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Unitary Representations of Non-Compact Supergroups

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Abstract. We give a general theory for the construction of oscillator-like unitary irreducible representations (UIRs) of non-compact supergroups in a super Fock space. This construction applies to all non-compact supergroups G whose coset space G/K with respect to their maximal compact subsupergroup K is "Hermitean supersymmetric". We illustrate our method with the example of $SU(m, p/n+q)$ by giving its oscillator-like UIRs in a "particle state" basis as well as "supercoherent state basis". The same class of UIRs can also be realized over the "super Hilbert spaces" of holomorphic functions of a Z variable labelling the coherent states.

1. Introduction

Lie groups play a fundamental role in the formulation of modern physical theories. The continuous symmetries observed in nature find expression in terms of them. The theory of their unitary representations is a well-established chapter of mathematics in the compact case. The rotation group and essentially all internal symmetry groups such as isospin $SU(2)_I$ or colour $SU(3)_C$ are compact. On the other hand, most space-time symmetry groups such as the Lorentz group, Poincaré group and conformal group are non-compact. The general theory of the unitary irreducible representations (UIRs) of non-compact Lie groups is however not yet at the same stage of completion as the compact case [1].

About a decade ago a new kind of symmetry principle entered physics, namely supersymmetry [2]. The novel feature of this symmetry is that it operates between bosons and fermions which have different space-time (or spin and statistics) properties. The generators of supersymmetry transformations form a Lie superalgebra whose *even* subalgebra is an ordinary Lie algebra. The *odd* generators corresponding to transformations between bosons and fermions close into the even subalgebra under anti-commutation. A complete classification of the simple

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Lie superalgebras was given by Kac [3]. A labelling of the irreducible representations of compact supergroups was first developed by Kac in [4] using the method of highest weights and Dynkin's diagrams. Some properties of the so-called "typical" representations were also given [4], and further developed in later investigations [5]. Another approach to the representation theory of supergroups was introduced in [6, 7]. Using Young supertableaux techniques many properties of typical, as well as atypical, representations of certain classes of compact supergroups were obtained. Furthermore, oscillator-like representations of some compact supergroups have been introduced [8] and discussed in [7]. The relation between Kac-Dynkin diagrams and supertableaux has also been obtained [7] and analyzed in detail [10]. Some of these results have already found applications in physics [7, 8].

Recently, a set of remarkable non-compact internal symmetry groups have been discovered in extended supergravity theories [11, 12]. Motivated by a study of the unitary realizations of these non-compact internal symmetry groups, a general theory of oscillator-like UIRs of non-compact groups with a Jordan structure with respect to their maximal compact subgroups was given in [13–15]. This general construction applies to all non-compact groups G whose coset spaces G/K with respect to their maximal compact subgroups K are Hermitean symmetric.

Our aim in this article is twofold. First we extend the construction of oscillator-like UIRs of non-compact groups to the case of non-compact supergroups G with a Jordan structure with respect to their maximal compact subsupergroup K. In this case the coset space G/K becomes a *super* Hermitean symmetric space. Second we show how the same class of UIRs can be realized over supercoherent states as induced representations. The coherent states are parametrized by a "superspace" of bosons and fermions which we label collectively by a rectangular super matrix Z. Then we show that Z undergoes a generalized linear fractional super-transformation under the action of G. In the coherent state basis it becomes evident how the same unitary representations can also be realized over the Hilbert spaces of holomorphic functions of Z, showing that these UIRs belong to the holomorphic discrete series both for the ordinary as well as super non-compact groups. Our coherent state construction of these induced representations is related to the orbit theory approach used in modern treatments of representations of semi-simple non-compact Lie groups.

The use of boson and fermion annihilation and creation operators to construct representations of ordinary Lie groups has a long history in physics. In fact, the first unitary representation of the Lorentz group was constructed by E. Majorana using boson operators. Later this method was extended and applied in physics by many authors, notably Schwinger, Goshen and Lipkin, Dirac, Gell-Mann, Gürsey, Neeman, Biedenharn, Barut, Nambu, Moshinsky, etc. References to these authors and others can be found in the books and review articles listed in [16]. Oscillator-like representations have also been studied by mathematicians, starting with the work of Bargmann [17]. Two of the more recent mathematical works on the subject are those of Kashiwara and Vergne [18] and Howe [19] to which we refer the reader for further references. In fact, the general theory of the construction of oscillator-like UIRs of non-compact groups with a Jordan structure that was given in [14] is related to the notion of dual pairs developed by Howe [19]. The present work extends these methods to the construction of the unitary representations of superalgebras and supergroups by the simultaneous use of Bose and Fermi quantum oscillators. As such it unifies the construction of oscillator-like unitary representations of compact and non-compact Lie groups as well as supergroups [20].

The plan of the paper is as follows. In Sect. 2 we give the construction of the Lie superalgebras of certain classes of non-compact supergroups with a Jordan structure in terms of boson and fermion annihilation and creation operators. In Sect. 3 we study the properties of the operator representing the unitary supergroup action in the super Fock space of boson and fermion operators. In Sect. 4 we give the explicit construction of the oscillator-like UIRs of the non-compact supergroup $SU(m, p/n+q)$ in the corresponding super Fock space. In Sect. 5 the same representations are given in terms of coherent states constructed out of boson and fermion operators. In the last section we formulate the general theory of oscillator-like UIRs of non-compact supergroups with a Jordan structure and give the list of such supergroups whose Lie superalgebras can be constructed from Jordan superalgebras via the generalized Tits-Koecher construction.

2. Construction of Lie Superalgebras in Terms of Boson and Fermion Operators

The construction of ordinary Lie algebras in terms of boson or fermion annihilation and creation operators has a long tradition in physics. It has often been used to study the representations of the corresponding group of physical interest. Recently a general construction of the oscillator-like unitary irreducible representations (UIRs) of non-compact groups has been given [10, 11]. This method is applicable to all non-compact groups G whose coset space G/K with respect to its maximal compact subgroup is Hermitean symmetric. It can also be applied to the construction of the finite dimensional unitary representations of the corresponding compact groups $\tilde{G}$. If one uses fermion annihilation and creation operators for the construction of the Lie algebras, one can then obtain the unitary representations of *compact groups only*, whereas in terms of boson annihilation and creation operators one can obtain the UIRs of both compact as well as non-compact groups [14]. Again recently the Lie superalgebras of certain classes of compact supergroups have been constructed using *boson and fermion* annihilation and creation operators [7–9]. In this section we shall extend this construction to the Lie superalgebras of certain non-compact Lie supergroups. We define a non-compact supergroup as a supergroup whose even subgroup is non-compact.

Consider a set of boson and fermion annihilation and creation operators $a_i(a^{i\dagger})$ and $\alpha_\mu(\alpha^{\mu\dagger})$, respectively:

$$\begin{gathered}
[a_i, a^{j\dagger}] = \delta_i^{\ j}\,; \qquad i,j = 1,2,\ldots,m\,,\\
\{\alpha_\mu, \alpha^{\nu\dagger}\} = \delta_\mu^{\ \nu}\,; \qquad \mu,\nu = 1,2,\ldots,n\,,\\
[a_i, \alpha_\mu] = 0 = [a_i, \alpha^{\mu\dagger}]\,,\\
\{\alpha_\mu, \alpha_\nu\} = 0 = [a_i, a_j]\,.
\end{gathered} \tag{2.1}$$

We shall denote the boson and fermion operators generically as $\xi_A(\xi^{A\dagger})$, $A=1,2,\ldots,m+n$, where

$$\xi_i \equiv a_i, \quad i=1,2,\ldots,m,$$
$$\xi_{m+\mu} \equiv \alpha_\mu, \quad \mu=1,2,\ldots,n,$$

and write symbolically

$$[\xi_A, \xi^{B\dagger}\} = \delta_A{}^B, \tag{2.2}$$

where the product [, } is to be understood as an anticommutator between any two fermionic components and as a commutator otherwise. Of the bilinear operators $\xi_A\xi^{B\dagger}$, the bosonic bilinears $a_i a^{j\dagger}$ and fermionic bilinears $\alpha_\mu \alpha^{\nu\dagger}$ generate the Lie algebras of U(m) and U(n) under commutation, respectively. The Bose-Fermi bilinears $a_i\alpha^{\mu\dagger}$ and $\alpha_\nu a^{j\dagger}$ close into the set $a_i a^{j\dagger}$ and $\alpha_\nu\alpha^{\mu\dagger}$ under anticommutation:

$$\begin{aligned} &\{a_i\alpha^{\mu\dagger}, \alpha_\nu a^{j\dagger}\} = \delta_i{}^j \alpha^{\mu\dagger}\alpha_\nu + \delta^\mu_\nu a^{j\dagger} a_i, \\ &\{a_i\alpha^{\mu\dagger}, a_j\alpha^{\nu\dagger}\} = 0 = \{\alpha_\mu a^{i\dagger}, \alpha_\nu a^{j\dagger}\}. \end{aligned} \tag{2.3}$$

Thus, considering the boson-fermion bilinears $a_i\alpha^{\mu\dagger}$ and $\alpha_\mu a^{i\dagger}$ as the odd generators and $a_i a^{j\dagger}$ and $\alpha_\mu\alpha^{\nu\dagger}$ as the even generators, one finds that the operators $\xi_A\xi^{B\dagger}$ form the Lie superalgebras U(m/n) with the Lie superproduct. Throughout this paper the Lie superproduct will mean an anticommutator between any two odd generators and a commutator otherwise.

The Lie superalgebra of U(m/n) generated by $\xi_A\xi^{B\dagger}$ can now be enlarged to the Lie superalgebra Osp($2n/2m$) with the inclusion of di-annihilation and di-creation operators of the form $\xi_A\xi_B$ and $\xi^{A\dagger}\xi^{B\dagger}$. The even part of the resulting Lie superalgebra is the Lie algebra of O($2n$)⊗Sp($2m$, ℝ), where the symmetric group Sp($2m$, ℝ) is non-compact, with the maximal compact subgroup U(m), and O($2n$) is compact. The form of the non-compactness of the even subgroup is determined in a superHermitean basis as defined in Sect. 3. We shall denote this "non-compact superalgebra" as Osp($2n/2m$, ℝ).

If we consider a pair of bose-fermi operators $\xi_A(\xi^{A\dagger})$ and $\eta_A(\eta^{A\dagger})$:

$$\begin{aligned} &[\xi_A, \xi^{B\dagger}\} = \delta_A{}^B = [\eta_A, \eta^{B\dagger}\}, \\ &[\eta_A, \xi^{B\dagger}\} = 0 = [\eta_A, \xi_B\}, \end{aligned} \tag{2.4}$$
$$A, B = 1, 2, \ldots, m, m+1, \ldots, m+n,$$

then the superalgebra U(m/n) generated by $T_A{}^B = \xi_A\xi^{B\dagger} + \eta_A\eta^{B\dagger}$ can be enlarged to other superalgebras by the inclusion of bilinear operators of the form $\xi\eta$. For example, the supersymmetric bilinear operators $S_{AB} = \xi_A\eta_B + \eta_A\xi_B$ and $S^{AB} = \xi^{A\dagger}\eta^{B\dagger} + \eta^{A\dagger}\eta^{B\dagger}$ extend the superalgebra U(m/n) generated by $T_A{}^B$ to the superalgebra Osp($2n/2m$, ℝ). If, instead of the S_{AB} and S^{AB}, we consider the superantisymmetric operators

$$A_{AB} = \xi_A\eta_B - \eta_A\xi_B\,; \qquad A^{AB} = \xi^{A\dagger}\eta^{B\dagger} - \eta^{A\dagger}\xi^{B\dagger},$$

then they, together with $T_A{}^B$, generate the Lie superalgebra of the non-compact supergroup Osp($2n^*/2m$) whose even subgroup is O($2n$)*⊗Usp($2m$). Here, O($2n$)* is non-compact with a maximal compact subgroup U(n) and Usp($2m$) is compact.

If, instead of extending the Lie superalgebra U(m/n) generated by $T_A{}^B$ with di-creation and di-annihilation operators, one considers an extension by bilinear operators of the form $\xi^\dagger\eta$ and $\eta^\dagger\xi$, then one obtains the Lie superalgebras of "*compact*" supergroups in the corresponding superHermitean basis[1]. For example, the operators

$$(\xi_A\eta^{B\dagger}+\xi_B\eta^{A\dagger}) \quad \text{and} \quad (\eta_A\xi^{B\dagger}+\eta_B\xi^{A\dagger})$$

extend the superalgebra U(m/n) generated by $T_A{}^B$ to the Lie superalgebra of Osp($2n/2m$). Similarly, the antisymmetric bilinear operators

$$(\xi_A\eta^{B\dagger}-\xi_B\eta^{A\dagger}) \quad \text{and} \quad (\eta_A\xi^{B\dagger}-\eta_B\xi^{A\dagger})$$

extend U(m/n) to the Lie superalgebra of Osp($2m/2n$).

If the bose-fermi operators ξ and η transform differently, i.e.,

$$[\xi_A, \xi^{B\dagger}\} = \delta_A{}^B\,; \qquad A, B = 1, 2, \ldots, m+n, m \mp n,$$

and (2.5)

$$[\eta_M, \eta^{N\dagger}\} = \delta_M{}^N\,; \qquad M, N = 1, \ldots, p+q, p \mp q,$$

where the first p components of η are bosonic and the last q components fermionic, then the bilinear operators

$$I_A{}^B = \xi_A\xi^{B\dagger} - \frac{1}{m-n}\delta_A{}^B(\xi_C\xi^{C\dagger}) \tag{2.6}$$

and

$$J_M{}^N = \eta_M\eta^{N\dagger} - \frac{1}{p-q}\delta_M{}^N(\eta_Q\eta^{Q\dagger}) \tag{2.7}$$

generate the Lie superalgebras SU(m/n) and SU(p/q) under the Lie superproduct. Note that $I_A{}^B$ and $J_M{}^N$ are supertraceless, i.e.,

$$\begin{aligned} \mathrm{Str}(I_A{}^B) &\equiv \sum_{C=1}^{m+n} I_C{}^C(-1)^{g(C)} = 0\,, \\ \mathrm{Str}(J_M{}^N) &\equiv \sum_{Q=1}^{p+q} J_Q{}^Q(-1)^{g(Q)} = 0\,, \end{aligned} \tag{2.8}$$

where $g(A)$ denotes the grade of an element A of the Lie superalgebra and $g(A)=0$ for even indices A and $g(A)=1$ for odd indices A.

The di-creation and di-annihilation operators $T^{AM}=\xi^{A\dagger}\eta^{M\dagger}$ and $T_{AM}=\eta_M\xi_A$ and the operators $I_A{}^B$ and $J_M{}^N$, together with the U(1) generator

$$N = \frac{1}{m-n}\xi^{A\dagger}\xi_A + \frac{1}{p-q}\eta^{M\dagger}\eta_M \tag{2.9}$$

close under the Lie superproduct and form the Lie superalgebra of the non-compact supergroup SU($m, p/n+q$). The even subgroup of SU($m, p/n+q$) is

1 That is, if the corresponding compact forms of these supergroups exist. If the compact forms do not exist then the method explained in the next section will lead to non-unitary finite dimensional representations in this case

$S(U(m,p)\otimes U(n+q))$, where $SU(m,p)$ is non-compact with a maximal compact subgroup $S(U(m)\times U(p))$.

If we have R generations of bose-fermi operators $\xi(r)$ and $\eta(r)$ $(r=1,\ldots,R)$ that "supercommute" with each other

$$\begin{aligned} [\xi_A(r), \xi^{B\dagger}(s)\} &= \delta_A{}^B \delta_{rs}, \\ [\eta_M(r), \eta^{N\dagger}(s)\} &= \delta_M{}^N \delta_{rs}, \\ [\xi_A(r), \eta_M(s)\} = 0 &= [\xi_A(r), \eta^{M\dagger}(s)\}, \qquad r,s=1,2,\ldots,R, \end{aligned} \tag{2.10}$$

then one can construct the Lie superalgebra $SU(m,p/n+q)$ similarly by summing over the generation index

$$\begin{aligned} I_A{}^B &= \boldsymbol{\xi}_A\cdot\boldsymbol{\xi}^{B\dagger} - \frac{1}{m-n}\delta_A{}^B(\boldsymbol{\xi}_C\cdot\boldsymbol{\xi}^{C\dagger}), \\ J_M{}^N &= \boldsymbol{\eta}_M\cdot\boldsymbol{\eta}^{N\dagger} - \frac{1}{p-q}\delta_M{}^N(\boldsymbol{\eta}_P\cdot\boldsymbol{\eta}^{P\dagger}), \\ N &= \frac{1}{m-n}\boldsymbol{\xi}^{A\dagger}\cdot\boldsymbol{\xi}_A + \frac{1}{p-q}\boldsymbol{\eta}^{M\dagger}\cdot\boldsymbol{\eta}_M, \\ T_{AM} &= \boldsymbol{\eta}_M\cdot\boldsymbol{\xi}_A, \\ T^{AM} &= \boldsymbol{\xi}^{A\dagger}\cdot\boldsymbol{\eta}^{M\dagger}, \end{aligned} \tag{2.11}$$

where $\boldsymbol{\xi}\cdot\boldsymbol{\eta}\equiv\sum_{r=1}^{R}\xi(r)\,\eta(r)$, etc. This applies to all the above constructions. If we have R pairs of operators that supercommute with each other, then the construction of the respective superalgebras is achieved by summing over the generation index r. This corresponds to taking a direct sum of R copies of the same superalgebra. However, as we shall see later, the possible *unitary irreducible representations* (UIRs) of the non-compact supergroups that can be constructed will be determined by the number R of generations of Bose-Fermi operators.

3. Superhermitean Basis of Lie Superalgebras and the Unitary Supergroup Action

In the following sections we shall give the construction of oscillator-like UIRs of non-compact supergroups in the super Fock space of boson and fermion operators. Before going into this explicit construction, let us explain how we choose the generators of a unitary supergroup action in the super Fock space. Denoting the generators of the even subgroup by H_a and the odd generators by A_α we can represent the supergroup action in the Fock space by the operator

$$\hat{U}(g) = e^{iw^a H_a + i\theta^\alpha A_\alpha},$$

where w^a and θ^α are real "bosonic" and "fermionic" (e.g., Grassmann) parameters, respectively. Since U mixes bosons with fermions, the consistency of the transformations requires that θ^α be taken as anticommuting with the odd generators A_α [21],

$$\{\theta^\alpha, A_\beta\} = 0 .$$

This is equivalent to taking the θ^α as anticommuting with the fermion operators. On the other hand, for $\hat{U}(g)$ to be a unitary operator in the super Fock space, we must have

$$\hat{U}^\dagger = \hat{U}^{-1} = e^{-iw^a H_a + i\theta^\alpha A_\alpha^\dagger},$$

which implies that we choose the generators such that $H_a^\dagger = H_a$, $A_\alpha^\dagger = -A_\alpha$. Thus the even generators H_a are Hermitean operators and the odd generators A_α are antiHermitean operators. In this basis all the structure constants of the Lie superalgebra are pure imaginary numbers, except for those relating two odd generators to an even one, which are pure real. We shall refer to this basis as the superHermitean basis. Throughout the paper it will be implicitly assumed that we are working in such a basis and the form of non-compactness of the supergroup (i.e., the Killing form) is to be determined in such a basis. As an example, let us give the superHermitean basis of the non-compact supergroup SU(2, 1/1) whose even subgroup is SU(U(2, 1) × U(1)). In this case we take

$$\xi = \begin{pmatrix} a_1 \\ a_2 \\ \alpha \end{pmatrix}, \qquad \eta = b,$$

where

$$\begin{aligned} &[a_i, a^{j\dagger}] = \delta_i^{\,j}, \qquad i,j = 1, 2, \\ &\{\alpha, \alpha^\dagger\} = 1, \\ &[b, b^\dagger] = 1, \\ &[a_i, b] = 0 = [\alpha, b]. \end{aligned} \tag{3.1}$$

The compact SU(2/1) subgroup is generated by $I_A^{\,B}$ as defined by Eq. (2.6). The "non-compact" generators are $\xi_A b$ and $\xi^{A\dagger} b^\dagger$. To go to the superHermitean basis we must choose Hermitean combinations of the even generators and antiHermitean combinations of the odd ones. For the even generators and their parameters we thus choose

$$\begin{aligned} W_i^j &= I_i^{\,j} + I_j^{\,i} = W_j^i \leftrightarrow w_i^{\,j}, \\ V_i^j &= i(I_i^{\,j} - I_j^{\,i}) = -V_j^i \leftrightarrow v_i^{\,j}, \\ N &= a^{i\dagger} a_i + \alpha^\dagger \alpha + b^\dagger b \leftrightarrow \varrho, \end{aligned} \tag{3.2}$$

$$\begin{aligned} W_i^4 &= a_i b + a^{i\dagger} b^\dagger \leftrightarrow w_i^{\,4}, \\ V_i^4 &= i(a_i b - a^{i\dagger} b^\dagger) \leftrightarrow v_i^{\,4}, \end{aligned} \tag{3.3}$$

and for the odd generators we choose the following antiHermitean operators and their Grassmann parameters

$$\begin{aligned} P_i{}^3 &= a_i\alpha^\dagger - \alpha a^{i\dagger} \leftrightarrow \theta_i{}^3, \\ Q_i{}^3 &= i(a_i\alpha^\dagger + \alpha a^{i\dagger}) \leftrightarrow \varepsilon_i{}^3, \\ P_3{}^4 &= (\alpha b - \alpha^\dagger b^\dagger) \leftrightarrow \theta_3{}^4, \\ Q_3{}^4 &= i(\alpha b + \alpha^\dagger b^\dagger) \leftrightarrow \varepsilon_3{}^4. \end{aligned} \tag{3.4}$$

The unitary operator representing the action of SU(2, 1/1) in our super Fock space can thus be written as

$$\hat{U}(g) = e^{i\bar{\psi}\mathscr{H}\psi}, \tag{3.5}$$

where

$$\psi = \begin{pmatrix} \xi \\ \eta^\dagger \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \alpha \\ b^\dagger \end{pmatrix}; \quad \bar{\psi} = \psi^\dagger \gamma, \; \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathscr{H} = \left(\begin{array}{cc|c|c} w_1{}^1 + \varrho & w_1{}^2 - iv_1{}^2 & \theta_1{}^3 - i\varepsilon_1{}^3 & w_1{}^4 - iv_1{}^4 \\ w_1{}^2 + iv_1{}^2 & w_2{}^2 + \varrho & \theta_2{}^3 - i\varepsilon_2{}^3 & w_2{}^4 - iv_2{}^4 \\ \hline \theta_1{}^3 + i\varepsilon_1{}^3 & \theta_2{}^3 + i\varepsilon_2{}^3 & w_1{}^1 + w_2{}^2 + \varrho & -\theta_3{}^4 - i\varepsilon_3{}^4 \\ \hline -w_1{}^4 - iv_1{}^4 & -w_2{}^4 - iv_2{}^4 & \theta_3{}^4 + i\varepsilon_3{}^4 & -\varrho \end{array}\right). \tag{3.6}$$

Note that in writing $\hat{U}(g)$ in this form we used the fact that the Grassmann parameters θ, ε anticommute with the fermion operators. The unitarity of $\hat{U}$ follows from the hermiticity of $\bar{\psi}\mathscr{H}\psi$:

$$\begin{aligned} (\bar{\psi}\mathscr{H}\psi)^\dagger &= \psi^\dagger \mathscr{H}^\dagger \gamma^\dagger \psi \\ &= \psi^\dagger \gamma \mathscr{H} \psi = \bar{\psi}\mathscr{H}\psi, \end{aligned}$$

since $\gamma\mathscr{H}\gamma = \mathscr{H}^\dagger$, where Hermitean conjugation interchanges the orders of fermionic parameters as well as all the operators. Here $\mathscr{H}$ is also supertraceless as it must be! In fact, $\mathscr{H}$ corresponds simply to the four-dimensional fundamental representation of the Lie superalgebra SU(2, 1/1), where the representation matrices are multiplied with their parameters (Bose or Fermi). This is a very general feature of the unitary group actions in super Fock spaces, which we explain below.

Consider for example the fundamental representation of the Lie superalgebra $SU(m/n)$ $(m \neq n)$. Let $\mathscr{H}$ represent the matrix of the generators in the fundamental representation multiplied with their parameters. It can then be written in the form of an $(m+n)\times(m+n)$ Hermitean matrix:

$$\mathscr{H}(m/n) = \begin{matrix} m \\ n \end{matrix} \overset{\quad m \qquad n}{\left(\begin{array}{c|c} H_m & \theta \\ \hline \theta^\dagger & H_n \end{array}\right)} = \mathscr{H}^\dagger(m/n)$$

where H_m and H_n are $m \times m$ and $n \times n$ Hermitean matrices and θ is a $m \times n$ matrix whose entries are anticommuting complex Grassmann numbers. $\theta^\dagger$ is the

Hermitean conjugate of θ. Furthermore, $\mathscr{H}$ is supertraceless:

$$\mathrm{Str}\,\mathscr{H} = \mathrm{Tr}\,H_m - \mathrm{Tr}\,H_n = 0\,. \tag{3.7}$$

These matrices close under *commutation*

$$[\mathscr{H}_1, \mathscr{H}_2] = i\mathscr{H}_3\,. \tag{3.8}$$

$\mathscr{H}_3$ is seen to preserve the hermiticity and supertracelessness properties provided one takes care of the order of Grassmann numbers in a product. Hermitean conjugation is defined such that it changes the order of fermions in a product in addition to complex conjugation and transposition of matrices. Thus we have the property of closure for SU(m/n) with only commutators and no anticommutators as long as we insist on combining the parameters with the matrix representation of the generators.

We can now try to combine the parameters with the "quantum" generators constructed from the Bose-Fermi operators. The following combination will be covariant [7] under SU(m/n):

$$\begin{aligned} \xi^{A\dagger}\mathscr{H}_A{}^B\xi_B &= \xi_B\xi^{A\dagger}\mathscr{H}_A{}^B(-1)^{g(B)} \\ &= I_B{}^A\mathscr{H}_A{}^B(-1)^{g(B)} = \mathrm{Str}[I\mathscr{H}]\,, \end{aligned} \tag{3.9}$$

where the supergenerators $I_A{}^B$ constructed in the previous section appear in the supertrace with the matrix $\mathscr{H}$ which contains the parameters (bosonic as well as fermionic). The closure property is now seen to take the simple form

$$[\xi^\dagger\mathscr{H}_1\xi, \xi^\dagger\mathscr{H}_2\xi] = \xi^\dagger[\mathscr{H}_1, \mathscr{H}_2]\xi = i\xi^\dagger\mathscr{H}_3\xi\,. \tag{3.10}$$

Again, since we have combined the generators with their corresponding parameters, closure is achieved with only commutators and no anticommutators. This property is essential in order to understand how a supergroup is constructed from the superalgebra by a simple exponentiation or by taking an infinite number of infinitesimal products

$$\hat{U}(\mathscr{H}) = e^{i\xi^\dagger\mathscr{H}\xi} = \lim_{n\to\infty}\left(1 + \frac{i}{n}\xi^\dagger\mathscr{H}\xi\right)^n. \tag{3.11}$$

One can verify that since closure and Jacobi identities are satisfied with the ordinary Lie product, the "quantum" operators $\hat{U}$ form a group.

$$\hat{U}(\mathscr{H}_1)\,\hat{U}(\mathscr{H}_2) = \hat{U}(\mathscr{H}_3)\,. \tag{3.12}$$

$\hat{U}(\mathscr{H})$ is a unitary operator in the super Fock space since $\xi^\dagger\mathscr{H}\xi$ is Hermitean.

$$\hat{U}(\mathscr{H}) = \hat{U}(-\mathscr{H}) = \hat{U}^{-1}(\mathscr{H})\,. \tag{3.13}$$

We define the supergroup element in the fundamental representation by the matrix

$$\begin{aligned} U_A{}^B &= (e^{i\mathscr{H}})_A{}^B \\ U_1U_2 &= U_3\,. \end{aligned} \tag{3.14}$$

It is easy to show that

$$\hat{U}^\dagger \xi_A \hat{U} = (e^{i\mathscr{H}})_A{}^B \xi_B$$
$$\hat{U}^\dagger \xi^{A\dagger} \hat{U} = \xi^{B\dagger} (e^{-i\mathscr{H}})_B{}^A \tag{3.15}$$
$$\hat{U}^\dagger I_A{}^B \hat{U} = (e^{i\mathscr{H}})_A{}^{A'} I_{A'}{}^{B'} (e^{-i\mathscr{H}})_{B'}{}^B .$$

Thus ξ_A is covariant, $\xi^{A\dagger}$ is contravariant and the generators $I_A{}^B$ transform just as they should under the group action. We should emphasize that the orders of the factors as written on the right-hand side are of crucial importance. If one attempts to change their order one has to keep in mind the fact that the odd generators (or equivalently the fermion operators) anticommute with Grassmann parameters. Even though the above equations formally appear like the familiar ordinary Lie group transformations, they are actually somewhat different and their content should carefully be understood.

4. Oscillator-Like Unitary Irreducible Representations of the Non-Compact Supergroup SU($m, p/n+q$)

In this section we shall illustrate our method of constructing oscillator-like unitary irreducible representations of non-compact supergroups with the example of SU($m, p/n+q$). This method is applicable to all non-compact supergroups whose coset spaces with respect to their maximal compact supersubgroup are Hermitean supersymmetric. The general theory will be given in Sect. 6.

Consider now R pairs of Bose-Fermi operators $\boldsymbol{\xi}_A(\boldsymbol{\xi}^{A\dagger})$ and $\boldsymbol{\eta}^M(\boldsymbol{\eta}^\dagger_M)$ satisfying the supercanonical relations

$$[\xi_A(r), \xi^{B\dagger}(s)\} = \delta_A{}^B \delta_{rs} ,$$
$$[\eta^M(r), \eta^\dagger_N(s)\} = \delta^M{}_N \delta_{rs} , \tag{4.1}$$
$$A, B = 1, \ldots, m+n ;\; M, N = 1, 2, \ldots, p+q ;\; r, s = 1, \ldots, R ,$$

where the annihilation operators $\boldsymbol{\xi}$ transform covariantly under SU(m/n) and the annihilation operators $\boldsymbol{\eta}$ transform contravariantly under SU(p/q). The Lie superalgebra SU($m, p/n+q$) than can be constructed from these operators has a three-dimensional graded structure (in a split basis):

$$L = L^{-1} \oplus L^0 \oplus L^{+1} ,$$

where L^0 denotes the Lie superalgebra of the maximal compact subsupergroup S(U(m/n)$\otimes$U(p/q)), and L^- and L^+ spaces correspond to the "non-compact" generators. Explicitly, we have

$$L^0 \cong I_A{}^B \oplus K_M{}^N \oplus N ,$$
$$L^{-1} \cong L_A{}^M ; \quad L^+ \cong L^A{}_M ,$$

where

$$I_A{}^B = \boldsymbol{\xi}_A \cdot \boldsymbol{\xi}^{B\dagger} - \frac{1}{m-n} \delta_A{}^B (\boldsymbol{\xi}^{C\dagger} \cdot \boldsymbol{\xi}_C) , \qquad m \neq n$$
$$K_M{}^N = -\boldsymbol{\eta}^\dagger_M \cdot \boldsymbol{\eta}^N + \frac{1}{p-q} \delta_M{}^N (\boldsymbol{\eta}^Q \cdot \boldsymbol{\eta}^\dagger_Q) , \qquad p \neq q \tag{4.2}$$

$$N=\frac{1}{m-n}\xi^{A\dagger}\cdot\xi_A+\frac{1}{p-q}\eta^M\cdot\eta_M^\dagger,$$
$$L_A{}^M=\xi_A\cdot\eta^M;\qquad L^A{}_M=\eta_M^\dagger\cdot\xi^{A\dagger}. \tag{4.3}$$

Note that the grading is achieved with the generator $\frac{1}{2}N$, i.e.,

$$[\tfrac{1}{2}N,I_A{}^B]=0=[\tfrac{1}{2}N,K_M{}^N]$$
$$[\tfrac{1}{2}N,L_A{}^M]=-L_A{}^M;\qquad [\tfrac{1}{2}N,L^A{}_M]=+L^A{}_M, \tag{4.4}$$

and L^- space consists of di-annihilation operators and L^+ of di-creation operators.

Consider the super Fock space $\mathscr{F}$ formed by taking a tensor product of the Fock spaces of all the Bose-Fermi operators ξ and η. The vacuum state $|0,0\rangle$ in $\mathscr{F}$ is then simply the tensor product state of all the individual vacua and is annihilated by all the annihilation operators

$$\xi_A|0,0\rangle=0;\qquad \eta^M|0,0\rangle=0. \tag{4.5}$$

Now, if we have a set of states $|K^{A\dots}_{M\dots}\rangle$ in $\mathscr{F}$ that transforms according to some representation of the maximal compact subsupergroup $S(U(m/n)\times U(p/q))$ and are annihilated by all the operators $L_A{}^M$ belonging to the L^- space, then the infinite set of states obtained by repeated application of the L^+ operators $L^A{}_M$ on the states $|K^{A\dots}_{M\dots}\rangle$:

$$|K^{A\dots}_{M\dots}\rangle,\ L^B{}_N|K^{A\dots}_{M\dots}\rangle,\ L^C{}_P L^B{}_N|K^{A\dots}_{M\dots}\rangle,\dots \tag{4.6}$$

forms the basis of a unitary representation of the non-compact supergroup $SU(m,p/n+q)$. The proof of this statement will be given in Sect. 6. The remarkable property of these representations is that if $|K^{A\dots}_{M\dots}\rangle$ is chosen so as to transform irreducibly under the maximal compact subsupergroup $S(U(m/n)\times U(p/q))$, then the resulting unitary representation of $SU(m,p/m+q)$ is also irreducible. They are the analogues of the highest weight representations of ordinary compact and non-compact groups [1, 14]. Thus the set of UIRs that we can construct this way is determined by the set of states $|K\rangle$ in $\mathscr{F}$ that transforms irreducibly under $S(U(m/n)\times U(p/q))$ and is annihilated by the operators in the L^- space. Clearly any state of the form

$$(\xi^{A\dagger})^k|0,0\rangle\equiv\underbrace{\xi^{A\dagger}\xi^{B\dagger}\dots\xi^{C\dagger}}_{K\text{ times}}|0,0\rangle \tag{4.7}$$

or of the form

$$(\eta_M^\dagger)^k|0,0\rangle,\qquad k=0,1,2,\dots \tag{4.8}$$

is annihilated by the operator $L_A{}^M=\xi_A\cdot\eta^M$ of the L^- space. A linear combination of pure ξ and pure η states will also be annihilated by $L_A{}^M$. However, since they are not irreducible representations of $S(U(m/n)\times U(p/q))$, we will not consider them.

The state $\xi^{A\dagger}|0,0\rangle$ transforms like the contravariant fundamental representation of $SU(m/n)$, whose first m components are bosonic and the last n components fermionic. Following [6, 7], we shall represent it by a dotted superbox

(corresponding to the supergeneralization of ordinary boxes in Young tableaux)

$$\xi^{A\dagger}\,|0,0\rangle \cong |\boxed{\diagup},0\rangle, \qquad A,B=1,\dots,m+n. \tag{4.9}$$

Then the state $\xi^{A\dagger}\xi^{B\dagger}\,|0,0\rangle$ corresponds to the supersymmetrized representation $\boxed{\diagup}\boxed{\diagup}$ of $\mathrm{SU}(m/n)$:

$$\xi^{A\dagger}\xi^{B\dagger}\,|0,0\rangle \cong |\boxed{\diagup}\boxed{\diagup},0\rangle. \tag{4.10}$$

Similarly, the state $(\xi^{A\dagger})^k\,|0,0\rangle$ transforms like the irreducible representation

$$\underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\underset{3}{\boxed{\diagup}}\boxed{\diagup}\cdots\underset{k}{\boxed{\diagup}}$$

of $\mathrm{SU}(m/n)$ $(k=1,2,\dots)$. If we have only one set of operators ξ, these are the most general irreducible representations that can be constructed in the super Fock space. On the other hand, if we have R sets of operators $\xi(r)$ $(r=1,\dots,R)$, then the set of irreducible representations that one can construct is richer. For example, if $R=2$, we have

$$(\xi^{A\dagger}(1))^{\ell_1}(\xi^{B\dagger}(2))^{\ell_2}\,|0,0\rangle \cong \underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\cdots\underset{\ell_1}{\boxed{\diagup}} \otimes \underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\underset{3}{\boxed{\diagup}}\cdots\underset{\ell_2}{\boxed{\diagup}}. \tag{4.11}$$

Thus the irreducible representations one obtains in this case are those that are contained in the tensor product of ℓ_1 supersymmetric and ℓ_2 supersymmetric representations of $\mathrm{SU}(m/n)$ (ℓ_1 and ℓ_2 being positive integers). In the general case

$$\begin{aligned}&(\xi^{A\dagger}(1))^{\ell_1}(\xi^{B\dagger}(2))^{\ell_2}\dots(\xi^{C\dagger}(R))^{\ell_R}\,|0,0\rangle\\ &\approx \underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\cdots\underset{\ell_1}{\boxed{\diagup}} \otimes \underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\cdots\underset{\ell_2}{\boxed{\diagup}} \otimes\dots\otimes \underset{1}{\boxed{\diagup}}\underset{2}{\boxed{\diagup}}\cdots\underset{\ell_R}{\boxed{\diagup}}.\end{aligned} \tag{4.12}$$

Thus, if we have R generations of the operators $\xi(r)$ $(r=1,\dots,R)$, then the states that transform irreducibly under $\mathrm{SU}(m/n)$ correspond to dotted supertableaux that can have up to R rows. These irreducible representations can be projected out of the set (4.12) by using appropriate projection operators. Note that all states constructed from only dotted (or only undotted) superboxes in a supertableau are irreducible [6, 7].

The same arguments apply to the multiple action of the operators $\eta^\dagger_M$ on the vacuum. In this case one obtains the representations of $\mathrm{SU}(p/q)$. One can use any one of the irreducible representations obtained by multiple applications of $\xi^{A\dagger}(r)$ only [or of $\eta^\dagger_M(r)$ only] followed by a suitable projection operator to construct a "lowest" state $|K\rangle$ on which to build the basis of a UIR of $\mathrm{SU}(m,p/n+q)$. Such states $|K\rangle$ will also be irreducible under the maximal compact subsupergroup $\mathrm{S}(\mathrm{U}(m/n)\times\mathrm{U}(p/q))$. The irreducible representations obtained by the action of $\xi^{A\dagger}(r)$ alone transform like the representation $[(\dot r_1,\dot r_2,\dot r_3,\dots),1]$ of $\mathrm{S}(\mathrm{U}(m/n)\times\mathrm{U}(p/q))$, and those obtained from $\eta^\dagger_M$ transform as $[1,(r_1,r_2,r_3,\dots)]$, where $(r_1,r_2,\dots)$ [or $(\dot r_1,\dot r_2,\dots)$] denote the supertableaux of an irreducible representation which has $r_i(\dot r_i)$ super(dotted) boxes in its i^{th} row. Note that there is no limit to the number of rows in a supertableau.

The operators $L^A{}_M=\xi^{A\dagger}\cdot\eta^\dagger_M$ of the L^+ space transforms like the representation $[(\dot 1,0,\dots),(1,0,0,\dots)]$ of $\mathrm{S}(\mathrm{U}(m/n)\times\mathrm{U}(p/q))$ and $(L^A{}_M)^\ell\approx[(\dot\ell,0,0,\dots),(\ell,0,0,\dots)]$. Now, starting from a "lowest" state $|K^{A\dots}_{M\dots}\rangle$ transforming irreducibly, say like the representation $[(\dot m_1,\dot m_2,\dots),1]$, under $\mathrm{S}(\mathrm{U}(m/n)\times\mathrm{U}(p/q))$ and annihilated by the operators $L_A{}^M$ of L^- space, we can construct an infinite tower of states by applying powers of the operator $L^A{}_M$ of the L^+ space. They transform as

$$\begin{aligned}(L^A{}_M)^\ell |K\rangle &= (\xi^{A\dagger}\cdot\eta^\dagger_M)^\ell |K\rangle \\ &\simeq [(\dot{\ell},0,0,\ldots),(\ell,0,0,\ldots)]\otimes[(\dot{m}_1,\dot{m}_2,\ldots),1], \quad \ell=1,2,\ldots, \end{aligned}\tag{4.13}$$

and form the basis of a UIR of the non-compact supergroup $SU(m,p/n+q)$ in the Fock space $\mathscr{F}$. Thus, for each such lowest state $|K\rangle$ we obtain a UIR of $SU(m,p/n+q)$. By choosing the generation number R large enough, we can construct any representation of $SU(m/n)$ with dotted supertableaux only and any representation of $SU(p/q)$ with undotted supertableaux only as a lowest state. For $n=0=q$ these representations reduce to the oscillator-like UIRs of $SU(m,p)$ [14]. For $m=0=p$ they give the finite dimensional UIRs of the compact group $SU(n+q)$. The state $|K\rangle$, annihilated by the L^- operator, corresponds to the highest weight (up to a reflection) of the irreducible representation thus obtained. To write down the unitary $\hat{U}$ representing the group action of $SU(m,p/n+q)$ in the super Fock space $\mathscr{F}$ we must exponentiate the generators in a superHermitean basis multiplied with their respective parameters. As explained in the previous section, this leads to the representation

$$\hat{U}=e^{i\bar{\psi}\mathscr{M}\psi}, \tag{4.14}$$

where

$$\psi=\begin{pmatrix}\xi_A\\ \eta^\dagger_M\end{pmatrix}; \qquad \bar{\psi}=\psi^\dagger\gamma=(\xi^{A\dagger}\ -\eta^M),$$

and

$$\gamma=\left(\begin{array}{c|c}1_{m+n} & 0\\ \hline 0 & -1_{p+q}\end{array}\right),$$

and $\mathscr{M}$ is the supertraceless $(m+n+p+q)\times(m+n+p+q)$ matrix representing the $(m+n+p+q)$ dimensional fundamental representation of the Lie superalgebra of $SU(m,p/n+q)$ multiplied with their respective parameters. The $\mathscr{M}$ can be represented in the form

$$\mathscr{M}=\left(\begin{array}{c|c}\mathscr{H}(m/n) & -iv\\ \hline -iv^\dagger & \mathscr{H}(p/q)\end{array}\right), \tag{4.15}$$

where $\mathscr{H}(m/n)$ and $\mathscr{H}(p/q)$ are $(m+n)$ and $(p+q)$ dimensional Hermitean matrices representing the Lie superalgebras of $U(m/n)$ and $U(p/q)$ multiplied with their respective parameters such that

$$\operatorname{Str}\mathscr{M}=\operatorname{Str}\mathscr{H}(m/n)+\operatorname{Str}\mathscr{H}(p/q)=0\ ;$$

v is a $(m+n)\times(p+q)$ matrix of the form

$$v=\begin{array}{c}\\ m\\ n\end{array}\!\!\begin{array}{c}\begin{array}{cc}p & q\end{array}\\ \left(\begin{array}{c|c}V & \lambda\\ \hline \chi & W\end{array}\right)\end{array}, \tag{4.16}$$

where V and W are $(m\times p)$ and $(n\times q)$ complex matrices, respectively. Here λ and χ are $(m\times q)$ and $(n\times p)$ matrices whose entries are complex Grassmann numbers.

Unitarity of $\hat{U}$ follows from the identity $(\gamma\mathcal{M})^\dagger = \mathcal{M}\gamma$ or $\mathcal{M}^\dagger = \gamma\mathcal{M}\gamma$. The operator $\hat{H} = \bar{\psi}\mathcal{M}\psi$ is Hermitean and

$$\hat{U}^\dagger = (e^{i\hat{H}})^\dagger = e^{-i\hat{H}} = \hat{U}^{-1}.$$

5. Coherent State Representations of the Non-Compact Supergroup SU($m, p/n+q$)

In this section we shall give the UIRs of the non-compact SU($m, p/n+q$) in terms of supercoherent states constructed out of the "particle states" in the super Fock space $\mathcal{F}$. For each oscillator-like UIR there exists such a coherent state basis. In the coherent state basis the analyticity properties of these representations will become more evident, showing that they indeed belong to the "holomorphic discrete series."

The unitary operator $\hat{U}$ representing the supergroup SU($m, p/n+q$) action in our super Fock space $\mathcal{F}$ is now given by Eq. (4.14)

$$\hat{U} = e^{i\bar{\psi}\mathcal{M}\psi} \equiv \hat{U}(\mathcal{M}), \tag{5.1}$$

where ψ, $\mathcal{M}$ were defined by Eqs. (4.14) and (4.15). The $(m+n+p+q)$ dimensional non-unitary fundamental representation of SU($m, p/n+q$) is given simply by the exponential of the matrix $\mathcal{M}$:

$$U = e^{i\mathcal{M}}. \tag{5.2}$$

The operator ψ transforms covariantly under the action of SU($m, p/n+q$) as

$$\hat{U}^\dagger(\mathcal{M})\psi\hat{U}(\mathcal{M}) = (e^{i\mathcal{M}}\psi) = U\psi, \tag{5.3}$$

and $\bar{\psi}$ contravariantly as

$$\hat{U}^\dagger(\mathcal{M})\bar{\psi}\hat{U}(\mathcal{M}) = \bar{\psi}e^{-i\mathcal{M}} = \bar{\psi}U^{-1}. \tag{5.4}$$

The supermatrix U can now, in general, be decomposed as [22]

$$U = th, \tag{5.5}$$

where h is a group element in the maximal compact subgroup S(U(m/n) × U(p/q)), and t sits on the coset space SU($m, p/n+q$)/S(U(m/n × U(p/q)), with h represented as

$$h = \exp\left[i\begin{pmatrix} \mathcal{H}(m/n) & 0 \\ 0 & \mathcal{H}(p/q) \end{pmatrix}\right], \tag{5.6}$$

and t as

$$t = \exp\left[\begin{pmatrix} 0 & v \\ v^\dagger & 0 \end{pmatrix}\right] = t(v), \tag{5.7}$$

where $\mathcal{H}(m/n)$, $\mathcal{H}(p/q)$ and v were defined by Eq. (4.15). Now $t(v)$ can be rewritten in the form

$$t(v) = \begin{pmatrix} \dfrac{1}{\sqrt{1-ZZ^\dagger}} & Z\dfrac{1}{\sqrt{1-Z^\dagger Z}} \\ \dfrac{1}{\sqrt{1-Z^\dagger Z}}Z^\dagger & \dfrac{1}{\sqrt{1-Z^\dagger Z}} \end{pmatrix} \equiv t(Z), \tag{5.8}$$

and h as

$$h=\left(\begin{array}{c|c}\mathrm{U}(m/n) & 0\\ \hline 0 & \mathrm{U}(p/q)\end{array}\right). \tag{5.9}$$

Note that the $(m+n)\times(p+q)$ supermatrix Z has the same graded structure as v, and is related to v by

$$Z=\frac{\tanh\sqrt{vv^\dagger}}{\sqrt{vv^\dagger}}\,v\,. \tag{5.10}$$

Similarly, the unitary operator $\hat{U}$ can be decomposed as

$$\hat{U}=\hat{t}(Z)\hat{h}\,, \tag{5.11}$$

where

$$\begin{aligned}\hat{t}(Z)&=\exp i\bar{\psi}\left(\begin{array}{c|c}0 & -iv\\ \hline -iv^\dagger & 0\end{array}\right)\psi\,,\\ \hat{t}(Z)&=\exp(\eta^T v^\dagger\xi-\xi^\dagger v\eta^{\dagger T})\,,\\ \hat{t}(Z)&=\exp(\eta_Q v^{\dagger QA}\xi_A-\xi^{A\dagger}v_{AQ}\eta^{\dagger Q})\,,\\ \hat{t}(Z)^\dagger&=\hat{t}(Z)^{-1}\,.\end{aligned} \tag{5.12}$$

Note that the order in the exponential is important since v is a rectangular supermatrix involving Grassmann parameters anticommuting with the odd generators (or equivalently with the Fermi operators).

The super Z variable appearing in $t(Z)$ parametrizing the supercoset space $\mathrm{SU}(m,p/n+q)/\mathrm{S}(\mathrm{U}(m/n)\times\mathrm{U}(p/q))$ transforms non-linearly under $\mathrm{SU}(m,p/n+q)$. Consider for example the left action g of $\mathrm{SU}(m,p/n+q)$ on $t(Z)$:

$$gt(z)=t(z')\,h(z,g)\,, \tag{5.13}$$

where $g=e^{i\mathscr{M}}$. Writing the matrix g representing a general group element of $\mathrm{SU}(m,p/n+q)$ in the fundamental representation as

$$g=e^{i\mathscr{M}}=\begin{array}{c}\\ m+n\\ p+q\end{array}\!\!\begin{array}{c}\begin{array}{cc}m+n & p+q\end{array}\\ \left(\begin{array}{c|c}\alpha & \beta\\ \hline \gamma & \delta\end{array}\right)\end{array}, \tag{5.14}$$

we find that Z undergoes a linear fractional transformation

$$g:Z\to Z'=(\alpha Z+\beta)(\gamma Z+\delta)^{-1}\,, \tag{5.15}$$

and

$$h(g,Z)=\left(\begin{array}{c|c}U_{m/n}(g,Z) & 0\\ \hline 0 & U_{p/q}(g,Z)\end{array}\right),$$

where

$$U_{p/q}(g,Z)=(1-Z'^\dagger Z')^{1/2}(\gamma Z+\delta)(1-Z^\dagger Z)^{-1/2}\,, \tag{5.16}$$

and

$$U_{m/n}(g,Z)=(1-Z'Z'^\dagger)^{1/2}(\alpha+\beta Z^\dagger)(1-ZZ^\dagger)^{-1/2}\,. \tag{5.17}$$

Similarly, for the operators $\hat{U}(g)$ and $\hat{t}(z)$, we find

$$\begin{aligned} \hat{U}(g) &= \exp i\bar{\psi}\mathcal{M}\psi\,, \\ \hat{U}(g)\,\hat{t}(z) &= \hat{t}(z')\,\hat{h}(z,g)\,, \end{aligned} \tag{5.18}$$

where

$$\hat{h}(z,g)=\exp\{\xi^\dagger(\ln U_{m/n}(g,z))\xi-\eta(\ln U_{p/q}(g,z))\eta^\dagger\}\,. \tag{5.19}$$

Note that the unitarity of $\hat{h}(z,g)$ follows from the unitarity of $U_{m/n}$ and $U_{p/q}$.

Consider now a "lowest state" $|K^{A\cdots}_{M\cdots}\rangle$ that transforms irreducibly under $S(U(m/n)\times U(p/q))$ and is annihilated by the operators $L_A{}^M$ of the L^- space. We define the supercoherent state $|K^{A\cdots}_{M\cdots}, Z\rangle$ labelled by the rectangular supermatrix Z as

$$e^{\xi_B{}^\dagger Z_B{}^N \eta_N{}^\dagger}|K^{A\cdots}_{M\cdots}\rangle \equiv |K^{A\cdots}_{M\cdots};Z\rangle\,. \tag{5.20}$$

The coherent states $|Z\rangle$ are clearly a linear combination of the particle states discussed in the last section and form an overcomplete basis of the UIR determined uniquely by $|K^{A\cdots}_{M\cdots}\rangle$. We will now relate them to the coset space $SU(m,p/n+q)/S(U(m/n)\times U(p\times q))$ by using the identity

$$\hat{t}(z)=e^{\xi^\dagger Z\eta^\dagger}\,e^{\xi^\dagger\ln(1-ZZ^\dagger)^{1/2}\xi-\eta\ln(1-Z^\dagger Z)^{-1/2}\eta^\dagger}\,e^{-\eta Z^\dagger\xi}\,, \tag{5.21}$$

which in the fundamental representation corresponds to the decomposition

$$t(z)=\begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix}\begin{pmatrix}(1-ZZ^\dagger)^{1/2} & 0\\ 0 & (1-Z^\dagger Z)^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ Z^\dagger & 1\end{pmatrix}. \tag{5.22}$$

For example, consider the case when the lowest state $|K\rangle$ transforms like the representation $[(0,0,\ldots),(1,0,0,\ldots)]$ of $S(U(m/n)\times U(p/q))$, i.e., it transforms like a covariant vector under $U(p/q)$. Then

$$\begin{aligned} |K_M;Z\rangle &= e^{\xi^\dagger Z\eta^\dagger}|K_M\rangle \\ &= \hat{t}(z)\,e^{\eta Z^\dagger\xi}\,e^{-\xi^\dagger\ln(1-ZZ^\dagger)^{1/2}\xi+\eta\ln(1-Z^\dagger Z)^{-1/2}\eta^\dagger}|K_M\rangle \\ &= \hat{t}(z)\,[(1-Z^\dagger Z)^{-1/2}]_M{}^{M'}\,|K_{M'}\rangle\,. \end{aligned} \tag{5.23}$$

Under the action of $\hat{U}(g)$ we get

$$\begin{aligned} \hat{U}(g)\,|K_M,Z\rangle &= [(1-Z^\dagger Z)^{-1/2}]_M{}^{M'}\,\hat{U}(g)\,\hat{t}(z)\,|K_{M'}\rangle \\ &= [(1-Z^\dagger Z)^{-1/2}]_M{}^{M'}\,\hat{t}(z')\,\hat{h}(z,g)\,|K_{M'}\rangle \\ &= [(1-Z^\dagger Z)^{-1/2}]_M{}^{M'}\,\hat{t}(z')\,[U^{-1}_{p/q}(z,g)]_{M'}{}^{M''}\,|K_{M''}\rangle\,, \end{aligned} \tag{5.24}$$

where

$$Z'=(\alpha Z+\beta)(\gamma Z+\delta)^{-1}\,,$$

and

$$U^{-1}_{p/q}(z,g)=(1-Z^\dagger Z)^{1/2}\,(\gamma Z+\delta)^{-1}\,(1-Z'^\dagger Z')^{-1/2}\,.$$

Thus, using Eq. (5.21) we obtain

$$\begin{aligned} \hat{U}(g)\,|K_M;Z\rangle &= [(\gamma Z+\delta)^{-1}]_M{}^{M'}\,[(1-Z'^\dagger Z')^{-1/2}]_{M'}{}^{M''}\,\hat{t}(Z')\,|K_{M''}\rangle\,, \\ \hat{U}(g)\,|K_M;Z\rangle &= [(\gamma Z+\delta)^{-1}]_M{}^{M'}\,|K_{M'};(\alpha Z+\beta)(\gamma Z+\delta)^{-1}\rangle\,. \end{aligned} \tag{5.25}$$

This shows that the coherent states $|K_M, Z\rangle$ transform among themselves under the unitary supergroup action and hence form a UIR of $SU(m,p/n+q)$. The extension of the above analysis to the general case is straightforward. One needs only be careful about the order of the representation matrices of $S(U(m/n)\times U(p/q))$, which are uniquely determined by the transformation properties of the Bose-Fermi operators ξ and η. Thus, for each "lowest" state $|K\rangle$ transforming irreducibly under the maximal compact subsupergroup $S(U(m/n)\times U(p/q))$ and annihilated by $L_A{}^M$ we obtain a UIR in the coherent state basis by the above method.

The analyticity properties of the UIRs constructed in the previous section become evident in the overcomplete coherent state basis. For example, the expansion coefficients of a coherent state in the particle basis $\langle K|(L^+)^\ell|K;Z\rangle$, $\ell=1,2,\ldots$, are all polynomial functions of Z. The coherent state representations of ordinary groups and their analyticity properties have been extensively studied [15, 16]. What the above analysis shows is that our supercoherent states have the same analyticity properties as the ordinary coherent states. Just as in the case of ordinary non-compact Lie groups [17–19, 25], one can realize the above UIRs on the super Hilbert space of the analytic functions of Z. In fact, our formalism can be used to define such "super Hilbert spaces" and study their properties.

6. Oscillator-Like UIRs of Non-Compact Supergroups with a Jordan Structure

The Lie superalgebras of non-compact supergroups constructed in the previous sections all have a Jordan structure with respect to the Lie superalgebra of their maximal compact subsupergroups, i.e., they can be decomposed in the form

$$L=L^{-1}\oplus L^0\oplus L^{+1}, \tag{6.1}$$

where L^0 is the Lie superalgebra of the maximal compact subsupergroup. It contains the generator Q of an Abelian U(1) factor which gives the grading

$$\begin{aligned} &L^0=H\oplus Q,\\ &[Q,H]=0,\\ &[Q,L^{+1}]=L^{+1},\\ &[Q,L^{-1}]=-L^{-1}. \end{aligned} \tag{6.2}$$

There is a conjugation $\dagger$ in L such that

$$(L^{+1})^\dagger\cong L^{-1}, \qquad (L^0)^\dagger\cong L^0,$$

and

$$[L^{+1},L^{-1}\}\cong L^0\,; \qquad [L^{+1},L^{+1}\}=0,$$
$$[L^0,L^{+1}\}\cong L^{+1}\,; \qquad [L^0,L^{-1}\}\cong L^{-1}.$$

In [14] a general theory of the construction of oscillator-like unitary representations of non-compact groups with a Jordan structure in the corresponding bosonic Fock spaces was given. In this section we shall extend this general construction to non-compact supergroups with a Jordan structure with respect to their maximal compact subsupergroups. The UIRs of $SU(m,p/n+q)$ given in the previous sections is a special example of this general construction.

Consider now the Lie superalgebra with a Jordan structure as in Eq. (6.1) constructed from the bilinears of Bose-Fermi annihilation and creation operators. Choose a set of states $|K_{A\ldots}^{M\ldots}\rangle$ in the super Fock space $\mathscr{F}$ of the Bose-Fermi operators that are annihilated by all the operators belonging to the L^{-1} space:

$$L^{-1}|K_{A\ldots}^{M\ldots}\rangle=0, \tag{6.3}$$

and which transforms as some representation of the maximal compact subsupergroup K generated by L^0. Then the infinite set of states obtained by applying the operators L^{+1} on the states $|K_{A\ldots}^{M\ldots}\rangle$ form the basis of a unitary representation of the non-compact supergroup G generated by L:

$$|K_{A\ldots}^{M\ldots}\rangle, L^+|K_{A\ldots}^{M\ldots}\rangle, L^+L^+|K_{A\ldots}^{M\ldots}\rangle, \ldots. \tag{6.4}$$

If the states $|K_{A\ldots}^{M\ldots}\rangle$ transform like an irreducible representation of K, then the infinite set of states thus generated forms the basis of a UIR of the non-compact supergroup G. The proof of this statement is identical to the case of ordinary non-compact groups [14] which we outline below.

Any Casimir operator of G must commute with the U(1) generator Q that gives the grading. This means that every term in the Casimir invariant must contain an equal number of L^+ and L^- operators. For example, the quadratic Casimir operator must be of the form

$$C_2=L^{+1}L^{-1}+L^{-1}L^{+1}+F_2(L^0), \tag{6.5}$$

where $F_2(L^0)$ involves the linear and quadratic Casimir invariants of the maximal compact subsupergroup. Now, by commutations and/or anticommutations, one can bring all the L^- operators to the right of each term, i.e.,

$$C_2=2L^{+1}L^{-1}+F_2'(L^0). \tag{6.6}$$

Thus, on the lowest state $|K_{A\ldots}^{M\ldots}\rangle$ we have

$$C_2|K_{A\ldots}^{M\ldots}\rangle=F_2'(L^0)|K_{A\ldots}^{M\ldots}\rangle. \tag{6.7}$$

Since $|K_{A\ldots}^{M\ldots}\rangle$ transforms irreducibly under the compact subgroup K, it must be an eigenstate of $F_2'(L^0)$ and hence of C_2

$$C_2|K_{A\ldots}^{M\ldots}\rangle=\lambda_2|K_{A\ldots}^{M\ldots}\rangle. \tag{6.8}$$

From $[L^+,C_2]=0$ we then find that all the higher states $(L^+)^k|K_{A\ldots}^{M\ldots}\rangle$ are eigenstates of C_2 with the same eigenvalue λ_2. One can similarly show that all the higher Casimir invariants are diagonalized by $|K_{A\ldots}^{M\ldots}\rangle$ and consequently by the states $(L^{+1})^\ell|K_{A\ldots}^{M\ldots}\rangle$ ($\ell=1,2,\ldots$), proving the irreducibility of the resulting unitary representation of G. The UIRs of G that can be constructed by this method are then determined by the set of lowest states $|K_{A\ldots}^{M\ldots}\rangle$ that can be constructed in the super Fock space which transforms irreducibly under K and is annihilated by L^{-1}. This in turn is determined by the number R of Bose-Fermi operators that enter the construction of L. In general, the generator Q which gives the grading in L can be chosen as half the Bose-Fermi number operator. Then the space L^{+1} *corresponds* *to di-creation operators and* L^{-1} to di-annihilation operators. In such cases there always exist lowest states $|K\rangle$ in the super Fock space $\mathscr{F}$ that transform irredicibly and are annihilated by L^{-1} operators. However, in some special cases the L^{-1} space involves not only di-annihilation operators, but also di-creation operators. In such cases there may not exist such lowest states $|K\rangle$ in $\mathscr{F}$. If this is the case, then our method leads to reducible unitary representations.

The Lie superalgebras of non-compact supergroups OSp($2n^*/2m$) and OSP($2n/2m$, $\mathbb{R}$) constructed in Sect. 2 have a Jordan structure with respect to their maximal subsuperalgebra U(m/n) and their oscillator-like UIRs can be constructed by the above method. One would like to know if there are other non-compact supergroups with a Jordan structure with respect to their maximal compact subsupergroups in addition to OSp($2n^*/2m$), OSp($2n/2m$, $\mathbb{R}$), and SU($m, p/n+q$). Of the ordinary simple Lie algebras, all have a Jordan structure with respect to some suitable subalgebra except for G_2, F_4, and E_8. The simple Lie algebras with a Jordan structure can be constructed from Hermitean Jordan triple systems by the so-called Tits-Koecher method [26]. This construction establishes a mapping between the elements, say in the L^+ space of the Lie algebra, and the elements of a Hermitean Jordan triple system. The Tits-Koecher construction of Lie algebras has also been extended to the construction of Lie superalgebras from super Jordan triple systems [27–29]. In particular one can construct a class of Lie superalgebras from Jordan superalgebras. Below we give the complete list of Jordan superalgebras J^s as classified by Kac [30] and the resulting superalgebras L^0 and L [27–29].

J^s	Dimension of J^s	L^0	$L=L^{-1}+L^0+L^+$
A^s	$(m+n)^2$	SU(m/n)⊕SU(m/n)⊕Q	SU($2m/2n$)
BC^s	$\frac{m(m+1)}{2}+n(2n-1)+2mn$	SU($m/2n$)⊕Q	OSp($4n/2m$)
D^s	$m+2n-1$	OSp($m/2n$)⊕Q	OSp($m+2/2n$)
P^s	$2n^2$	SU(n/n)⊕Q	$P(2n-1)$
Q^s	$2n^2$	$Q(n-1)\oplus Q(n-1)\oplus Q\oplus F$	$Q(2n-1)$
D_t^s	4	SU(1/2)⊕Q	$D(2,1;t)$
J_3^8	27	$E_6\oplus Q$	E_7
F^s	10	OSp(2/4)⊕Q	$F^s(4)$
K	3	SU(1/2)	SU(2/2)

where Q stands for the SO(2) generator that gives the grading and F stands for the generator of one parameter Lie superalgebra [similar to U(1)]. Note that we have denoted the Lie superalgebras with their "compact forms." Now, one constructs the Lie superalgebras L listed above as bilinear Bose-Fermi operators in such a way that in a superHermitean basis L^0 generates a maximal compact subsupergroup K. Then one can construct the unitary representations of the non-compact group G generated by L in the super Fock space of these Bose-Fermi operators. The corresponding coherent states can be labelled by the elements Z of the underlying Jordan superalgebra, which undergo a linear fractional transformation under the action of the non-compact group G [29]. In fact, for all the UIRs of non-compact supergroups with a Jordan structure, a coherent state basis exists. Thus they can equivalently be realized over super Hilbert spaces of analytic functions of the super Z variable labelling the coherent state.

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References

1. For a review of the theory of unitary representations of non-compact groups see: Schmid, W.: Proc. of the International Congress of Mathematicians, Helsinki (1978); Academia Scientiarum Fennica (Helsinki, 1980)
2. Wess, J., Zumino, B.: Supergauge transformations in four dimensions. Nucl. Phys. B **70**, 39 (1974) Volkov, D.V., Akulov, V.P.: Is the neutrino a Goldstone particle? Phys. Lett. **46** B, 109 (1973)
3. Kac, V.G.: Adv. Math. **26**, 8 (1977) See also, Freund, P.G.O., Kaplansky, I.: Simple supersymmetries. J. Math. Phys. **17**, 228 (1976)
4. Kac, V.G.: Representations of classical Lie superalgebras. In: Differential geometrical methods in mathematical physics. Bleuler, K., Petry, H.R., Reetz, A. (eds.). Berlin, Heidelberg, New York: Springer 1978
5. Ne'eman, Y., Sternberg, S.: Internal supersymmetry and unification. Proc. Natl. Acad. Sci. (USA) **77**, 3127 (1980) Marcu, M.: The representations of Spl(2,1) – an example of representations of basic superalgebras. J. Math. Phys. **21**, 1277 (1980) Dondi, P.H., Jarvis, P.D.: Assignments in strong-electroweak unified models with internal and spacetime supersymmetry. Z. Physik C **4**, 201 (1980)
6. Balantekin, A.B., Bars, I.: Dimension and character formulas for Lie supergroups, J. Math. Phys. **22**, 1149 (1981), Representations of supergroups. J. Math. Phys. **22**, 1810 (1981) and Branching Rules for the Supergroup SU(N/M) from those of SU(N+M). J. Math. Phys. **23**, 1239 (1982)
7. For a comprehensive review and further references to the representations of compact supergroups see: Bars, I.: Supergroups and their Representations, Yale preprint YTP-8-25, Proceedings of School on Supersymmetry in Physics, Mexico (December 1981) (to be published)
8. Balantekin, A.B., Bars, I., Iachello, F.: Nucl. Phys. A **370**, 284 (1981)
9. Balantekin, A.B.: Ph. D. Thesis, Yale University (1982) (unpublished)
10. Bars, I., Morel, B., Ruegg, H.: CERN preprint TH-3333 (1982) to appear in J. Math. Phys. (1983)
11. Cremmer, E., Ferrara, S., Scherk, J.: SU(4) invariant supergravity theory. Phys. Lett. **74** B, 61 (1978)
12. Cremmer, E., Julia, B.: The SO(8) supergravity. Nucl. Phys. B **159**, 141 (1979)
13. Günaydin, M., Saçlioğlu, C.: Bosonic construction of the Lie algebras of some non-compact groups appearing in the supergravity theories and their oscillator-like unitary representations. Phys. Lett. **108** B, 180 (1982)
14. Günaydin, M., Saçlioğlu, C.: Oscillator-like unitary representations of non-compact groups with a Jordan structure and the non-compact groups of supergravity. Commun. Math. Phys. **87**, 159–179 (1982)
15. For a discussion of the relevance of the unitary representations given in [13] and [14] to the attempts to extract a realistic grand unified theory from supergravity theories, see: Günaydin, M.: Unitary realizations of non-compact groups of supergravity, talk presented at the Second Europhysics Study Conference on Unification of Fundamental Interactions, Erice, Sicily (1981); CERN preprint TH-3222 (1981)
16. Loebl, E.M. (ed.): Group theory and its applications, Vols. I–III. New York: Academic Press 1968 Wybourne, B.G.: Classical groups for physicists. New York: Wiley 1974 Dyson, F.J.: Symmetry groups in nuclear and particle physics. New York: Benjamin 1966 Gürsey, F. (ed.): Group theoretical concepts and methods in elementary particle physics. New York: Gordon and Breach 1964
17. Bargmann, V.: Ann. Math. **48**, 568 (1947); Commun. Pure Appl. Math. **14**, 187 (1961)
18. Kashiwara, M., Vergne, M.: Invent. Math. **44**, 1 (1978)

19. Howe, R.: Classical invariant theory. Yale Univ. preprint, unpublished; and Transcending classical invariant theory. Yale Univ. preprint (unpublished)
20. R. Howe has suggested the possibility of extending the dual pair notion to the case of superalgebras (private communication). See also [19]
21. We should note that when the generators are represented by ordinary matrices rather than oscillators, θ^α will, of course, commute with those matrices
22. Gürsey, F., Marchildon, L.: Spontaneous symmetry breaking and nonlinear invariant Lagrangians: Applications to SU(2)⊗U(2) and OSp(1/4). Phys. Rev. D **17**, 2038 (1978); The graded Lie groups SU(2, 2/1) and OSp(1/4). J. Math. Phys. **19**, 942 (1978)
23. For a review of coherent states and their applications, see:
Perelomov, A.M.: Sov. Phys. Usp. **20**, 703 (1977)
24. For a study of the analyticity properties of coherent state representations of Lie groups and further references on the subject, see:
Onofri, E.: A note on coherent state representations of Lie groups. J. Math. Phys. **16**, 1087 (1974)
25. Gelfand, I.M., Graev, M.I., Vilenkin, N.Y.: Generalized functions, Vol. 5. New York: Academic Press 1968. For an operator treatment of the unitary representations of some non-compact groups related to the Poincaré group à la Gelfand et al., see:
Gürsey, F.: Representations of some non-compact groups related to the Poincaré group. Yale Univ. mimeographed notes (1971)
Bars, I., Gürsey, F.: Operator treatment of the Gelfand-Naimark basis for SL(2, *C*)*. J. Math. Phys. **13**, 131 (1972)
Gürsey, F., Orfanidis, S.: Conformal invariance and field theory in two dimensions. Phys. Rev. D **7**, 2414 (1973)
26. Tits, J.: Nederl. Akad. van Wetenschapp **65**, 530 (1962)
Koecher, M.: Am. J. Math. **89**, 787 (1967)
27. Bars, I., Günaydin, M.: Construction of Lie algebras and Lie superalgebras from ternary algebras. J. Math. Phys. **20**, 1977 (1979)
28. Bars, I.: Proceedings of the 8th Intern. Coll. on Group Theoretical Methods. Annals of Israel Physical Society, Vol. 3 (1980)
29. Günaydin, M.: Proceedings of the 8th Intern. Coll. on Group Theoretical Methods. Annals of Israel Physical Society, Vol. 3 (1980), p. 279
30. Kac, V.: Commun. Algebra **5**, 1375 (1977)

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