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The Dimension Spectrum of Some Dynamical Systems

P. Collet,[1] J. L. Lebowitz,[2] and A. Porzio[1]

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We analyze the dimension spectrum previously introduced and measured experimentally by Jensen, Kadanoff, and Libchaber. Using large-deviation theory, we prove, for some invariant measures of expanding Markov maps, that the Hausdorff dimension $f(\alpha)$ of the set on which the measure has a singularity α is a well-defined, concave, and regular function. In particular, we show that this is the case for the accumulation of period doubling and critical mappings of the circle with golden rotation number. We also show in these particular cases that the function f is universal.

KEY WORDS: Hausdorff dimension spectrum; partition function; period doubling; critical circle map; universality.

1. INTRODUCTION

In recent work[1,2,8,12,15–17] a new method for the description of singularities of measures has been discussed. The main idea is to estimate the Hausdorff dimension of the set where the measure has a given power law singularity. For simplicity we shall work in dimension one, although many results extend to higher dimension. Let μ be a nonatomic Borel probability measure on the real line. We define two functions α^+ and α^- by

$$\alpha^+(x) = \limsup_{\substack{|I| \to 0 \\ x \in \mathrm{Int}\{I\}}} \log \mu(I)/\log |I|$$

and

$$\alpha^-(x) = \liminf_{\substack{|I| \to 0 \\ x \in \mathrm{Int}\{I\}}} \log \mu(I)/\log |I|$$

[1] Centre de Physique Théorique, Ecole Polytechnique, F91128 Palaiseau, France.
[2] Department of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903.

where $|I|$ denotes the length of the interval I and $\mathrm{Int}\{\cdot\}$ denotes the interior. In some sense these two functions describe the singularity of the measure μ at x. If one considers instead of μ the cumulative distribution function of the measure, then α^+ and α^- are related to the Hölder exponents at x.

The following question is now rather natural. For a positive real number α, define the sets $B_\alpha^\pm$ by

$$B_\alpha^\pm = \{x \mid \alpha^\pm(x) = \alpha\}$$

What can we say about these sets? Note first that in general they can be quite small when measured by μ. In the cases we shall discuss below, there is only one value of α such that $\mu(B_\alpha^\pm) \neq 0$ (and both sets are in fact of full measure). Nevertheless, one can try to analyze these sets from a different point of view. In the very interesting seminal paper by Halsey *et al.*[15] it was proposed to look at the Hausdorff dimension of such sets obtained from experimental measurements as a function of α. Somewhat surprisingly, it was indeed possible to extract such a function from the experiment. This function turned out to be smooth and in good agreement with the numerical computations for the invariant measure associated to critical mappings of the circle with golden rotation number. It is the purpose of this paper to prove rigorously the existence of such a smooth function $f(\alpha)$ for some invariant measure μ of expanding Markov maps (including the above situation): $f(\alpha)$ is the Hausdorff dimension of the set B_α^+ and also B_α^- for this measure. Note that without some restrictions on the measure μ nothing interesting can be said. In fact, it is easy to construct measures for which the above function behaves wildly.

Our approach is based on an idea introduced in Ref. 15, which roughly goes as follows. If A_n^0 is the dyadic partition of $\mathbb{R}$ by intervals of length 2^{-n}, the partition function $Z_U^n(\beta)$ is defined for $\beta \in \mathbb{R}$ by

$$Z_U^n(\beta) = \sum_{A \in A_n^0} \mu(A)^\beta = \sum_{A \in A_n^0} e^{-\beta E(A)} \tag{1}$$

the sum being taken over all atoms A for which $\mu(A) \neq 0$. The free energy of μ for this uniform partition is defined (when it exists) by

$$F(\beta) = \lim_{n \to +\infty} -n^{-1} \log_2 Z_U^n(\beta)$$

The analogy with statistical mechanics is then used to relate the Legendre transform of $F(\inf_\beta[\beta t - F(\beta)])$ to the distribution of the numbers $\mu(I)$ for $I \in A_n^0$, i.e., to $f(\alpha)$.

We give rigorous proofs of the above connections when μ is an invariant measure for an expanding Markov map on an interval or a circle.

We also prove that the invariant measures associated to the dynamical systems for period doubling and for critical circle mappings with golden rotation number satisfy our hypothesis. For the latter case we have to construct explicitly this invariant measure (in the former case it is already known). We also prove for both cases the universality of the function f. This means for the period doubling that any function in the universality class of the quadratic (unimodal real analytic) fixed point will have the same function f. A similar result holds for circle maps. The argument is based on the rapid convergence of the associated potentials.

We observe here that while the results for the period doubling case and for the circle maps are very similar, the proof of the existence of the free energy is much more difficult for the first case. The problem comes from the fact that the invariant measure for period doubling is concentrated on a Cantor set. This leads to difficulties with the free energy for negative values of β due to the fact that $E(A)$ in (1) may be atypically large. This will happen when A is almost contained in the complement of the support of μ. Our remedy for this comes from the observation that any such interval A must have a left or a right neighboring interval which has a normal weight. We can therefore modify the partitions A_n^0 by joining together such intervals and we redefine $Z^n(\beta)$ in terms of the new partition. This in fact does not change the free energy for $\beta \geqslant 0$. The circle map has no such problems: the support of the invariant measure is the whole circle.

We note that this problem may also arise in the determination of f from experimental data. As mentioned earlier, f could be determined very readily for the "critical circle map" experiment. An attempt for a period doubling experiment encountered difficulties apparently similar to those mentioned above (A. Libchaber, private communication). It is possible that our theoretical remedy will also work for the experiment.

It is tempting to interpret $E(A)$ as the energy of the configuration A of some statistical mechanical system. This analogy can be carried out completely for the case of the usual Cantor set if one uses the triadic partition. The associated system of statistical mechanics is the semi-infinite Bernoulli shift with three symbols and weights (1/2, 0, 1/2). We do not know how to carry out this analogy directly for the more general case we are looking at. Nevertheless, although we do not know how to define the Gibbs states of our system, we can do the thermodynamics. This is due in part to the relation with the (well-known) statistical mechanical approach to invariant measures of dynamical systems.[27]

The proof of the existence of the free energy is carried out in Section 2. We also prove there that F is differentiable and independent of the precise partition by showing that it is the inverse of another, more intrinsic free energy G_D previously used in the theory of dynamical systems.[27] In fact, an analysis of f based entirely on the function G_D was developed independently by Rand[28] for axiom A-like maps. Our result shows in particular that the uniform partition used in the analysis of the experiment give the same result as the dynamical partition. Unfortunately, this can be proven only after the existence of F has been established the hard way.

In Section 3 we use the large-deviation theorem[5,26] to prove that f as introduced above is in fact the Legendre transform of F. The upper bound on the Hausdorff dimension follows directly from the large deviation theorem, while the lower bound is constructed using Frostman's Lemma.[9,18] The application to the two examples mentioned above is given in Section 4, where the construction of the invariant measure for critical circle maps with golden rotation number is carried out in detail.

2. FREE ENERGY FOR EXPANDING MARKOV MAPS

2.1. Existence

We first recall the definition of an expanding Markov map. Let K be a closed interval or the circle. Let $K_1,..., K_p$ be a finite covering of K by p closed intervals with disjoint interiors. Let D be a subset (nonempty) of $\{1,..., p\}$. Then g is a Markov map of K if g is defined and continuous on $\bigcup_{j\in D} K_j$, and if $r\in D$ and $\mathrm{Int}\{K_q \cap g(K_r)\} \neq \varnothing$, then $K_q \subset g(K_r)$. Let g^n denote the nth iterate of g. We shall assume that there is a closed subset K_0 of K such that for any large enough integer n and for any integer q in D, $g^n(K_q) = K_0$. We shall also assume that g is regular and expanding in the following sense:

(i) g is C^2 on each K_j for $j\in D$ and there are two finite numbers $\rho \geqslant \chi > 1$ such that if $g'(x)$ is defined,

$$\rho \geqslant |g'(x)| \geqslant \chi$$

and (ii)

$$\sup_{j\in D} \sup_{x\in K_j} |g''_{K_j}(x)| \leqslant \rho$$

It follows from the Markov property that the boundaries of the intervals K_j, $j\in D$, are preperiodic points. We can also assume that if $j\notin D$, K_j is a connected component of the complement of $\bigcup_{r\in D} K_r$. We now recall the well-known distortion lemma for expanding Markov maps (see, for example, Ref. 13).

Lemma 2.1. There is a finite constant $\gamma \geqslant 1$ such that if $t \in \mathbb{N}$ and if I and J are two subintervals of K, with the property that $g^s(I) \cup g^s(J) \subset K_q$ $\forall s \in \mathbb{N}$, $0 \leqslant s \leqslant t$, then

$$\gamma^{-1}\, |I|/|J| \leqslant |\, g^t(I)|/|\, g^t(J)| \leqslant \gamma\, |I|/|J|$$

We shall assume that a g-invariant Borel measure μ has been defined on K with the following properties

(i) $\mu(K_r) = 0$ if $r \notin D$.

(ii) If I and J are two subsets of some interval K_s, $s \in D$, then

$$\mu(I)/\mu(J) = \mu(g(I))/\mu(g(J))$$

The above assumptions together with the set of numbers $(\mu(K_s))_{s\in D}$ completely determine the Borel measure μ (there is of course a finite number of consistency relations to be satisfied). Since we shall only be interested in the properties of the measure μ, we can assume that

$$D = \{r \mid \mu(K_r) > 0 \text{ and } g \text{ is defined on } K_r\}$$

We can also assume that $K_0 = \bigcup_{r\in D} K_r$.

An example of a transformation satisfying these conditions in which the measure μ is concentrated on the usual Cantor set is obtained as follows:

$$K = [0, 1], \qquad p = 3, \qquad K_1 = [0, \tfrac{1}{3}], \qquad K_2 = [\tfrac{1}{3}, \tfrac{2}{3}]$$

$$K_3 = [\tfrac{2}{3}, 1], \qquad D = \{1, 3\}, \qquad g_{|K_1}(x) = 3x$$

$$g_{|K_3}(x) = 3(x - \tfrac{2}{3}), \qquad \mu(K_1) = \mu(K_3) = \tfrac{1}{2}, \qquad \mu(K_2) = 0$$

The cases we are interested in and treat later in detail, i.e., period doubling and critical maps of the circle, do not satisfy the hypothesis on the derivative, because the maps have a critical point. We shall see, however, that the corresponding measures are also invariant and ergodic for other transformations that do satisfy the conditions, and as seen from the definitions, the free energies depend only on the invariant measure and not on the transformation.

Let l_0 denote the smallest distance between the points in $\bigcup_{r\in D} \partial K_r$. The following lemma describes the singularity of μ near a boundary point.

Lemma 2.2. There are four finite positive numbers C, $\sigma_1 > \sigma_2 > 0$, $0 < \varepsilon_0 < l_0 \rho^{-1}/2$, such that if $r \in D$, and b is a boundary point of K_r and $0 \leqslant \varepsilon \leqslant \varepsilon_0$, then

$$C^{-1}\varepsilon^{\sigma_1} \leqslant \mu([b-\varepsilon, b+\varepsilon] \cap K_r) \leqslant C\varepsilon^{\sigma_2}$$

Proof. We denote by I_ε the set $K_r \cap [b-\varepsilon, b+\varepsilon]$. Let q be the smallest integer such that $\mathrm{Int}\{g^q(I_\varepsilon)\}$ contains a boundary point of some K_s for $s \in D$. If ε_0 is small enough, we can assume that $g^q(b)$ is a periodic point of g. If $0 \leqslant j < q$, we have $g^j(I_\varepsilon) \subset K_{r_j}$ for some $r_j \in D$. Therefore

$$\mu(I_\varepsilon) = \mu(g^q(I_\varepsilon)) \prod_{j=0}^{q} \frac{\mu(K_{r_j})}{\mu(g(K_{r_j}))}$$

Note that since μ is g-invariant, and $\mu(K_{r_j}) > 0$, we have $\mu(g(K_{r_j})) > 0$. Moreover, there is a finite constant $C_1 > 0$ such that for any $r \in D$

$$C_1^{-1} \leqslant \mu(K_r)/\mu(g(K_r)) \leqslant C_1$$

The result follows at once from the estimate $q = O(1) \log \varepsilon^{-1}$, which is a consequence of

$$\varepsilon\chi^q \leqslant |g^q(I_\varepsilon)| \leqslant \varepsilon\rho^q \quad \blacksquare$$

We now choose once for all an integer n_0 and a positive number l such that

$$2^{-n_0} < l_0/2, \qquad l_0\gamma^{-1}\rho^{-1} > 4l, \qquad l < \varepsilon_0 < l_0\rho^{-1}/2$$

and

$$C^{-1} l^{2\sigma_1} < \inf_{s \in D} \mu(K_s)/3$$

where the above constant C is the same as in Lemma 2.2. We shall denote by S the set $\bigcup_{j \in \{1,\dots,p\}} \partial K_j$.

For an interval J let t_J be the largest integer such that for $0 \leqslant t \leqslant t_J$, g^t is differentiable on J. We shall say that J is a regular interval if

$$|g^{t_J}(J)| \geqslant l$$

We denote by A_n^0 the partition of K by closed intervals of length 2^{-n} (we can assume that K has length 1). Strictly speaking, this is not a partition, because of the overlap of the boundaries. However, this will have no effect, since the measure μ is nonatomic. The sequence 2^{-n} has been chosen for simplicity; any exponentially decreasing sequence would give the same result. As already mentioned in the introduction, we have to use a partition different from A_n^0 because for negative temperature the contributions of the exceptionally small atoms may dominate in the partition function. Notice, however, that these atoms do not contribute at positive temperature, and it is easy to show that in this last regime the two partitions give rise to the same free energy. In general, for $\beta < 0$ we have to introduce a new partition

A_n, which consists of atoms that are more regular than those of A_n^0 (in the above sense). A_n will be defined only for n large enough, i.e., for $n \geqslant n_0$. Let I be an atom of A_n^0 that is not regular. If $\mu(I \cap K_0) = 0$, we do not modify I. Otherwise we can write $I = I' \cup I''$, where I' and I'' are closed, $\mathrm{Int}\{I'\} \cap \mathrm{Int}\{I''\} = \varnothing$, and the point $\{y\} = I' \cap I''$ is a preimage of a preperiodic point of g, i.e., there is an integer t such that $g^t(y) \in S$ (we assume that t is the smallest such integer). Assume now that $\mu(I' \cap K_0) \neq 0$ (the same construction has to be applied to I''). From the definition of n_0 we deduce that $I' \subset K_j$ for some $j \in D$. Let J be the atom of A_n^0 adjacent to I' with $y \notin J$; we have also $J \in K_j$. We now claim that J is regular. If not, there is a smallest integer t_0 such that:

(i) $g^{t_0}(\mathrm{Int}\{J\}) \cap S \neq \varnothing$ and $|g^{t_0}(J)| < l$;

(ii) or $g^{t_0}(\mathrm{Int}\{J\}) \cap K_0 = \varnothing$.

In the first case, from the Distortion Lemma and $2\gamma l < l_0$ we cannot have $t = t_0$. If $t < t_0$, we consider the family of intervals $g^{t+\tau}(I' \cup J)$. As long as $|g^{t+\tau}(I' \cup J)| < l_0$, we can apply g. Therefore, if τ_0 is the largest integer such that $|g^{t+\tau_0}(I' \cup J)| < l_0$, we have $t_0 > t + \tau_0$, while from the Distortion Lemma, we have

$$|g^{t+\tau_0}(J)| > \gamma^{-1}\,|J|\,|g^{t+\tau_0}(I' \cup J)|/|I' \cup J| \geqslant l_0\gamma^{-1}\rho^{-1}/2 > l$$

which is a contradiction. If $t > t_0$, we can repeat the argument with I and J exchanged. The first case is therefore excluded. In the second case, if $t_0 \geqslant t$, we argue as before. If $t_0 < t$, g^s cannot be defined on I' for $s > t_0$ and we have again a contradiction.

We now define a partition $(A_n^1)_{n \geqslant n_0}$ as follows. If I is a nonregular atom of A_n^0, we write as above $I = I' \cup I''$ and we take $J \cup I'$ for an atom of A_n^1 where J is as above. In other words, the atoms of A_n^1 are of the form

$$I_1 \cup J \cup I_2$$

where J is a regular atom of A_n^0, and I_1 and I_2 are two closed subintervals (eventually empty) of nonregular atoms of A_n^0 adjacent to J. Note that from the definition of l, the atoms of A_n^1 have the following properties:

(i) If $I \in A_n^1$, then $2^{-n} \leqslant |I| \leqslant 3 \cdot 2^{-n}$.

(ii) If $I \in A_n^1$, $\mu(I) \neq 0$ and for $0 \leqslant q < t_I$, $g^q(I) \cap S = \varnothing$.

We shall now slightly modify A_n^1 to arrive at A_n. We have to do this because although $g^{t_I}(I)$ is not too small, it may still have a small measure. Again, if $I \in A_n^1$ and $\mu(I) = 0$, we do not change I. If $\mu(I) > 0$, let as before q

be the smallest integer such that $g^q(\mathrm{Int}\{I\}) \cap S \neq \varnothing$. Let I_1, I_2 be two closed intervals such that $I = I_1 \cup I_2$, $\mathrm{Int}\{I_1\} \cap \mathrm{Int}\{I_2\} = \varnothing$, and

$$g^q(\mathrm{Int}\{I_1\}) \cap S = g^q(\mathrm{Int}\{I_2\}) \cap S = \varnothing$$

There are now different cases. Let

$$\theta = C^{-1} l^{\sigma_1} \rho^{-\sigma_1\{1 + [-\log(\gamma^{-1}l)/\log \chi]\}}$$

Case 1. $\mu(g^q(I)) > \theta$. In this case we do not modify I.

Case 2. $\mu(g^q(I)) \leqslant \theta$. In this case it follows from Lemma 2.2 that either $\mu(I_2) = 0$ or $\mu(I_1) = 0$. We shall consider the first situation; the second is treated similarly. Let J be the element of A_n^1 that is adjacent to I_1. Let q_1 be the smallest integer such that $g^{q_1}(\mathrm{Int}\{J\}) \cap S \neq \varnothing$. We claim that $\mu(g^{q_1}(J)) > \theta$. To prove this fact, assume first that $q_1 \leqslant q$. From the Distortion Lemma, we have

$$|g^{q_1}(I)| \geqslant \gamma^{-1} |g^{q_1}(J)| \geqslant \gamma^{-1} l$$

Therefore

$$q \leqslant q_1 + [-\log(\gamma^{-1} l)/\log \chi] + 1$$

If $\mu(g^{q_1}(J)) \leqslant \theta$, we deduce from Lemma 2 that

$$d(S, \partial g^{q_1}(J)) \leqslant l \rho^{-\{1 + [-\log(\gamma^{-1}l)/\log \chi]\}}$$

From the expansiveness of g and the upper bound on $q - q_1$, we get $d(S, \partial g^q(J)) \leqslant l$. Using Lemma 2 and $l < l_0/2$, we get

$$\mu(g^q(I)) \geqslant C^{-1} l^{\sigma_1} > \theta$$

which is a contradiction. If $q > q_1$, we can apply the same argument with I and J exchanged.

We now modify the partition as follows. I_2 becomes an atom of A_n (of measure 0). To the atom J we add I_1, and eventually a similar piece on the other side. It is easy to show that this partition A_n has the following properties.

Proposition 2.3. If $I \in A_n$, and $\mu(I) > 0$, we have:

(i) $2^{-n} \leqslant |I| \leqslant 5 \cdot 2^{-n}$.

(ii) If q is the smallest integer such that $g^q(\mathrm{Int}\{I\}) \cap S \neq \varnothing$, then $\mu(g^q(I)) \geqslant \theta$, where θ is as above.

(iii) A_{n+1} is a refinement of A_n.

We are now ready to define the partition function of our system by

$$Z^n(\beta) = \sum_{\substack{I \in A_n \\ \mu(I) > 0}} \mu(I)^\beta$$

Before proving the existence of the thermodynamic limit, we shall establish some rough estimates on $Z^n(\beta)$.

Lemma 2.4. There is a constant C_0 such that if p, $n \in \mathbb{N}$, if $J \in A_n$ and $I \in A_{n+p}$ satisfy $I \subset J$ and $\mu(I) > 0$, then

$$\mu(I)/\mu(J) \geqslant C_0 b^{1 + p/\log_2 \chi}\quad .$$

where $b = \inf_{j \in D} \mu(K_j)$.

Proof. We have

$$\mu(I)/\mu(J) = \mu(g^{t_J}(I))/\mu(g^{t_J}(J)) \geqslant \mu(g^{t_J}(I))$$

From $|g^{t_J}(J)| \geqslant l$ and the Distortion Lemma, we derive

$$|g^{t_J}(I)| \geqslant \gamma^{-1}\,|I|\,|g^{t_J}(J)|/|J| \geqslant \gamma^{-1} l 2^{-p}$$

This implies $t_I - t_J \leqslant 1 + \log 2^p\gamma/\log \chi$. Therefore we get, using Proposition 2.3,

$$\mu(g^{t_J}(I)) \geqslant \theta \prod_{1}^{t_I - t_J - 1} \frac{\mu(K_{s_{j-1}})}{\mu(K_{s_j})}$$

where s_j is the index in D such that $g^j(g^{t_J}(I)) \in K_{s_j}$ for $0 \leqslant j \leqslant t_I - t_J - 1$. The result follows now from the definition of b if we define C_0 by

$$C_0 = \theta b^{1 + \log \gamma/\log \chi} \quad \blacksquare$$

Corollary 2.5. If p and n are integers and C_0 is the above constant, we have for $\beta > 0$

$$C_0^\beta b^{p\beta/\log_2 \chi} Z^n(\beta) \leqslant Z^{n+p}(\beta) \leqslant 5^{p\beta} Z^n(\beta)$$

and the reversed inequalities for $\beta < 0$.

Proof. The upper bound follows at once from Proposition 2.3, since each atom of A_n contains at most 5^p atoms of A_{n+p}. The lower bound is also obvious from Lemma 2.4 if we associate to each atom of A_n a subatom of A_{n+p} of nonzero measure. $\blacksquare$

We are now ready to prove the existence of the thermodynamic limit.

Theorem 2.6. For any $\beta \in \mathbb{R}$

$$F(\beta) = \lim_{n \to \infty} -n^{-1} \log_2 Z^n(\beta)$$

exists and defines a convex function F of β.

Proof. We shall prove that for n large enough, the sequence $-\log_2 Z^n(\beta)$ is subadditive. Namely, there is a number n_1 and a constant $C_2(\beta)$ such that if n and m are larger than n_1, we have

$$Z^{n+m}(\beta) \leqslant C_2(\beta)\, Z^n(\beta)\, Z^m(\beta) \tag{2.1}$$

The result then follows by standard arguments from the existence of a lower bound on $\log_2 Z^n(\beta)$ (which follows at once from Corollary 2.5).

We can write

$$Z^{n+m}(\beta) = \sum_{J \in A_n} \mu(J)^\beta \sum_{\substack{I \in A_{n+m} \\ I \subset J}} [\mu(I)/\mu(J)]^\beta$$

If I and J are as in the above sum, we have

$$\mu(I)/\mu(J) = \mu(g^{t_J}(I))/\mu(g^{t_J}(J))$$

From the Distortion Lemma we deduce

$$\gamma^{-1}\, |I|\, |g^{t_J}(J)|/|J| \leqslant |g^{t_J}(I)| \leqslant \gamma\, |I|\, |g^{t_J}(J)|/|J|$$

which implies

$$\gamma^{-1} l 2^{-m} \leqslant |g^{t_J}(I)| \leqslant \gamma 2^{-m}$$

We now define an integer q_1 by

$$q_1 = 1 + [\log_2 \gamma^2 l^{-2}]$$

where [ ] denotes the integer part. From now on, we shall assume that n and m are bigger than $n_1 = n_0 + 2q_1$. Assume first $\beta > 0$. For $J \in A_n$, the family $\mathscr{F}_J$ of subsets

$$\{ g^{t_J}(I) \mid I \in A_{n+m}, I \subset J \}$$

is in some sense finer than the partition A_{n-q_1}, i.e., each element intersects at most two atoms of A_{n-q_1}. Moreover, each atom of A_{n-q_1} contains at most 4^{q_1} elements of the family $\mathscr{F}_J$. This implies

$$\sum_{\substack{I \in A_{n+m} \\ I \subset J}} \mu(I)^\beta \leqslant 2^\beta 4^{q_1} Z^{m-q_1}(\beta)$$

and the subadditivity follows immediately from Corollary 2.5. To deal with the case $\beta < 0$ we use a similar argument. Note, however, that we have to use estimates in the reversed direction. Each element of the family $\mathscr{F}_J$ contains at most 4^{q_1} atoms of the partition A_{n+q_1}. Moreover, each atom of A_{n+q_1} intersects at most two elements of $\mathscr{F}_J$. Assume that $Y \in \mathscr{F}_J$ and $R \in A_{n+q_1}$ are such that $Y \cap R \neq \varnothing$. If $\mu(Y) > 0$, it follows easily from the Distortion Lemma that $\mu(R)/\mu(Y) \leqslant \theta^{-1}$. This implies $\mu(Y)^\beta \leqslant \theta^{-\beta}\mu(R)^\beta$. The result follows now as in the case $\beta > 0$. ∎

2.2. Regularity

The above construction does not yield directly any information about the regularity of F. In particular, we do not know whether F has a singularity corresponding to a phase transition, e.g., a discontinuity of the first derivative. Such behavior is known to occur for some transformations with critical points, e.g., for the absolutely continuous invariant measure of the map $x \to 1 - 2x^2$. We shall prove here that this does not in fact occur under our hypothesis. We do this by showing that our free energy is in fact the inverse of another free energy G_D obtained from a partition for which $\mu(A)$ is the same for all the atoms A. That free energy, called the dynamical free energy, has been investigated in Ref. 27 and is known in our case to be C^1.

Let $\mathscr{P}_0$ be the partition of K defined by the intervals $K_1, ..., K_p$ used in the covering of K. We shall denote by $\mathscr{P}_n$ the partition

$$\mathscr{P}_n = \bigvee_0^n g^{-j}\mathscr{P}_0$$

The dynamical free energy $G_D(x, y)$ is defined for x and y in $\mathbb{R}$ by

$$G_D(x, y) = \lim_{n \to \infty} n^{-1} \log_2 \sum_{\substack{A \in \mathscr{P}_n \\ \mu(A) > 0}} |A|^y \, \mu(A)^x$$

It was shown in Refs. 27 and 30 that G_D exists on $\mathbb{R}^2$, is C^2, and $D_2 G_D \neq 0$ on $\mathbb{R}^2$. For the following proposition we only need the existence of G_D.

Proposition 2.7. $\forall \beta \in \mathbb{R}$, we have

$$G_D(\beta, F(\beta)) = 0$$

Proof. Let ε be a positive real number small enough, and let $\beta \in \mathbb{R}$. There is an integer $N(\varepsilon)$ such that if n is larger than $N(\varepsilon)$, we have

$$2^{-n(F(\beta)+\varepsilon)} \leqslant \sum_{\substack{I \in A_n \\ \mu(I) \neq 0}} \mu(I)^\beta \leqslant 2^{-n(F(\beta)-\varepsilon)}$$

Let m be an integer smaller than n (one should imagine $n \gg m \gg 1$). We have

$$\sum_{A \in \mathscr{P}_m} \sum_{\substack{I \in A_n \\ I \subset A}} \mu(I)^\beta \leqslant \sum_{\substack{I \in A_n \\ \mu(I) \neq 0}} \mu(I)^\beta \leqslant \sum_{A \in \mathscr{P}_m} \sum_{\substack{I \in A_n \\ \mu(I \cap A) > 0}} \mu(I)^\beta$$

We now observe that if $A \in \mathscr{P}_m$, we have

$$|A| \geqslant l_0 \rho^{-m}$$

We consider first the lower bound. Note that if $A \in \mathscr{P}_m$, $g^m(A) = K_s$ for some $s \in D$ if $\mu(K_s) > 0$. Therefore, if $A \in \mathscr{P}_m$,

$$\sum_{\substack{I \in A_n \\ \mu(I \cap A) > 0}} \mu(I)^\beta = \mu(A)^\beta \sum_{\substack{I \in A_n \\ \mu(I \cap A) > 0}} \mu(g^m(I))^\beta / \mu(K_s)^\beta$$

From the distortion Lemma, if $I \in A_n$, $\mu(I \cap A) > 0$, and $n > m \log_2 \rho - \log \gamma^{-1} l_0$, we have, with $g^m(A) = K_s$

$$\gamma^{-1} l_0 \, |I|/|A| \leqslant \gamma^{-1} \, |I| \, |K_s|/|A| \leqslant |g^m(I)| \leqslant \gamma \, |I| \, |K_s|/|A| \leqslant |I|/|A|$$

Let

$$n_1 = n + [-\log_2 |A|] + 10 \, [-\log_2 \gamma^{-1} l_0] + 10$$

Then, as in the proof of Theorem 2.6, we have

$$\sum_{\substack{I \in A_n \\ \mu(I \cap A) > 0}} \mu(g^m(I))^\beta \leqslant C_7(\beta) \, Z^{n_1}(\beta) \leqslant C_7(\beta) \, 2^{-n_1(F(\beta) - \varepsilon)}$$

where $C_7(\beta)$ is a constant which does not depend on n, m, or A. We get

$$2^{-n(F(\beta) + \varepsilon)} \leqslant \sum_{A \in \mathscr{P}_m} \mu(A)^\beta \, C_8(\beta) \, 2^{-n(F(\beta) - \varepsilon)} \, 2^{(\log_2 |A|) F(\beta)}$$

for some new constant $C_8(\beta)$. Therefore

$$2^{-2n\varepsilon} \leqslant C_8(\beta) \sum_{A \in \mathscr{P}_m} \mu(A)^\beta \, |A|^{F(\beta)}$$

Let now R be a fixed, positive, real number larger than $\log_2 \rho$; we let m and n tend to $+\infty$ in such a way that $n/m \to R$. Taking the log of the above equality, we get

$$G_D(\beta, F(\beta)) \geqslant -2R\varepsilon$$

and since ε is arbitrary,

$$G_D(\beta, F(\beta)) \geqslant 0$$

The estimate for the upper bound is done similarly. If $A \in \mathscr{P}_m$ with $\mu(A) > 0$, we denote by U the set

$$U = \bigcup_{\substack{I \in A_n \\ I \subset A}} I$$

Let n_2 be the smallest integer such that

$$\bigcup_{j \in D} K_j \subset g^{n_2}(U)$$

(by this we mean the image by g^{n_2} of the subset of U contained in the domain of g^{n_2}). If n is larger than $m + n_2 + [-\log \gamma^{-1} l]$ and $I \subset U$, then $I \in A_n$ is in the domain of g^{n_2}, and we have

$$g^j(\mathrm{Int}\{I\}) \cap S = \varnothing$$

for $0 \leqslant j \leqslant n_2$. Using the Distortion Lemma, we can conclude as above that for some finite, positive constant $C_9(\beta)$ we have

$$\sum_{\substack{I \in A_n \\ I \subset A}} \mu(I)^\beta \geqslant \mu(A)^\beta \, C_9(\beta) \, Z^{n_1}(\beta)$$

and the result follows as before. ▮

Corollary 2.8. If $D_2 G_D \neq 0$, F is C^1.

The proof follows from the inverse function theorem.

3. HAUSDORFF DIMENSIONS OF THE SINGULARITIES

In this section, we shall analyze the Hausdorff dimension of the sets where the singularity of the measure μ takes a given value. In other words, we shall give an expression for the Hausdorff dimension of the sets $B_t^{\pm}$. As we shall see, these two sets have the same Hausdorff dimension. Note, however, that this does not imply that the sets are the same; they may, for example, have different Hausdorff measure. The large-deviation results that we are about to use are not precise enough for the analysis of such fine details. An analysis of the speed of convergence to the thermodynamic limit may provide more information on these singularity sets. The main result of this section can be formulated as follows. Let f denote the Legendre transform of the free energy F. Then, using the same assumptions and definitions as before, we have the following result.

Theorem 3.1. B_t^+ and B_t^- have the same Hausdorff dimension, which is equal to $f(t)$.

The proof will be given in two distinct steps. We shall first establish an upper bound for the Hausdorff dimension. This is rather easy, using the result of large-deviation theory. We shall count the number of balls of a given radius that are needed to cover $B_t^\pm$. From the previous definitions and the results of Section 2, the answer to this question is exactly provided by the large-deviation theory.

In a second step, we shall prove a lower bound. This turns out to be more difficult. We shall in fact give lower bounds on the Hausdorff dimension of subsets of $B_t^\pm$. We shall use a result due to Frostman, which can be formulated as follows. Let $D_r(P)$ denote the ball in $\mathbb{R}^n$ with radius r, centered at P.

Lemma 3.2 (Frostman). Let L be a Borel subset of $\mathbb{R}^n$ and assume that there is a Borel probability measure ν on $\mathbb{R}^n$ such that:

(i) $\nu(L)=1$.

(ii) There are two positive, finite numbers C and δ such that

$$\nu(D_r(P)) \leqslant Cr^\delta$$

Then the δ Hausdorff measure of L is positive.

We shall use a weaker version of the above theorem, which says that under the same hypothesis the Hausdorff dimension of L is at least δ. The proof of Frostman's Lemma as stated above is an easy exercise. Deeper results can be found in Refs. 9 and 18. To prove a lower bound on the Hausdorff dimension of $B_t^\pm$ we shall construct a measure as above. The interesting fact is that this measure does not have to be related to the dynamical system. It will be constructed by giving explicit weights to atoms of the partitions at different scales. These weights will again be chosen according to the statistics of the large deviations.

We now start proving upper bounds on the Hausdorff dimensions. For convenience, we shall split the proof of Theorem 3.1 into several lemmas. If A is a subset of K, we shall denote by $\mathrm{HD}(A)$ its Hausdorff dimension and by $\mathrm{HDM}_\delta(A)$ its δ Hausdorff measure (see Ref. 10). From our hypothesis on F it follows that f is $-\infty$ except on an interval $]t_1, t_2[$, where it is a C^1 concave function with a maximum at some point t_0 which corresponds to $\beta=0$ for F.

Lemma 3.3:

(i) If $t_1 < t < t_0$, then $\mathrm{HD}(B_t^-) \leqslant f(t)$.

(ii) If $t_2 > t > t_1$, then $\mathrm{HD}(B_t^+) \leqslant f(t)$.

Proof. We shall start by proving (i). The result will follow from the equality

$$\mathrm{HDM}_\tau(B_t^-) = 0 \qquad \forall \tau > f(t)$$

More precisely, we shall show that for any (small) positive number ε and for any positive number δ small enough, we have

$$\mathrm{HDM}_{\tau,\delta}(B_t^-) \leqslant \varepsilon$$

where $\mathrm{HDM}_{\tau,\delta}(A)$ is defined by

$$\mathrm{HDM}_{\tau,\delta}(A) = \inf_{\substack{A \subset \bigcup_i B_i \\ \mathrm{diam}(B_i) \leqslant \delta}} \sum_i [\mathrm{diam}(B_i)]^\tau$$

Here the B_i are balls and $\mathrm{diam}(B_i)$ denotes their diameter. The final result will follow from $\mathrm{HDM}_\tau(A) = \lim_{\delta \to 0} \mathrm{HDM}_{\tau,\delta}(A)$. Since τ is larger than $f(t)$ and f is continuous, we can find a positive number ε such that

$$\tau > f(t + 2\varepsilon) + \varepsilon$$

We now impose an upper bound on δ and apply the large-deviation theorems as follows.[5,26] We first observe that the cardinality $\#(A_n)$ of the partition A_n defined in Section 2 is bounded by

$$2^n/5 \leqslant \#(A_n) \leqslant 2^n$$

We equip the discrete set A_n with a probability measure that is simply the counting measure and define on A_n a random variable X_n by

$$\begin{aligned} X_n(I) &= 0 && \text{if} \quad \mu(I) = 0 \\ X_n(I) &= \log \mu(I) && \text{otherwise} \end{aligned}$$

The first choice is of course arbitrary; a more natural value for $X_n(I)$ would be $-\infty$, but this is not very convenient from the point of view of measure theory.

It follows at once from the definitions that $\forall \beta \geqslant 0$, we have

$$\mathbb{E}(\exp \beta X_n) = Z^n(\beta)/\#(A_n)$$

We can now apply the large-deviation theorem to conclude that for a given positive ε, there is an integer $n_0(\varepsilon)$ such that if $n > n_0(\varepsilon)$, the number of dyadic intervals I of length 2^{-n} satisfying

$$\mu(I) \geqslant |I|^{t+2\varepsilon}$$

is bounded by

$$2^{n[f(t+2\varepsilon)+\varepsilon]}$$

If $\beta < 0$, we proceed similarly, with X_n defined by

$$X_n(I) = \begin{cases} +\infty & \text{if} \quad \mu(I) = 0 \\ \log \mu(I) & \text{otherwise} \end{cases}$$

Let m be an integer larger than $n_0(\varepsilon) + 4$ and take $\delta = 2^{-m}$. If a point x belongs to B_t^-, there is a sequence of open intervals $(J_r)_{r \in \mathbb{N}}$ such that

$$\{x\} = \bigcap_r J_r, \qquad |J_r| \to 0, \qquad \log \mu(J_r)/\log |J_r| \to t \quad \text{as } r \to +\infty$$

Note, however, that the intervals J_r may not belong to any of our partitions A_n. For r large enough, we have

$$\mu(J_r) \geqslant |J_r|^{t+\varepsilon} \qquad \text{and} \qquad \delta \geqslant 2\,|J_r|$$

We now observe that there are at most six intervals $I_1,\ldots, I_6$ in some A_n such that $J_r \subset I_1 \cup \cdots \cup I_6$ and

$$6 \inf_i |I_i| \geqslant |J_r| \geqslant \sup_i |I_i|/6$$

Therefore

$$\sup_{i=1,\ldots,6} \mu(I_i) \geqslant \mu(J_r)/6 \geqslant |J_r|^{t+\varepsilon}/6 \geqslant 6^{-2-t-\varepsilon} \sup_{i=1,\ldots,6} |I_i|^{t+\varepsilon}$$

This implies that at least one of the intervals I satisfies

$$\mu(I) \geqslant |I|^{t+2\varepsilon}$$

if m is large enough. We therefore get

$$\mathrm{HDM}_{\tau,\delta}(B_t^-) \leqslant \sum_{n=m}^{+\infty} 2^{n[f(t+2\varepsilon)+\varepsilon]}\, 6^{\tau} 2^{-n\tau}$$

since each J_r is covered by a ball of radius $6 \cdot 2^{-n}$ centered in the middle of I. It follows that for m large enough

$$\mathrm{HDM}_{\tau,2^{-m}}(B_t^-) \leqslant \varepsilon$$

The proof of (ii) is rather similar. We again use the large-deviation theorem (with $\beta<0$) to estimate the number of intervals I of length 2^{-n} satisfying

$$\mu(I) \leqslant |I|^{t+2\varepsilon}$$

For $x \in B_t^+$, we introduce again a sequence $(J_r)_{r\in\mathbb{N}}$ of intervals, which satisfy

$$\mu(J_r) \leqslant |J_r|^{t+\varepsilon}$$

for r large enough. There is an interval I in some A_n such that $I \subset J_r$ and $|I| \geqslant |J_r|/6$. Therefore

$$\mu(I) \leqslant \mu(J_r) \leqslant |J_r|^{t+\varepsilon} \leqslant 6^{t+\varepsilon}\,|I|^{t+\varepsilon}$$

and J_r is contained in a ball of radius $6\,|I|$ centered in the middle of I. The upper bound follows as before. ▌

Remark. We have in fact shown the following more general result (although we shall not use it).

Lemma 3.4. Let $\tilde{B}_t$ be the set of points x such that t is an accumulation point of

$$\log \mu(J)/\log |J| \qquad \text{when} \quad |J| \to 0$$

Then if $t \neq t_0$, $\mathrm{HD}(\tilde{B}_t) \leqslant f(t)$.

The next lemma provides a lower bound on the dimension of B_t^+ and B_t^-.

Lemma 3.5. Let $t \neq t_0$ be a given positive number. There is a set V_t such that

(i) $\mathrm{HD}(V_t) \geqslant f(t)$

(ii) $V_t \subset B_t^+ \cap B_t^-$.

Proof. We shall consider first the case $t_1 < t < t_0$ and we fix t in that range. To simplify notation we shall write V for $V_t(\varepsilon)$ when there is no ambiguity.

We shall construct recursively a sequence $(F_j)_{j\in\mathbb{N}}$ of families of intervals. This construction will depend on two sequences of positive real numbers $(r_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$ satisfying, for $j \to \infty$,

$$r_j \to \infty, \qquad r_{j+2}\Big/\sum_{l=1}^{j} r_l \to 0, \qquad \delta_j \to 0$$

We shall also require that $r_{j+1} - r_j > 3 \log l^{-1}/\log \chi$, and that the sequence δ_j is nonincreasing. These two sequences will be chosen suitably later, and we define now the families F_j for $j \in \mathbb{N}$. The family F_0 is the family with only one element K. Assume the families $F_1, F_2, ..., F_{p-1}$ have already been defined, we shall now construct the family F_p. Let J be an element of F_{p-1} and let q be the smallest integer such that $|g^q(J)| \geqslant l$ (see Section 2 for the definition of l and of a regular interval). It will follow from the recursive assumptions that J is regular. We shall collect in F_p all the subintervals I of J with the following properties:

1. $g^q(I)$ belongs to A_{r_p}.
2. $|g^q(I)|^{t+\delta_p} \leqslant \mu(g^q(I)) \leqslant |g^q(I)|^{t-\delta_p}$.
3. The distance from $g^q(I)$ to the boundary of $g^q(J)$ is larger than $2^{-\sqrt{r_p}}$.

It should become clear later that from our choice of the sequences $(r_p)_{p \in \mathbb{N}}$ and $(\delta_j)_{j \in \mathbb{N}}$, F_p is not empty.

The set V is defined by

$$V = \bigcap_{p=0}^{\infty} \left(\bigcup_{I \in F_p} I \right)$$

We shall now compute the values of the function T^+ and T^- on V. Let x be a point in V and let H be an open interval containing x. Let p be the largest integer such that $H \subset I$, where $I \in F_p$. Let I' be the atom of F_{p+1} that contains x. We have, using the Distortion Lemma,

$$1 \geqslant d(x, \partial I')/|I'| \geqslant \gamma^{-1} 2^{-\sqrt{r_{p+2}}}$$

This implies

$$|H| \geqslant |I'| \, \gamma^{-1} 2^{-\sqrt{r_{p+2}}}$$

since otherwise H would be contained in I' (contradicting the maximality of p). We have from fact 2 in the definition of F_p

$$\begin{aligned} \mu(H) &\leqslant (\mu(I)/\mu(I')) \times \mu(I') \\ &\leqslant \mu(g^q(I))/\mu(g^q(I')) \times \mu(I') \\ &\leqslant |g^q(I')|^{-t-\delta_p} \mu(I') \\ &\leqslant O(1) \, 2^{r_{p+1}(t+\delta_p)} \mu(I') \end{aligned}$$

We shall now construct a lower bound on $\mu(H)$. The difficulty here is that x may be very near to the boundary of H. We have to deal with this situation because we have not imposed any restriction on the intervals that appear in the definitions of the functions $T^{\pm}$. Let I'' be the atom of F_{p+2} containing x. There are two cases.

Case 1. $d(x, \partial H)/|H| > 2^{-r_{p+2}/2}$. In this case, $I'' \subset H$ and we have of course

$$\mu(H) \geqslant \mu(I'')$$

Case 2. $d(x, \partial H)/|H| < 2^{-r_{p+2}/2}$. Let q_1 be the largest integer such that for $0 < q < q_1$, g^q is differentiable on H, and $g^{q_1}(H) \cap S \neq \emptyset$. Let q' be the corresponding integer for I'. Note that

$$|g^{q'}(I')| \geqslant l \qquad \text{and} \qquad d(g^{q'}(x), \partial g^{q'}(I')) \geqslant 2^{-\sqrt{r_{p+2}}}$$

Let y be the unique point in H such that $g^{q_1}(y) \in S$, and z the unique point in I' such that $g^{q'}(z) \in S$. There are now four cases.

Case 2.1. $q_1 = q'$ and $y \neq z$. Let U denote the closed interval with boundaries y and z. Since $x \in V$ and $x \in \operatorname{Int}\{U\}$, we must have $\mu(U) > 0$. Let $U_1 = U \cap H$ and $U_2 = U \cap I'$; we have of course $\mu(H) \geqslant \mu(U_1)$, and also

$$\mu(U_1) = \frac{\mu(g^{q'}(U_1))\,\mu(U_2)}{\mu(g^{q'}(U_2))\,\mu(I')}\,\mu(I')$$

Since $d(g^{q'}(x), \partial g^{q'}(I')) \geqslant 2^{-\sqrt{r_{p+2}}}$, we get, using Lemma 2.2,

$$\begin{aligned}\mu(U_2)/\mu(I') &\geqslant \mu(g^{q'}(U_2))/\mu(g^{q'}(I'))\\ &\geqslant C^{-1}\,|g^{q'}(U_2)|^{\sigma_1} \geqslant C^{-1}\,2^{-\sigma_1\sqrt{r_{p+2}}}\end{aligned}$$

Similarly, from $|g^{q'}(U_1)| \geqslant 2^{-\sqrt{r_{p+2}}}$ we get

$$\mu(g^{q'}(U_1)) \geqslant C^{-1}\,2^{-\sigma_1\sqrt{r_{p+2}}}$$

Combining these estimates, we have

$$\mu(H) \geqslant C^{-2}\,2^{-2\sigma_1\sqrt{r_{p+2}}}\mu(I')$$

Case 2.2. $q_1 = q'$, $y = z$. We denote by U the segment with boundary points y and x. We now observe that if I'' denotes the interval of $F_{r_{p+2}}$ containing x, and if q_2 is the smallest integer such that $|g^{q_2}(I')| > l$, we have $q_2 \leqslant q'$, and $q' - q_2 \leqslant \log l^{-1}/\log \chi$. If r_{p+2} is large enough so that

$$3\rho^{-\log l/\log \chi}\,2^{-r_{p+2}} < l$$

we have

$$|g^{q'}(I'')| \leqslant l$$

On the other hand, if q_3 is the smallest integer such that

$$|g^{q_3}(I'')| \geqslant l$$

we have

$$q_3 \leqslant q_1 + r_{p+2}/\log_2 \chi \leqslant q' + r_{p+2}/\log_2 \chi$$

From

$$d(g^{q_3}(x), \partial g^{q_3}(I'')) > 2^{-\sqrt{r_{p+3}}}$$

and $g^{q'}(I'') \cap S = \varnothing$, we get

$$|g^{q'}(x) - g^{q'}(y)| \geqslant \gamma^{-1} \rho^{-r_{p+2}/\log_2 \chi}\, 2^{-\sqrt{r_{p+3}}}$$

We can proceed as in the previous case, i.e.,

$$\begin{aligned}\mu(H) \geqslant \mu(U) &= \mu(g^{q'}(U))/\mu(g^{q'}(I')) \times \mu(I')\\ &\geqslant C^{-1} \gamma^{-\sigma_1} \rho^{-\sigma_1 r_{p+2}/\log_2 \chi}\, 2^{-\sigma_1 \sqrt{r_{p+3}}} \mu(I')\end{aligned}$$

Note that $\mu(g^{q'}(U)) \neq 0$ since $\mu(I') \neq 0$, and there is a number p' such that if $I_1 \in A_{r_p}$, and $x \in I_1$, then $|I_1| < |U|/2$, and $\mu(I_1) > 0$.

Case 2.3. $q_1 < q'$. Let U be the interval with boundary points y and z, and $U_1 = U \cap H$. Since $q_1 < q'$, we have $y \notin I'$. There are now two cases.

Case 2.3.1. $z \notin H$. We set $U_2 = I' \cap U$, and we use the same argument as in Case 2.2

$$\begin{aligned}\mu(U_1) &= [\mu(g^{q_1}(U_1))/\mu(g^{q_1}(U_2))] \times [\mu(g^{q'}(U_2))/\mu(g^{q'}(I'))]\ \mu(I')\\ &\geqslant \mu(I')\, C^{-2}\, 2^{-\sigma_1 \sqrt{r_{p+2}}} \gamma^{-\sigma_1} \rho^{-\sigma_1 r_{p+2} \log \chi}\, 2^{-\sigma_1 \sqrt{r_{p+3}}}\end{aligned}$$

Case 2.3.2. $z \in H$. We denote by U_2 the interval with boundary points z and x. We have

$$\mu(U_1) = [\mu(g^{q_1}(U_1))/\mu(g^{q_1}(U_2))] \times [\mu(g^{q'}(U_2))/\mu(g^{q'}(I'))] \times \mu(I')$$

As in Case 2.2, we have

$$\begin{aligned}|g^{q'}(U_1)| &\geqslant \gamma^{-1} \rho^{-r_{p+2}/\log_2 \chi}\, 2^{-\sqrt{r_{p+3}}}\\ |g^{q'}(U_2)| &\geqslant \gamma^{-1} \rho^{-r_{p+2}/\log_2 \chi}\, 2^{-\sqrt{r_{p+3}}}\end{aligned}$$

Hence,

$$\mu(U_1) \geqslant C^{-2} \gamma^{-2\sigma_1} \rho^{-2\sigma_1 r_{p+2}/\log_2 \chi}\, 2^{-2\sigma_1 \sqrt{r_{p+3}}} \mu(I')$$

Case 2.4. $q_1 > q'$. Let U be as above, and let $U_2 = U \cap I'$. We now have $z \notin H$. There are again two cases.

Case 2.4.1. $y \notin I'$. We set $U_1 = H \cap U$, and we have

$$\begin{aligned}\mu(U_1) &= [\mu(g^{q_1}(U_1))/\mu(g^{q_1}(U_2))] \times [\mu(g^{q'}(U_2))/\mu(g^{q'}(I'))]\ \mu(I') \\ &\geqslant C^{-2}\, 2^{-\sigma_1\sqrt{r_p+3}}\, \gamma^{-\sigma_1} \rho^{-\sigma_1 r_p + 2\log\chi}\, 2^{-\sigma_1\sqrt{r_p+2}} \mu(I')\end{aligned}$$

Case 2.4.2. $y \in I'$. We denote by U_1 the interval with boundary points y and x, and we get

$$\mu(U_1) \geqslant C^{-2}\, \gamma^{-2\sigma_1} \rho^{-2\sigma_1 r_p + 2\log_2\chi}\, 2^{-2\sigma_1\sqrt{r_p+3}} \mu(I')$$

Combining all the cases, we have shown the following inequalities:

$$O(1)\, 2^{r_{p+1}(t+\delta_p)} \mu(I') \geqslant \mu(H) \geqslant C_2 2^{-2\sigma_1 r_p + 2\log_\chi \rho}\, 2^{-2\sigma_1\sqrt{r_p+3}} \mu(I')$$

We shall now estimate $\mu(I)$ [and $\mu(I')$ similarly]. Let J be the element of F_{p-1} containing I. Let u be the smallest integer such that $|g^u(J)| \geqslant l$. To estimate $\mu(g^u(J))$, we let s_1 be the smallest integer such that

$$g^{u+s_1}(J) \cap S \neq \varnothing$$

We have of course $s_1 \leqslant \log l^{-1}/\log\chi$. We first observe that

$$d(g^u(x), \partial g^u(J)) \geqslant 2^{-\sqrt{r_p}}$$

Therefore

$$d(g^{u+s_1}(x), \partial g^{u+s_1}(J)) \geqslant \chi^{s_1}\, 2^{-\sqrt{r_p}}$$

We can write $g^{u+s_1}(J) = U_1 \cup U_2$, $\operatorname{Int}\{U_1\} \cap \operatorname{Int}\{U_2\} = \varnothing$, $\operatorname{Int}\{U_j\} \cap S = \varnothing$, for $i = 1, 2$, and $x \in U_1$. As before, we must have $\mu(U_1) > 0$, and Lemma 2.2 implies

$$\mu(U_1) \geqslant C^{-1} \chi^{\sigma_1 s_1}\, 2^{-\sigma_1\sqrt{r_p}}$$

We conclude that

$$\mu(g^u(J)) \geqslant C_3^{-1}\, 2^{-\sigma_1\sqrt{r_p}}$$

where

$$C_3^{-1} = [\inf_{\substack{\mu(K_i)\neq 0 \\ i\in D}} \mu(K_i)]^{\log l^{-1}/\log\chi}\, C^{-1}$$

Therefore

$$C_3^{-1} 2^{-\sigma_1\sqrt{r_p}} \mu(g^u(I)) \leqslant \mu(I)/\mu(J) \leqslant C_3 2^{\sigma_1\sqrt{r_p}} \mu(g^u(I))$$

Note that the lower bound is rather poor. We have used it for the symmetry of the formula. From condition 2 in the definition of F_p, we get

$$C_3^{-1} 2^{-\sigma_1 \sqrt{r_p}} \, |g^u(I)|^{t+\delta_p} \leqslant \mu(I)/\mu(J) \leqslant C_3 2^{\sigma_1 \sqrt{r_p}} \, |g^u(I)|^{t-\delta_p}$$

If we define C_4 by

$$C_4 = l^{-(t+\delta_0)} \, \gamma^{(t+\delta_0)} C_3$$

we get from the Distortion Lemma

$$\begin{aligned} &2^{-\sigma_1 \sqrt{r_p}} C_4^{-1} (|I|^{\delta_p + t} / |J|^{\delta_{p-1} + t}) \, |J|^{\delta_{p-1} - \delta_p} \\ &\qquad \leqslant \mu(I)/\mu(J) \leqslant C_4 (|I|^{t-\delta_p} / |J|^{t - \delta_{p-1}}) \, |J|^{\delta_p - \delta_{p-1}} \, 2^{\sigma_1 \sqrt{r_p}} \end{aligned}$$

Again from the Distortion Lemma and from the definition of F_p, we have

$$\gamma^{-1} \rho^{-1} 2^{-r_p}/3 \leqslant |I|/|J| \leqslant 3 \gamma l^{-1} \, 2^{-r_p}$$

Therefore,

$$C_5^{-p} \, 2^{-\Sigma_0^p r_j} \leqslant |I| \leqslant C_5^p 2^{-\Sigma_0^p r_j}$$

where

$$C_5 = 3 \gamma \rho l^{-1} (\inf_{U \in F_0} |U|)^{-1}$$

Combining recursively the above inequalities, we get, with $C_8 = C_4 C_5$,

$$\begin{aligned} &C_5^{-1-\delta_0} C_8^{-p} \, 2^{-\sigma_1 \Sigma_0^p \sqrt{r_j}} \, 2^{-\Sigma_0^{p-1} r_j \delta_j} \, 2^{-\delta_p \Sigma_0^{p-1} r_j} \, |I|^{t+\delta_p} \\ &\qquad \leqslant \mu(I) \leqslant C_5^{1+\delta_0} \, C_8^p \, 2^{\sigma_1 \Sigma_0^p \sqrt{r_j}} 2^{\Sigma_0^{p-1} r_j \delta_j + \delta_p \Sigma_0^{p-1} r_j} \, |I|^{t-\delta_p} \end{aligned}$$

We now use the fact that there is a constant $a > 5$ such that

$$\sum_0^p r_j \geqslant a p \delta_p / 5$$

$$r_p \Big/ \sum_0^{p-1} r_j \leqslant a/5, \qquad (r_{p+2})^{1/2} \Big/ \sum_0^{p-1} r_j \leqslant a/5$$

and we deduce for $p \geqslant 1$ with

$$\omega_p = a \delta_p + a \left(r_p + \sum_0^{p-1} r_j \delta_j + \sum_0^{p-1} \sqrt{r_j} \right) \Big/ \sum_0^{p-1} r_j$$

that

$$|I|^{t+\omega_p} \leqslant \mu(I) \leqslant |I|^{t-\omega_p}$$

Using the similar estimate for I', and the various upper and lower bounds, we get

$$C_2\, 2^{-\sigma_1 r_{p+2}\log_2 \rho} 2^{-\sigma_1\sqrt{r_{p+3}}}\, |I'|^{t+\omega_{p+1}} \leqslant \mu(H) \leqslant O(1)\, 2^{r_{p+1}(t+\delta_p)}\, |I'|^{t-\omega_{p+1}}$$

i.e.,

$$C_2 2^{-\sigma_1 r_{p+2}\log_2 \rho}\, 2^{-\sigma_1\sqrt{r_{p+3}}}\, \gamma^{-(t+a\delta_{p+1})}\, 2^{-(t+a\delta_{p+1})r_{p+1}}\, |H|^{t+\omega_{p+1}}$$
$$\leqslant \mu(H) \leqslant O(1)\, \gamma^{t-\omega_{p+1}}\, 2^{(t-a\delta_{p+1})\sqrt{r_{p+2}}}\, 2^{r_{p+1}(t+\delta_p)}\, |H|^{t-\omega_{p+1}}$$

This implies

$$|H|^{t+2\omega_{p+1}} \leqslant \mu(H) \leqslant |H|^{t-2\omega_{p+1}}$$

Therefore, since $p \to +\infty$ when $|H| \to 0$, implying $\omega_p \to 0$, we get

$$T^+(x) = T^-(x) = t$$

We shall now obtain a lower bound on the Hausdorff dimension of V. Each family F_p induces a finite partition of the set V. These partitions induce on V a σ-algebra $\mathscr{B}$, and it is easy to verify that $\mathscr{B}$ is the Borel σ-algebra.[14] We shall now construct a Borel measure ν on V by specifying the value of ν on each atom of each family F_p. This is done recursively as follows. We first set $\nu(K) = 1$. Assuming ν has already been defined for the elements of $F_0,..., F_{p-1}$, we define ν on F_p as follows. If I belongs to F_p, there is a unique interval J in F_{p-1} containing I. We set

$$\nu(I) = \nu(J)/\#\{H \in F_p \mid H \subset J\}$$

In other words, $\nu(\cdot \mid J)$ when restricted to F_p is the counting measure. It is easy to verify that ν extends to a unique Borel probability measure on K with $\nu(K) = 1$. We shall now estimate $\nu(I)$ for I in F_p. As before, there is a unique element J in F_{p-1} containing I, and we denote by u the smallest integer such that

$$|g^u(J)| \geqslant l$$

In order to estimate $\nu(I)/\nu(J)$, we have to count the number of intervals Y in A_{r_p} such that:

1. $Y \subset g^u(J)$
2. $|Y|^{t+\delta_p} \leqslant \mu(Y) \leqslant |Y|^{t-\delta_p}$
3. $d(Y, \partial g^u(J)) \geqslant 2^{-\sqrt{r_p}}$

Since we only need an upper bound on $\nu(I)$, we shall only produce a lower bound on the number of atoms of A_{r_p} satisfying 1–3. This estimate will follow again from the large-deviation theorem. However, we shall apply this theorem to $g^u(J)$ instead of K, and this will require some more estimates. We shall in fact show that the free energy is the same. Let U be the union of the intervals in A_{r_p} satisfying 1 and 3. We have $|U| \geqslant l/2$. Let q be the smallest integer such that

$$g^q(U) \cap S \neq \varnothing$$

We observe that $q \leqslant 2 \log l^{-1}/\log \chi$. We can write $g^q(U) = U_1 \cap U_2$, where

$$(\mathrm{Int}\{U_1\} \cup \mathrm{Int}\{U_2\}) \cap S = \varnothing$$

and $\mu(U_1) \geqslant \theta/2$ (see Lemma 2.4). By Lemma 2.2, this implies

$$|U_1| \geqslant l^\zeta$$

with $\zeta > 0$ independent of l for l small enough. Let q_1 be the smallest integer such that $g^{q_1}(\bar{U}_1) \cap S$ consists of at least two points. Since $q_1 \leqslant 6\zeta \log l^{-1}/\log \chi$, we deduce that $q_1 < r_j\ \forall j > 1$. Therefore,

$$g^{q_1}(\bar{U}_1) = K_m$$

for some index m belonging to D. We now reduce U_1 such that $g^{q_1}(\bar{U}_1) = K_m$. Note that $\mu(\bar{U}_1) > 0$ from Lemma 2.2.

Let q_2 denote the smallest integer such that $\inf_{m \in D} |g^{q_2}(K_m)| = 1$. We now consider the following situation. Let W be an interval such that $|W| > l^\zeta 6$ and for some integer $q_3 < q_2 + 12\zeta \log l^{-1}/\log \chi$, we have

$$g^{q_3}(W) = K$$

Let

$$Z_W^{(n)}(\beta) = \sum_{\substack{Y \in A_n \\ Y \subset W, \mu(Y) \neq 0}} \mu(Y)^\beta$$

where n is larger than $n_3 = 1 + [\zeta \log_2 l^{-1} \rho^{q_3}]$. Let

$$n_4 = 1 + [\zeta \log \rho l^{-2} \gamma/\log \chi]$$

For $Y \in A_n$, $Y \subset W$, we can cover Y by intervals $I_1,\dots, I_s$ belonging to A_{n+n_4}, $0 < s < 3 \cdot 2^{n_4}$. Assume $\mu(Y) \neq 0$; then we have two cases:

Case 1. There is an interval I_j such that $\mu(I_j) \neq 0$, and $\mathrm{Int}\{I_j\} \subset Y$. In this case we have

$$\mu(I_j) \leqslant \mu(Y) \leqslant 2^{n_4} \sup_{\substack{L \in A_{n+n_4} \\ L \cap Y \neq \varnothing}} \mu(L)$$

Case 2. If $I_j \in A_{n+n_4}$ and $\mu(I_j) \neq 0$, then $I_j \not\subset Y$. As was already explained in the proof of Theorem 2.6, this situation cannot occur, because of Lemma 2.2. This shows that for some constant $C_5(\beta)$ we have

$$Z_W^{(n)}(\beta) \leqslant C_5(\beta)\, Z^{(n)}(\beta)$$

for any $\beta \in \mathbb{R}$.

We now prove the lower bound in a similar way. We first observe that

$$|g^{q_3}(Y)| < l \quad \forall Y \in A_n, \qquad Y \subset W$$

if $n > n_3$. Next, we have, if $\mu(Y) \neq 0$,

$$b^{q_3}\mu(g^{q_3}(Y)) \leqslant \mu(Y) \leqslant b^{-q_3}\mu(g^{q_3}(Y))$$

where the constant b was defined in Lemma 2.4. We now cover $g^{q_3}(Y)$ by atoms of A_{n+n_4}. We need at most

$$3 \cdot 2^{n_4}\, 2^{[\log \gamma]+1+q_3[\log \rho]}$$

such atoms (from the Distortion Lemma), and we conclude as before that if $\mu(Y) \neq 0$, there is an interval $I_j \in A_{n+n_4}$ with $\mu(I_j) \neq 0$ and $I_j \subset g^{q_3}(Y)$. We have

$$\mu(I_j) \leqslant \mu(g^{q_3}(Y)) \leqslant 3 \cdot 2^{n_4}\, 2^{1+[\log \gamma]+q_3[\log \rho]} \sup_{\substack{L \in A_{n+n_4} \\ L \cap g^{q_3}(Y) \neq \varnothing}} \mu(L)$$

We now have

$$Z_W^{(n)}(\beta) \geqslant \sum_{\substack{Y \in A_n \\ Y \subset W \\ \mu(Y) \neq 0}} b^{q_3\beta}\mu(g^{q_3}(Y))^{\beta}$$

Since W is a union of atoms of A_n and $g^{q_1}(W) = K$, we obtain the required lower bound:

$$Z_W^{(n)}(\beta) \geqslant C_6(\beta)\, Z^{(n)}(\beta)$$

Therefore,

$$-n^{-1} \log Z_W^{(n)}(\beta) \xrightarrow[n \to \infty]{} F(\beta)$$

Note also that the above convergence is uniform in W (since C_5 and C_6 can be chosen independent of W).

From the large-deviation theorem we conclude that for any given positive number ε there is a positive integer $n(\varepsilon)$ such that if $n > n(\varepsilon)$, we have for $t > t_0$

$$\#\{Y \in A_n \mid \mu(Y) > 0,\ Y \subset W, \text{ and } \mu(Y) \leqslant |Y|^{t+\varepsilon}\} \leqslant 2^{n[f(t+\varepsilon)+\eta\varepsilon]}$$

and

$$\#\{Y \in A_n \mid \mu(Y) > 0,\ Y \subset W, \text{ and } \mu(Y) \leqslant |Y|^{t-\varepsilon}\} \geqslant 2^{n[f(t-\varepsilon)-\eta\varepsilon]}$$

where $\eta = |f'(t)|/2$. We may, and will, assume that $n(\varepsilon) \to \infty$ if $\varepsilon \to 0$. Therefore

$$\begin{aligned}\#\{Y \in A_n \mid \mu(Y) > 0,\ Y \subset W, \text{ and } |Y|^{t-\varepsilon} \leqslant \mu(Y) \leqslant |Y|^{t+\varepsilon}\}\\ \geqslant 2^{n[f(t)-3\eta\varepsilon]}\,\eta\varepsilon_1/2\end{aligned}$$

provided ε is smaller than some fixed positive number ε_1 and $n\varepsilon \geqslant \varepsilon_1$. A similar estimate holds for $t < t_0$. We now choose the sequences $(r_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$. Let n_6 be defined by

$$n_6 = \sup(\varepsilon_1^{-1}, 6\zeta \log \rho l^{-1}/\log \chi)$$

We set $r_j = jn_6$, and we define δ_j recursively as follows. We set $\delta_1 = \varepsilon_1$. Assume $\delta_1, \delta_2, \ldots, \delta_{j-1}$ have already been defined. Using the integer-valued function $n(\cdot)$, provided by the large-deviation theorem, we have an integer $n(\delta_{j-1})$. If $n(\delta_{j-1}) \geqslant r_j$, we set $\delta_j = \delta_{j-1}$. If $n(\delta_{j-1}) < r_j$, we define δ_j to be the smallest number ε such that $n(\varepsilon) \leqslant r_j$ and $\varepsilon r_j \geqslant \varepsilon_1$ (we may still have $\varepsilon = \delta_{j-1}$). We have $\delta_j \to 0$ if $j \to \infty$. It is now easy to verify that all our assumptions on the two sequences $(\delta_j)_{j\in\mathbb{N}}$ and $(r_j)_{j\in\mathbb{N}}$ are satisfied. Combining the above results, we get for $I \in F_p$

$$\nu(I) \leqslant (\eta\varepsilon_1/2)^{-p}\, 2^{-f(t)\sum_0^p r_j + 3\eta\sum_0^p \delta_j r_j}$$

Let now B_ξ be an interval of length ξ and such that $\nu(B_\xi) > 0$. In particular, we have $B_\xi \cap V \neq \emptyset$. As explained in the first part, if p is the largest integer such that $B_\xi \subset I$ for some I in F_p, we have

$$|I| \geqslant |B_\xi| \geqslant \gamma^{-1}\, 2^{\sqrt{r_{p+2}}}\, |I'|$$

for some $I' \in F_{p+1}$ with $\nu(I' \cap B_\xi) > 0$.

We have the following estimates (see above)

$$|I| \leqslant C_5 2^{-\sum_0^p r_j}, \qquad |I'| \geqslant C_5^{-1} 2^{-\sum_0^p r_j}$$

Therefore

$$\begin{aligned} \nu(B_\xi) &\leqslant |B_\xi|^{f(t)+O(1)[(p+r_{p+1}+\sqrt{r_{p+2}})/\Sigma_0^p r_j]+O(1)\omega_p} \\ &\leqslant C_7\, |B_\xi|^{f(t)} \end{aligned}$$

where C_7 is a finite constant which is uniform in p and ξ. The result follows from Frostman's lemma.[9,18] ∎

The theorem is now an obvious consequence of the above results.

Remark. It is easy to derive from the above analysis that $f(t_0)$ is the Hausdorff dimension of the support of the measure μ, and that t_1 defined by $f(t_1)=t_1$ is the Hausdorff dimension of that measure.

4. APPLICATIONS

4.1. Period Doubling

In this section, we shall give an application of the above theory to the accumulation point of period doubling. A real analytic mapping ϕ of the interval $[-1, 1]$, has been constructed in Refs. 19, 4, and 8. This map satisfies

$$\phi(\phi(\lambda x)) = -\lambda\phi(x), \qquad \phi(0)=1, \qquad \phi(x)=\phi(-x), \qquad \phi(1)=-\lambda$$

The dynamical system associated to ϕ has periodic orbits of periods 2^q for any integer q, and an invariant Cantor set Ω. Moreover, there is on Ω a unique invariant measure μ which describes the asymptotic behavior of almost any trajectory.[4,24] It was observed in Ref. 23 that if one defines K and g by

$$\begin{aligned} &K=K_1\cup K_2\cup K_3, \qquad K_1=[-\lambda, \lambda^2], \qquad K_2=[\lambda^2, \phi(\lambda)] \\ &K_3=[\phi(\lambda), 1], \qquad g|_{K_1}(x)=-x/\lambda, \qquad g|_{K_3}(x)=-\phi(x)/\lambda \end{aligned}$$

then ϕ and g have the same invariant set Ω and the same invariant measure μ. Moreover, the hypotheses of Section 2 are satisfied [in particular, $\mu(K_1)=\mu(K_3)=1/2$].

More generally, for unimodal maps of the interval, a renormalization transformation $\mathscr{R}$ is defined by (see Ref. 3 for more details)

$$\mathscr{R}(\psi)(x)=\psi(1)^{-1}\,\psi(\psi(\psi(1)x))$$

and ϕ is a fixed point of $\mathscr{R}$. If ψ belongs to the stable manifold of ϕ, then $\mathscr{R}^q\psi$ converges to ϕ exponentially fast in C^2. It is easy to generalize the above analysis to this new situation. One defines for $q \in \mathbb{N}$ sets K_q and maps g_q by

$$K_q = K_{q,1} \cup K_{q,2} \cup K_{q,3}$$

where

$$K_{q,1} = [\mathcal{R}^q(\psi)(1), (\mathcal{R}^q(\psi))^3(1)]$$
$$K_{q,2} = [(\mathcal{R}^q(\psi))^3(1), (\mathcal{R}^q(\psi))^2(1)]$$
$$K_{q,3} = [(\mathcal{R}^q(\psi))^2(1), 1]$$
$$g_q|_{K_{q,1}}(x) = x/\mathcal{R}^q(\psi)(1), \qquad g_q|_{K_{q,3}}(x) = \mathcal{R}^q(\psi)(x)/\mathcal{R}^q(\psi)(1)$$

Theorem 2.6 can be extended to this situation, and one can show that the free energy F is the same for all the maps ψ on the stable manifold of ϕ. It is easy to see that if $A \in \mathcal{P}_m$ (the dynamical partition introduced in Section 2.2), then $\mu(A) = 2^{-m}$. Therefore

$$\sum_{\substack{A \in \mathcal{P}_m \\ \mu(A) > 0}} |A|^y \, \mu(A)^x = 2^{-mx} Z_D^m(y)$$

where

$$Z_D^m(\beta) = \sum_{\substack{A \in \mathcal{P}_m \\ \mu(A) > 0}} |A|^\beta$$

In Ref. 30 it was shown that a C^2 function $F_D(\beta)$ is defined by

$$F_D(\beta) = \lim_{m \to +\infty} m^{-1} \log_2 Z_D^m(\beta)$$

Therefore, as in Section 2.2, we get $G_D(x, y) = F_D(y) - x$. From this equality we conclude that

$$F_D(F(\beta)) = \beta$$

i.e., F is the inverse function of F_D, and therefore differentiable.

The above equation can be easily generalized to the case of a sequence of maps converging exponentially fast to the map g. Since we know from Ref. 30 that the function F_D is universal, we obtain another proof that F is universal and therefore also its Legendre transform f. In other words, all the maps on the stable manifold of the fixed point ϕ have the same dimension spectrum.

4.2. Critical Maps of the Circle

We shall now give an application to the case of critical maps of the circle with golden rotation number. The precise hypotheses are as follows.[26] We shall denote by M the set of pairs of maps (ξ, η) satisfying:

(a) The numbers $\xi(0)$ and $\eta(0)$ satisfy $0 < \xi(0) = \eta(0) + 1 < 1$.

(b) ξ is an increasing map from $[\eta(0), 0]$ to $[\eta(0), \xi(0)]$. Similarly, η is an increasing map from $[0, \xi(0)]$ to $[\eta(0), \xi(0)]$.

(c) ξ is analytic in a complex neighborhood of $[0, \xi(0)]$. η is analytic in a complex neighborhood of $[0, \xi(0)]$. Both maps $\xi(z)$ and $\eta(z)$ are analytic functions of z^3. They have a critical point at $z=0$, and we shall assume that their third derivative in 0 is nonzero.

(d) $\xi \circ \eta = \eta \circ \xi$ whenever both sides of this equality are defined.

(e) ξ and η have only one critical point (namely 0) in the complex neighborhoods introduced in condition (c).

(f) $\xi(\eta(0)) > 0$; see condition (g) below.

Under the above hypotheses, the couple (ξ, η) can be used to construct a diffeomorphism ϕ of the circle by setting

$$\phi\,|_{[\eta(0),0]} = \xi \qquad \text{and} \qquad \phi\,|_{[0,\xi(0)]} = \eta$$

We shall also impose on ϕ the following condition:

(g) The rotation number of the diffeomorphism ϕ constructed with ξ and η is the golden number σ.

We refer the reader to Refs. 25 and 15 for a justification of these hypotheses.

A particularly interesting case is obtained as follows. Let h be a homeomorphism of the circle with rotation number σ. We can lift this homeomorphism to a map of the real line also denoted by h. We shall assume that the circle has length one, and we choose the lifting that satisfies $0 < h(0) < 1$. Assume h is a real analytic function of z^3. Assume also that h has only one critical point in $[-1/2, 1/2]$. Then it is easy to verify that the couple (ξ, η) defined by

$$\xi(x) = h(x) \qquad \text{and} \qquad \eta(x) = h(x) - 1$$

belongs to M.

The renormalization transformation is a map $\mathscr{R}$ from M into itself given by

$$\mathscr{R}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\alpha^{-1}\eta(\alpha.)\\ \alpha^{-1}\xi\circ\eta(\alpha.)\end{pmatrix}$$

where $\alpha = \eta(0) - \xi(\eta(0))$.[7,11,22,25,29] Note that from the above hypothesis we have $-1 < \alpha < 0$. We shall use the following theorem from renormalization theory.

Theorem 4.1. If (ξ, η) belongs to M, then $\mathcal{R}^n(\xi, \eta)$ converges exponentially fast in C^2 to an element (ξ_*, η_*) of M which is a fixed point of $\mathcal{R}$.

For a computer-assisted proof of this theorem see Refs. 29 and 22. For a direct proof see Ref. 7. The above result is formulated sometimes using maps of the circle instead of pairs of maps as above. For our purposes the above formulation is more convenient. If (ξ, η) belongs to M, we shall denote by (ξ_r, η_r) the couple $\mathcal{R}^r(\xi, \eta)$. As noted before, an element (ξ, η) of M defines a homeomorphism of the circle. This homeomorphism is C^0 conjugated to the rotation by the angle σ, and is therefore uniquely ergodic.[31] We shall now give a construction of this unique ergodic invariant measure. To do so, we first construct recursively a family $\mathcal{P}_p^r$, r, $p \in \mathbb{N}$, of finite partitions of the intervals $J_r = [\eta_r(0), \xi_r(0)]$. For $r \in \mathbb{N}$, we set

$$\mathcal{P}_0^r = \{J_r\}, \qquad \mathcal{P}_1^r = \{[\eta_r(0), \eta_r(\xi_r(0))],]\eta_r(\xi_r(0)), \xi_r(0)]\}$$

Assume now that the partitions $\mathcal{P}_q^r$ have been already constructed for $r \in \mathbb{N}$, we first note the following important relations:

Lemma 4.2. Let $\alpha_r = \eta_r(0) - \xi_r(\eta_r(0))$. Then:

(i) $\xi_r(\eta_r(0)) = \alpha_r \eta_{r+1}(0)$.

(ii) $\eta_r(\eta_r(\xi_r(0))) = \alpha_r \xi_{r+1}(\eta_{r+1}(0))$.

(iii) $\alpha_r^{-1}[\eta_r(0), \eta_r(\xi_r(0))] = J_{r+1}$.

(iv) $\eta_r(]\alpha_r \eta_{r+1}(0), \xi_r(0)]) = \alpha_r[\eta_{r+1}(0), \xi_{r+1}(\eta_{r+1}(0))[$.

The proof is immediate from the definition of $\mathcal{R}$.

We now define $\mathcal{P}_p^r$ as follows:

$$\mathcal{P}_p^r|_{[\eta_r(0), \eta_r(\xi_r(0))]} = \alpha_r \mathcal{P}_{p-1}^{r+1}$$
$$\mathcal{P}_p^r|_{]\eta_r(\xi_r(0)), \xi_r(0)]} = \eta_r^{-1}(\mathcal{P}_p^r|_{\alpha_r[\eta_{r+1}(0), \xi_{r+1}(\eta_{r+1}(0))[})$$

Note that the above definition is consistent, since it is easy to show recursively that $\eta_r(\xi_r(0))$ is a boundary point for $\mathcal{P}_p^r$ if $p \geqslant 1$. We now have the following result: let α_* denote the scaling factor for the fixed point, i.e., $\alpha_* = \eta_*(0) - \xi_*(\eta_*(0))$ (this number has a modulus smaller than 1); then we have the following result.

Lemma 4.3. For any fixed couple $(\xi, \eta) \in M$, there is a positive constant C_1 such that

$$\sup_r \sup_{I \in \mathscr{P}_p^r} |I| \leqslant C_1 |\alpha_*|^p$$

Proof. We first observe that if r is large enough,

$$\eta'_r|_{[\eta_r(\xi_r(0)), \xi_r(0)]} > 1$$

This is true for the fixed point[27,22] and follows from Theorem 4.1 in the general case for r large enough. Therefore, the largest atom of $\mathscr{P}_p^r$ is contained in the interval $[\eta_r(0), \eta_r(\xi_r(0))]$. The length of this largest atom is therefore $|\alpha_r|$ times the length of the largest atom of $\mathscr{P}_{p-1}^{r+1}$. Using iteratively this argument, we conclude that for r large enough, the largest atom of $\mathscr{P}_p^r$ has a length bounded by

$$O(1) \prod_{j=0}^{p-1} |\alpha_{r+j}|$$

Therefore, for large r, the result follows from the exponential convergence in Theorem 4.1. If r is not large enough, the first few contributions are absorbed in the constant C_1 and then we can apply the above argument. ∎

For a fixed r, it is easy to verify recursively that the sequence of partitions $(\mathscr{P}_p^r)_{p \in \mathbb{N}}$ is increasing. This sequence of increasing partitions defines a σ-algebra $\mathscr{B}$. It follows easily from Lemma 4.3 (see Ref. 14) that $\mathscr{B}$ is the Borel σ-algebra.

For a fixed couple $(\xi, \eta) \in M$, we shall now define recursively a sequence of measures μ_r. For every r we first set

$$\mu_r([\eta_r(0), \eta_r(\xi_r(0))]) = \sigma \tag{4.1}$$

$$\mu_r(]\eta_r(\xi_r(0)), \xi_r(0)]) = \sigma^2 \tag{4.2}$$

We also impose

$$\mu_r(I) = \sigma \mu_{r+1}(\alpha_r^{-1} I) \qquad \text{if} \quad I \in \mathscr{P}_p^r|_{[\eta_r(0), \eta_r(\xi_r(0))]} \tag{4.3}$$

$$\mu_r(I) = \mu_r(\eta_r(I)) \qquad \text{if} \quad I \in \mathscr{P}_p^r|_{[\eta_r(\xi_r(0)), \xi_r(0)]} \tag{4.4}$$

Theorem 4.4. The constraints (4.1)–(4.4) define for each r a unique measure μ_r invariant under the homeomorphism associated to (ξ_r, η_r).

Proof. We note first that uniqueness follows from the unique ergodicity. Therefore we only have to show that μ_r is a well-defined invariant measure. The above constraints can be used recursively to define weights for all the sets in $\bigcup_0^\infty \mathscr{P}_p^r$. If we can show that this family of weights is consistent, it will follow that μ_r is a well-defined measure.[14] To show the coherence of the above definitions, it is enough to show that if B is an atom of $\mathscr{P}_p^r$ that is a union of atoms $B_1,..., B_s$ of atoms of $\mathscr{P}_{p+1}^r$ (which is a finer partition), then

$$\mu_r(B) = \mu_r(B_1) + \cdots + \mu_r(B_s)$$

If B is in $[\alpha_r \eta_{r+1}(0), \xi_r(0)]$, we can apply η_r to B and the sets B_i without changing the weights. Therefore we can assume that B is in $[\eta_r(0), \alpha_r \eta_{r+1}(0)]$. We can now apply α_r^{-1} and we are reduced to the proof of the similar statement for the measure μ_{r+1} but for atoms of the partitions $\mathscr{P}_{p-1}^{r+1}$ and $\mathscr{P}_p^{r+1}$. We can now proceed recursively until we reach $p=0$. In this case the partitions to be considered are $\mathscr{P}_0^{r+p}$ and $\mathscr{P}_1^{r+p}$, and the result follows immediately from the relation $\sigma^2 + \sigma = 1$.

We now show that μ_r is an invariant measure. We first observe that (4.5) is valid for any Borel set I because the partitions $(\mathscr{P}_p^r)_{p \in \mathbb{N}}$ generate the Borel σ-algebra. Therefore, it is enough to show that the measure is invariant on all the atoms of the partitions $\mathscr{P}_p^r$. The proof is recursive, i.e., we shall show that for all $r \in \mathbb{N}$, the measure μ_r is invariant on the atoms of $\mathscr{P}_q^r$, $0 \leqslant q \leqslant p-1$, that do not contain 0 (the case $p=2$ is easy to check using the definition of μ_r). Assume now $K \in \mathscr{P}_p^r$, and $K \subset [\eta_r(0), 0]$. We have

$$\mu_r(\xi_r(K)) = \mu_r(\eta_r(\xi_r(K))) = \mu_r(\eta_r(\xi_r(\alpha_r L)))$$

since $\xi_r(K) \subset [\alpha_r \eta_{r+1}(0), \xi_r(0)]$, and $L = \alpha_r^{-1} K \in \mathscr{P}_{p-1}^{r+1}$. Therefore [using (4.4) and the recursion hypothesis] we have

$$\begin{aligned}
\mu_r(\xi_r(K)) &= \mu_r(\alpha_r \eta_{r+1}(L)) = \sigma \mu_{r+1}(\eta_{r+1}(L)) \\
&= \sigma \mu_{r+1}(L) = \sigma \mu_{r+1}(\alpha_r^{-1} K) \\
&= \mu_r(K)
\end{aligned}$$

Similarly, if $K \subset [0, \alpha_r \eta_{r+1}(0)]$, we have

$$\begin{aligned}
\mu_r(\eta_r(K)) &= \mu_r(\eta_r(\alpha_r L)) = \mu_r(\alpha_r \xi_{r+1}(L)) \\
&= \sigma \mu_{r+1}(\xi_{r+1}(L)) = \sigma \mu_{r+1}(L) \\
&= \mu_r(K)
\end{aligned}$$

where $L = \alpha_r^{-1} \in \mathscr{P}_{p-1}^{r+1}$. ∎

Notice that from the uniqueness of the invariant measure it also follows that μ_r is the Bowen–Ruelle measure.

We now start the application of the results of the previous sections. Let $(\xi, \eta) \in M$, and μ the associated invariant measure. We define a map g by

$$g\,|_{[\eta(0), \eta(\xi(0))]}(x) = \alpha_*^{-1} x, \qquad g\,|_{[\eta(\xi(0)), \xi(0)]}(x) = \alpha_*^{-1} \eta(x)$$

Let $\mathscr{P}_n$ be the partition constructed as in Section 2 for the above map g. We have the following result.

Theorem 4.5. For any real number β the following limit exists and defines a universal function F:

$$F(\beta) = \lim_{n \to \infty} -n^{-1} \log_2 \sum_{I \in \mathscr{P}_n} \mu(I)^\beta$$

This theorem follows directly from the results in Section 2 if (ξ, η) is the fixed point of the renormalization. If not, we have to use, as in the case of the period doubling, the exponentially fast convergence of the successive renormalizations.

We shall now show that F is C^1. As before, this will be done by coding the problem to a problem of statistical mechanics. For a fixed element $(\xi, \eta) \in M$ we shall first introduce a coding of the points of the circle. We shall denote by T the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Ω will denote the phase space of the subshift of $\{0, 1\}^{\mathbb{N}}$ associated with the incidence matrix T. More precisely, Ω is the space of sequences $(\omega_j)_{j \in \mathbb{N}}$, with $\omega_j \in \{0, 1\}$ and the constraint that a 1 is followed by a 0. It is easy to show that Ω is invariant by the shift $\mathscr{S}$.

Proposition 4.6. There is a countable set $E \subset [\eta(0), \xi(0)]$ and a continuous map $h: \Omega \to [\eta(0), \xi(0)]$ such that:

(i) h^{-1} is injective (on $[\eta(0), \xi(0)]$).

(ii) If $x \in E$, $\mathrm{Card}\{h^{-1}(x) = 2\}$.

(iii) $h \circ \mathscr{S}^n = R_n \circ h$ on $\Omega \backslash h^{-1}(E)$, where R_n is a map from $[\eta_n(0), \xi_n(0)]$ into $[\eta(0), \xi(0)]$ which is equal to $R^{(0)} \circ \cdots \circ R^{(n-1)}$, and the map $R^{(j)}$ from $[\eta_j(0), \xi_j(0)]$ to $[\eta_{j+1}(0), \xi_{j+1}(0)]$ is defined by

$$R^{(j)}(x) = \begin{cases} \alpha_j^{-1} x & \text{if} \quad x \in [\eta_j(0), \eta_j(\xi_j(0))] \\ \alpha_j^{-1} \eta_j(x) & \text{if} \quad x \in \,]\eta_j(\xi_j(0)), \xi_j(0)] \end{cases}$$

Proof. If we consider the fixed point of the renormalization, this is a well-known result in the theory of expanding Markov maps.[27] If we are not at the fixed point, but on the stable manifold, the same result follows from the exponentially fast convergence to the fixed point. ▮

We shall now construct on Ω a Gibbs state indexed by two real numbers u and v whose jth potential is given by the function

$$\omega \to u \log |R^{(j)'}(h(\omega))| + v\theta(\mathcal{S}^j \omega)$$

where

$$\theta(\omega) = \begin{cases} \log \sigma & \text{if } \omega_0 = 0 \\ 2 \log \sigma & \text{if } \omega_0 = 1 \end{cases}$$

Notice that $R^{(j)'}$ converges exponentially fast to the function R' associated to the fixed point of the renormalization transformation. Since $\log |R'|$ is bounded above and below, it follows from standard results[27] that to each $(\xi, \eta) \in M$ and $\beta \in \mathbb{R}$ we can associate a unique Gibbs state with the above potentials. Moreover, for a fixed couple (u, v) all these Gibbs measures are equivalent to the Gibbs measure of the fixed point of the renormalization transformation. We shall denote by $G_D(u, v)$ the (universal) free energy. Note that G_D is C^∞ in (u, v). The partition $\mathcal{P}_n^0$ introduced above is easily seen to be identical with the partition $\mathcal{P}$ of Section 2. The following result is now the analog of a theorem of Vul *et al.*[30] for the case of period doubling.

Proposition 4.7. For any element (ξ, η) of M we have

$$\lim_{n \to +\infty} n^{-1} \log \sum_{I \in \mathcal{P}_n^0} |I|^\beta = F_D(\beta)$$

Proof. It is easy to verify that if I is an atom of $\mathcal{P}_n^0$, then there is a sequence $\varepsilon_1, \ldots, \varepsilon_n$ of 0 and 1 such that

$$I = \{x \mid h^{-1}(x)_j = \varepsilon_j \text{ for } j = 1, \ldots, n\}$$

From Lemma 2.1 and Theorem 4.1 it follows that there is a finite constant C_3 which is independent of n such that if x and y belongs to I, then

$$C_3^{-1} |R_n'(x)| \leqslant |R_n'(y)| \leqslant C_3 |R_n'(x)|$$

(this is obvious from the chain rule). From the equality $R_n(I) = [\eta(0), \xi(0)]$, we deduce that for some finite constant C_2 independent of n, we have for any atom I of $\mathcal{P}_n^0$ the bound

$$C_2^{-1} |R_n'(x)|^{-1} \leqslant |I| \leqslant C_2 |R_n'(x)|^{-1} \qquad \forall x \in I$$

The result now follows from our definition of the potentials for the Gibbs state of free energy $G_D(u, v)$. ▌

It is easy to verify that $D_2 G_D \neq 0$. We can now apply Proposition 2.4 and Corollary 2.5 to conclude that F is C^1. From the results in Section 3 we conclude that the Hausdorff dimension of the singularities of μ are given by the universal function which is the Legendre transform of F.

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