Commun. Math. Phys. 91, 87–101 (1983)

Communications in Mathematical Physics

Potentials for the Supersymmetric Nonlinear σ-Model

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Abstract. The most general structure for potential terms compatible with $N=1$, $N=2$, and $N=4$ supersymmetry in the nonlinear σ-model in two space-time dimensions is determined. The differential geometry of the internal manifold of the model plays an important role in the method used and in the results. An interesting application of nontrivial dimensional reduction is found.

1. Introduction

A strong connection has been established between extended supersymmetry in the nonlinear σ-model in two space-time dimensions and the differential geometry of the internal manifold M on which the model is defined. $N=2$ supersymmetry requires that M is a Kahler manifold [1, 2], $N=3$ supersymmetry implies $N=4$, and $N=4$ requires that M is hyperKahler. This connection between complex differential geometry and supersymmetry strongly constrains renormalization counterterms [3, 4], and there are strong indications that at least the $N=4$ theories are ultraviolet finite to all orders in perturbation theory [2, 3].

In this paper we consider the inclusion of potential terms with a coupling constant of the dimensions of mass in the model. One motivation for this arises from the infrared problems of massless scalars in two dimensions. In the $O(n)$ and CP^n models there is spontaneous generation of mass [5, 6] due to asymptotic freedom, but the resolution of the infrared difficulty is unclear for $N=4$ models which are ultraviolet finite. The potential gives massive excitations at the classical level which circumvents the infrared problems.

In the bosonic σ-model the potential $V(\phi)$ can be an arbitrary function on M. In $N=1$ supersymmetry one can add an arbitrary superpotential $W(\phi)$, as is well known, but additional parity non-conserving terms are possible if M possesses

* Research supported in part by the National Science Foundation under Grants PHY82-15249, PHY82-03669, PHY80-19754, and by the Harvard Society of Fellows
** Research supported in part by the National Science Foundation under Grant PHY81-07394

Killing vectors. $N=2$ and $N=4$ supersymmetry place additional constraints on the allowed potential terms which involve functions and Killing vectors which are holomorphic with respect to the complex structures of M. The Killing vector terms in the potential arise from nontrivial dimensional reduction [7] of the nonlinear σ-model in 3 or 4 dimensions. The supersymmetry algebra is typically modified by the presence of potentials and includes central charges whose action on the fields is that of Killing vectors of M.

Two methods are used to obtain the potential terms. In the first approach, an extension of [2], a general ansatz for the Lagrangian and transformation rules is made, and the requirement of supersymmetric invariance is used to constrain the unknown quantities of the ansatz. This approach leads to the most general results on allowed potential terms. The second approach is a novel modification of superspace methods in which extended supersymmetry of the resulting potential terms is very simple to prove, but the potential structure is not the most general possible.

2. Summary of the Massless Case

We begin by reviewing the way in which supersymmetry is implemented in the nonlinear σ-model without potential.

In the ordinary bosonic σ-model, there are n scalar fields $\phi^i(x)$ which are interpreted as functions from two-dimensional Minkowski space-time into a Riemannian manifold M with metric g_{ij} and standard connection $\Gamma^i{}_{jk}$. The action is

$$S=\tfrac{1}{2}\int d^2x g_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j\,, \tag{1}$$

and is invariant under coordinate reparameterizations of M.

It is always possible to find an $N=1$ supersymmetric extension of (1) [8]. The simplest way is to introduce n two-component anticommuting Majorana spinor fields $\psi^i{}_\alpha(x)$, and n real auxiliary scalars $F^i(x)$, and join them in a superfield

$$\Phi^i(x,\theta)=\phi^i(x)+\bar\theta\psi^i(x)+\tfrac{1}{2}\bar\theta\theta F^i(x)\,, \tag{2}$$

where θ_α, $\alpha=1,2$, is a Grassmann coordinate of superspace. With supercovariant derivative and γ-matrices defined by

$$D_\alpha=\frac{\partial}{\partial\bar\theta_\alpha}-i(\gamma^\mu\theta)_\alpha\partial_\mu\,,$$

$$\gamma^0=\sigma_y,\ \gamma^1=i\sigma_x,\ \gamma^5=\gamma^0\gamma^1=\sigma_z,\ \bar\psi=\psi^T\gamma^0\,, \tag{3}$$

one can write the superspace action

$$S[\Phi]=\frac{1}{4i}\int d^2\theta d^2x g_{ij}(\Phi)\bar D\Phi^i D\Phi^j\,, \tag{4}$$

which is invariant under supersymmetry transformations

$$\delta\Phi^i=\bar\varepsilon_\alpha\left(\frac{\partial}{\partial\bar\theta_\alpha}+i(\not\partial\theta)_\alpha\right)\Phi^i \tag{5}$$

with spinor parameter ε_α.

After performing the θ integration and eliminating auxiliary fields, we find the component action

$$S[\Phi, \Psi] = \tfrac{1}{2}\int d^2x\{g_{ij}\partial_\mu\phi^i\partial_\mu\phi^j + ig_{ij}\bar{\psi}^i \not{D}\psi^j + \tfrac{1}{6}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l\}\,,$$
$$D_\mu\psi^j = \partial_\mu\psi^j + \Gamma^j{}_{kl}(\partial_\mu\phi)^k\psi^l\,, \tag{6}$$

and transformation rules

$$\delta\phi^i = \bar{\varepsilon}\psi^i\,, \qquad \delta\psi^i = -i\not{\partial}\phi^i\varepsilon - \Gamma^i{}_{jk}\bar{\varepsilon}\psi^j\psi^k\,. \tag{7}$$

It is important to note that supersymmetry transformations commute with infinitesimal coordinate transformations on M which are given by

$$\Delta\phi^i = \xi^i(\phi)\,, \qquad \Delta\psi^i = \partial_j\xi^i\psi^j\,. \tag{8}$$

This property will be useful later. The component coordinate transformations (8) are deduced from the fact that the superfield Φ^i transforms as $\Delta\Phi^i = \xi^i(\Phi)$ (which implies that the auxiliary fields F^i have complicated transformation properties).

It is known that the action (5) admits a second supersymmetry if and only if M is a Kahler manifold [1, 2]. This means that there exists a tensor $f^i{}_j$ on M which satisfies

$$f^i{}_j f^j{}_k = -\delta^i{}_k\,, \tag{9a}$$

$$g_{ij} f^i{}_k f^j{}_l = g_{kl}\,, \tag{9b}$$

$$D_i f^j{}_k = 0\,, \tag{9c}$$

(9a) implies that the dimension n of the manifold is even. From (9a)–(9c) it follows that M can be covered smoothly with complex coordinate charts $(z^\alpha, z^{\bar{\alpha}})$ such that transition functions in overlapping coordinate patches are holomorphic. In complex coordinates the line element ds^2 can be written as

$$ds^2 = 2g_{\alpha\bar{\beta}}\,dz^\alpha\,dz^{\bar{\beta}}\,. \tag{10}$$

The two-form

$$F = ig_{\alpha\bar{\beta}}\,dz^\alpha \wedge dz^{\bar{\beta}} \tag{11}$$

is closed which implies that locally

$$g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha\,\partial z^{\bar{\beta}}}K(z, \bar{z})\,, \tag{12}$$

where $K(z, \bar{z})$ is the Kahler potential. Further local properties of Kahler manifolds will be used in the following sections. See [9] for an elementary discussion.

If M is Kahler, the second supersymmetry transformation is

$$\delta\phi^i = \bar{\varepsilon} f^i{}_j\psi^j\,, \qquad \delta(f^i{}_j\psi^j) = -i\not{\partial}\phi^i\varepsilon - \Gamma^i{}_{jk} f^j{}_l f^k{}_m\bar{\varepsilon}\psi^l\psi^m\,. \tag{13}$$

Finally one can show that each supersymmetry beyond (7) requires an independent Kahler structure $f^{(a)i}{}_j$ which satisfies (9a)–(9c) and

$$f^{(a)i}{}_k f^{(b)k}{}_j + f^{(b)i}{}_k f^{(a)k}{}_j = -2\delta^i{}_j\delta^{ab}\,. \tag{14}$$

It is clear that $N=3$ implies $N=4$ because if $f^{(1)}$ and $f^{(2)}$ are two Kahler structures which satisfy (13), so is $f^{(3)i}{}_j = f^{(1)i}{}_k f^{(2)k}{}_j$. If M is an irreducible manifold, then $N=4$ supersymmetry can only be satisfied if there are three such complex structures which satisfy the algebra of quaternions, and $N=4$ is the maximal supersymmetry. When $N=4$ supersymmetry is realized the manifold is hyperKahler and its dimension is a multiple of 4. For each of the 3 extended supersymmetries one has transformation rules of the form (12) with each of the 3 independent complex structures $f^{(a)i}{}_j$ inserted. In the absence of potentials, irreducible manifolds are the only ones of physical interest since the fields ϕ^i and ψ^i would otherwise split into two or more sets without mutual interactions.

The results above for extended supersymmetry were derived [2] from the most general ansatz for the transformation rules consistent with general coordinate invariance, Lorentz invariance, and dimensional considerations. However, it is interesting to note that there is a simple method to prove invariance of (6) under the extended supersymmetry transformation (13) using the following superfield arguments. Let us define a new superfield:

$$\Phi'^i = \phi^i + \bar{\theta} f^i{}_j \psi^j + \tfrac{1}{2}\bar{\theta}\theta F'^i . \tag{15}$$

The action $S[\phi, f\psi]$ constructed from $S[\Phi']$ in (4) by θ integration and elimination of F'^i is automatically invariant under (13). Then the action (6) will be invariant under both (7) and (13) if we can establish that

$$S[\phi, f\psi] = S[\phi, \psi] . \tag{16}$$

However (16) is true because the transformation $\psi^i \to f^i{}_j \psi^j$ is a discrete symmetry of $S[\phi, \psi]$ in (6), a fact which follows simply from (9b) and the properties

$$D_\mu(f^i{}_j \psi^j) = f^i{}_j D_\mu \psi^j , \tag{17a}$$

$$R_{mnpq} f^m{}_i f^n{}_j f^p{}_k f^q{}_l = R_{ijkl} , \tag{17b}$$

which are consequences of (9b) and (9c) in a Kahler manifold. Since the transformation $\psi^i \to \gamma_5 \psi^i$ is also a discrete symmetry of the model, we could obtain identical results for supersymmetry from the modified superfield

$$\Phi''^i = \phi^i + \bar{\theta} f^i{}_j \gamma_5 \psi^j + \tfrac{1}{2}\bar{\theta}\theta F''^i . \tag{18}$$

As we will see in later sections, the modified superfield methods based on (15) and (18) give two different supersymmetric potential structures. Both are special cases of the most general structure obtained from a general invariant ansatz.

3. Potentials for an $N=1$ Riemannian σ-Model

In order to determine the most general potential structure we consider the following ansatz for the action and transformation rules:

$$\begin{aligned} S[\phi, \psi] = \tfrac{1}{2}\int d^2x \{ & g_{ij}\partial_\mu \phi^i \partial_\mu \phi^j + i g_{ij}\bar{\psi}^i \not{D} \psi^j \\ & + \tfrac{1}{6} R_{ijkl}\bar{\psi}^i \psi^k \bar{\psi}^j \psi^l - m^2 V(\phi) - m W_{ij}(\phi)\bar{\psi}^i \psi^j \\ & - m W^5{}_{ij}(\phi)\bar{\psi}^i \gamma_5 \psi^j \} , \end{aligned} \tag{19}$$

$$\delta\phi^i = \bar{\varepsilon}\psi^i , \tag{20a}$$

$$\delta\psi^i = i\not{\partial}\phi^i \varepsilon - \Gamma^i{}_{jk}\bar{\varepsilon}\psi^j \psi^k - m H^i(\phi)\varepsilon - m G^i(\phi)\gamma_5 \varepsilon , \tag{20b}$$

where we have augmented (6) and (7) by the most general set of terms which involve positive powers of a dimensional constant m and are invariant under proper Lorentz transformations. Since this structure involves operators of dimension zero and one, it remains closed under renormalization, in the general sense of Friedan [10]. The ansatz is reparameterisation invariant if $V(\phi)$ is a scalar, $H^i(\phi)$ and $G^i(\phi)$ are vector fields on M, and W_{ij} and $W^5{}_{ij}$ are tensor fields which are, respectively, symmetric and anti-symmetric.

We now require that (19) is invariant under (20a) and (20b) and work order by order in m to determine the necessary constraints on the ansatz. Since the ansatz is reparameterisation invariant, the variation of S is a scalar. This makes calculation easy since we need not follow in detail terms involving the connection $\Gamma^i{}_{jk}$.

To $\mathcal{O}(1)$ invariance is already known. The $\mathcal{O}(m)$ terms in the variation split into two pieces which must vanish independently since they are linear and trilinear in ψ. The $\mathcal{O}(m)$ linear terms are

$$\delta S = im\bar{\varepsilon}\int d^2x[D_iH_j-\gamma_5D_iG_j-W_{ij}+\gamma_5W^5_{ij}]\partial\!\!\!/\phi^i\psi^j\,. \tag{21}$$

The scalar terms vanish if the anti-symmetric part of D_iH_j vanishes implying that locally $H_j=D_jW$, and if one further requires that $W_{ij}=D_iD_jW$. Thus $W(\phi)$ is the standard superpotential which is expected in these models. The pseudoscalar terms in (21) vanish if the symmetric part of D_iG_j vanishes, which means that G_j in a Killing vector of M, and one must further require that $W^5{}_{ij}$ is the curl of G_j. These results may be summarized by

$$\begin{aligned} W_{ij}&=D_iD_jW\,, & H_i&=D_iW\,, \\ W^5{}_{ij}&=D_iG_j\,, & D_iG_j+D_jG_i&=0\,. \end{aligned} \tag{22}$$

Before studying trilinear terms in ψ we study the $\mathcal{O}(m^2)$ term which is linear. Here one finds scalar and pseudoscalar terms which must vanish separately giving the conditions

$$\tfrac{1}{2}D_iV-(D_iD_jW)(D^jW)+(D_iG_j)G^j=0\,, \tag{23a}$$

$$G^iD_iD_jW+(D_\alpha G^i)D^jW=0\,. \tag{23b}$$

The first condition may be integrated to express the scalar potential as

$$V(\phi)=g^{ij}(D_iWD_jW+G_iG_j)\,, \tag{24}$$

where an irrelevant constant is assumed to vanish. Hence the scalar potential $V(\phi)$ is related to the superpotential in the standard way, but there are also contributions from the Killing vector which are new. Condition (23b) states simply that the Lie derivative $\pounds_G\,\partial W$ vanishes which means that the superfield has constant Lie derivative, *viz.*

$$\pounds_G W=G^i\,\partial_i W=\text{const}\,. \tag{25}$$

One must now study the $\mathcal{O}(m)$ trilinear fermion terms in the variation of $S[\phi,\psi]$ which come from

$$-\tfrac{1}{2}m\delta(D_iD_jW)\bar{\psi}^i\psi^j-\tfrac{1}{2}m\delta(D_iG_j)\bar{\psi}^i\gamma_5\psi^j+\tfrac{1}{12}R_{ijkl}\delta(\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l)\,, \tag{26}$$

where only linear terms in m are kept. Again there are scalar and pseudoscalar terms which must vanish separately. The vanishing of the scalar terms follows without calculation from superspace since these terms are independent of G_i and follow from the addition of a standard superpotential term $mW(\Phi)/2i$ to the action (4). The pseudoscalar terms require some work, and it is useful to use the identity

$$R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l = \tfrac{3}{2} R_{ijkl}\bar{\psi}^i\gamma_5\psi^j\bar{\psi}^k\gamma_5\psi^l \tag{27}$$

before taking the $\delta\psi$ variation. This identity is established using Fierz rearrangement and the first Bianchi identity. The vanishing of the pseudoscalar term can then be shown, if the Killing property of G and the Ricci identity for $[D_i, D_j]G_k$ is used.

This completes the determination of the most general potential structure allowed in $N=1$ supersymmetry. This structure involves a scalar superpotential $W(\phi)$ and a Killing vector G_i, and the Lie derivative $\pounds_G W = G^i\,\partial_i W$ is constant. The Lagrangian and transformation rules are

$$\begin{aligned}\pounds = \tfrac{1}{2}\{&g_{ij}\,\partial_\mu\phi^i\,\partial^\mu\phi^j + ig_{ij}\bar{\psi}^i \not{D}\psi^j + \tfrac{1}{6}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l \\ &- m^2 g^{ij}(D_iWD_jW + G_iG_j) - mD_iD_jW\bar{\psi}^i\psi^j - mD_iG_j\bar{\psi}^i\gamma_5\psi^j\}\,,\end{aligned} \tag{28}$$

$$\delta\phi^i = \bar{\varepsilon}\psi^i\,, \tag{29a}$$

$$\delta\psi^i = -\mathrm{i}\not{\partial}\phi^i\varepsilon - \Gamma^i{}_{jk}\bar{\varepsilon}\psi^i\psi^k - \mathrm{m}\mathrm{D}^i\mathrm{W}\,\varepsilon - \mathrm{m}\mathrm{G}^i\gamma_5\varepsilon\,. \tag{29b}$$

The commutator algebra of (29a) and (29b) gives a surprise. One finds by calculation that

$$\begin{aligned}&[\delta_1, \delta_2]\,\phi^i = 2i(\bar{\varepsilon}_i\gamma^\mu\varepsilon_2)\,\partial_\mu\phi^i + 2m(\bar{\varepsilon}_1\gamma_5\varepsilon_2)\,G^i\,,\\ &[\delta_1, \delta_2]\,\psi^i = 2i(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2)\,\partial_\mu\psi^i + 2m(\bar{\varepsilon}_1\gamma_5\varepsilon_2)\,\partial_j G^i\psi^j\,,\end{aligned} \tag{30}$$

where the equation of motion for ψ^i is assumed to be satisfied. The first term is the expected space-time translation, and the second term is a central charge which corresponds to a coordinate transformation with the Killing vector field. The central charge is implemented canonically which is not the case for the coordinate reparameterisation (8) and the Lagrangian is invariant because $\pounds_G g_{ij} = 0$, which is equivalent to the Killing condition.

It is possible to understand the central charge and the G_i terms in the general potential structure from the viewpoint of dimensional reduction from an $N=1$ σ-model in $2+1$ dimensions (with coordinates x^0, x^1, x^5). Here one simply performs dimensional reduction in the manner of Scherk and Schwarz [7] so that the x^5 dependence of field configurations corresponds to motion along orbits of the Killing vector G^i in M. Specifically field configurations $\phi(x^0, x^1, x^5)$ and $\psi(x^0, x^1, x^5)$ must satisfy

$$\begin{aligned}&\frac{\partial}{\partial x^5}\,\phi^i = mG^i(\phi)\,,\\ &\frac{\partial}{\partial x^5}\,\psi^i = m\,\partial_j G^i(\phi)\psi^j\,.\end{aligned} \tag{31}$$

It is then easy to see that all G^i terms in (28) are generated by dimensional reduction and the resulting Lagrangian is independent of x^5. The central charge in the algebra is the residual effect of the translation in the x^5-direction. It is interesting to see how the Scherk-Schwarz mechanism works in the geometrical context of the non-linear σ-model on a manifold with a continuous isometry.

It is also very simple to implement the dimensional reduction in superspace using the action (4) with spatial volume element $dx^0\,dx^1\,dx^5$. If superfield configurations are required to satisfy

$$\frac{\partial}{\partial x^5}\Phi^i = mG^i(\Phi), \tag{32}$$

which contains (31), then the component Lagrangian is (28).

4. Potentials for $N=2$ Kahler Manifold Models

Before analyzing the structure of potentials for $N=2$ σ-models we explain the concept of a holomorphic Killing vector on a Kahler manifold. See Bagger and Witten [11] for further discussion. A holomorphic Killing vector generates an isometry of M with coordinate changes that are holomorphic, *viz.*

$$\delta z^\alpha = V^\alpha(z), \qquad \delta\bar{z}^\alpha = V^{\bar{\alpha}}(\bar{z}) = \bar{V}^\alpha(z), \quad \text{or} \quad \partial_{\bar{\beta}}V^\alpha = \partial_\beta V^{\bar{\alpha}} = 0. \tag{33}$$

The condition for an isometry is just the Killing condition

$$D_\alpha V_\beta + D_\beta V_\alpha = 0, \tag{34a}$$

$$\partial_\alpha V_{\bar{\beta}} + \partial_{\bar{\beta}} V_\alpha = 0. \tag{34b}$$

Here (34a) is equivalent to (33), while (34b) is locally equivalent to the existence of a real scalar function $U(z,\bar{z})$, called the Killing potential, such that

$$\begin{aligned} V_\alpha &= i\,\partial_\alpha U, \\ V_{\bar{\alpha}} &= -i\,\partial_{\bar{\alpha}} U, \end{aligned} \tag{35}$$

which can be written in real form as $V_i = f^j{}_i\,\partial_j U$.

If $V^\alpha(z)$, $V^{\bar{\alpha}}(\bar{z})$ is a holomorphic Killing vector, then the Lie derivative of the Kahler potential is a Kahler gauge transformation, i.e.

$$\pounds_v K(z,\bar{z}) = V^\alpha\,\partial_\alpha K + \bar{V}^\alpha\,\partial_{\bar{\alpha}} K = f(z) + \bar{f}(\bar{z}). \tag{36}$$

Given the Killing vector, one may use the Kahler potential $K(z,\bar{z})$ to construct $U(z,\bar{z})$ [12]. Specifically

$$U(z,\bar{z}) = \tfrac{1}{2} i[V^\alpha\,\partial_\alpha K - f - V^{\bar{\alpha}}\,\partial_{\bar{\alpha}} K + \bar{f}]. \tag{37}$$

Let us take the example of the manifolds CP^n where, in Fubini-Study coordinates, the Kahler potential is

$$K(z,\bar{z}) = \ln(1 + \bar{z}^\alpha z^\alpha). \tag{38}$$

The holomorphic isometry group is $SU(n+1)$, and the general Killing vector is

$$V^\alpha = iC^{\alpha\beta} z^\beta + b^\alpha + (\bar{b}^\beta z^\beta) z^\alpha, \tag{39}$$

where $C^{\alpha\beta}$ is a Hermitean matrix and b^α is a complex "vector," which together comprise the $2n^2+n$ real parameters of $SU(n+1)$. One has

$$\pounds_V K = \bar{b}^\alpha z^\alpha + \bar{b}^\alpha \bar{z}^\alpha , \tag{40}$$

and applying (37) we find

$$U(z,\bar{z}) = -\frac{\bar{z}^\alpha C^{\alpha\beta} z^\beta + b\cdot z - b\cdot \bar{z}}{1+\bar{z}\cdot z} . \tag{41}$$

Returning to the question of potentials for the $N=2$ σ-model, we now seek conditions under which the general $N=1$ Lagrangian (28) is invariant under a second supersymmetry transformation. We pose the ansatz

$$\begin{aligned} &\delta\phi^i = \bar{\varepsilon} f^i{}_j \psi^j , \\ &\hat{\delta}(f^i{}_j\psi^j) = -i\,\partial\phi^i\varepsilon - \Gamma^i{}_{jk} f^j{}_l f^k{}_m (\bar{\varepsilon}\psi^l)\psi^m - m(2U^i - D^i W)\varepsilon - m\gamma_5 L^i\varepsilon , \end{aligned} \tag{42}$$

where the m independent terms coincide with (13) and, for convenience in presenting results, the scalar coefficient is taken as $2U^i - D^i W$, where U^i is a vector field on M.

Following step-by-step the arguments of the $N=1$ case, we find that vanishing of the $\mathcal{O}(m\psi)$ term requires the two conditions

$$2D_k U_l - (\delta^i{}_k \delta^j{}_l + f^i{}_k f^j{}_l) D_i D_j W = 0 , \tag{43a}$$

$$D_k L_l - f^i{}_k f^j{}_l D_k G_l = 0 . \tag{43b}$$

Since the second term in (43a) is symmetric, the antisymmetric part of $D_k U_l$ must vanish implying that U_l is the gradient of a scalar, $U_l = \partial_l U$. Now projecting (43a) with $\delta^k{}_i \delta^l{}_j - f^k{}_i f^l{}_j$, we find

$$(\delta^k{}_i \delta^l{}_j - f^k{}_i f^l{}_j) D_k D_l U = 0 . \tag{44}$$

This is equivalent to the statement that $V_i = f^j{}_i\,\partial_j U$ is a holomorphic Killing vector in real form. To see this we introduce a complex coordinate chart, where $f^i{}_j$ takes the form $f^\alpha{}_\beta = i\delta^\alpha{}_\beta$ and $f^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^{\bar{\alpha}}{}_{\bar{\beta}}$. Then the complex components of V_i are $V_\alpha = i\,\partial_\alpha U$ and $V_\alpha = -i\,\partial_\alpha U$, which implies (34b), while (44) gives $D_\alpha D_\beta U = 0$ and $D_{\bar{\alpha}} D_{\bar{\beta}} U = 0$, which mean that V^α and $V^{\bar{\alpha}}$ are holomorphic. Next we write (43a) in complex coordinates as

$$\partial_\alpha \partial_{\bar{\beta}} U - \partial_\alpha \partial_{\bar{\beta}} W = 0 , \tag{45}$$

which can be integrated to give

$$W(z,\bar{z}) = U(z,\bar{z}) + h(z) + \bar{h}(\bar{z}) , \tag{46}$$

where $h(z)$ is an arbitrary holomorphic function. Finally $V_i = f^j{}_k\,\partial_j U$ can be inverted to write $U^i = f^i{}_j V^j$, so that the scalar term in (42) can be written in terms of the Killing field.

To proceed further we must compute the $O(m\psi^3)$ terms which are the analogues of (26) with the transformation (42). There are scalar and pseudoscalar terms which must vanish separately. Using (44), (46), and (17b) to express the scalar terms so that ψ^j always appears contracted with $f^i{}_j$, we see that these terms vanish because they are of the same form as in the $N=1$ case, except for the changes $\psi^i \to f^i{}_j\psi^i$ and $W \to 2U - W$. The pseudoscalar terms can be rewritten, after manipulation similar to the $N=1$ case, as

$$\tfrac{1}{2} m R_{ijkl}(L^i - G^i)\bar{\varepsilon} f^j{}_m \psi^m \bar{\psi}^k \gamma_5 \psi^l = 0 . \tag{47}$$

For general spinor fields this vanishes only if

$$R^i{}_{jkl}(L^j - G^j) = 0 , \tag{48}$$

and this condition can now be analyzed geometrically. If M is irreducible there can be no zero eigenvector of the curvature terms since this would imply an invariant subspace of the representation of the holonomy group. Thus one must take $L^i = G^i$, and (44b) can be written as $D_\alpha G_\beta = D_{\bar{\alpha}} G_{\bar{\beta}} = 0$, which implies that G_i, which was already known from the $N=1$ analysis to be a Killing vector, must be holomorphic. If M is not irreducible there may be other possibilities than $L^i = G^i$, which are not studied here.

The final test for $N=2$ supersymmetry comes with the $O(m^2\psi)$ terms coming from the $\hat{\delta}$ variation of the potential in (28). Here one finds the three conditions

$$U^i D_i G^j - G^i D_i U^j = 0 , \tag{49a}$$

$$U^i \partial_i(h + \bar{h}) = \text{const} , \tag{49b}$$

$$G^i \partial_i(h + \bar{h}) = \text{const} . \tag{49c}$$

Upon introducing the holomorphic Killing vector $V^i = -f^i{}_j U^i$, one sees that (49a) is just the statement that V^i and G^i have vanishing Lie bracket and hence generate commuting isometries of M. (Note that one can take $V^i = G^i$ if one wishes.) On the other hand (48b) and (48c) imply that the Lie derivatives $\pounds_V h$ and $\pounds_G h$ are constant.

The results may now be summarized. Given a Kahler manifold M with matrix g_{ij} and complex structure $f^i{}_j$, then the most general potential structure consistent with $N=2$ supersymmetry involves a holomorphic function $h(z)$ and two commuting holomorphic Killing vectors G^i and V^i which leave $\partial_i h$ invariant. The Lagrangian and transformation rules are

$$\begin{aligned} L = \tfrac{1}{2}\{ & g_{ij}\partial_\mu\phi^i \partial_\mu\phi^j + i g_{ij}\bar{\psi}^i \not{D}\psi^j + \tfrac{1}{6} R_{ijkl}\bar{\psi}^i\bar{\psi}^k\psi^j\psi^l \\ & - m^2 g^{ij}(V_i V_j + \partial_i(h+\bar{h})\,\partial_j(h+\bar{h}) + G_i G_j) - m D_i D_j(U + \bar{h} + h)\bar{\psi}^i\psi^j \\ & - m D_i G_j \bar{\psi}^i \gamma_5 \psi^j\} , \end{aligned} \tag{50}$$

$$\delta\phi^i = \bar{\varepsilon}\psi^i , \qquad \delta\psi^i = -i\not{\partial}\phi^i\varepsilon - \Gamma^i{}_{jk}\bar{\varepsilon}\psi^j\psi^k - m D^i(U + h + \bar{h})\varepsilon - m G^i\gamma_5\varepsilon , \tag{51}$$

$$\begin{aligned} \hat{\delta}\phi^i &= \bar{\varepsilon} f^i{}_j \psi^j , \\ \hat{\delta}(f^i{}_j\psi^j) &= -i\not{\partial}\phi^i\varepsilon - \Gamma^i{}_{jk} f^i{}_l f^k{}_m(\bar{\varepsilon}\psi l)\psi^m - m D^i(U - h - \bar{h})\varepsilon - m\gamma_5 G^i\varepsilon , \end{aligned} \tag{52}$$

where U is the real potential of the Killing vector V^i, and (49b) has been used. These results can be rewritten in complex notation, but the transcription is somewhat elaborate for spinors (see [1] for the m-independent terms). Therefore we write only the scalar part of the Lagrangian, namely

$$L_{\text{scalar}} = g_{\alpha\bar{\beta}}\partial_\mu z^\alpha \partial_\mu z^{\bar{\beta}} - m^2 g^{\alpha\bar{\beta}}(V_\alpha + \partial_\alpha h + G_\alpha)(V_{\bar{\beta}} + \partial_{\bar{\beta}}\bar{h} + G_{\bar{\beta}}) + c . \tag{53}$$

The commutator algebra of the $N=2$ model is easily worked out with the result

$$[\delta_1,\delta_2]=[\hat{\delta}_1,\hat{\delta}_2]=2i\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\,\partial_\mu+2m\bar{\varepsilon}_1\gamma_5\varepsilon_2\pounds_G\,, \qquad (54)$$
$$[\delta_1,\hat{\delta}_2]=2m\bar{\varepsilon}_1\varepsilon_2\pounds_V\,,$$

where $\pounds_G$ and $\pounds_V$ are Lie derivatives whose action on fields is

$$\pounds_V\phi^i=V^i\,,\quad \pounds_V\psi^i=\partial_jV^i\psi^j\,,\quad \text{idem for}\quad \pounds_G\,. \qquad (55)$$

These Lie derivatives are commuting internal symmetries of the Lagrangian (50) and are central charges of the supersymmetry algebra.

The pseudoscalar central charge can be obtained from dimensional reduction as shown in Sect. 3, and it is natural to ask whether there is a similar provenance for the entire $N=2$ potential structure. The natural place to look is the supersymmetric σ-model in four dimensions whose superspace structure is well known to be

$$S[Z,\bar{Z}]=\int d^4x\{d^4\theta K(Z,\bar{Z})+d^2\theta h(Z)+d^2\bar{\theta}\bar{h}(\bar{Z})\}\,, \qquad (56)$$

where Z^α is a chiral superfield and $K(Z,\bar{Z})$ is the Kahler potential of M, and $h(Z)$ is the standard holomorphic superpotential, which is the same as that found here. If $V^\alpha(Z)$ and $G^\alpha(Z)$ are holomorphic Killing vectors of M, then Scherk-Schwarz dimensional reduction can be implemented by imposing

$$\frac{\partial}{\partial x^3}Z^\alpha=mG^\alpha(Z)\,, \qquad (57a)$$

$$\frac{\partial}{\partial x^4}Z^\alpha=mV^\alpha(Z)\,, \qquad (57b)$$

and their conjugates. The compatibility condition of (57a) and (57b) is just the condition that the Lie bracket of G and V vanishes. If $\pounds_Vh$ and $\pounds_Gh$ are constant, then the action (56) is independent of x_3 and x_4, and the resulting component Lagrangian in two dimensions must coincide with (50). The implicit conclusion of this argument is that the most general potential structure of the $N=2$ nonlinear σ-model in two dimensions can be obtained by dimensional reduction from four dimensions.

Let us now compare the previous results with those which can be obtained by the modified superfield shortcut discussed in Sect. 2. Thus we take the $d=2$, $N=1$ superfield action (4) with general potential $W(\Phi)$ added, and introduce the modified superfields Φ'^i and Φ''^i of (15) and (16). $N=2$ supersymmetry then requires the identity of the component actions

$$S[\phi,f\psi]=S[\phi,\psi]\,, \qquad (58a)$$

$$S[\phi,\gamma_5f\psi]=S[\phi,\psi]\,. \qquad (58b)$$

Referring to (28) (with $G_j=0$), one sees that (58a) and (58b) require, respectively

$$f^i{}_kf^j{}_lD_i\partial_jW=\pm D_k\partial_lW\,. \qquad (59+)\ (59-)$$

The first case requires that $f^j{}_i \partial_j W$ is a holomorphic Killing vector [see (44)], while the second case implies that W is the real part of an analytic function. Thus one obtains here the two special cases of the previous general results: i) $G^i=0$, $W=U$, and $V^i=f^j{}_i \partial_j W$ a holomorphic Killing vector, and ii) $G^i=0$, $W=h(z)+\bar{h}(\bar{z})$.

5. The HyperKahler Case: $N=4$

A hyperKahler manifold has real dimension $4n$ and has three covariantly constant complex structures which we denote by $I^i{}_j$, $J^i{}_j$, and $K^i{}_j$. Together with $\delta^i{}_j$ they are a basis for the quaternions. In a frame for the tangent space of M at any point, one may take the explicit complex structures

$$I=\begin{pmatrix} 0 & \mathbb{1}\times\mathbb{1}_n \\ -\mathbb{1}\times\mathbb{1}_n & 0 \end{pmatrix}, \quad J=\begin{pmatrix} 0 & i\sigma_y\times\mathbb{1}_n \\ i\sigma_y\times\mathbb{1}_n & 0 \end{pmatrix}, \quad K=\begin{pmatrix} i\sigma_y\times\mathbb{1}_n & 0 \\ 0 & -i\sigma_y\times\mathbb{1}_n \end{pmatrix}, \tag{60}$$

where, in this direct product notation, $\mathbb{1}$ and σ_y are the 2×2 identity and Pauli matrix and $\mathbb{1}_n$ is the $n\times n$ unit matrix.

One can introduce on M three distinct types of complex coordinate charts: i) I-holomorphic coordinates z^α, $z^{\bar{\alpha}}=\bar{z}^\alpha$ with $\alpha=1,2\ldots 2n$, such that in a coordinate basis I takes standard form $I^\alpha{}_\beta=i\delta^\alpha{}_\beta$, $I^{\bar{\alpha}}{}_{\bar{\beta}}=-i\delta^{\bar{\alpha}}{}_{\bar{\beta}}$; ii) J-holomorphic coordinates u^α, $u^{\bar{\alpha}}$ such that J takes standard form, and iii) K-holomorphic coordinates v^α, $v^{\bar{\alpha}}$, where K takes standard form.

A real function $H_I(z,\bar{z})$ which satisfies

$$D_i\partial_j H_I=-I^k{}_i I^l{}_j D_k\partial_l H_i \tag{61}$$

is the real part of an I-holomorphic function and can be represented as $H_I(z,\bar{z})=h_I(z)+\bar{h}_I(\bar{z})$. A real function $U_I(z,\bar{z})$ which satisfies

$$D_i\partial_j U_I=I^k{}_i I^l{}_j D_k\partial_l U_I \tag{62}$$

is the potential of an I-holomorphic Killing vector V_I given by $V_{Ii}=f^j{}_i\partial_j U_I$. J- and K-holomorphic functions and J- and K-holomorphic Killing vectors are defined analogously.

It follows from previous work that the general $N=1$ Lagrangian (28) is $N=4$ supersymmetric if and only if it is invariant under the transformation (52) for each of the complex structures I,J,K. This requires the compatibility conditions that the $N=1$ real superpotential $W(\phi)$ be simultaneously expressed in the form (46) for I,J,K, *viz.*

$$\begin{aligned} W(\phi)&=U_I(z,\bar{z})+h_I(z)+\bar{h}_I(\bar{z}) \\ &=U_J(u,\bar{u})+h_J(u)+\bar{h}_J(\bar{u}) \\ &=U_K(v,\bar{v})+h_K(v)+\bar{h}_K(\bar{v}), \end{aligned} \tag{63}$$

and that G_i be simultaneously I-, J-, and K-holomorphic, i.e. G_i is a tri-holomorphic Killing vector. We will not explore these general compatibility conditions in detail but we will make some relevant comments below.

Instead of a general analysis we will pass to the more restrictive framework in which extended supersymmetry is derived by the modified superfield method. In this case one has $G_i=0$, and it is sufficient for $N=4$ supersymmetry that the Hessian of the superpotential $W(\phi)$ satisfies (59) for I, J, and K with appropriately chosen $\pm$ signs. It is easy to see that the quaternionic algebra, $IJ=K$ etc., restricts the allowed sign configurations to two, namely

i) $+++$, in which case W is a tri-holomorphic Killing potential, and there are three holomorphic Killing vectors, $V_{Ii}=I^j{}_i\,\partial_j W$ and $V_{Ji}=J^j{}_i\,\partial_j W$, and $V_{Ki}=K^j{}_i\,\partial_j W$.

ii) $--+$, in which case W is the real part of an analytic function for any two complex structures, here taken as I and J, and is the potential for a Killing vector holomorphic with respect to the remaining complex structure, here taken as K. Exclusion of $---$ reflects the general fact that there are no tri-holomorphic functions.

We also note that (59$\pm$) is equivalent to the statement that the Hessian of W commutes (anti-commutes) with the given complex structure. This means that

$$\begin{aligned} I^i{}_k D^k D_j W \mp D^i D_k W I^k{}_j &= 0\,, \\ J^i{}_k D^k D_j W \mp D^i D_k W J^i{}_j &= 0\,, \\ K^i{}_k D^k D_j W - D^i D_k W K^i{}_j &= 0\,, \end{aligned} \tag{64$\pm$}$$

where the upper signs correspond to the alternative i) above and the lower signs to the alternative ii). The third equation in (64$\pm$) is implied by the first two.

Let us now discuss the possibility i) briefly. Flat R^{4n} space has triholomorphic Killing potentials. The translation Killing potentials are trivially triholomorphic. To ascertain whether there is a triholomorphic subgroup of the SO$(4n)$ isotropy group of R^{4n}, we consider the quadratic potential $W=\frac{1}{2}S_{ij}\phi^i\phi^j$ in the standard Cartesian coordinates. Direct calculation shows that there are $n(2n-1)$ independent symmetric matrices S_{ij} which satisfy (64+). This suggests that for each complex structure there is an SO$(2n)$ subgroup of SO$(4n)$ which is determined by the triholomorphic potentials. Thus for flat R^{4n} one obtains with possibility i) only free field theories with mass.

For $n=1$ the unique triholomorphic potential on R^4 is $W=\frac{1}{2}\delta_{ij}\phi^i\phi^j$. The fact that δ_{ij} is the only symmetric matrix which commutes with I, J, and K in four dimensions is a consequence of Schur's lemma for real representations [13]. On a non-trivial four dimensional hyperKahler manifold M, it is an immediate consequence that the Hessian of a triholomorphic potential must satisfy

$$D_i D_j W = f(\phi) g_{ij}\,, \tag{65}$$

where g_{ij} is the metric tenor of M and $f(\phi)$ is a scalar function. After further differentiation and antisymmetrization one obtains

$$R_{ijk}{}^l\,\partial_l W = (g_{jk}\,\partial_i f - g_{ik}\,\partial_j f)\,. \tag{66}$$

If we contract with g^{ik} and use the fact that hyperKahler manifolds are Ricci-flat, we find that $f(\phi)$ is constant. Thus (66) implies

$$R_{ijk}{}^l\,\partial_l W = 0\,, \tag{67}$$

which is not possible to satisfy on irreducible manifolds because it implies that the representation of the holonomy group is reducible. Thus triholomorphic Killing potentials cannot exist for interesting 4-dimensional hyperKahler manifolds. For higher dimensional hyperKahler manifolds the question of existence of triholomorphic Killing potentials remains open, since the implications of Schur's lemma is weaker than (65) and a line of argument similar to that above does not seem to work.

To investigate the alternative ii) we again directly count the quadratic potentials $W=\frac{1}{2}S_{ij}\phi^i\phi^j$ on flat R^{4n} which satisfy (64). One finds that there are $n(2n+1)$ allowed potentials, which indicates that the relevant subgroup of SO$(4n)$ is Sp(n). To discuss non-trivial hyperKahler manifolds, we consider two four-dimensional Euclidean self-dual gravitational instanton metrics, namely a) the Taub-NUT [14] and b) the Eguchi-Hansen [14] instantons. Both metrics are asymptotically locally Euclidean, so that any solution of (64−) should asymptotically approach a solution on flat R^4. However the Euclidean time coordinate of the Taub-NUT metric is periodically identified. Since this simple topological requirement cannot be satisfied by the quadratic potentials of R^4 we conclude that Taub-NUT space does not support $N=4$ supersymmetric potentials (except possibly potentials which vanish relative to those of R^4 which seems unlikely). In Eguchi-Hansen space any solution of (64−) in pseudo-Euclidean coordinates must be invariant under a Z_2 group of reflection of diametrically opposite points in the surface of the boundary 3-spheres. Since the quadratic potentials of flat R^4 satisfy the required reflection symmetry, we would expect that there are three independent solutions of (64−).

Indeed such $N=4$ superpotentials have been found by Jourjine [16] who solved (64−) explicitly using the real "hyperspherical" parameterization of the Eguchi-Hansen metric. The $N=2$ supersymmetric nonlinear σ-model in 4 space-time dimensions on the Calabi series [17] of hyperKahler manifolds (of which the lowest case coincides with Eguchi-Hansen) was first formulated in components [18] and then by superspace methods [19]. Rôcek and Townsend have used the superspace formulation to find allowed potentials in the form of analytic functions of chiral $N=1$ superfields in four dimensions [20]. Upon trivial dimensional reduction to two-dimensions, one would find an $N=4$ nonlinear σ-model, in a manifest $N=2$ superspace form with a chiral superfield potential.

6. Renormalization

Since the renormalization properties of the supersymmetric nonlinear σ-model are one of its most interesting features, we whish to give a brief qualitative discussion of the renormalization of the potential structures derived in the previous sections.

Let us consider the $N=1$ model of (28) with $G_i=0$. In this case there is an $N=1$ superspace formulation as in (4) with the additional superpotential term $W(\Phi)/2i$. Further the $N=2$ and $N=4$ models with $G_i=0$ can all be placed in this form. One can compute quantum corrections to the potential using superspace perturbation theory and the normal coordinate expansion on the manifold M using the method previously used to compute the quantum corrections to the metric [4]. It is important to note that this computational procedure is universal, since results can be obtained for a general manifold M, metric g_{ij} and potential W. It is known that in 1-loop order the bare superpotential is corrected by a counterterm proportional to the Laplacian $g^{ij}D_i\partial_j W(\Phi)$, and in a higher order one expects further counterterms linear in covariant derivatives of $W(\Phi)$ contracted with the metric and curvature tensor of M. Such counterterms have been studied more explicitly by Jourjine [12].

With the above in view, consider the $N=2$ model of (50) with the restriction ($U=0$, $G_i=0$) to a purely holomorphic superpotential $W=h+\bar{h}$. In this case there is an $N=2$ superfield formulation with purely chiral superpotential $h(z^\alpha)$. The non-renormalization theorems [21] of $N=1$ supersymmetry in 4 dimensions apply in the present situation, and imply that there are no radiative corrections to the $N=2$ superpotential. Universality implies that the counterterms of the $N=1$ model must vanish when $W(\Phi)$ is restricted to the real part of a holomorphic function. For example, the one-loop counterterm certainly obeys this requirement.

For $N=4$ models on hyperKahler manifolds it is significant that the potentials in the framework of the modified superfield and the explicit solutions on the Calabi manifolds are holomorphic functions. There is an $N=2$ chiral superfield formulation and, again, no radiative corrections to the classical potential. Since there are excellent arguments to the effect that there are no metric counterterms in the $N=4$ models [2, 3], these models appear to be entirely ultraviolet finite. Jourjine [12, 16] also applied the non-renormalization theorem and concluded that the holomorphic potentials for $N=2$ models have no radiative corrections, but he drew somewhat different conclusions about the $N=4$ models.

Let us now consider the case of $N=2$ models with a potential which includes the contribution from a holomorphic Killing vector. In this case there is only an $N=1$ superfield formulation, and there is a one-loop counterterm proportional to $g^{ij}D_i\partial_j U$. For general Kahler manifolds this quantity is not the potential of a holomorphic Killing vector (although there are special cases, such as the CP^n models [see (41)], where $g^{ij}D_i\partial_j U$ is either a trivial constant or proportional to the classical potential U). Thus we seem to have a situation where the radiative corrections break $N=2$ supersymmetry down to $N=1$ supersymmetry. It also appears that when a Killing vector G_i with pseudoscalar fermion couplings is included in the $N=1$ model, the counterterms cannot be described in terms of Killing vectors and that even $N=1$ supersymmetry is broken by radiative corrections. We have not investigated this supersymmetry breakdown mechanism, and suggest that it may be an interesting problem for future work. The mechanism may be related to the non-trivial dimensional reduction procedure which leads to these models.

Acknowledgements. L. Alvarez-Gaumé thanks the theory group of Princeton University for hospitality where part of this research was done. The authors are grateful to Prof. E. Witten for suggesting that central charges arise from dimensional reduction, to Prof. M. Rôcek for help in understanding tri-holomorphic Killing vectors, and to Mr. A. Jourjine for suggesting the line of argument leading to (67). Jourjine was a collaborator in the early stages of our work where potentials were studied in the context of the modified superfield method. We refer readers to his preprints [12, 16] which discuss this early approach and some further considerations. We are not in agreement with Jourjine on the role of dimensional reduction. He argues [12] that holomorphic Killing potentials cannot be obtained by trivial dimensional reduction, whereas we argue that such potentials are generated by non-trivial (Scherk-Schwarz) dimensional reduction for $N=2$ and by trivial dimensional reduction for $N=4$.

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Communicated by A. Jaffe

Received May 12, 1983