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The Phase Transition in the One-Dimensional Ising Model with $1/r^2$ Interaction Energy

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Abstract. We prove the existence of a spontaneous magnetization at low temperature for the one-dimensional Ising Model with $1/r^2$ interaction energy.

1. Introduction, Basic Ideas and Main Results

It has been known for some time that the one-dimensional Ising model exhibits a phase transition when the forces are sufficiently long range. If the interaction energy is given by

$$J(i-j) \equiv J(r) \geqq c\left[\frac{\ln \ln (|r|+3)}{r^2+1}\right],$$

then there is a spontaneous magnetization at low temperature. This result is due to Dyson [2, 4] and was obtained by comparison to a hierarchical model. On the other hand if

$$\lim_{N \to \infty} [\ln(N)]^{-1/2} \sum_{n=1}^{N} J(r) r \to 0,$$

Rogers and Thompson [7] showed that the spontaneous magnetization vanishes for all temperatures. The same result is expected if the exponent 1/2 is replaced by 1. See [3, 8] for other related results.

In this paper we establish a phase transition when $J(r) = 1/r^2$. This is a borderline case which has been discussed by Anderson and Yuval [1] in connection with the Kondo problem. Thouless has also studied this model and predicted a discontinuity in the spontaneous magnetization as a function of temperature—the Thouless

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effect [10]. Simon and Sokal [9] have rigorously established this discontinuity assuming

i) there is a spontaneous magnetization, for $\beta = T^{-1}$ large, and

ii) the spin–spin correlation $\langle \sigma_x \sigma_y \rangle(\beta) - \langle \sigma_x \rangle(\beta)^2$ has a uniform power fall-off for $\beta > \beta_c$.

Some time ago Dyson established the Thouless effect in a hierarchical model [4].

We shall apply an energy–entropy argument similar to the one we developed for the two-dimensional Coulomb gas [5] to establish the existence of a spontaneous magnetization for the $1/r^2$ model at low temperature, $T = \beta^{-1} \ll 1$, thus establishing i) above. The simplest form of this argument is due to Landau and Lifshitz [6]. In order to explain their idea we first set up our notation. Let

$$H_L(\sigma) = \sum_{i<j} |i-j|^{-2}(1 - \sigma_i \sigma_j) \tag{1.1}$$

denote the energy of a configuration, $\sigma = \{\sigma_i\}_{i \in \mathbb{Z}}$, of Ising spins $\sigma_i = \pm 1$. We impose the boundary condition

$$\sigma_i = +1, \quad \text{for } |i| \geqq L,$$

where $2L$ is the length of a finite subsystem, and we shall let L tend to ∞. It is convenient to introduce the lattice $\mathbb{Z}^*$ of nearest neighbor bonds, $b = (i, i+1)$, $i \in \mathbb{Z}$. (Note that $\mathbb{Z}^* \approx \mathbb{Z} + 1/2$ if we identify b by its midpoint.)

Each configuration σ of spins completely specifies a subset $\Gamma \equiv \Gamma(\sigma) \subseteq \mathbb{Z}_L^*$, where $\mathbb{Z}_L^* \equiv \mathbb{Z}^* \cap [-L, L]$, which is the set of spin flips, i.e.

$$b \in \Gamma \quad \text{iff} \quad \tau_b \equiv \sigma_i \sigma_{i+1} = -1.$$

Note that our choice of boundary conditions implies that the cardinality of $\Gamma(\sigma)$ (i.e. the number of spin flips in $\Gamma(\sigma)$) is *even*. Conversely, each even subset $\Gamma \subseteq \mathbb{Z}_L^*$ of spin flips determines a unique configuration $\sigma = \sigma(\Gamma)$ of spins. Subsets of a configuration Γ of spin flips are denoted by $\gamma, \gamma', \gamma_1, \gamma_2, \ldots$. Given some $\gamma \subset \Gamma$, let $b_-(\gamma)$ be the smallest and $b_+(\gamma)$ the largest bond belonging to γ, and let $d(\gamma)$ be the diameter of γ, i.e. the total number of bonds of $\mathbb{Z}^*$ lying between the left endpoint of $b_-(\gamma)$ and the right endpoint of $b_+(\gamma)$. (It is assumed that $\mathbb{Z}^*$ is equipped with its natural order.)

The basic energy–entropy argument may now be described as follows: Consider the elementary configurations, $\Gamma = \{b_-, b_+\} \subset \mathbb{Z}^*$, whose energy is given by

$$H(\Gamma) = 4 \sum_{\substack{i < b_- \\ b_- < j < b_+}} |i-j|^{-2} \geqq C_1 \ln d(\Gamma),$$

for some positive constant C_1. Here, $i < b$ means that i is smaller than or equal to the left endpoint of b, $i > b$ means that i is larger than or equal to the right endpoint of b. [For the reader familiar with [5] we note that $H(\Gamma)$ is proportional to the electrostatic energy, with respect to the *two-dimensional* Coulomb potential, of a dipole of length $d(\Gamma)$ in the plane.] The entropy of the class of elementary configurations Γ with diameter $d(\Gamma) = l$ is $l-1$, because there are $l-1$ such configurations for which $\sigma_0 = -1$. In the approximation in which only elementary configurations are included one concludes that for $C_1\beta \geqq 3$

$$\tfrac{1}{2}\langle 1-\sigma_0\rangle_L^+(\beta) \leqq \sum_{l=2}^{L} e^{-C_1\beta \ln l}(l-1) < \tfrac{1}{2},$$

uniformly in L, hence

$$\langle \sigma_0\rangle^+(\beta) \equiv \lim_{L\to\infty} \langle \sigma_0\rangle_L^+(\beta) > 0.$$

[Here $\langle\cdot\rangle_L^+(\beta)$ denotes the expectation in the equilibrium state of the model at inverse temperature β with boundary conditions $\sigma_i = +1$, for $|i| \geqq L$. The limit $L\to\infty$ exists, by correlation inequalities [11].]

The above argument is similar to the Peierls argument for the two-dimensional Ising model. To make it rigorous, we must consider general configurations of spin flips. This makes our rigorous energy–entropy arguments somewhat involved.

We now establish some further notation and definitions. Each configuration Γ of spin flips is partitioned into disjoint subsets $\gamma_1, \gamma_2, \ldots$, called "*primitive*" (or "*connected*") *contours* in such a way that the following *Condition D* (*D* for "distance") holds:

a) The cardinality of each γ_α is even, $\bigcup_{\alpha \geqq 1} \gamma_\alpha = \Gamma$, and

$$\gamma_\alpha \cap \gamma_{\alpha'} = \emptyset, \quad \alpha \neq \alpha'.$$

b) dist $(\gamma_\alpha, \gamma_{\alpha'}) \geqq M[\min(d(\gamma_\alpha), d(\gamma_{\alpha'}))]^{3/2}$, for $\alpha \neq \alpha'$. (1.2)

c) If γ is a subset of some γ_α (called a "constituent" of γ_α) satisfying the inequality

$$\text{dist}(\gamma, \gamma_\alpha \sim \gamma) \geqq 2M\, d(\gamma)^{3/2} \tag{1.3}$$

then card (γ) is odd, [we say that γ is *charged*], for all α.

In b) and c) of Condition D, M is a constant independent of Γ and γ, to be chosen later.

In order to establish the existence of a partition of each configuration Γ into primitive contours $\{\gamma_1, \gamma_2, \ldots\}$ satisfying Condition D, we choose the *finest* partition $\{\gamma_\alpha\}_{\alpha=1,2,\ldots}$ of Γ satisfying a) and b). Then c) is automatically fulfilled (see also Sect. 2 of [5]). The uniqueness of $\{\gamma_\alpha\}$ will not concern us—we arbitrarily assign to each Γ an arbitrary, but fixed partition satisfying Condition D. We briefly comment on the construction of $\{\gamma_\alpha\}$ in the appendix [For readers familiar with [5] we note that the γ_α correspond to the neutral multipoles, or molecules, ρ, introduced in Sect. 2 of [5]. Charges in the Ising model studied here are defined modulo 2-even, odd. Thus, each γ_α can be interpreted as a neutral molecule of spin flips.]

Condition b) ensures that neutral molecules, γ_α, are far separated, and hence their total energy is nearly additive, i.e.

$$H(\gamma_\alpha \cup \gamma_{\alpha'}) \approx H(\gamma_\alpha) + H(\gamma_{\alpha'}), \quad \alpha \neq \alpha'.$$

[Recall that in the nearest neighbor Ising model the energies of disjoint contours are exactly additive.] In Sect. 4 we show

Theorem A. *Let $\Gamma \subset \mathbb{Z}_L^*$ be an arbitrary configuration of spin flips, and let γ be* *a primitive contour of* Γ. *Then*

$$\begin{aligned}\delta H(\Gamma;\Gamma\sim\gamma) &\equiv H(\Gamma)-H(\Gamma\sim\gamma)\\ &\geqq H(\gamma)(1-\text{const}\,M^{-1}(\ln M)^3),\end{aligned} \tag{1.4}$$

for M sufficiently large.

Property c) in Condition D is our primitivity (or connectivity) condition and will be crucial in the energy estimates, (i.e. in the proofs of Theorem B, below, and Theorem 2.2).

Now we estimate the probability that $\sigma_0=-1$ in terms of our primitive contours:

$$\tfrac{1}{2}\langle 1-\sigma_0\rangle_L^+=\frac{\sum_{\Gamma} e^{-\beta H_L(\Gamma)}\chi_0(\Gamma)}{\sum_{\Gamma} e^{-\beta H_L(\Gamma)}}, \tag{1.5}$$

where Γ ranges over all allowed configurations, and $\chi_0(\Gamma)=0$ if $\sigma_0(\Gamma)=1, \chi_0(\Gamma)=1$ if $\sigma_0(\Gamma)=-1$. Here $\sigma_0(\Gamma)$ is the value of the spin σ_0 in the configuration Γ. Note that if $\{\gamma_\alpha\}$ are the primitive contours of Γ then $\chi_0(\Gamma)=0$ unless there is some contour γ_α separating 0 from $\pm L$. Given a set γ of spin flips, let $I(\gamma)\subset\mathbb{R}$ denote the interval spanned by the endpoints of γ. Thus $\chi_0(\Gamma)=0$, unless $0\in I(\gamma_\alpha)$, for some α. Let $\alpha=1$ label the primitive contour of minimal diameter enclosing 0. Then by Theorem A

$$\begin{aligned}\tfrac{1}{2}\langle 1-\sigma_0\rangle_L^+(\beta) &\leqq \frac{\sum\limits_{\substack{\Gamma\\ 0\in I(\gamma_1)}} e^{-(\beta/2)H(\gamma_1)}e^{-\beta H(\Gamma\sim\gamma_1)}}{\sum\limits_{\Gamma} e^{-\beta H(\Gamma)}}\\ &\leqq \sum_{\substack{\gamma_1\\ 0\in I(\gamma_1)}} e^{-(\beta/2)H(\gamma_1)}\end{aligned} \tag{1.6}$$

if M is chosen sufficiently large; see (1.4).

The last inequality follows because, given any $\Gamma, \Gamma\sim\gamma_1$ also appears in the denominator. To estimate the sum over γ_1 we need rather involved energy–entropy arguments similar to those in [5].

In order to estimate the energy and entropy of primitive contours γ, we introduce a sequence of length scales, $2^n, n=0,1,2,\ldots$. Let

$$n_0=[\ln_2 d(\gamma)]+1, \qquad (\ln_2(\cdot)\equiv\log_{\text{base }2}(\cdot)),$$

where $[x]$ is the integer part of a non-negative number x. For every $n\leqq n_0$, let $N_n(\gamma)$ be the minimum number of open intervals of length 2^n needed to cover γ. For $n>n_0$ we set $N_n(\gamma)=0$. We define

$$N(\gamma)\equiv\sum_{n=0}^{\infty} N_n(\gamma). \tag{1.7}$$

The quantity $N(\gamma)$ measures both the energy of a primitive contour γ and the entropy of the family of all primitive contours, γ, such that $0\in I(\gamma)$ and $N(\gamma)$ takes some given value.

Our principal estimates on the energy and entropy of primitive contours may now be stated as follows.

Theorem B. *Let* $\{\gamma_\alpha\}_{\alpha=1,2,3,\ldots}$ *be a partitioning of a configuration* Γ *of spin flips into primitive contours satisfying Condition D. There exists a constant* $\varepsilon > 0$ *independent of* Γ *such that for* M *sufficiently large*

$$H(\Gamma) - H(\Gamma \sim \gamma_\alpha) \geqq \tfrac{1}{2} H(\gamma_\alpha) \geqq \varepsilon N(\gamma_\alpha), \tag{1.8}$$

for every α.

Theorem C. *Let* $\mathscr{C}_L(R)$ *be the collection of subsets* $\gamma \subseteq \mathbb{Z}_L^*$ *such that* $N(\gamma) \leqq R$, $R = 1, 2, 3, \ldots$, *and* $0 \in I(\gamma)$. *There exists a constant* C_2 *independent of* R *and* L *such that*

$$\operatorname{card} \mathscr{C}_L(R) \leqq e^{C_2 R}. \tag{1.9}$$

Theorems B and C permit us to estimate the sum on the right side of (1.6) uniformly in L:

$$\begin{aligned} \tfrac{1}{2}\langle 1 - \sigma_0 \rangle_L^+ &\leqq \sum_{0 \in I(\gamma_1)} e^{-\beta \varepsilon N(\gamma_1)} \\ &\leqq \sum_{R > 1} e^{-\beta \varepsilon R} e^{C_2 (R+1)} \\ &\ll 1, \quad \text{for } \beta \gg 1, \end{aligned}$$

uniformly in L. Thus we have proved

$$\langle \sigma_0 \rangle^+(\beta) \equiv m \equiv \lim_{L \to \infty} \langle \sigma_0 \rangle_L^+(\beta) > 0, \tag{1.10}$$

for $\beta \gg 1$.

Next, we show that $m = 0$, for small β. This actually follows from the results in [12]. Here, we sketch a proof based on Simon's inequality [13] in a form given in [14]: Let I_o denote the interval $[-l_o, l_o]$, $l_o = 0, 1, 2, 3, \ldots$. Let $j \notin I_o$. Then in the thermodynamic limit ($L = \infty$, the existence of which follows from [11])

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \leqq \beta \sum_{\substack{i \in I_o \\ k \notin I_o}} \langle \sigma_0 \sigma_i \rangle^0(\beta) |i - k|^{-2} \langle \sigma_k \sigma_j \rangle^+(\beta), \tag{1.11}$$

where $\langle \cdot \rangle^0(\beta)$ is the equilibrium state at inverse temperature β with boundary conditions $\sigma_n = 0$ when $|n| > l_o$. Since $\langle \sigma_0 \sigma_i \rangle^0(\beta) \leqq 1$, (1.11) implies that for sufficiently small β

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \to 0, \quad \text{as } |j| \to \infty; \tag{1.12}$$

see [13]. (Choose e.g. $I_o = \{0\}$. Then (1.12) holds if $\beta < \left(\sum_{i \neq 0} |i|^{-2} \right)^{-1}$. For more details we refer to [13] and Sect. 3 of [14]). By the Griffiths inequality [11]

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \geqq \tanh(\beta |j|^{-2}) \approx \beta |j|^{-2} \tag{1.13}$$

Next, let β_c be the supremum over all those β for which

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \leqq \text{const}\, |j|^{-\varepsilon}, \tag{1.14}$$

for some $\varepsilon > 0$. Let $\beta < \beta_c$. In (1.11) we may choose $I_o = I_o(j) \equiv \left[-\frac{|j|}{2}, \frac{|j|}{2} \right]$. It then follows from (1.11) and (1.14) by iteration that

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \leqq C(\beta)|j|^{-2}, \tag{1.15}$$

for some finite constant $C(\beta)$. Thus, for $\beta < \beta_c$

$$\langle \sigma_0 \sigma_j \rangle^+(\beta) \approx |j|^{-2}, \quad \text{as } |j| \to \infty.$$

From Newman's Gaussian inequality (e.g. [14], and refs. given there) it then follows that all connected correlations fall off at least like $1/[\text{distance}]^2$ if $\beta < \beta_c$.

If $\beta \gg 1$ one cannot use these arguments, because the correlations in (1.11) are not connected. It is conceivable, however, that our definition of primitive contours and Theorems A through C would permit one to prove convergence of a low temperature expansion for connected correlations if $\beta \gg 1$ and $M = M(\beta)$ is chosen conveniently. We pose this as an open problem.

The remainder of our paper is organized as follows. In Sect. 2 we prove Theorem C. The proof is quite easy in comparison to its higher dimensional analogue [5]. This is because we can exploit the natural order of $\mathbb{Z}^*$. In Sect. 2 we also introduce a new measure of γ, $N'(\gamma)$, which counts the number of *far separated, odd* (i.e. "charged") constituents of γ, and we show that if γ satisfies Condition D, c), see (1.3), then

$$H(\gamma) \geqq \tfrac{1}{2} N'(\gamma).$$

The following section is devoted to proving that N and N' are equivalent, i.e. that there exists a constant C such that

$$CN'(\gamma) \geqq N(\gamma) \geqq N'(\gamma).$$

These two inequalities and Theorem A yield Theorem B.

In the final section, we show that the interaction energy between a primitive contour and the remaining contours of an arbitrary configuration of spin flips is relatively small. Thereby, we establish Theorem A. As shown above, see (1.6) and (1.8)–(1.10), this will complete our proof of the existence of a phase transition and spontaneous magnetization at low temperature.

2. Entropy Estimate and a Lower Bound on $H(\gamma)$

An arbitrary collection of spin flips $\gamma \subseteq \mathbb{Z}_L^*$ may be specified by an increasing sequence of integers $\{i_k\}_{k=1,2,\ldots}, i_k < i_{k+1}$. We define the logarithmic length, $L(\gamma)$, of γ by

$$L(\gamma) = \sum_{k=1,2,\ldots} \{[ln_2(i_{k+1} - i_k)] + 1\}. \tag{2.1}$$

Lemma 2.1. *For any collection of spin flips* $\gamma \subseteq \mathbb{Z}_L^*$

$$N(\gamma) \geqq L(\gamma), \tag{2.2}$$

where $N(\gamma)$ *is defined by (1.7).*

Proof. We define

$$l_k = [\ln_2(i_{k+1} - i_k)], k = 1, 2, \dots. \tag{2.3}$$

Let $\mathcal{I}_l(\gamma)$ be a minimal collection of open intervals of length 2^l needed to cover γ. By minimal, we mean that $\mathcal{I}_l(\gamma)$ contains the smallest possible number of intervals, i.e.

$$\operatorname{card} \mathcal{I}_l(\gamma) = N_l(\gamma),$$

see (1.7). For every $l = 0, \dots, l_k, \mathcal{I}_l(\gamma)$ necessarily contains an interval covering i_k which does not cover i_{k+1}. Lemma 2.1 follows by summation over k. □

Proof of Theorem C. Clearly every γ is determined by fixing i_1 and specifying $\ln_2(i_{k+1} - i_k)$. By (2.1)–(2.3) and the assumption $N(\gamma) \leqq R$, we have

$$\Sigma l_k + 1 \leqq L(\gamma) \leqq R. \tag{2.4}$$

There are less than 2^{R+1} ways of specifying (in order) integers $l_k + 1 \geqq 1$ which satisfy (2.4). [In fact it is easy to see that there are precisely 2^{L-1} way of choosing a sequence of integers $n_k \geqq 1$ such that $\Sigma n_k = L$.] Furthermore, there are less than 2^{l_k+1} integers $z \geqq 1$ such that $[\ln_2 z] = l_k$, since

$$2^{l_k} \leqq z \leqq 2^{l_k+1}.$$

Thus we conclude that there are less than

$$2^{R+1} \cdot 2^{\Sigma_k l_k + 1} \leqq e^{(2R+1)\ln 2}$$

collections of spin flips γ with $L(\gamma) \leqq N(\gamma) \leqq R$ and with i_1 fixed. If we require that $0 \in I(\gamma)$ then there are fewer than $d(\gamma)$ possible choices for i_1 and thus Theorem B follows after noting that

$$d(\gamma) \leqq 2^{L(\gamma)} \leqq e^{R \cdot \ln 2}. \quad \square$$

Now, we turn to the definition of $N'(\gamma)$. Let $\mathcal{I}'_n(\gamma)$ be the subcollection of intervals, I', of length 2^n contained in $\mathcal{I}_n(\gamma)$ (defined in the proof of Lemma 2.1) which are isolated in the sense that

$$\operatorname{dist}(I', I) \geqq 2M2^{3n/2} \equiv 2^{b+3n/2}, \tag{2.5}$$

for all $I \in \mathcal{I}_n(\gamma), I \neq I'$. If $\mathcal{I}_n(\gamma)$ consists of a single interval we set $\mathcal{I}'_n(\gamma) = \emptyset$. We define

$$N'(\gamma) \equiv |\gamma| + \sum_{n \geqq 1} |\mathcal{I}'_n(\gamma)|. \tag{2.6}$$

Here $|S|$ denotes the cardinality of the set S. Let Γ be an arbitrary configuration of spin flips, and let $\gamma \subset \Gamma$ be an arbitrary primitive contour in a partition of Γ into primitive contours satisfying Condition D, Sect. 1. Then by (1.3) $I' \cap \gamma$ is charged for any $I' \in \mathscr{I}'_n(\gamma)$. More precisely, $|I' \cap \gamma|$ is odd. Thus $|\mathscr{I}'_n(\gamma)|$ is a lower bound for the number of charged blocks of spin flips (i.e. ones containing an odd number of spin flips) on a scale 2^n. The following theorem shows that $N'(\gamma)$ is a natural measure of the energy, $H(\gamma)$, of γ.

Theorem 2.2. *If Γ satisfies (1.3) (Condition D, c)) then*

$$H(\gamma) \geqq \tfrac{1}{2} N'(\gamma). \tag{2.7}$$

Proof. Note that for any configuration Γ of spin flips, $\sigma_i \sigma_j = -1$ if and only if

$$|[i,j] \cap \Gamma| \quad \text{is odd.} \tag{2.8}$$

Let $\chi_\Gamma(i,j) = 1$ if (2.8) holds and $\chi_\Gamma(i,j) = 0$ otherwise. Then

$$H(\Gamma) = 2 \sum_{i<j} |i-j|^{-2} \chi_\Gamma(i,j). \tag{2.9}$$

Now, let Γ be given by γ. If in (2.9) we consider the subsum for which $|i-j| = 1$ we have

$$\sum_{\substack{|i-j|=1 \\ i<j}} |i-j|^{-2} \chi_\gamma(i,j) = |\gamma|. \tag{2.10}$$

Next, let I'_n be an interval in $\mathscr{I}'_n(\gamma)$ and I_{n+1} an interval in $\mathscr{I}_{n+1}(\gamma)$ covering I'_n. We may then choose I_{n+1} such that I'_n and I_{n+1} are centered at the same point which we may for convenience suppose to be the origin. Let

$$D_n \equiv D_n(I'_n) \equiv \{i,j | i < 0 < j, i,j \in I_{n+1} \sim I'_n\}.$$

If i and j belong to D_n then, by the definition of $\mathscr{I}'_n(\gamma)$, $|[i,j] \cap \gamma|$ is odd. Thus $\chi_\gamma(i,j) = 1$. It is then easy to show that

$$2 \sum_{\substack{i,j \in D_n \\ i<j}} |i-j|^{-2} \chi_\gamma(i,j) = 2 \sum_{\substack{i,j \in D_n \\ i<j}} |i-j|^{-2} \geqq \tfrac{1}{2}, \tag{2.11}$$

for each D_n. It follows from the definition of $\mathscr{I}'_n(\gamma)$ and D_n that the sets

$$D_n(I'), I' \in \mathscr{I}'_n(\gamma), \quad n = 1, 2, 3, \ldots$$

are *disjoint*. By (2.10) and (2.11)

$$\begin{aligned} H(\gamma) &\geqq 2|\gamma| + 2 \sum_{n \geqq 1} \sum_{I' \in I'_n(\gamma)} \sum_{\substack{i,j \in D_n(I') \\ i<j}} |i-j|^{-2} \\ &\geqq \tfrac{1}{2} N'(\gamma). \quad \square \end{aligned}$$

3. The Equivalence of N and N'

Theorem 3.1. *There is a constant C independent of M such that*

$$N'(\gamma) \leqq N(\gamma) \leqq C(\ln M)^2 N'(\gamma), \tag{3.1}$$

for any finite subset $\gamma \subset \mathbb{Z}^*$.

Proof. Define $\mathscr{I}''_n \equiv \mathscr{I}_n \sim \mathscr{I}'_n$, and set

$$f(n) \equiv [\tfrac{2}{3}(n-b-2)], \tag{3.2}$$

where $2M \equiv 2^b$. We claim that if n is such that $2^{f(n)} < d(\gamma)$—so that $\mathscr{I}_{f(n)}(\gamma)$ contains at least two intervals—then

$$\begin{aligned} N_n(\gamma) &\leqq \tfrac{1}{2}|\mathscr{I}''_{f(n)}(\gamma)| + N'_{f(n)}(\gamma) \\ &\leqq \tfrac{1}{2} N_{f(n)}(\gamma) + N'_{f(n)}(\gamma), \end{aligned} \tag{3.3}$$

where $N_n(\gamma) \equiv |\mathscr{I}_n(\gamma)|$, $N'_n(\gamma) \equiv |\mathscr{I}'_n(\gamma)|$. We note that if n is such that $2^{f(n)} \geqq d(\gamma)$ then, by the definition of $\mathscr{I}_n(\gamma), N_n(\gamma) = 0$, so (3.3) holds trivially. We shall iterate (3.3) to obtain (3.1). To establish our claim, let I_1 be an interval in $\mathscr{I}''_{f(n)}(\gamma)$. By the definition of $\mathscr{I}''_{f(n)}$ there exists an interval I_2 in $\mathscr{I}''_{f(n)}(\gamma)$ such that

$$\begin{aligned} \operatorname{dist}(I_1, I_2) &< 2M 2^{(3/2)f(n)} \\ &\leqq 2^b 2^{n-b-2} = 2^{n-2}. \end{aligned}$$

Hence I_1 and I_2 can be covered by a single interval of length 2^n. Also if I_1, I_2 and I_3 belong to $\mathscr{I}''_{f(n)}(\gamma)$ and are such that $\operatorname{dist}(I_i, I_2) \leqq 2M2^{(3/2)f(n)}$, $i = 1, 3$ then $I_1 \cup I_2 \cup I_3$ can be covered by a single interval of length 2^n, provided M is large enough. Thus at most $\frac{1}{2}|\mathscr{I}''_{f(n)}(\gamma)|$ intervals of size 2^n suffice to cover all the intervals in $\mathscr{I}''_{f(n)}(\gamma)$, and (3.3) follows.

Let $\delta \equiv b - 2$. Clearly, (3.3) can be applied only if

$$f(n) = [\tfrac{2}{3}(n-\delta)] \geqq 0, \quad \text{i.e. } n \geqq \delta. \tag{3.4}$$

For each n we now iterate (3.3) $l(n)$ times, where $l \equiv l(n)$ is the maximal number for which

$$f^{l(n)}(n) > 0.$$

Here f^m denotes the m-fold composition of f with itself. This yields

$$\begin{aligned} N_n(\gamma) &\leqq \sum_{m=1}^{l} 2^{-m+1} N'_{f^m(n)}(\gamma) + 2^{-l} N_{f^l(n)}(\gamma) \\ &\leqq \sum_{m=1}^{l} 2^{-m+1} N'_{f^m(n)}(\gamma) + 2^{-l}|\gamma|. \end{aligned} \tag{3.5}$$

Here we have used the fact that $|\gamma| \geqq N_n(\gamma)$, for all n. Now, we make two elementary assertions which are easily checked (see Sect. 3 of [5] for details):

1)
$$l(n) \geqq \begin{cases} 0, 0 \leqq n \leqq n_0, \\ [(\ln_2(3/2))^{-1} \cdot \ln_2(n/n_0)], \text{ otherwise,} \end{cases} \tag{3.6}$$

where

$$n_0 \equiv 2(\tfrac{3}{2} + \delta) \leqq \text{const} \ln M.$$

2) Let

$$S_{m,j} \equiv \{n | f^m(n) = j\}.$$

Then

$$|S_{m,j}| < 6(\tfrac{3}{2})^m. \tag{3.7}$$

By (3.5)

$$\begin{aligned} N(\gamma) &= \sum_{n=0}^{\infty} N_n(\gamma) \\ &\leqq \sum_{n=0}^{\infty} \left\{ 2^{-l(n)}|\gamma| + \sum_{m=1}^{l(n)} 2^{-m+1} N'_{f^m(n)}(\gamma) \right\} \\ &\leqq E|\gamma| + F \sum_{j=1}^{\infty} N'_j(\gamma) \leqq (E+F) N'(\gamma), \end{aligned}$$

where

$$E = \sum_{n=0}^{\infty} 2^{-l(n)} \leqq n_0 + \sum_n \left(\frac{n_0}{n} \right)^p \leqq \text{const} (\ln M)^2,$$

where $1 < p \equiv (\ln_2 \frac{3}{2})^{-1} < 2$. Here, we have used (3.6). The bound on F follows by summing over n with $f^m(n) = j$ fixed and using (3.7), i.e.

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=1}^{l(n)} 2^{-m+1} N'_{f^m(n)}(\gamma) &\leqq \sum_{j=0}^{\infty} \left(\sum_{m=1}^{\infty} 2^{-m+1} |S_{m,j}| \right) N'_j(\gamma) \\ &\leqq 36 \sum_{j=0}^{\infty} N'(\gamma). \quad \square \end{aligned}$$

Remark. Theorem 3.1, (3.1) and Theorem 2.2, (2.7) clearly imply the lower bound on $H(\gamma_\alpha)$ stated in Theorem B, (1.8), with $\varepsilon = \text{const} (\ln M)^{-2}$.

4. Interaction Energy: The Proof of Theorem A

Let Γ be an arbitrary even configuration of spin flips, and let $\{\gamma, \gamma_2, \gamma_3, \ldots\}$ be a partition of Γ into primitive contours satisfying Condition *D*, Sect. 1. We set

$$\Gamma' = \bigcup_{\alpha \geqq 2} \gamma_\alpha$$

and specify γ by the positions $\{i_k\}_{k=1,2,\ldots}$ of all spin flips contained in γ, where the sites i_k belong to $\mathbb{Z}^*$, and $i_k < i_{k+1}$, $k = 1, 2, 3, \ldots$.

We define $W(\gamma, \Gamma')$ to be $(-1) \times$ interaction energy between γ and Γ' which is given by

$$-W(\gamma, \Gamma') = H(\Gamma) - H(\Gamma') - H(\gamma). \tag{4.1}$$

Using (2.9) and (4.1) we see that

$$\begin{aligned} W(\gamma, \Gamma') &= 2 \sum_{i<j} |i-j|^{-2} \{\chi_{\Gamma'}(i,j) + \chi_\gamma(i,j) - \chi_\Gamma(i,j)\} \\ &= 4 \sum_{i<j} |i-j|^{-2} \chi_\gamma(i,j) \chi_{\Gamma'}(i,j). \end{aligned} \tag{4.2}$$

Theorem 4.1. *If $\Gamma = \gamma \cup \gamma_2 \cup \gamma_3 \cup \ldots$ satisfies Condition D, Sect. 1, then there is a* *constant C_3 independent of M such that*

$$0 < W(\gamma, \Gamma') \leqq C_3 M^{-1} \ln M \cdot L(\gamma), \tag{4.3}$$

where $L(\gamma)$ is the logarithmic length of γ defined in (2.1), and M is the constant appearing in Condition D, (1.2) and (1.3).

Remark. By (2.2) and Theorem 3.1,

$$L(\gamma) \leqq N(\gamma) \leqq C(\ln M)^2 N'(\gamma).$$

Thus by Theorem 2.2

$$C_3 M^{-1} \ln M \cdot L(\gamma) \leqq 2C_3 C M^{-1} (\ln M)^3 H(\gamma).$$

Using (4.1) and (4.3) we conclude that

$$H(\Gamma') + H(\gamma) - H(\Gamma) \leqq \text{const}\, M^{-1} (\ln M)^3 H(\gamma),$$

i.e.

$$\begin{aligned} \delta H(\gamma \cup \Gamma'; \Gamma') &\equiv H(\Gamma) - H(\Gamma') \\ &\geqq H(\gamma)(1 - \text{const}\, M^{-1} (\ln M)^3). \end{aligned}$$

Hence Theorem A is proven, and this yields the upper bound on $(\frac{1}{2})H(\gamma_i)$ in Theorem B, (1.8), provided M is large enough.

Proof of Theorem 4.1. Let I_k denote the interval $[i_k, i_{k+1}]$, where $\{i_k\}_{k=1,2,3,\ldots}$ defines γ. Note that by (1.2) if $\gamma_\alpha \cap I_k \neq \emptyset$, for some $\alpha \geqq 2$ then $I(\gamma_\alpha) \subset I_k$; [see Condition D, b). We recall that $I(\gamma_\alpha) \subset \mathbb{R}$ is the interval spanned by the endpoints of γ_α]. In order to bound W we define for each k three sets of pairs (i,j) of sites, A_k, B_k and C_k, where

$$\begin{aligned} A_k &\equiv \{(i,j) \mid i \in I(\gamma_\alpha), \text{ for some } \gamma_\alpha \text{ such that } I(\gamma_\alpha) \subset I_k, \text{ and } j \notin I_k\}, \\ B_k &\equiv \{(i,j) \mid i \in I_k \text{ and } j \in I(\gamma_\alpha), \text{ for some } \gamma_\alpha \text{ such that } I(\gamma_\alpha) \subset I_k^c\}, \\ C_k &\equiv \{(i,j) \mid i \in I_k \text{ and } j \in I(\gamma_\alpha), \text{ with} \\ &\qquad \text{dist}(j, I_k) \geqq M d(\gamma)^{3/2}, \text{ for some } \gamma_\alpha \text{ such that } I(\gamma_\alpha) \supset I_k\}. \end{aligned} \tag{4.4}$$

[The sets C_k deal with the events where $I(\gamma_\alpha) \supset I_k$. Hence by (1.2) $d(\gamma_\alpha) \geqq M d(\gamma)^{3/2}$ and $\text{dist}(\gamma_\alpha, \gamma) \geqq M d(\gamma)^{3/2}$.]

We define

$$\chi^*_{\gamma,\Gamma'} \equiv \sum_k (\chi_{A_k} + \chi_{B_k} + \chi_{C_k}),$$

where χ_X is the characteristic function of the corresponding set defined above, $X = A_k, B_k, C_k$.

Now, we claim that

$$\chi_\gamma(i,j) \chi_{\Gamma'}(i,j) \leqq \chi^*_{\gamma,\Gamma'}(i,j) + \chi^*_{\gamma,\Gamma'}(j,i). \tag{4.5}$$

Clearly the left side of (4.5) vanishes if both i and j belong to I_k, for some k, since then $[i,j] \cap \gamma = \emptyset$ which is an even set. Similarly if both i and j are contained in the complement of $I(\gamma)$ the left side of (4.5) vanishes. Thus we may suppose that $i \in I_k$, for some k, and $j \notin I_k$. Now, suppose that the right side of (4.5) vanishes. Then the conditions

$$i \notin I(\gamma_\alpha), \quad \text{for all } \gamma_\alpha \subset I_k,$$

and

$$j \notin I(\gamma_\alpha), \quad \text{for all } \gamma_\alpha \subset I_k^c,$$

$\alpha = 2, 3, \ldots$, and

$$\operatorname{dist}(j, I_k) < Md(\gamma)^{3/2} \tag{4.6}$$

must all be fulfilled simultaneously. We now observe that if $\gamma_\alpha \cap [i,j] \neq \emptyset$ then (4.6) and (1.2) imply $\gamma_\alpha \subset [i,j]$. Thus we conclude that $|\Gamma' \cap [i,j]|$ is even, hence the left side of (4.5) vanishes, and our claim is established.

In order to prove (4.3) it suffices therefore to show that

$$\sum_{i<j} |i-j|^{-2} \chi_{X_k}(i,j) \leqq \text{const}\, M^{-1} \ln M \cdot \{[\ln_2(i_{k+1} - i_k)] + 1\}, \tag{4.7}$$

for $X_k = A_k, B_k, C_k$ and all k. For convenience suppose $i_k = 0$, $i_{k+1} = l$. First, we consider the case where $X_k = C_k$. We bound the sum over i on the left side of (4.7) by $d(\gamma)$ times the maximum over $i \in I_k$ which is less than

$$d(\gamma) \left(\sum_{j:|j| \geqq Md(\gamma)^{3/2}} |j|^{-2} \right) \leqq \frac{\text{const}}{M}. \tag{4.8}$$

For the case $X_k = A_k$ we define U_r to be the union of all intervals $I(\gamma_\alpha)$ such that

$$\gamma_\alpha \subset I_k \text{ and } 2^r < d(\gamma_\alpha) \leqq 2^{r+1}.$$

By (1.2) (Condition D, b)) such intervals are sparse:

$$\begin{aligned} &\operatorname{dist}[\{0,\ l\}, I(\gamma_\alpha)] \geqq M 2^{3r/2}, \\ &\operatorname{dist}[I(\gamma_\alpha), I(\gamma_{\alpha'})] \geqq M 2^{3r/2}, \end{aligned} \tag{4.9}$$

for $\alpha \neq \alpha'$. Using these inequalities, we can bound the left side of (4.7) by

$$\begin{aligned} 2 \sum_r \sum_{j<0<i<l} |i-j|^{-2} \chi(\{i | i \in U_r\}) &\leqq \text{const} \sum_r \sum_{0<i<l} |i|^{-1} \chi(\{i | i \in U_r\}) \\ &\leqq \text{const}\, M^{-1} \ln l \sum_r 2^{r+1} 2^{-3r/2} \\ &\leqq \text{const}\, M^{-1} \ln l. \end{aligned} \tag{4.10}$$

The factor of 2 in the first line of (4.10) takes care of a similar sum ranging over $0 < i < l < j$.

Finally, we consider the case where $X_k = B_k$. The left side of (4.7) is then bounded by a sum of two terms, denoted by Σ_1 and Σ_2, where Σ_1 is the sum over all $j \in I(\gamma_\alpha)$, for all γ_α for which $\operatorname{dist}(I(\gamma_\alpha), [0, l]) \geqq Ml$, and Σ_2 is the sum over all $j \in I(\gamma_\alpha)$, for all γ_α for which $\operatorname{dist}(I(\gamma_\alpha), [0, l]) < Ml$. Thus

$$\Sigma_1 \leqq 2 \sum_{\substack{0<i<l \\ j \leqq -Ml}} |i-j|^{-2} \leqq \text{const}\, M^{-1}. \tag{4.11}$$

Next, we bound Σ_2. Let U_r' be the union of all intervals $I(\gamma_\alpha)$ such that

$$I(\gamma_\alpha) \subset [-Ml, -M2^{3r/2}] \quad \text{and} \quad 2^r \leqq d(\gamma_\alpha) \leqq 2^{r+1},$$

where $r \leqq [\frac{2}{3}\ln_2 l]$. Then Σ_2 is bounded by

$$2\sum_r \sum_{-Ml<j<0<i<l} |i-j|^{-2}\chi(\{j|j\in U_r'\}) \leqq \text{const}\, M^{-1}\ln(Ml)\sum_r 2^{r+1}2^{-3r/2}$$
$$\leqq \text{const}\, M^{-1}\ln M(\ln l+1). \tag{4.12}$$

The argument leading to this bound is very similar to the one used in (4.10).

Inequality (4.7) follows from (4.8), (4.10), (4.11) and (4.12). With (4.5) this completes the proof of Theorem 4.1. □

Appendix

In this appendix, we sketch the construction of a partition of an arbitrary, even configuration $\Gamma \subseteq \mathbb{Z}_L^*$ of spin flips into primitive contours $\{\gamma_\alpha\}_{\alpha=1,2,3,\ldots}$ in such a way that Condition D, Sect. 1, is satisfied. The construction proceeds inductively over a sequence of length scales $2^n, n=0,1,2,\ldots$.

On scale 2^0 we first group adjacent spin flips (i.e., ones separated by a distance of 2^0) in pairs, in an arbitrary way. This yields a partition of Γ into subsets $\{\varphi_\mu^0\}_{\mu=1,2,3,\ldots}$, where each φ_μ^0 consists of a single spin flip or a nearest neighbor pair of spin flips. Next, we regroup adjacent subsets, $\varphi_\mu^0, \varphi_{\mu'}^0$, (i.e., $\text{dist}(\varphi_\mu^0, \varphi_{\mu'}^0) = 2^0$) in pairs, in an arbitrary way. For finite L, finitely many sweeps of pairing operations suffice to provide us with a partition of Γ into subsets $\{\gamma_\alpha^0\}_{\alpha=1,2,3,\ldots}$ with the property that $\text{dist}(\gamma_\alpha^0, \gamma_{\alpha'}^0) > 2^0$, for $\alpha \neq \alpha'$. For every γ_α^0 we define

$$\overline{\gamma_\alpha^0} = \{j|\text{dist}(j, \gamma_\alpha^0) < M\, d(\gamma_\alpha^0)^{3/2}\}.$$

We define

$$\mathscr{P}^{00} = \{\gamma_\alpha^0 \,|\, |\gamma_\alpha^0| \text{ is even, and } \overline{\gamma_\alpha^0} \cap \gamma_{\alpha'}^0 = \emptyset, \quad \text{for } \alpha \neq \alpha'\},$$

and inductively,

$$\mathscr{P}^{0n} = \left\{\gamma_\alpha^0 \,|\, |\gamma_\alpha^0| \text{ is even, } \gamma_\alpha^0 \in \left(\Gamma \sim \bigcup_{l=0}^{n-1} \mathscr{P}^{0l}\right), \overline{\gamma_\alpha^0} \cap \gamma_{\alpha'}^0 = \emptyset,\right.$$
$$\left.\text{for } \gamma_{\alpha'}^0 \in \left(\Gamma \sim \bigcup_{l=0}^{n-1} \mathscr{P}^{0l}\right), \alpha \neq \alpha'\right\}.$$

Finally, we set

$$\mathscr{P}_0 \equiv \mathscr{P}^0 = \bigcup_{n=0}^{\infty} \mathscr{P}^{0n}. \tag{A.1}$$

It is easy to check that Condition D is satisfied for $\mathscr{P}^0$ and that $\text{dist}(\gamma, \gamma') \geqq Md(\gamma)^{3/2}$, for all $\gamma \in \mathscr{P}^0$, $\gamma' \in \Gamma \sim \mathscr{P}^0$.

We now suppose that on scale 2^k, after k induction steps, we have arrived at a partition of Γ with the following properties:

$$\Gamma = \mathscr{P}_k \cup \{\gamma_\mu^k\}_{\mu=1,2,3,\ldots},$$

where $\mathscr{P}_k = \{\gamma_\alpha\}_{\alpha=1,2,3,\ldots}$ satisfies Condition D, and

$$\operatorname{dist}(\gamma_\alpha, \gamma_\mu^k) \geqq Md(\gamma_\alpha)^{3/2},$$

for all $\gamma_\alpha \in \mathscr{P}_k$, $\gamma_\mu^k \in \Gamma \sim \mathscr{P}_k$. Moreover

$$\operatorname{dist}(\gamma_\mu^k, \gamma_\nu^k) > 2^k,$$

for γ_μ^k, γ_ν^k in $\Gamma \sim \mathscr{P}_k$, $\mu \neq \nu$.

In order to do the induction step, i.e., increase the distance scale from 2^k to 2^{k+1}, we regroup the subsets $\{\gamma_\mu^k\}_{\mu=1,2,3,\ldots}$ into pairs $\varphi_\sigma^k = \gamma_\mu^k \cup \gamma_\nu^k$, in an arbitrary way, but subject to the rule that

$$\operatorname{dist}(\gamma_\mu^k, \gamma_\nu^k) \leqq 2^{k+1},$$

for two subsets forming a pair. For finite L, finitely many sweeps of such pairing operations suffice to partition $\Gamma \sim \mathscr{P}_k$ into new, larger subsets $\{\gamma_\mu^{k+1}\}_{\mu=1,2,3,\ldots}$, with the property that

$$\operatorname{dist}(\gamma_\mu^{k+1}, \gamma_\nu^{k+1}) > 2^{k+1}, \quad \text{for } \mu \neq \nu. \tag{A.2}$$

Let $\overline{\gamma_\mu^{k+1}} \equiv \{j | \operatorname{dist}(j, \gamma_\mu^{k+1}) < Md(\gamma_\mu^{k+1})^{3/2}\}$. We define

$$\mathscr{P}^{k+1,0} \equiv \{\gamma_\mu^{k+1} \,|\, |\gamma_\mu^{k+1}| \text{ is even}, \overline{\gamma_\mu^{k+1}} \cap \gamma_\nu^{k+1} = \emptyset, \quad \text{for } \mu \neq \nu\},$$

and inductively

$$\mathscr{P}^{k+1,n} \equiv \left\{ \gamma_\mu^{k+1} \,|\, |\gamma_\mu^{k+1}| \text{ is even}, \gamma_\mu^{k+1} \in \left(\Gamma - \left(\mathscr{P}_k \cup \bigcup_{l=0}^{n-1} \mathscr{P}^{k+1,l} \right) \right), \right.$$

$$\left. \overline{\gamma_\mu^{k+1}} \cap \gamma_\nu^{k+1} = \phi \quad \text{for } \gamma_\nu^{k+1} \notin \left(\mathscr{P}_k \cup \bigcup_{l=0}^{n-1} \mathscr{P}^{k+1,l} \right) \right\}.$$

Then we define

$$\mathscr{P}^{k+1} \equiv \bigcup_{n=0}^{\infty} \mathscr{P}^{k+1,n}$$

(this union is finite for $L < \infty$), and

$$\mathscr{P}_{k+1} \equiv \mathscr{P}_k \cup \mathscr{P}^{k+1}.$$

By (A.2), $\Gamma \sim \mathscr{P}_{k+1} = \emptyset$ if k is such that $2^{k+1} \geqq L$, i.e., the induction terminates after finitely many steps when $L < \infty$. It is straightforward to check that $\mathscr{P}_\infty (= \mathscr{P}_k$, for $k \geqq [\ln_2 L] + 1)$ is a partition of Γ satisfying Condition D. For more details concerning a closely related, but more difficult problem see Sect. 2 of [5].

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