

## The Riemannian Geometry of the Configuration Space of Gauge Theories

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**Abstract.** We state some new results about the configuration space of pure Yang-Mills theory. These results come from the study of the kinetic energy term of the Lagrangian of the theory. This term defines a riemannian metric on the space of non-equivalent gauge potentials. We develop a riemannian calculus on the configuration space, compute the riemannian connection, the curvature tensor, and solve for the geodesics, etc. We show that the Gribov ambiguity is more than an artefact of the choice of a gauge condition, and is related to the existence of conjugate points on the geodesics, and is thus an intrinsic feature of the theory.

### Introduction

Pure Yang-Mills theory yields one of the most interesting examples of a singular Lagrangian in field theory (i.e. in a system with an infinite number of degrees of freedom). A serious analysis of this system requires the use of Dirac's formalism for systems with constraints [1–3].

Our analysis differs from that of [2] in that, instead of introducing second class constraints to eliminate the gauge freedom, we directly define the theory on the space of non-equivalent gauge potentials (quotient space by the action of the group of gauge transformations, which we call orbit space). This analysis leads us to the definition of the *true* configuration space  $\mathfrak{M}$  of the theory, and to an effective Lagrangian defined on  $\mathfrak{M}$ . This Lagrangian contains a kinetic part (the electric part) and a velocity independent potential term (the magnetic part). The kinetic term provides a riemannian metric on  $\mathfrak{M}$  by saying that it is of the form  $\frac{1}{2} \cdot \{\text{square of the velocity computed with that metric}\}$ . Consequently the classical motion is the motion of a point in an infinite dimensional riemannian manifold in a potential.

The study of the configuration space is essential, first because it is the space of classical physical states of the system, secondly because the quantum functional integral is to be defined on paths on this space. The metric defined on  $\mathfrak{M}$  is a

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powerful way of investigating this structure. Furthermore two very strong facts show the central role played by this metric in Yang-Mills theory:

1. This metric is the most natural one if we take into account the whole geometrical content of the theory.

2. This metric is already used in the conventional quantization: it was shown in [4] that there is a deep link between the usual Faddeev-Popov determinant [5] and the determinant of this metric. We see there a sign of its importance in the definitions to come of a path integral [6].

The paper is organized as follows:

**1.** We cast the ingredients of the theory into a geometrical framework. **2.** We briefly recall Dirac's analysis of the Yang-Mills Lagrangian, and give the true configuration space together with the effective Lagrangian defined on it. **3.** We show how the metric which appears in this analysis is actually coded into the structure of the theory (more precisely in the action of the group of gauge transformations on the set of all gauge potentials). **4.** We display a coordinate system, which is merely defined by a gauge condition, express the metric in this coordinate system, and introduce some interesting related operators. **5.** We compute the riemannian connection, the curvature tensor, the sectional curvature which is found to be *positive*, and recall some formulae concerning determinants. **6.** We solve for the geodesics and give a normal coordinate system. **7.** We show the existence of focal points on any geodesics, and show in what sense the Gribov "horizon" is the conjugate locus of some point. **8.** Conclusion. **9.** An Appendix.

## 1. Proper Geometrical Setting [7, 8]

Gauge potentials are time dependent connections in a principal bundle  $\mathbf{P}$  with base space  $\mathbf{M}$  ( $\mathbf{M}$  = space = compact riemannian manifold of dimension  $d$ , without boundary) and group  $\mathbf{G}$ . We shall suppose  $\mathbf{G}$  is a connected compact simple Lie group).

We will denote by  $\mathcal{C}$  the set of connections in  $\mathbf{P}$ . The group  $\mathcal{G}$  of gauge transformations is the group of automorphisms of  $\mathbf{P}$  which induce the identity on  $\mathbf{M}$  [8]. A gauge transformation can be realized as a section of the bundle  $\mathbf{B}$  associated to  $\mathbf{P}$ , with fiber  $\mathbf{G}$  endowed with an adjoint action of  $\mathbf{G}$  on itself. Locally a gauge transformation is given by a mapping  $\mathbf{g}:\mathbf{M}\rightarrow\mathbf{G}$ . We will also use the bundle  $\mathbf{E}$  associated to  $\mathbf{P}$  with fiber  $\mathcal{A}(\mathbf{G})$  (the Lie algebra of  $\mathbf{G}$ ) and adjoint action of  $\mathbf{G}$  on  $\mathcal{A}(\mathbf{G})$ .

If  $\omega$  is a connection in  $\mathbf{P}$ ,  $\nabla$  will be the corresponding covariant derivative in  $\mathbf{E}$ . If we denote by  $\mathbf{A}^k$  the space of  $k$ -forms on  $\mathbf{M}$  with values in  $\mathbf{E}$ ; then

$$\nabla:\mathbf{A}^0\rightarrow\mathbf{A}^1.$$

We shall write the action of a gauge transformation  $\mathbf{g}$  on a connection  $\omega$  as:

$$\omega\rightarrow{}^{\mathbf{g}}\omega=\omega+\mathbf{g}^{-1}\nabla\mathbf{g}.$$

It is remarkable that the difference  $\tau$  of two connections  $\omega$  and  $\omega'$  is *not* a connection but a covariant object i.e.:

$${}^{\mathbf{g}}\tau=\mathbf{g}^{-1}\tau\mathbf{g}.$$

In other words  $\tau \in \mathbf{A}^1$ .

Moreover, if  $\omega$  is a connection and  $\tau$  any element of  $\mathbf{A}^1$ , then  $\omega + \tau$  is a connection. Consequently  $\mathcal{C}$  is an affine space modelled on  $\mathbf{A}^1$ . We will denote by  $\mathbf{T}_\omega(\mathcal{C})$  the tangent space to  $\mathcal{C}$  at  $\omega$ .

The metric on  $\mathbf{M}$  and the Killing metric on  $\mathbf{G}$  (tr) give rise to a scalar product in  $\mathbf{A}^s$ , defined with the help of the Hodge operator.

$$(\alpha, \beta) = \int_M \text{tr}(\alpha \wedge * \beta).$$

As a consequence, for  $\not\neq 1$  we have a scalar product on the tangent space to  $\mathcal{C}$ .

It is essential to notice that the metric on  $\mathcal{C}$  thus constructed is *invariant by gauge transformations*. We denote by  $\nabla^*$  the adjoint of  $\nabla$  with respect to the scalar product  $(\cdot, \cdot)$ : If  $\xi \in \mathbf{A}^0$  and  $\tau \in \mathbf{A}^1$ , then

$$(\tau, \nabla \xi) = (\nabla^* \tau, \xi).$$

Let  $\square_\omega$  be the covariant Laplacian acting on  $\mathbf{A}^0$ :  $\square = \nabla^* \nabla$ . Generically (e.g. if the connection is irreducible and, in any case, in an open dense set of  $\mathcal{C}$ ), the operator  $\square_\omega$  has trivial kernel:  $\square \xi = 0 \Rightarrow \xi = 0$ . We then denote by  $G_\omega$  its inverse.

## 2. Dirac's Formalism

The Yang-Mills action is

$$S = \frac{1}{4} \int dt \int_M dv \text{tr}(F^{\mu\nu} F_{\mu\nu}).$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \mu, \nu = 0, 1, 2, 3.$$

The Lagrangian  $L$  can be written

$$L = \frac{1}{2}(\dot{A} - \nabla A_0, \dot{A} - \nabla A_0) - V,$$

if we define

$$\dot{A} = \frac{\partial A_i}{\partial t} \cdot dx^i$$

and

$$V = \frac{1}{4} \int_M dv \text{tr}(F_{ij} F^{ij}), \quad i, j = 1, 2, 3.$$

Computing the canonically conjugate momenta  $\not\mu_\mu$  of  $A_\mu$ , we get

$$\not\mu = \not\mu_i dx^i = \dot{A} - \nabla A_0$$

and  $\not\mu_0 = 0$ , which is the *primary constraint* [1]; this leads to the Hamiltonian

$$H_0 = \frac{1}{2}(\not\mu, \not\mu) + V + (\not\mu, \nabla A_0) + (\lambda, \not\mu_0),$$

with an arbitrary Lagrange multiplier  $\lambda$ .

We thus get the *secondary constraint* (Gauss condition)

$$\not\mu_0 = 0 = \{H_0, \not\mu_0\} = \nabla^* \not\mu,$$

where  $\{\cdot, \cdot\}$  denotes the Poisson bracket. Then the total Hamiltonian is

$$H_T = H_0 + (\mu, \nabla^* \not{A}),$$

where  $\mu$  is an arbitrary Lagrange multiplier. This Hamiltonian gives, among others, the following equations of motion:

$$\begin{aligned} \not{A}_0 &= 0, \\ \dot{A}_0 &= \lambda. \end{aligned}$$

The time evolution of  $A_0$  being arbitrary and  $\not{A}_0$  being zero, following Dirac we can discard these unphysical variables, and write the Hamiltonian  $H$ :

$$H = \frac{1}{2}(\not{A}, \not{A}) + V + (\xi, \nabla^* \not{A}),$$

where again  $\xi$  is arbitrary.

The equation of motion reads

$$\dot{A} = \{H, A\} = \not{A} + \nabla \xi. \quad (1)$$

We see here that the time evolution of  $A$  contains an arbitrary part  $\nabla \xi$ , which corresponds to an infinitesimal arbitrary gauge transformation. Thus in order to get a well defined time evolution, we need to remove the unphysical degrees of freedom corresponding to gauge transformations. A way to do this is to quotient the space of gauge potentials by the group of gauge transformations.

In order to get the Lagrangian on the true configuration space, we first write the Lagrangian  $L$  deduced from  $H$ . From Eq. (1) and the Gauss condition we get:

$$\not{A} = (\mathbb{1} - \nabla G \nabla^*) \dot{A},$$

thus

$$L = \frac{1}{2}(\Pi \dot{A}, \Pi \dot{A}) - V, \quad (2)$$

where

$$\Pi = \mathbb{1} - \nabla G \nabla^*.$$

This Lagrangian consists of two parts. The first term  $E = \frac{1}{2}(\Pi \dot{A}, \Pi \dot{A})$  is gauge invariant, quadratic in the velocity  $\dot{A}$ , and appears as the square of the gauge covariant object  $\Pi \dot{A}$ , defined by  $(\cdot, \cdot)$ , and  $V$  is the gauge invariant, velocity independent potential. We will see that  $L$  is actually a non-singular Lagrangian on the space of non-gauge equivalent potentials.

### 3. The Metric on $\mathfrak{M}$

The space of non-equivalent gauge potentials is obtained by quotienting the space of all connections  $\mathcal{C}$  by the action of the group of gauge transformations. *Modulo certain restrictions*, this space is a manifold [10, 11, see also the Appendix]. We will denote this manifold by  $\mathfrak{M}$  and refer to it as the orbit space. The trajectories under gauge transformations are the orbits. We will denote by  $\mathbf{p}$  the projection:  $\mathbf{p}: \mathcal{C} \rightarrow \mathfrak{M}$ .

Moreover, this manifold can be equipped with a weak Riemannian metric which is induced from the gauge invariant metric on  $\mathcal{C}$ . We will see that this metric is precisely the one appearing in [Eq. (2)].

If  $\omega$  is a generic connection we define  $\chi_\omega: \mathbf{A}^1 \rightarrow \mathbf{A}^0$  by

$$\chi_\omega = G_\omega \nabla_\omega^*.$$

We then have a splitting of the tangent space  $\mathbf{T}_\omega(\mathcal{C})$  into two orthogonal subspaces [12]

$$\mathbf{T}_\omega(\mathcal{C}) = \mathbf{H}_\omega \oplus \mathbf{V}_\omega,$$

$\mathbf{V}_\omega =$  vertical subspace = tangent space to the orbit = Image of  $\nabla_\omega$ ,  $\mathbf{H}_\omega =$  horizontal subspace = kernel of  $\chi_\omega$ . There is a projection operator  $\Pi_\omega$  on  $\mathbf{H}_\omega$  [orthogonal projection in  $\mathbf{T}_\omega(\mathcal{C})$ ] such that

$$\Pi_\omega = \mathbb{1} - \nabla_\omega G_\omega \nabla_\omega^*,$$

$$\Pi_\omega^* = \Pi_\omega,$$

$$\Pi_\omega^2 = \Pi_\omega.$$

The induced metric is then defined on  $\mathfrak{M}$  in a standard way. If  $\mathbf{a}$  is a point in  $\mathfrak{M}$  and  $\omega$  some point in  $\mathbf{p}^{-1}(\mathbf{a})$ , and  $X$  some vector in  $\mathbf{T}_\omega(\mathfrak{M})$ , there exists a unique vector  $\tau$  in  $\mathbf{H}_\omega$  projecting onto  $X$ , i.e.  $\mathbf{p}_*(\tau) = X$ .  $\tau$  is the horizontal lift of  $X$  at  $\omega$ .

If  $X$  and  $Y$  belong to  $\mathbf{T}_\omega(\mathfrak{M})$  and  $\tau_x$  and  $\tau_y$  are their respective lifts at  $\omega$ , we define the metric  $\mathcal{G}(\cdot, \cdot)$  on  $\mathfrak{M}$  by:

$$\mathcal{G}(X, Y) = (\tau_x, \tau_y).$$

Suppose  $\dot{\omega}$  is a tangent vector at  $\omega$ , not necessarily horizontal,  $\dot{\omega}$  projects on  $\mathfrak{M}$  onto a vector  $X = \mathbf{p}_* \dot{\omega}$ ,  $X$  can be lifted horizontally to a vector  $\tau_x$  at  $\omega$ , which is nothing else than  $\tau_x = \Pi_\omega \dot{\omega}$ . By this procedure we have removed from  $\dot{\omega}$  its vertical part, which is unphysical [see Eq. (1)]. Now

$$\mathcal{G}(\mathbf{p}_* \dot{\omega}, \mathbf{p}_* \dot{\omega}) = (\Pi_\omega \dot{\omega}, \Pi_\omega \dot{\omega}),$$

or

$$\frac{1}{2} \dot{\omega}_{\text{physical}}^2 = \frac{1}{2} (\Pi_\omega \dot{\omega}, \Pi_\omega \dot{\omega}).$$

This was already noticed from Faddeev's analysis in [4].

*Remark.* The orbit space  $\mathfrak{M}$  not only possesses a weak riemannian structure, but is also a Hausdorff metric space [9, see also Sect. 6].

#### 4. Coordinate System on $\mathfrak{M}$

Given a point  $\mathbf{a}_0$  in  $\mathfrak{M}$ , we will define a coordinate system in a neighbourhood of  $\mathbf{a}_0$ . In order to do this we choose a point  $\omega_0$  in the orbit  $\mathbf{p}^{-1}(\mathbf{a}_0)$  and define the affine subspace  $\mathcal{S}_0$  of  $\mathcal{C}$  by:

$$\mathcal{S}_0 = \{\omega \in \mathcal{C} | \nabla_0^*(\omega - \omega_0) = 0\}.$$

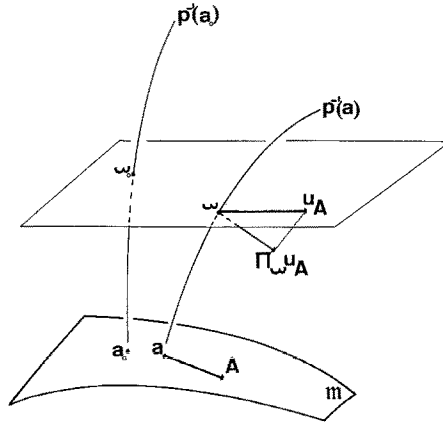


Fig. 1

$\mathcal{S}_0$  is generated by the horizontal space  $\mathbf{H}_0$ . If  $\mathbf{a}$ , in  $\mathfrak{M}$  is sufficiently close to  $\mathbf{a}_0$ , then the orbit  $\mathbf{p}^{-1}(\mathbf{a})$  cuts  $\mathcal{S}_0$  in a unique point  $\omega(\mathbf{a})$  which will be the coordinate of  $\mathbf{a}$ .

This choice of coordinate amounts to “fixing the gauge” by a covariant background gauge condition. It applies only locally in  $\mathfrak{M}$  [15].

Notice that we do not solve the gauge condition. If  $A$  and  $B$  are vectors tangent to  $\mathfrak{M}$  at  $\mathbf{a}$ , then the components of  $A$  and  $B$  are two vectors  $u_A$  and  $u_B$  [in  $T_\omega(\mathcal{S})$ ] which belong to  $\mathcal{S}_0$  (i.e.  $\nabla_0^* u_A = \nabla_0^* u_B = 0$ ), and which project respectively on  $A$  and  $B$ . Clearly

$$\mathbf{p}_*(\Pi_\omega u_A) = \mathbf{p}_*(u_A),$$

$$\mathbf{p}_*(\Pi_\omega u_B) = \mathbf{p}_*(u_B).$$

Consequently  $\Pi_\omega u_A$  is the horizontal lift of  $A$  at  $\omega$  (respectively  $\Pi_\omega u_B$  is the lift of  $B$ ). Hence

$$\varphi(A, B) = (\Pi_\omega u_A, \Pi_\omega u_B),$$

or

$$\varphi(A, B) = (u_A, \Pi_0 \Pi_\omega \Pi_0 u_B).$$

In this coordinate system the metric is  $\Pi_0 \Pi_\omega \Pi_0$ . Notice that the coordinate system becomes singular when  $\nabla_0^* \nabla_\omega$  has a kernel, i.e. when part of the orbit through  $\omega$  is tangent to  $\mathcal{S}_0$ . This is precisely the point where the Gribov ambiguity appears [16].

Let us introduce a few relevant operators (related to the above coordinate system). Let  $\gamma = \nabla_0^* \nabla_\omega$  (the Faddeev-Popov operator). If  $\omega$  is on  $\mathcal{S}_0$ , then  $\gamma$  is self-adjoint. If  $\omega$  is sufficiently close to  $\omega_0$ , then  $\gamma$  has an inverse [11, and see Sect. 7]. From now on  $\omega \in \mathcal{S}_0$ .

Let  $\mathbf{P}$  be the projection on  $\mathbf{H}_\omega$  parallel to  $\mathbf{V}_0$

$$\mathbf{P} = \mathbb{1} - \nabla_\omega \gamma^{-1} \nabla_\omega^*.$$

Let  $\mathbf{P}^*$  be its adjoint.  $\mathbf{P}^*$  is the projection on  $\mathbf{H}_0$  parallel to  $\mathbf{V}_\omega$ .

$$\mathbf{P}^* = \mathbb{1} - \nabla_\omega \gamma^{-1} \nabla_\omega^*.$$

We have

$$\begin{array}{ll}
 \mathbf{P}^2 = \mathbf{P} & \mathbf{P}^{*2} = \mathbf{P}^* \\
 \mathbf{P}\Pi_0 = \mathbf{P} & \Pi_0\mathbf{P}^* = \mathbf{P}^* \\
 \Pi_0\mathbf{P} = \Pi_0 & \mathbf{P}^*\Pi_0 = \Pi_0 \\
 \mathbf{P}\Pi_\omega = \Pi_\omega & \Pi_\omega\mathbf{P}^* = \Pi_\omega \\
 \Pi_\omega\mathbf{P} = \mathbf{P} & \mathbf{P}^*\Pi_\omega = \mathbf{P}^*,
 \end{array}$$

and

$$\Pi_0\Pi_\omega\Pi_0\mathbf{P}^*\mathbf{P} = \mathbf{P}^*\mathbf{P}\Pi_0\Pi_\omega\Pi_0 = \Pi_0.$$

Therefore on  $\mathcal{S}_0$ :  $\mathbf{P}^*\mathbf{P}$  is the inverse of the metric.

We denote by  $\mathbf{K}_\tau: \mathbf{A}^0 \rightarrow \mathbf{A}^1$  the commutation with  $\tau \in \mathbf{A}^1$ ,

$$\mathbf{K}_\tau(\xi) = [\tau, \xi].$$

$\mathbf{K}_\tau^*: \mathbf{A}^1 \rightarrow \mathbf{A}^0$  is its adjoint. The metric  $\Pi_0\Pi_\omega\Pi_0$  can be written

$$\Pi_0(\mathbb{1} - \mathbf{K}_\tau G_\omega \mathbf{K}_\tau^*)\Pi_0,$$

where  $\tau = \omega - \omega_0$ . The coordinate system is seen to be a geodesic coordinate system since the metric is of the form  $1 + 0(\tau^2)$ .

### 5. Riemannian Calculus on $\mathfrak{M}$

In what follows, we denote by  $X$  (respectively  $Y, Z \dots$ ) vector fields having in the coordinate system given by  $\mathcal{S}_0$  as above, constant coordinate  $X$  (with  $\nabla_0^* X = 0$ ) respectively  $Y \dots$ . Notice that the Lie bracket of these vector fields is zero.

Let  $D_X Z$  be the riemannian covariant derivative of  $Z$  along  $X$ . We compute  $D_X Z$  by the standard formula

$$2\mathcal{G}(Y, D_X Z) = X \cdot \mathcal{G}(Y, Z) + Z \cdot \mathcal{G}(Y, X) - Y \cdot \mathcal{G}(X, Z),$$

from which, by a straightforward claculation, we get

$$\begin{aligned}
 D_X Z = & \frac{1}{2}\mathbf{P}^*\mathbf{P}(-\chi_\omega^* \mathbf{K}_X^* \Pi_\omega Z - \Pi_\omega \mathbf{K}_X \chi_\omega Z - \chi_\omega^* \mathbf{K}_Z^* \Pi_\omega X - \Pi_\omega \mathbf{K}_Z \chi_\omega X + [\chi_\omega X, \Pi_\omega Z] \\
 & + [\chi_\omega Z, \Pi_\omega X]).
 \end{aligned} \tag{3}$$

To first order in  $\tau = \omega - \omega_0$ , we get

$$D_X Z = -\Pi_0(\mathbf{K}_X G_0 \mathbf{K}_\tau^* Z + \mathbf{K}_Z G_0 \mathbf{K}_\tau^* X). \tag{4}$$

From [Eq. (4)] we get the Riemann curvature tensor,

$$\begin{aligned}
 R(X, Y) &= [D_X, D_Y] - D_{[X, Y]} \quad (\text{sign convention taken from [8]}), \\
 R(X, Y)Z &= \Pi_0 \{ -2\mathbf{K}_Z G_\omega \mathbf{K}_X^*(Y) - \mathbf{K}_Y G_\omega \mathbf{K}_X^*(Z) + \mathbf{K}_X G_\omega \mathbf{K}_Y^*(Z) \}.
 \end{aligned}$$

The sectional curvature on the direction of the 2-plane generated by two orthonormal vectors  $X$  and  $Y$  is

$$\begin{aligned} \mathfrak{R}(X, Y) &= (R(X, Y)Y, X) \\ \mathfrak{R}(X, Y) &= 3(\mathbf{K}_X^*(Y), G_\omega \mathbf{K}_X^*(Y)). \end{aligned}$$

*The sectional curvature is non-negative.*

In addition, we would like to recall some formulae relating the determinant of the metric and the Faddeev-Popov determinant [4]. Formally

$$(\det \varphi)^{1/2} = \frac{\det \gamma}{(\det \square_0)^{1/2} (\det \square_\omega)^{1/2}}. \tag{5}$$

This equation comes from the identity of the spectra (for the eigenvalues differing from unity) of the two operators  $\varphi: \mathcal{S}_0 \rightarrow \mathcal{S}_0$  and  $\Gamma: \mathbf{A}^0 \rightarrow \mathbf{A}^0$ , with  $\Gamma = G_\omega \nabla_\omega^* \nabla_0 G_0 \nabla_0^* \nabla_\omega$ .

For a proof and implications of [Eq. (5)], in particular for the distinctions between the canonical and covariant quantization, see [4].

### 6. Geodesics

Before going ahead with the study of geodesics, we will show an elementary but very useful proposition:

**Proposition.** *If a straight line in  $\mathcal{C}$  cuts one orbit perpendicularly, then it cuts perpendicularly all other orbits it meets.*

*Proof.* Let  $\omega_0 + t\tau$  be a line in  $\mathcal{C}$  such that  $\nabla_0^* \tau = 0$ . Then  $\nabla_{\omega_0+t\tau}^* \tau = \nabla_0^* \tau + t\mathbf{K}_\tau^*(\tau) = \nabla_0^* \tau = 0$ .

This is also a consequence of a more general proposition [17].

In the coordinate system  $\mathcal{S}_0$ , the equation for geodesics reads

$$\left. \begin{aligned} \ddot{\tau} + D_t \dot{\tau} &= 0 \\ \nabla_0^* \tau &= 0 \end{aligned} \right\}. \tag{6}$$

[We suppose  $\tau = \omega - \omega_0 = \tau(t)$  and  $\left( \dot{\tau} = \frac{d\tau}{dt}, \quad \ddot{\tau} = \frac{d^2\tau}{dt^2} \right)$ ].

From Eq. (3), the above proposition, and inspection of [Eq. (6)], we see that the straight lines  $\omega = \omega_0 + t\mathbf{v}$  with constant  $\mathbf{v}$  satisfy [Eq. (6)]. Consequently *the radial geodesics are the straight lines through  $\omega_0$* . In other words, the coordinates introduced in Sect. 6 give a *normal coordinate system at  $\omega_0$* .

A distance function  $d$  was introduced on  $\mathfrak{M}$  in [9].  $d$  is computed as the minimum length of the segment joining two generic points  $\omega$  and  $\omega'$  on different orbits. Let  $\omega' = \omega + \tau$ . If we naively apply a calculation of variation, we see that any minimizing  $\tau$  should verify:

$$\nabla_\omega^* \tau = \nabla_{\omega'}^* \tau = 0,$$

i.e. any minimizing segment ought to be perpendicular to both orbits.



Justifying this variational approach is beyond the scope of this paper, but a reasonable assumption would be that the distance defined in [9] actually is the *geodesic distance* on  $\mathfrak{M}$ .

Notice that this assumption contains the idea that the horizontal space  $\mathcal{S}_0(\omega_0)$  being any generic point in  $\mathcal{C}$ ) meets all orbits.

### 7. Focal Points

The knowledge of the normal coordinate system  $\mathcal{S}_0$  determines completely the exponential map around  $\omega_0$ . If  $\tau$  is a unit horizontal vector at  $\omega_0$

$$\exp(t\tau) = \mathbf{p}(\omega_0 + t\tau).$$

The exponential is singular in  $\omega = \omega_0 + \lambda\tau$  if  $\exists \mathbf{v} \in \mathbf{T}_\omega(\mathcal{C}) \cap \mathcal{S}_0$ , such that  $\mathbf{v} \neq 0$  but  $\mathbf{p}_*(\mathbf{v}) = 0$ , i.e.  $\Pi_\omega \mathbf{v} = 0$ , that is to say  $\exists \mathbf{v} \in \mathcal{S}_0$ , of the form  $\nabla_\omega \xi$ . Therefore  $\exists \xi$  such that  $\nabla_0^* \nabla_\omega \xi = 0$ , i.e.  $\gamma(\lambda) = \square_0 + \lambda \nabla_0^* K_\tau$ , has a non-trivial kernel.

According to [11], the operator  $\nabla_0^* K_\tau$  is self adjoint non-nonnegative, and since  $\square_0$  is self adjoint positive, there exists a smallest finite  $\lambda$  such that  $\gamma(\lambda)$  has a non-trivial kernel, and this kernel is finite dimensional since  $\gamma(\lambda)$  is elliptic.

The point  $\omega = \omega_0 + \lambda\tau$  is said to be on the Gribov horizon around  $\omega_0$  (in the direction  $\tau$ ). [For details on the problems arising from the existence of non-generic connections, see the Appendix.]

Generically, suppose  $\omega$  is on the horizon of  $\omega_0$  and  $\nabla_0^* \nabla_\omega \xi = 0$ ;  $\mathbf{v} = \nabla_\omega \xi$  is a vector at  $\omega$  in  $\mathcal{S}_0$  such that  $\mathbf{p}_*(\mathbf{v}) = 0$ . Consequently the exponential is singular at  $\omega$ . Therefore  $\omega$  is the first *conjugate* point of  $\omega_0$  in the direction of the unit vector  $\tau$ . This can be seen in terms of Jacobi fields on  $\mathfrak{M}$ . We construct an infinitesimal variation of the geodesic joining  $\omega_0$  and  $\omega$  as follows: Let  $\mathbf{v} = \nabla_\omega \xi$  as above,  $t$  be the canonical parameter on the geodesic. At  $\omega(t) = \omega_0 + t\tau$  define  $\mathbf{J}(t) = \frac{t}{\lambda} \cdot \mathbf{v}$ .  $J$  is the coordinate of a Jacobi field on  $\mathfrak{M}$ , vanishing at  $\mathbf{p}(\omega_0)$  and  $\mathbf{p}(\omega)$ .

### 8. Conclusion

The simplicity of the results we get on the orbit space reveals its extraordinary richness and leads us to believe that many properties encountered in the finite dimensional case extend to this infinite dimensional case. For instance, in cases where all connections are generic [e.g. if  $\mathbf{P}$  is a non-trivial  $SU(2)$  bundle on  $\mathbf{M}$  and  $H^2(\mathbf{M}, \mathbb{Z}) = 0$ ] the *orbit space is geodesically complete* (i.e. all geodesics can be prolonged to infinity). We could hope for a generalization to this case of a theorem on cut locus [8]. This would solve the Gribov problem.

On the other hand the important question of non-generic connections arises. There are already results on this program. We give an account of the situation in the Appendix.

The study of Yang-Mills theories has led us to consider a natural (non-relativistic) dynamical system (with infinitely many degrees of freedom). This system is defined by the geodesic motion of a point on the orbit space. Given initial

conditions, we may describe this motion by some straight lines in the total space  $\mathcal{C}$  of connections. However the true motion (on the physically relevant space) is obtained after projecting these lines on the orbit space. This projection is a highly non-trivial operation: geodesics may have a very intricate shape (e.g. double points, etc.). The question arises as to whether the motion is ergodic, is completely integrable, or is of some other kind. Some preliminary results (existence of infinitely many conserved quantities) suggest non-ergodicity, and this problem is under investigation.

## Appendix

We stated that  $\mathfrak{M} = \mathcal{C}/\mathcal{G}$  is a manifold “under certain restrictions.” The first restriction is on the space of connections (and gauge transformations) in order that  $\mathcal{C} \xrightarrow{\mathfrak{p}} \mathfrak{M}$  be a principal fiber bundle. We would need to take connections (respectively gauge transformations) belonging to adequate Sobolev spaces. [12]. The second restriction comes from the necessity of having a free action of  $\mathcal{G}$  on  $\mathcal{C}$ . A connection  $\omega$  is left unchanged by  $\mathfrak{g} \in \mathcal{G}$  if

$$\mathfrak{g}\omega = \omega + \mathfrak{g}^{-1}\nabla_{\omega}\mathfrak{g} = \omega,$$

or equivalently  $\mathfrak{g}^{-1}\nabla\mathfrak{g} = 0$ . This implies that  $\mathfrak{g}$  commutes with the holonomy group  $\mathbf{N}(\omega)$  of  $\omega$ . That is  $\mathfrak{g}$  belongs to the centralizer  $\mathfrak{c}(\omega)$  of  $\mathbf{N}(\omega)$ .  $\omega$  is called *generic* if  $\mathfrak{C}(\omega)$  reduces to the center of  $\mathcal{G}$ .  $\omega$  is generic if  $\square_{\omega}$  has trivial kernel. In particular, all irreducible connections are generic if the structure group is simple. Since the center  $\mathcal{Z}$  of  $\mathcal{G}$  has trivial action on  $\mathcal{C}$ , we replace the group  $\mathcal{G}$  by  $\mathcal{G}/\mathcal{Z}$ .

For a non-generic connection the inverse  $G_{\omega}$  of  $\square_{\omega}$  is not defined. The induced metric cannot be defined directly. (For example, if a geodesic hits a non-generic point we do not know how it behaves.) However it is known that generic connections form an open dense set in  $\mathcal{C}$  (and thus fill  $\mathcal{C}$ ), but the precise location of the non-generic points is still to be found. Some positive results have been obtained on the *structure* of the quotient space of all connections: it is a stratified space [10,11].

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