H_ω more precisely. In doing this the following definition is convenient.

Definition. We call a real function W on $\mathbb{R}^d$ an *admissible potential* for the operator

$H_\omega := -\Delta + V_\omega(V_\omega$ as defined in (17)) if $W(x) = \Sigma c_i f(x - x_i + u_i)$ with nonrandom $c_i \in \mathbb{R}$, $u_i \in \mathbb{R}^d$ and the following conditions are satisfied:

1) $c_i \in \operatorname{supp} P_{q_i}$, where P_{q_i} is the (common) distribution of q_i.
2) $u_i \in \operatorname{supp} P_{\xi_i}$, where P_{ξ_i} is the (common) distribution of ξ_i.
3) W is locally square *integrable*.
4) $-\Delta + W$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.
5) $\displaystyle\sum_{|i| \geq M} c_i \sup_{x \in C_0} |f(x - x_i + u_i)| < \infty$ for M large enough.

Remark. For almost every $\omega \in \Omega$ V_ω is an admissible potential; conditions 3) and 5) follow from Lemma 1, condition 4) from Theorem 1.

Now we prove the following theorem which allows an investigation of the spectrum of H_ω by means of admissible potentials.

Theorem 3. *If W is an admissible potential for H_ω the spectrum $\sigma(-\Delta + W)$ of $-\Delta + W$ is contained in the set Σ, which is the spectrum of H_ω almost surely. Thus*
$$\Sigma = \bigcup_{W \in A} \sigma(-\Delta + W)$$
where the union is taken over the class A of all admissible potentials.

Proof. Let $A := \left\{ \omega \in \Omega; \underset{N \in \mathbb{N}}{\forall} \underset{k \in \mathbb{N}}{\forall} \underset{x_0(N,k,\omega) \in \mathbb{Z}^d}{\exists} \right.$

$$\left. \int_{B_N} |W(x) - V(x + x_0(N,k,\omega), \omega)|^2 dx < 1/k \right\}.$$

where $B_N := \{x; |x| \leq N\}$.

In Lemma 2 we will show that $P(A) = 1$. Take now $\lambda \in \sigma(-\Delta + w)$, then by the Weyl criterion (see e.g. Weidmann [22] Theorem 7.22) there exists a sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^d)$ such that: $\|\varphi_k\| = 1$ ($\|\ \|$ denotes the L^2-norm) and $\|(-\Delta + W)\varphi_k - \lambda\varphi_k\| < 1/k$. Choose N_k large enough in such a way that $\operatorname{supp}\varphi_k \subset B_{N_k}$ and take $\omega \in A \cap \Omega_0 (\Omega_0 = \{\omega \in \Omega; \sigma(H_\omega) = \Sigma\})$. By Theorem 2 and Lemma 2 we have $P(A \cap \Omega_0) = 1$. Choose $x_0(N_k, k^2\|\varphi_k\|_\infty^2, \omega)$, where $\|\varphi_k\|_\infty = \sup_{x \in \mathbb{R}^d} |\varphi_k(x)|$; by the definition of A we have:

$$\int_{B_N} |W(x) - V_\omega(x + x_0(N_k, k^2\|\varphi_k\|_\infty^2, \omega))|^2 dx < \frac{1}{k^2\|\varphi_k\|_\infty^2}.$$

If we define the new sequence of C_0^∞-functions:

$$\psi_k(x) := \varphi_k(x - x_0(N_k, K^2\|\varphi_k\|_\infty^2, \omega)),$$

then: $\|(-\Delta + V_\omega)\psi_k - \lambda\psi_k\|$
$$= \|(-\Delta + V_\omega(x + x_0(N_k, k^2\|\varphi_k\|_\infty^2, \omega)))\varphi_k - \lambda\varphi_k\|$$
$$\leq \|(-\Delta + W)\varphi_k - \lambda\varphi_k\| + \|\{W(x) - V_\omega(x + x_0(N_k, k^2\|\varphi_k\|_\infty^2, \omega))\}\varphi_k\|$$
$$\leq \frac{1}{k} + \frac{1}{k} = 2k.$$

Hence, again by the Weyl criterion, $\lambda \in \sigma(-\Delta + V_\omega) = \Sigma$ since ω belongs by assumption also to Ω_0. $\qquad\square$

It remains only to show the following lemma:

Lemma 2. *Let W be an admissible potential and define*

$$A = \left\{ \omega \in \Omega ; \ \underset{N \in \mathbb{N}}{\forall} \ \underset{k \in \mathbb{N}}{\forall} \ \exists x_0(N,k,\omega) \in \mathbb{Z}^d \right.$$

$$\left. \int_{B_N} |W(x) - V_\omega(x + x_0(N,k,\omega))|^2 \, dx < \frac{1}{k} \right\}.$$

Then $P(A) = 1$.

Proof. Let $A_{N,k} = \left\{ \omega \in \Omega; \ \exists x_0(\omega) \int_{B_N} |W(x) - V_\omega(x + x_0(\omega))|^2 dx < \frac{1}{k} \right\}$. Since $A_{N+1,k} \subset A_{N,k}$, if we define $A_k = \bigcap_{N=1}^\infty A_{N,k}$, then $P(A_k) = \lim_{N \to \infty} P(A_{N,k})$. By definition $A = \bigcap_{k=1}^\infty A_k$, and since $A_{k+1} \subset A_k$ we get $P(A) = \lim_{k \to \infty} P(A_k)$. Thus in order to show that $P(A) = 1$ it suffices to show that $P(A_{N,k}) = 1$ for any $N,k \in \mathbb{N}$. Clearly $A_{N,k}$ is invariant under the ergodic shift $\{T_i\}_{i \in \mathbb{Z}^d}$ on Ω, so that $P(A_{N,k})$ is either equal to one or to zero. So it is enough to show that $P(A_{N,k}) > 0$, or $P(C_{N,k}) > 0$, where $C_{N,k} \subset A_{N,k}$ is the set $\left\{ \omega \in \Omega; \int_{B_N} |W(x) - V(x,\omega)|^2 dx < 1/k \right\}$.

Let M_1 be a positive constant large enough such that:

$$\int_{B_N} \left| \sum_{|i| > M_1} c_i f(x - x_i + u_i) \right|^2 dx < 1/4k.$$

This constant M_1 always exists since:

$$\int_{B_N} \left| \sum_{|i| > M} c_i f(x - x_i + u_i) \right|^2 dx$$

$$\leq \sum_{|j|=1}^{k(N)} \int_{C_0 + x_j} \left| \sum_{|i| > M_1} c_i f(x - x_i + u_i) \right|^2 dx$$

$$= \sum_{|j|=1}^{k(N)} \int_{C_0} \left| \sum_{|i| > M_1} c_i f(x - x_j - x_i + u_i) \right|^2 dx$$

$$\leq \sum_{|j|=1}^{k(N)} \left(\sum_{|i| > M_1} |c_i| \sup_{x \in C_0} |f(x + x_j - x_i + u_i)| \right)^2$$

and this last term can be made arbitrarily small because of point (5) of the definition of admissible potential.

Next, consider the set

$$D = \left\{ \omega \in \Omega; \ \exists M \int_{B_N} \left(\sum_{|i| > M} |q_i(\omega)| \, |f(x - x_i + \xi_i(\omega))| \right)^2 dx < \frac{1}{4k} \right\}.$$

Since V_ω is almost surely a locally square integrable function $P(D) = 1$, if we define now the set D_M by:

$$D_M = \left\{ \omega; \int_{B_N} \left(\sum_{|i| > M} |q_i(\omega)| \, |f(x - x_i + \xi_i(\omega))| \right)^2 dx < \frac{1}{4k} \right\},$$

then $D_M \subset D_{M+1}$ and $D = \bigcup D_M$; hence $P(D) = \lim_{M \to \infty} P(D_M)$ which implies that there exists an integer M_2 such that $P(D_M) \geq \frac{1}{2}$. Let now $M = \max \{M_1, M_2\}$ and take $\omega \in D_M$; we can then estimate $\int_{B_N} |W(x) - V_\omega(x)|^2 dx$ as follows:

$$\int_{B_N} \left| W(x) - V_\omega(x) \right|^2 dx$$
$$\leq 4 \int_{B_N} \left| \sum_{|i| \geq M} c_i f(x - x_i + u_i) \right|^2 dx$$
$$+ 4 \int_{B_N} \left| \sum_{|i| \geq M} q_i(\omega) f(x - x_i + \xi_i(\omega)) \right|^2 dx$$
$$+ 2 \int_{B_N} \left| \sum_{|i| \leq M} c_i (f(x - x_i + u_i) - f(x - x_i + \xi_i(\omega))) \right.$$
$$\left. + (c_i - q_i(\omega)) f(x - x_i + \xi_i(\omega)) \right|^2 dx$$
$$\leq \frac{1}{2k} + 2C_{2M} \sum_{|i| \leq M} c_i^2 \int_{B_N} |f(x - x_i + u_i) - f(x - x_i + \xi_i(\omega))|^2 dx$$
$$+ 2C_{2M} \sup_{|i| \leq M} |q_i(\omega) - c_i|^2 \sum_{|i| \leq M} \int_{\mathbb{R}^d} |f(x - x_i + \xi_i(\omega))|^2 dx, \qquad (18)$$

where C_{2M} is a positive constant depending only on M.

The last term of (18) can be chosen with positive probability less than $1/4k$ since by assumption $c_i \in \text{supp } p_q$ and the q_i's are independent, and since $f \in L^2(\mathbb{R}^d)$ because $f \in L^2_{\text{loc}}(\mathbb{R}^d)$ and furthermore f is such that $\sum_{|i| > K_0} \sup_{x \in C_0} |f(x - x_i)| < \infty$.

For the estimate of the first term of (18) we need the following remark: Since the map $x \to U_x (U_x f(z) = f(z + x))$ is strongly continuous from $\mathbb{R}^d$ into the bounded operators on $L^2(\mathbb{R}^d)$, for any ε, there exists a $\delta(\varepsilon, x, f)$ such that:

$$\int_{B_N} |f(x + z) - f(z + y)|^2 dz < \infty$$

for any y such that $|x - y| < \delta(\varepsilon, x, f)$. Thus in the first term of (18) if we make $|u_i - \xi_i(\omega)|$ small enough, which again is possible with positive probability since $u_i \in \text{supp } P_\xi$ and the ξ_i are independent (and also independent of the q_i), we have:

$$2C_{2M} \sum_{|i| < M} c_i^2 \int_{B_N} |f(x - x_i + u_i) - f(x - x_i + \xi_i(\omega))|^2 dx < \frac{1}{4k}.$$

Hence with positive probability we have that:

$$\int_{B_N} |W(x) - V_\omega(x)|^2 dx < \frac{1}{k}$$

i.e. $P(C_N, k) > 0$ hence $P(A_{N,k}) = 1$. $\qquad \square$

Remarks.

1) It is not difficult to give an alternative proof of the fact that $\sigma(H_\omega)$ is a nonrandom set by using the above lemma. In particular one can show that for each $\omega \in A$ $\sigma(H_\omega) = \Sigma$.

2) Each $\lambda \in \Sigma$ can be obtained by the construction in the theorem. On the other hand we can ensure that $x_0(N, k, \omega)$ goes to infinity as N goes to infinity. Thus the sequence $\{\psi_k\}$ can be chosen as weakly convergent to zero and moreover orthogonal. Again by the Weyl criterion this means that each spectral value $\lambda \in \Sigma$ belongs to the essential spectrum of H_ω.

By the previous theorem the spectrum of H_ω looks very large because the class of admissible potentials is very large. But the following theorem tells us that for knowing the spectrum Σ it is enough to know the spectra of all periodic admissible potentials.

Denote by P the class of all admissible potentials which are also periodic in the sense that $W \in P$ if there exists a basis $\{a_i\}_{i=1}^d$ of $\mathbb{R}^d$ such that

$$W(x + a_i) = W(x) \qquad \forall \; x \in \mathbb{R}^d.$$

Theorem 4. *In the hypotheses of Theorem 3:* $\Sigma = \overline{\bigcup_{W \in P} \sigma(-\Delta + W)}$

Proof. It will be enough to show the following: if W is an admissible potential, $W \in A$, then there exists a sequence of periodic admissible potentials $W_n \in P$ such that $-\Delta + W_n \to -\Delta + W$ in the sense of strong resolvent convergence. From this it follows that $\overline{\bigcup_{n \in \mathbb{N}} \sigma(-\Delta + W_n)} \supset \sigma(-\Delta + W)$ (see e.g. Reed–Simon I, VIII 2h [20]). But since the union of the spectra of all the admissible potentials contains Σ we have $\overline{\bigcup_{W \in P} \sigma(-\Delta + W)} \supset \Sigma$; but from Theorem 3 we know that $\bigcup_{W \in P} \sigma(-\Delta + W) \subset \Sigma$ so that $\Sigma = \overline{\bigcup_{W \in P} \sigma(-\Delta + W)}$.

We now prove that any operator $H = -\Delta + W$, $W \in A$, is the strong resolvent limit of operators $H_N = -\Delta + W_N$, $W_N \in P$ for any $N \in \mathbb{N}$. Since by definition $C_0^\infty(\mathbb{R}^d)$ is a core for $-\Delta + W$ for any $W \in A$, it suffices to show that there are periodic potentials $W_n \in P$ such that $-\Delta + W_n$ converges strongly on $C_0^\infty(\mathbb{R}^d)$ to $-\Delta + W$, $W \in A$. For this it is enough to show that for any compact set K and any $\varepsilon > 0$ there exists a periodic admissible potential $\bar{W}$ such that:

$$\left[\int_K |W(x) - \bar{W}(x)|^2 dx \right]^{1/2} < \varepsilon. \tag{19}$$

Since any compact set $K \subset \mathbb{R}^d$ can be covered by a finite number of unit cubes, it is enough to check (19) with $K = C_0$. Let $W(x) = \Sigma c_i f(x - x_i + u_i)$ be given and define

$$\tilde{c}_i = \tilde{c}_{i + M_1 k} = c_i \qquad \text{for } |i| < M_1 \qquad k \in \mathbb{Z}^d,$$

$$\tilde{u}_i = \tilde{u}_{i + M_1 k'} = u_i \qquad \text{for } |i| < M_1 \qquad k' \in \mathbb{Z}^d,$$

where M_1 is a positive constant such that

$$\sum_{|i| > M_1} |c_i| \sup_{x \in c_0} |f(x - x_i + u_i)| < \varepsilon/3. \tag{20}$$

This constant always exists since W is admissible. Without loss of generality we can assume M_2 greater than $N_0 + 1$, where N_0 is such that $|\xi_i| < N_0$ almost surely (see Remark after Lemma 1). Clearly the potential $\tilde{W} = \Sigma \tilde{c}_i f(x - x_i + \tilde{u}_i)$ is periodic of period M_1. Furthermore since $|\tilde{u}_i|$ is uniformly bounded by N_0 (see Remark after Lemma 1) by assumption (1) on f, it follows that there exists a constant M_2 such that:

$$\sup_{|i| \leq M_1} |c_i| \sum_{|i| > M_2} \sup_{x \in C_0} |f(x - x_i)| < \varepsilon/3.$$

Define
$$\bar{c}_i = \bar{c}_{i + M_2 k} = \min \{c_i, \tilde{c}_i\} \quad \text{for } |i| < M_1 \ \ k \in \mathbb{Z}^d,$$
$$\bar{u}_i = \bar{u}_{i + M_2 k'} = u_i \quad \text{for } |i| < M_1 \ \ k' \in \mathbb{Z}^d,$$

and let $\bar{W} = \Sigma \bar{c}_i f(x - x_i + \bar{u}_i)$. Then

$$\left(\int_{c_0} |W(x) - \bar{W}(x)|^2 dx \right)^{1/2}$$

$$\leq \sup_{x \in C_0} \left| \sum_{|i| > M_1} (c_i f(x - x_i + u_i) - \bar{c}_i f(x - x_i + \bar{u}_i)) \right|$$

$$\leq \sum_{|i| > M_1} |c_i| \sup_{x \in C_0} |f(x - x_i + u_i)| +$$

$$+ \sum_{|i| > M_2} |\bar{c}_i| \sup_{x \in C_0} |f(x - x_i + u_i)|$$

$$+ \sum_{M_1 < |i| \leq M_2} |\bar{c}_i| \sup_{x \in C_0} |f(x - x_i + u_i)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{M_1 \leq |i| < M_2} |c_i| \sup_{x \in C_0} |f(x - x_i + \bar{u}_i)| < \varepsilon.$$

Remark 1. From the above theorem it follows that in principle the spectrum of the random operator H_ω has band structure, i.e. the set Σ is a union of closed intervals with possibly gaps between them.

Remark 2. The spectrum of a possible pure crystal (i.e. $q_i = \bar{q}$, $\xi_i = \bar{\xi}$ for any $i \in \mathbb{Z}^d$) is always contained in the spectrum of the random mixture. For example if $0 \in \text{supp } P_{q_0}$, then the set $[0, +\infty[\subset \Sigma$.

Remark 3. The spectrum Σ depends only on the support of the distribution P_{q_0} of each q_i. For a similar result in the discrete case (when the Hilbert space is $l^2(\mathbb{Z}^d)$) see Kunz–Souillard [13].

Remark 4. Actually the random variables q_i and ξ_i need not to be independent; it is enough that they form a stationary metrically transitive random field such that the support of the conditional distribution $P(q_0 | q_i i \neq 0)$ of q_0 given q_i ($i \neq 0$) is equal to the support of $P(q_0)$ and the same for the ξ_i.

Section 3

In this section we give a simple condition for the existence of gaps

in the spectrum of the random operator $H_\omega = -\Delta + \Sigma q_i(\omega) f(x - x_i)$ on $L^2(\mathbb{R}^d)$, which is useful when dealing with concrete examples. Let $W(x) = \Sigma \lambda_i f(x - x_i)$ be a periodic admissible potential and let $\{a_i\}_{i=1}^d$ be a basic of the vector space $\mathbb{R}^d$ such that

$$W(x + a_i) = W(x) \qquad \forall x \in \mathbb{R}^d.$$

Clearly the Hamiltonian $H_w = -\Delta + w$ is invariant under the group $\Lambda = \{n a_i, n \in \mathbb{Z}^d\}$, so that it can be decomposed as follows:

$$H_w = \int_{\hat{\Lambda}} H_\omega(k) dk = \frac{1}{|B|} \int_B H_w(k) dk,$$

where $\hat{\Lambda}$ is the dual group of Λ and B is the Brillouin zone (see e.g. Reed–Simon IV [20] and Avron–Grossmann–Rodriguez [3]. $H_\omega(k)$ are called the reduced Block Hamiltonians; from the above integral decomposition it follows that the spectrum of H_w is the union of the spectra $H_w(k)$. Furthermore, since $W(x) \in L^2_{loc}(\mathbb{R}^d)$, it follows that for $d \leq 3$ the reduced Hamiltonians have compact resolvent and thus their spectrum consists only of isolated eigenvalues $E_n(W, k)$, labeled by the discrete parameter $n \in \mathbb{N}$, of finite multiplicity (see Avron–Grossmann–Rodriguez, Th. 3, 1 [3]).

Assumption A. The eigenvalues $E_n(W, k)$ of the reduced Hamiltonians $H_W(k)$, $W = \Sigma \lambda_i f(x - x_i)$, are such that for any $n \in \mathbb{N}$ and any $k \in B$ there exist two numbers $\underline{\lambda}^{(n,k)}$, $\overline{\lambda}^{(n,k)}$, $\lambda_{\min} \leq \underline{\lambda}^{(n,k)} \leq \lambda_i \leq \overline{\lambda}^{(n,k)} \leq \lambda_{\max}$, where $\lambda_{\min}(\lambda_{\max})$ is the inf (sup) of the set supp P_{q_0}, P_{q_0} being the distribution function of each of the random variables q_i, or $\lambda_{\min} = -\infty (\lambda_{\max} = +\infty)$ if the inf (sup) of supp P_{q_0} does not exist, such that:

$$E_n(\underline{W}, k) \leq E_n(W, k) \leq E_n(\overline{W}, k), \text{ or}$$

$$E_n(\underline{W}, k) \geq E_n(W, k) \geq E_n(\overline{W}, k),$$

$$\text{with } \underline{W}(x) = \underline{\lambda}^{(n,k)} \Sigma f(x - x_i), \overline{W}(x) = \overline{\lambda}^{(n,k)} \Sigma f(x - x_i).$$

Remark. By the mini-max principle Assumption A holds if for example $f(x)$ has a definite sign.

Theorem 5. *If Assumption A is satisfied and the open interval (α, β) does not belong to the spectrum of the periodic Hamiltonian $-\Delta + \lambda \Sigma f(x - x_i)$ for any $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$, then $(\alpha, \beta) \cap \Sigma = \varnothing$, where Σ is the spectrum of $H_\omega = -\Delta + \Sigma q_i(\omega) f(x - x_i)$ almost surely.*

Proof. By Theorem 4 it is enough to show that $(\alpha, \beta) \cap \sigma(-\Delta + W) \equiv D = \varnothing$ for any periodic admissible potential W. Assume $D \neq \varnothing$ and let $E_0 \in D$. By the integral decomposition of $-\Delta + W$ it follows that $E_0 = E_{n_0}(W, k_0)$ for some $n_0 \in \mathbb{N}$ and $k_0 \in B$. By assumption A we have

$$E_{n_0}(\underline{W}, k_0) \leq E_{n_0}(W, k_0) \leq E_{n_0}(\overline{W}, k_0),$$

for some $W(x) = \underline{\lambda}^{(n_0, k_0)} \sum_{i \in \mathbb{Z}^d} f(x - x_i)$, and

$$\overline{W}(x) = \overline{\lambda}^{(n_0, k_0)} \sum_{i \in \mathbb{Z}^d} f(x - x_i),$$

with $\lambda^{(n_0, k_0)} \leq \bar{\lambda}^{(n_0, k_0)}$ or vice versa. Since $E_{n_0}(\lambda \Sigma f(x - x_i), k_0)$ is a continuous function of λ, (see e.g. Reed–Simon IV [20]) there exists a $\lambda^{(n_0, k_0)} \leq \tilde{\lambda} \leq \bar{\lambda}^{(n_0, k_0)}$ (or $\lambda^{(n_0, k_0)} \geq \tilde{\lambda}$ $\geq \bar{\lambda}^{(n_0, k_0)}$) such that

$$E_{n_0}(W, k_0) = E_{n_0}(\tilde{W}, k_0) \quad \text{when} \quad \tilde{W}(x) = \tilde{\lambda} \sum_{i \in \mathbb{Z}^d} f(x - x_i).$$

But this means that $E_0 = E_{n_0}(W, k_0) E_{n_0}(\tilde{W}, k_0)$ belongs to the spectrum of $-\Delta + \lambda \Sigma f(x - x_i)$ in contradiction with the hypothesis. $\square$

Corollary 1. *If the supp P_{q_0} is connected and if Assumption A holds, then the open interval (α, β) is a gap for the random Hamiltonian H_ω if and only if it is a gap for all the periodic Hamiltonians $H_\lambda = -\Delta + \lambda \Sigma f(x - x_i)$ $\lambda \in \text{supp} P_{q_0}$.*

Proof. It is a direct consequence of the previous theorem and of Theorem 4. $\square$

It follows from the above Corollary that, in some case, the study of the (possible) gaps in the spectrum Σ of H_ω is reduced to the study of the gaps of the "pure crystals" $-\Delta + \sum_{i\mathbb{Z}^d} f(x - x_i)$, where λ runs in the supp P_{q_0}. In the examples we shall consider it will turn out to be sufficient to study the gaps of the "pure crystals" corresponding to the values $\lambda = \lambda_{\max}$ and $\lambda = \lambda_{\min}$.

Section 4

In this last section we give first two one-dimensional examples of random operators of the Kronig–Penney type for which infinitely many gaps occur, and then we show that also for the random version of the periodic point interaction model in three dimensions treated by Grossmann, Høegh-Krohn, Mebkhout [9] still a gap is present in the nonrandom spectrum Σ. While the first example is only an application of the general result of Sects. 1–3, the other two examples require new proofs of results similar to that of the previous sections, since the above theorems cannot be applied directly because of the strong singularities of these Hamiltonians.

1) The Kronig–Penney model with a step potential in one dimension (see Kronig–Penney [12]).

Let $f(x) = \chi_{[0, a]}(x)$, $0 < a < x_1(\chi_A(\cdot)$ is the characteristic function of the set $A \subset \mathbb{R})$ and let $V(x) = \Sigma f(x - x_i)$ when the lattice $\{x_i\}_{i \in \mathbb{Z}}$ is assumed for simplicity to be $\mathbb{Z}$. Furthermore let $\{q_i\}_{i \in \mathbb{Z}}$ be independent, identically distributed random variables such that:

$$0 < M_1 \leq q_i(\omega) \leq M_2 < \infty \qquad \forall \omega \in \Omega, \forall i \in \mathbb{Z}.$$

and define $V_\omega(x) = \sum_{i \in \mathbb{Z}} q_i(\omega) f(x - x_i)$.

Propositional 1. *Denote by Σ the almost surely constant spectrum of $H_\omega = -d^2/dx^2 + V_\omega$. Then, for each of the points $E_n = (n\pi/(x_1 - a))^2, E_n < M_1$, there exists a neighborhood A_n such that $A_n \cap \Sigma = \varnothing$.*

Proof. In Flügge [6] it is shown that for any $0 < \lambda < +\infty$ the Hamiltonian

$-d^2/dx^2 + \lambda \sum_{i \in \mathbb{Z}} f(x - x_i)$ has gaps A_n^λ around the points $E_n < \lambda$. If we define $A_n = \bigcap_{M_1 \leq \lambda \leq M_2} A_n^\lambda$ we have that $A_n \neq \varnothing$ and that A_n is a common gap for all the Hamiltonian $-d^2/dx^2 + \lambda \sum_{i \in \mathbb{Z}} f(x - x_i)$ with $M_1 \leq \lambda \leq M_2$. Furthermore since $f(x)$ has a definite sign $(f(x) \geq 0)$ and $f(x) \leq 1$ Assumption A holds so that we get the statement by Theorem 5.

Remark. Let $A_n^\lambda \equiv (\alpha_n^\lambda, \beta_n^\lambda)$ be the gap around the point $(n\pi/(x_i - a))^2$ for the periodic Hamiltonian $-d^2/dx^2 + \lambda \sum_{i \in \mathbb{Z}} f(x - x_i) M_1 \leq \lambda \leq M_2$. By the general theory of one-dimensional periodic Schrödinger operators (see, e.g. Reed–Simon IV [20]), we know that α_n and β_n are then n^{th} eigenvalues of $-d^2/dx^2 + \lambda \sum_{i \in \mathbb{Z}} f(x - x_i)$ on $L^2([0x_1], dx)$ with respectively periodic and antiperiodic boundary conditions if n is even and vice versa if n is odd. Hence, by the mini-max principle $\alpha_n^\lambda, \beta_n^\lambda$ are monotone increasing functions of λ, so that the gap A_n for the random operator $-d^2/dx^2 + \Sigma q_i(\omega) f(x - x_i)$ is given by: $A_n = (\alpha_n^{M_2}, \beta_n^{M_1})$.

2) We now pass to consider a random Kronig–Penney model with point interactions formally defined by:

$$H_\omega = -\frac{d^2}{dx^2} + \sum_{i \in \mathbb{Z}} q_i(\omega) \delta(x - x_i), \tag{21}$$

where $\{q_i(\omega)\}_{i \in \mathbb{Z}}$ are independent identically distributed random variables satisfying $0 < c_1 \leq q_i(\omega) \leq c_2 < +\infty$ for any $i \in \mathbb{Z}$ and $\omega \in \Omega$. The Hamiltonian H_ω can be well defined as a sum of quadratic forms as follows: Denote by $Q = Q(-d^2/dx^2)$ the form domain of the operator $-d^2/dx^2$ on $L^2(\mathbb{R})$ and by Q_0 the function in Q with compact support. We define on Q_0 the quadratic form:

$$\beta_\omega(\varphi, \psi) = \sum_{i \in \mathbb{Z}} q_i(\omega) \varphi(x_i) \psi(x_i).$$

First we show: For any $a > 0$, there is a $b \in \mathbb{R}$ such that:

$$|\beta_\omega(\psi, \psi)| \leq a \langle \psi', \psi' \rangle + b \langle \psi, \psi \rangle \tag{22}$$

for any $\psi \in Q_0$ and $\omega \in \Omega$. In order to prove this, take a sequence $\{\psi_i\}_{i \in \mathbb{Z}}$ in $C_0^\infty(\mathbb{R})$ such that:

1) $\mathrm{supp}\ \psi_i \subset]x_{i-1}, x_{i+1}[$,

2) $\sum_{i \in \mathbb{Z}} \psi_i^2(x) = 1$,

3) $\sup_{x \in \mathbb{R}} \sum_{i \in \mathbb{Z}} |\psi_i'(x)|^2 < \infty$.

A sequence satisfying 2) and 3) is called a "local partition" in Morgan [16].

We have:

$$|\beta_\omega(\psi_i\psi, \psi_i\psi)| \leq c_2|\psi(x_i)|^2 \leq a \langle (\psi_i\psi)', (\psi_i\psi)' \rangle + b \langle \psi, \psi \rangle \qquad (23)$$

(see e.g. Reed–Simon II, X. 2[20]). By repeating now word by word the proof of Theorem 2.2 in Morgan [16] we get from the local estimate (23) the global one (22).

Now we show that β_ω can be defined on the whole space $Q(-d^2/dx^2)$. Let $\psi \in Q$, choose $\psi_n \in Q_0$, $\psi_n \to \psi$ in $L^2(\mathbb{R})$, $\psi_n' \to \psi'$ in $L^2(\mathbb{R})$, which is possible since Q_0 is dense in Q with respect to the norm $(\|\psi'\|^2 + \|\psi\|^2)^{1/2}$; moreover we can choose ψ_n in such a way that $\psi_n(x) = \psi(x)$ for $|x| \leq n$. Now:

$$\sum_{|i| \leq n} q_i(\omega)|\psi(x_i)|^2 = \sum_{|i| \leq n} q_i(\omega)|\psi_n(x_i)|^2 \leq C_2 \sum_{i \in \mathbb{Z}} \psi_n(x_i)|^2$$
$$\leq a\|\psi_n'\|^2 + b|\psi_n|^2.$$

The last expression is bounded independent of n, since ψ_n' and ψ_n are convergent, hence the norms are bounded. Thus $\beta_\omega(.,.)$ is well defined on $Q(-d^2/dx^2)$.

Furthermore, by continuity inequality (22) holds for all $\psi \in Q$. Hence we can apply the KLMN (Kata, Lax, Milgram, Nelson)-Theorem (see e.g. Reed–Simon II, X. 2 [20]) to get a well defined, unique selfadjoint operator H_ω on $L^2(\mathbb{R})$ associated with the closed quadratic form $\gamma_\omega(\psi, \psi) := \langle \psi', \psi' \rangle + \beta_\omega(\psi, \psi)$ on $Q(-d^2/dx^2)$. It is this operator we mean by the formal expression in (21).

In [11] we showed that the spectrum of H is a nonrandom set. We call this set Σ. In order to investigate Σ we have to prove analogs of Theorems 3 and 4 for the (very singular) Hamiltonian H_ω. Although it is possible to give proofs similar to those of Sect. 2, since H_ω can be shown to be essentially selfadjoint on a set of functions with compact supports, we prefer to give a more direct proof based on the explicit expression of the resolvent of H_ω. This method has the advantage that it can be extended immediately to the three-dimensional case (see the next example) and moreover the case of continuously distributed random variables $\{q_i\}_{i \in \mathbb{Z}}$ causes no further difficulty as it does for the previous method.

We call, as in Sect. 2, the formal expression $W = \Sigma \lambda_i \delta(x - x_i)$ an admissible potential if $\lambda_i \in \text{supp } P_{q_0}$ for all $i \in \mathbb{Z}$. In this case we also call the sequence $\{\lambda_i\}_{i \in \mathbb{Z}}$ admissible.

Now we give explicitly the resolvent of the operator

$$H_{\{\lambda_i\}} = -\frac{d^2}{dx^2} + \sum_{i \in \mathbb{Z}} \lambda_i \delta(. - x_i) \quad \lambda_i \text{ bounded.}$$

Faris [5] (in §5) computes the resolvent of the operator $H_{\lambda_0} = -\frac{d^2}{dx^2} + \lambda_0 \delta(.)$.

The resolvent is for $E \notin \sigma\left(-\frac{d^2}{dx^2}\right)$: $(H_{\lambda_0} - E)^{-1}\psi(x) = \int_{\mathbb{R}} G_E(x - y)\psi(y)\,dy$

$-\left(\frac{1}{\lambda_0} + \frac{1}{2\sqrt{E}}\right)^{-1} \int_{\mathbb{R}} G_E(y)\psi(y)\,dy\, G_E(x),$

where $G_E(x) = \frac{1}{2\sqrt{-E}} e^{-\sqrt{-E}|x|}$ is the free Green's function in one dimension

and $\sqrt{-E}$ is chosen such that $\mathrm{Re}(\sqrt{-E}) > 0$. In the case of finitely many δ's we get a similar expression:

$$\left(-\frac{d^2}{dx^2} + \sum_{i=-n}^{n} \lambda_i \delta(x - x_i) - E\right)^{-1} \psi(x) = \int_{\mathbb{R}} G_E(x - y)\psi(y)dy$$

$$- \sum_{i,j=-n}^{n} (T_E^{(n)-1})_{ij} \int G_E(y - x_j)\psi(y)dy\, G_E(x - x_i),$$

where $T_E^{(n)}$ is the $n \times n$-matrix given by $(T_E^{(n)})_{ij} = \delta_{ij}/\lambda_i + G_E(x_i - x_j)$. This expression as well as its calculation is in complete analogy with those given in Grossmann, Høegh-Krohn and Mebkhout [9] where a three-dimensional point interaction is considered.

Using Theorem 7–10 in Faris [5] we get easily that $-\dfrac{d}{dx^2} + \sum\limits_{i=n}^{} \lambda_i \delta(. - x_i)$ converges in the strong resolvent sense to $H_{\{\lambda_i\}} = -\dfrac{d^2}{dx^2} + \sum\limits_{i=-\infty}^{+\infty} \lambda_i \delta(. - x_i)$ as n goes to infinity.

Hence the resolvent of $H_{\{\lambda_i\}}$ is given by:

$$(H_{\{\lambda_i\}} - E)^{-1}\psi(x) = \int G_k(x - y)\psi(y)dy - \sum_{i,j \in \mathbb{Z}} (T_E^{-1})_{ij}$$

$$\int G_E(y - x_j)\psi(y)dy\, G_E(x - x_i), \tag{24}$$

where T_E is the bounded operator on $l^2(\mathbb{Z})$ given by: $(T_E)_{ij} = \delta_{ij}/\lambda_i + G_E(x_i - x_j)$. Indeed, that (24) is the correct limit for the weak resolvent convergence is easily obtained by computation observing that $\psi_i = \int G_E(x - x_j)\psi(x)dx$ is in $l^2(\mathbb{Z})$ that T_E and $T_F^{(n)}$ can be looked upon as bounded operators in $l^2(\mathbb{Z})$ and $T_E^{(n)-1} \to T_E^{-1}$ weakly in $l^2(\mathbb{Z})$ as $n \to \infty$. But since we know that $-\dfrac{d^2}{dx^2} + \sum\limits_{i=-n}^{n} \lambda_i(-x_i)$ converges in the strong resolvent sense, the operator in (24) is also the strong limit.

Proposition 2 *Let W be an admissible potential, then $\sigma(-\Delta + W) \subset \Sigma$.*

Proof. The proof is based on the following lemma:

Lemma 3. *Let $\mathrm{Im}\, E \neq 0$ and let $W = \{\lambda_i\}_{i \in \mathbb{Z}}$ be an admissible potential. Then there exists a sequence of $\omega_n \in \Omega_1$, when $\Omega_1 = \{\omega \in \Omega; \sigma(H_\omega) = \Sigma\}$, such that the operator $T_E(\omega_n)$ on $l_2(\mathbb{Z})$ with matrix elements*

$$\frac{1}{q_k(\omega_n)}\delta_{kj} + G_E(x_k - x_j)$$

converges in the strong resolvent sense to the operator T_E given by

$$\frac{1}{\lambda_k}\delta_{kj} + G_E(x_k - x_j).$$

Proof. Define $\Omega_n = \{\omega \in \Omega; |q_i(\omega) - \lambda_i| < 1/n \quad \forall |i| \leq n\}$.

Since $P(\Omega_1) = 1$ and $P(\Omega_n) > 0$, $\Omega_1 \cap \Omega_n \neq \emptyset$.

Hence we can pick $\omega_n \in \Omega_n \cap \Omega_1$ and compute for $\psi \in l_{2,0}(\mathbb{Z}) = \{\psi \in l_2(\mathbb{Z}); \psi(i) \neq 0$ only for finitely many $i \in \mathbb{Z}\}$ the quantity; $\|(T_E(\omega_n) - T_E)\psi\|_{l_2}(\mathbb{Z})$.

We have:

$$\sum_{k \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \left[\frac{1}{q_k(\omega_n)} \delta_{kj} + G_E(x_k - x_j) \right] - \left(\frac{1}{\lambda_k} \delta_{kj} + G_E(x_k - x_j) \right) \psi(j) \right|^2$$

$$= \sum_{k \in \mathbb{Z}} \left| \left(\frac{1}{q_k(\omega_n)} - \frac{1}{\lambda_k} \right) \psi(k) \right|^2.$$

By definition of ω_n and the fact that $\psi \in l_2, (\mathbb{Z})$ the last expression goes to zero as $n \to \infty$. Since $T_E(\omega_n)$ and T_E are bounded and symmetric, it follows that $T_E(\omega_n) \to T_E$ in the strong resolvent sense.

Now we give the proof of the proposition:

Proof of the Proposition. First we note that the p-space version of the resolvent of $H_{\{\lambda_i\}}$ is given by:

$$(H_{\{\lambda_i\}} - E)^{-1} \psi(p) = \frac{1}{p^2 - E} \psi(p) + \frac{1}{2\pi} \sum_{j,k \in \mathbb{Z}} (T_E)_{kj}^{-1} \frac{1}{2\pi} \int \frac{\psi(q)}{q^2 - E} e^{-ix_j q} dq \frac{1}{p^2 - E} e^{ix_k p}.$$

Take now $\psi \in C_0^\infty(R)$, then $\tilde{\psi}(j) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(q)}{q^2 - E} e^{-iqx_j} dq$ is an element of $l_2(\mathbb{Z})$. By the previous lemma: $(T_E^{-1} - T_E(\omega_n)^{-1}) \tilde{\psi} = : \tilde{\psi}_n$ tends to zero in $l_2(\mathbb{Z})$-norm. The Fourier transform $\varphi_n(q) = (1/2\pi) \Sigma \tilde{\psi}_n(j) e^{ix_j q}$ of $\tilde{\tilde{\psi}}_n$ hence tends to zero as a function in $L^2(0, 2\pi)$. But

$$\int_{\mathbb{R}} \frac{|\varphi_n(p)|^2}{(p^2 - E)^2} dp \leq \|\varphi_n\|_{L^2(0,2\pi)} \left\| \frac{1}{p^2 - E} \right\|_{L^2(\mathbb{R})} \to 0.$$

we have proved that $H_{\omega(n)} \to H_\omega$ in the strong resolvent sense. From this we get $\sigma(H_\omega) \subset \overline{\bigcup \sigma(H_{\omega(n)})} = \Sigma$ since $\omega(n) \in \Omega_1$ for all $n \in N$.

Call now $W = \Sigma \lambda_i \delta(. - x_i)$ an admissible periodic potential if $\lambda_{i+M} = \lambda_i$ for some $M \in \mathbb{Z}$, $\lambda_i \in \text{supp} P_{q_0}$. As in Sect. 2 we denote by P the class of periodic admissible potentials.

Proposition 3.

$$\Sigma = \overline{\bigcup_{W \in P} \sigma(H_W)}.$$

Proof. As before we need a preparatory lemma:

Lemma 4. *Let $W = \{\lambda_i\}_{i \in \mathbb{Z}}$ be an admissible periodic potential. Then there exist $W_n = \{\lambda_k^{(n)}\}_{k \in \mathbb{Z}} \in P$ such that $T_E(W_n) \to T_E(W)$ in the strong resolvent sense in $l_2(\mathbb{Z})$,*

where $T_E(W_n)$ and $T_E(W)$ have respectively matrix elements:

$$\frac{1}{\lambda_k^{(n)}}\delta_{kj} + G_E(x_k - x_j) \quad \text{and} \quad \frac{1}{\lambda_k}\delta_{kj} + G_E(x_k - x_j).$$

Proof. Choose $\lambda_k^{(n)} = \lambda_k$ if $|k| \leq n$ and $\lambda_{k+2n+1}^{(n)} = \lambda_k^{(n)}$ $\forall n \in \mathbb{Z}$.
Then, as in Lemma 3 it can be shown immediately that:

$$\| [T_E(W_n) - T_E(W)]\psi \|_{l_2(\mathbb{Z})} \to 0, \text{ as } n \to \infty \quad \forall \psi \in l_2,(\mathbb{Z}).$$

From the above lemma, precisely as in the previous proposition, we get that
$H_{W_n} \to H_W$ in the strong resolvent sense. This implies that $\sigma(H_W) \subset \bigcup \sigma(H_{W_n})$,
i.e. $\Sigma \subset \bigcup_{w \in P} \sigma(H_\omega)$; and by the previous proposition we get:

$$\Sigma = \overline{\bigcup_{W \in P} \sigma(H_\omega)}. \qquad \qquad \square$$

Now we turn to the determination of the spectrum of the operator H. As it was
shown by Kronig and Penney [12] the spectrum of the operator H_λ formally defined
by

$$H_\lambda = -\frac{d^2}{dx^2} + \lambda \sum_{i=-\infty}^{\infty} \delta(x - x_i) \text{ is given by } \sigma(H_\lambda) = \bigcup [f_n(\lambda), (n\pi)^2],$$

where the f_n are continuous monotone increasing functions of $\lambda \geq 0$ and $f_n(\lambda) >$
$((n-1)\pi)^2$ for $\lambda > 0$ (see also Flügge [6]). It is a special property of this operator that
$\sigma(H_\lambda) \subset \sigma(H_\mu)$ whenever $\mu \leq \lambda$.

The reduced Bloch Hamiltonians of $H_{\{\lambda_i\}} = -d^2/dx^2 + \Sigma \lambda_i \delta(x - x_i)$, where
$\lambda_{i+N} = \lambda_i$ can be defined using the KLMN-theorem as the unique selfadjoint
operator associated to the closed form:

$$\langle H_{\{\lambda_i\}}(k)\psi, \psi \rangle := \langle \psi', \psi' \rangle + \sum_{i=0}^{N-1} \lambda_i \psi(x_i)\overline{\psi(x_i)}$$

on the form domain $Q((-d^2/dx^2)_k)$ of the Laplacian on $L^2 - 1/2, (N - 1/2)$ with
boundary conditions

$$\psi(-\tfrac{1}{2}) = e^{ik(N)}\psi(N - \tfrac{1}{2}) \text{ and } \psi'(-\tfrac{1}{2}) = e^{ikN}\psi'(W - \tfrac{1}{2}).$$

These reduced Hamiltonians have discrete eigenvalues (see e.g. Avron, Grossman
and Rodriguez [3]) and satisfies Assumption A of Sect. 3, where (i) can be checked by
the mini-max-principle and (ii) follows from the computation by Kronig and Penney
[12]. By repeating word by word the proof of Theorem 5 we can show that Theorem
5 holds with the Hamiltonian $-\Delta + \lambda \Sigma f(x - x_i)$ replaced by the Hamiltonian
$-d^2/dx^2 + \lambda \Sigma \delta(x - x_i)$. Hence we conclude that $]((n-1)\pi)^2, f_n(\lambda_{min})[$ where
$\lambda_{min} = \inf \{x \in \text{supp } P_{q_0}\} > 0$ is a gap for the spectrum Σ of the random operator

$$H_\omega := -\frac{d^2}{dx^2} + \Sigma q_{i(\omega)}\delta(x - x_i).$$

But from Proposition 2 we know that $\sigma(H_{\lambda_{min}}) \subset \Sigma$. Hence we have shown:

Proposition 4. *Let $\{q_i(\omega)\}_{i\in\mathbb{Z}}$ be independent distributed random variables satisfying $0 < c_1 \leq q_i(\omega) \leq c_2$ and let H_ω be formally given by $H_\omega := -d^2/dx^2 + \Sigma q_i(\omega)\delta(x - x_i)$, the exact meaning of which is given above.*
Then the spectrum Σ of almost all H_ω is given by

$$\Sigma = \sigma(H_{q_{\min}}),$$

where $H_{q_{\min}}$ is formally $-d^2/dx^2 + q_{\min}\Sigma\delta(x - x_i)$ and $q_{\min} = \inf\{q; q\in\text{supp } P_{q_0}\}$.

3) *Random point interactions in three dimensions.* As the last example we treat the random version of the point interactions model in three dimensions extensively investigated by Grossman, Høegh-Krohn and Mebkhout [9]. For other approaches to such operators see Albeverio and Høegh-Krohn [2] and Albeverio, Fenstad and Høegh-Krohn [1]. Let $\{q_i(\omega)\}_{i\in\mathbb{Z}^3}$ be independent identically distributed random variables such that $|q_i(\omega)| < M \,\forall i\in\mathbb{Z}^3$ and $\forall\omega\in\Omega$, for some positive constant M, and let H_ω be the selfadjoint operator on $L^2(\mathbb{R}^3)$ whose resolvent is given by:

$$(H - E)^{-1} = (p^2 - E)^{-1} \sum_{k,j\mathbb{Z}^3} \left[\left(q_k(\omega) - \frac{i\sqrt{E}}{4\pi}\right)\delta_{kj} - \tilde{G}_E(k - j)\right]_{\mathbb{Z}^3}^{-1} \text{ and}$$

$$H_\omega(2\pi)^{-3}\frac{e^{i(pk - qj)}}{(p^2 - E)(q^2 - E)}, \tag{25}$$

where $\text{Im } E \neq 0$, $\tilde{G}_E(k - j) = (1/4\pi)e^{i\sqrt{E}|k - j|}/|k - j|$, if $k \neq j$ and $\tilde{G}_E(0) = 0$ and $[(q_k(\omega) - i\sqrt{E}/4\pi)\delta_{kj} - \tilde{G}_E(k - j)]_{\mathbb{Z}^3}^{-1}$ is the inverse as an operator on $l_2(\mathbb{Z}^3)$. The sum in (25) is absolutely convergent in the sense that if we integrate with respect to $L^2(\mathbb{R}^3)$ functions of p and q respectively, then the sum is absolutely convergent. We remark that, as in the one-dimensional case, the operator on $l_2(\mathbb{Z}^3)$ given by $q_k(\omega)\delta_{kj} - \tilde{G}_E(k - j)$ is a bounded operator if $\text{Im } E \neq 0$.

In our previous paper [11] we proved that the spectrum of H is almost surely a nonrandom set Σ of the real line. As in the one-dimensional case, in order to study the set $\Sigma \subset R$, we need the analogs of Proposition 2 and 3 about the admissible potentials. We will call $W = \{\lambda_i\}_{i\in\mathbb{Z}^3}$ an admissible potential if $\lambda_i\in\text{supp } P_{q_0} \,\forall i\in\mathbb{Z}^3$ (P_{q_0} is the probability distribution of $q_0(\omega)$), and we will denote by H_ω the selfadjoint operator on $L^2(\mathbb{R}^3)$ whose resolvent is given by (25) with $q_k(\omega)$ replaced by λ_k.

Proposition 5. *Let W be an admissible potential; then $\sigma(H_w) \subset \Sigma$.*
The proof is omitted since it is identical to that of Proposition 2.

As before we will call $W = \{\lambda_k\}_{k\in\mathbb{Z}^3}$ a periodic admissible potential if there exists $L\in\mathbb{Z}^3$ such that $\lambda_{k+L} = \lambda_k \,\forall k\in\mathbb{Z}^3$, and we will denote by P the class of the periodic admissible potentials. By repeating the proof of Proposition 3 we get:

Proposition 6.

$$\Sigma = \overline{\bigcup_{w\in P} \sigma(H_\omega)}.$$

The following two results about the periodic point interactions (see [9], Th 5.4, 5.1) will turn out to be useful:

(i) Denote by H_λ the periodic Hamiltonian corresponding to the resolvent (25) with $q_k(\omega)$ replaced by $\lambda \forall k \in \mathbb{Z}^3$.
If $\lambda < \lambda_0$ for some constant λ_0, then the spectrum of $H_\lambda \sigma(H_\lambda)$ is given by:

$$\sigma(H_\lambda) = [E_0^\lambda, E_1^\lambda] \cup [0\,\infty],$$

with $E_1^\lambda < 0$ and E_0^λ, E_1^λ continuous monotone increasing functions of λ.

(ii) Let $A = \{n_1 a_1 + n_2 a_2 + n_3 a_3 ; (n_1, n_2, n_3) \mathbb{Z}^3\}$ where a_1, a_2, a_3 are three independent vectors in $\mathbb{Z}^3$; without loss of generality we can assume a_1, a_2, a_3 to be orthonormal, and let X be a finite subset of $\mathbb{Z}^3$. Let λ be a real function on $Y = A + X$ invariant under A, i.e. $\lambda_{a+x} = \lambda_x$, $x \in X$, $\alpha \in A$. Then the Hamiltonian H_λ whose resolvent is given by (31) with $q_k(\omega)$ replaced by $\lambda_k k \in \mathbb{Z}^3$ is invariant under translation in A so that:

$$H_\lambda = \frac{1}{|B|} \int_B H_\lambda(k)\,dk,$$

where $B = \{s_1 b_1 + s_2 b_2 + s_3 b_3, -1/2 < s_i \leq 1/2\}$, $(b_i, a_j) = 2\pi \delta_{ij}$, is the Brillouin zone. The reduced Hamiltonians $H_\lambda(k)$ are selfadjoint operators on $l_2(\Gamma)$, $\Gamma = \{n_1 b_1 + n_2 b_2 + n_3 b_3, (n_1, n_2, n_3) \in \mathbb{Z}^3\}$ is the orthogonal lattice, whose resolvent is given by:

$$(H_\lambda(k) - E)^{-1} = ((\gamma + k)^2 - E)^{-1}$$

$$- \sum_{x, y \in X} [\lambda_x \delta_{xy} - g_E(x - y, k)]^{-1} \frac{e^{i(\gamma + k)x - i(\gamma' + k)y}}{((\gamma + k)^2 - E)((\gamma' + k)^2 - E)} (2\pi)^{-3},$$

$\gamma, \gamma' \in \Gamma, k \in B$, where

$$g_E(x - y, k) = (2)^{-3} \sum_{\gamma \in \Gamma} \frac{e^{i(\gamma + k)(x - y)}}{(\gamma + k)^2 - E}$$

if $x - y \neq 0$ and

$$g_E(o, k) = \lim_{\omega \to \infty} (2\pi)^{-3} \left[\sum_{\substack{\gamma \in \Gamma \\ |\gamma + k| < \omega}} \frac{1}{|\gamma + k|^2 - E} - 4\pi\omega \right]$$

and $[\quad]^{-1}$ is the inverse of the $n \times n$ matrix

$$\lambda_x \delta_{xy} - g_E(x - y, k), x, y \in X, n = |X|.$$

Furthermore $H_\lambda(k)$ has discrete spectrum with eigenvalues $E_n(k, \lambda)$; the negative eigenvalues $E_n(k, \lambda)$ are the poles of $[\lambda_x \delta_{xy} - g_E(x - y, k)]^{-1}$. From this we get easily the following:

Proposition 7. *Let the Hamiltonian H_λ be as above (point (ii)) and suppose the periodic function λ on $Y = A + X \subset \mathbb{Z}^3$ be such that: $-\infty < \lambda^{\min} \leq \lambda_x \leq \lambda^{\max} \leq \lambda_0 \forall x \in X$ (for the definition of λ_0, see point (i)). Then for any $\tilde{E} \in \sigma(H_\lambda)$, $\tilde{E} < 0$, there exists a function $\bar{\lambda}$ on $\mathbb{Z}^3$, $\bar{\lambda}_j = \bar{\lambda} \forall j \in \mathbb{Z}^3$ such that: $\tilde{E} \in \sigma(H_\lambda)$.*

Proof. By the previous result (point (ii)) we know that $\tilde{E}$ is a pole of $[\lambda_x \delta_{xy} - g_E (x - y, k)]$, $x, y \in X$, i.e. there is one of the eigenvalues $\varepsilon_m^\lambda(\tilde{E}, k)$ of the $n \times n$ matrix

$[\lambda_x \delta_{xy} - g_E(x - y, k)]$ which is zero. Let us denote it by $\varepsilon^\lambda_{m_1}(\tilde{E}, k_1)$. Since by the mini-max-principle $\varepsilon^\lambda_m(\tilde{E}, k)$ are monotone increasing functions of $\{\lambda_x\}_{x \in X}$, in the sense that if $\mu \geqq \lambda$ pointwise in X, then $\varepsilon^\mu_n(\tilde{E}, k) \geqq \varepsilon_n(\tilde{E}, k)$, we get:

$$\varepsilon^{\lambda \min}_1(\tilde{E}, k_1) \leqq \varepsilon^\lambda_{m_1}(\tilde{E}, k_1) \leqq \varepsilon^{\lambda \max}_{m_1}(\tilde{E}, k_1),$$

and thus, by the continuity, there exists a $\bar{\lambda}(m_1, k_1, \tilde{E})$ such that $\varepsilon^\lambda_{m_2}(\tilde{E}, k_1) = 0$, where $\varepsilon^\lambda_m(E, k)$ is the mth eigenvalue of the matrix $[\lambda \delta_{xy} - g_E(x - y, k)]$. This means that $\tilde{E} \in \sigma(H^\lambda(k_2))$, i.e. $\tilde{E} \in \sigma(H^\lambda)$.

We are now in a position to prove the main result.

Proposition 8. *Let the random variables $\{q_i(\omega)\}_{i \in \mathbb{Z}^3}$ be such that supp $P_{q_0} \subset [q_{\min}, q_{\max}]$ with $q_{\max} < \lambda_0$.*
Then

$$(E^{q_{\max}}_1, 0) \cap \Sigma = \varnothing,$$

when $E^{q_{\max}}_1$ is the upper edge of the negative band in the spectrum of $H_{q_{\max}}$ Furthermore if $q_{\min}(q_{\max})$ is the inf (sup) of supp P_{q_0} and if

$$[E^{q_{\min}}_0, E^{q_{\min}}_1] \cap [E^{q_{\max}}_0, E^{q_{\max}}_1] \neq \varnothing,$$

then

$$\Sigma = \sigma(H_{q_{\min}}) \cup \sigma(H_{q_{\max}}).$$

Proof. From Proposition 6 we know that $\Sigma = \overline{\bigcup_{W \in P} \sigma(H_W)}$, and from Proposition 7 we have that any $\tilde{E} \in \sigma(H_W), \tilde{E} < 0, W \in P$ belongs to the spectrum of some $H_{\chi(\tilde{E})}$; hence $E^{q_{\min}}_0 \leqq \tilde{E} \leqq E^{q_{\min}}_1$ since both E^λ_0 and E^λ_1 are monotone increasing functions of λ. From this it follows that

$$(E^{q_{\max}}_1, 0) \cap \sigma(H_W) = \varnothing \quad \forall W \in P,$$

and thus $(E^{q_{\max}}_1, 0) \cap \Sigma = \varnothing$.
If now $q^{\min}(q^{\max})$ is the inf(sup) of supp P_{q_0},
then $\sigma(H_{q_{\min}}) \subset \Sigma$ and the same for $\sigma(H_{q_{\max}})$.
Since $\sigma(H_{q_{\min}}) \cup \sigma(H_{q_{\max}}) = [E^{q_{\min}}_0, E^{q_{\max}}_1] \cup [0, \infty[$,
by the previous discussion we get that

$$\sigma(H_W) \subset \sigma(H_{q_{\min}}) \cup \sigma(H_{q_{\max}}) \quad \forall W \in P,$$

hence $\Sigma \subset \sigma(H_{q_{\min}}) \cup \sigma(H_{q_{\max}})$ and by Proposition 5 we get

$$\Sigma = \sigma(H_{q_{\min}}) \cup \sigma(H_{q_{\max}}). \qquad \square$$

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