Commun. Math. Phys. 86, 1–48 (1982)

Communications in
Mathematical
Physics

Geometric Analysis of ϕ^4 Fields and Ising Models. Parts I and II

Michael Aizenman*

Department of Physics, Princeton University, Princeton, NJ 08544, USA**

Abstract. We provide here the details of the proof, announced in [1], that in $d>4$ dimensions the (even) ϕ^4 Euclidean field theory, with a lattice cut-off, is inevitably free in the continuum limit (in the single phase regime). The analysis is nonperturbative, and is based on a representation of the field variables (or spins in Ising systems) as source/sink creation operators in a system of random currents – which may be viewed as the mediators of correlations. In this dual representation, the onset of long-range-order is attributed to percolation in an ensemble of sourceless currents, and the physical interaction in the ϕ^4 field – and other aspects of the critical behavior in Ising models – are directly related to the intersection properties of long current clusters. An insight into the criticality of the dimension $d=4$ is derived from an analogy (foreseen by K. Symanzik) with the intersection properties of paths of Brownian motion. Other results include the proof that in certain respect, the critical behavior in Ising models is in exact agreement with the mean-field approximation in high dimensions $d>4$, but not in the low dimension $d=2$ – for which we establish the "universality" of hyperscaling.

Table of Contents

* A. P. Sloan Foundation Research Fellow. Supported in part by the National Science Foundation Grant PHY-78-25390-A02

** New address: Dept. of Mathematics (also in Physics Dept.), Rutgers University, New Brunswick, NJ 08903, USA

0010-3616/82/0086/0001/$09.60

Part II. ϕ^4 Fields

1. Introduction

Expansions in geometrically identified objects (like "contour" surfaces, strings, lines and sheets of flux) are common in perturbative treatments of various theories. In this note I present a *nonperturbative* geometric analysis of systems of Ising spins, and related theories. The class includes Ising models with pair interactions, $\mathbb{Z}(2)$ lattice gauge models, and the Euclidean ϕ^4 field theory – taken here as the continuum limit of lattice system with the action $\left(\frac{\lambda}{4!}\phi^4+B\phi^2+C\right)$ + the kinetic term.

The main results can be read off the table of contents. In essence, we can now prove, and explain, a number of basic features of the critical behavior of Ising models and ϕ^4 fields, in *high* and *low* dimensions. High – means above the dimension $d=4$, where various aspects of the critical behavior are shown to agree *exactly* with the predictions of the mean-field approximation, and where we prove that the continuum limits of the ϕ^4 Euclidean field theory, in the single phase region, are inevitably Gaussian (i.e. exhibit no physical interaction). For the low dimension $d=2$, we offer a simple explanation of the non-Gaussian behavior in Ising models, and a general proof of *hyperscaling.*

The multiple correlations of spins, or fields, in these systems can be represented as the result of a mediation by an ensemble of random currents, on which the spins act as external sources. In this representation, the effective interaction, measured by the deviation in the high-order correlations from the Gaussian behavior, can be traced to the probability of intersection of current clusters. The criticality of the dimension $d=4$ is then strongly suggested by the fact that the intersection probability of two Wiener paths (Brownian motion) vanishes exactly for $d\geqq 4$ (see Appendix I). Such intuition is present in the early ideas of Symanzik [2] (which were brought to my attention by T. Spencer).

The main results discussed here were announced in [1]. Since then they found an extension to the two-component ϕ^4 field in the work of Fröhlich [3], who used an independent argument.

In Part III, which appears separately, we discuss some well-known problems which have a natural formulation in terms of the properties of *random surfaces.* These, however, are not yet that tractable. Our main purpose there is to motivate, by partial results, a further study of stochastic-geometric methods. Included there is also a unified derivation of various "classical", and some new, correlation inequalities.

Readers who seek a quick access to the main *field-theoretical* results may start directly at Sect. 10, and – after reading Proposition 10.1, proceed to Sects. 13 and 14 (where use is made of Propositions 11.1 and 12.1).

Part I. Ising Systems

2. A System of Random Currents Associated with the Ising Model

The first part of this paper concerns equilibrium states of Ising models, with ferromagnetic pair interactions (at zero magnetic field). The systems consist of spin variables, $\sigma_x = \pm 1$, which are associated with the sites of a set, Λ. The interactions we consider are represented by the Hamiltonian function

$$H = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{x,y}\sigma_x\sigma_y, \tag{2.1}$$

with some fixed couplings $J_{x,y} \geqq 0$. Extension of our analysis to nonferromagnetic cases, and to higher than two-body interactions, is suggested by the methods used in Sects. 16 and 18.

The most familiar Ising model has the nearest-neighbor interaction on a d-dimensional cubic lattice, i.e. $\Lambda = \mathbb{Z}^d$ and

$$J_{x,y} = \delta_{|x-y|,1}.$$

However in this section, and in the next one, *the structure of Λ plays no role.*

States of the system are given by *probability* measures on the space of configurations – the "thermodynamic equilibrium", at the inverse temperature β, being described by the *Gibbs states.* Specifically, for a finite system the corresponding expectation value of functions of the spins are:

$$\langle f \rangle_\Lambda = \mathrm{tr}_\Lambda f(\sigma) e^{-\beta H(\sigma)} / \mathrm{tr}_\Lambda e^{-\beta H(\sigma)}, \tag{2.2}$$

where tr_Λ represents the average $\prod_{x\in\Lambda}\left(\frac{1}{2}\sum_{\sigma_x=\pm 1}\right)$. For infinite systems, e.g. $\Lambda = \mathbb{Z}^d$, the Gibbs states are defined by limits of this expression. The subscript in $\langle - \rangle_\Lambda$ would often be omitted.

The critical behavior in the above systems is related to the spontaneous breaking of the global symmetry, $\sigma \to -\sigma$. The symmetrical states defined by (2.2) correspond to the *free boundary conditions.* One may break the symmetry at the boundary, by restricting the average to spin configurations which take there only the value $+1$. These states are denoted by $\langle - \rangle_{\Lambda,+}$. (Other states are considered in Sects. 16 and 17.)

We shall now discuss another representation for the states $\langle - \rangle_\Lambda$. Ultimately, we are of course interested in the infinite volume limits, e.g. $\Lambda = \mathbb{Z}^d$. However, since the convergence in the standard setup is well understood, we shall discuss the alternative representation in finite volumes, and gloss over its direct formulation for infinite systems. The relations which would be derived for spin correlations hold, by continuity, in the infinite volume limit.

The starting point is a variant of the expansion of $\exp(-\beta H)$ in characters of $\mathbb{Z}(2)^\Lambda$, which has been used to set a *high temperature expansion* (although we approach it *nonperturbatively*, avoiding any problems of convergence). We regard pairs of sites as *bonds*, $b = \{x, y\}$, with $J_b \equiv J_{x,y}$. The *partition function*, of a finite system, is then

$$Z_\Lambda = \mathrm{tr}_\Lambda e^{-\beta H} = \mathrm{tr}_\Lambda \prod_{b \equiv \{x,y\}} e^{\beta J_b \sigma_x \sigma_y}. \tag{2.3}$$

Let us expand, for each bond,

$$\exp(\beta J_b \sigma_x \sigma_y) = \sum_{n_b = 0}^{\infty} (\sigma_x \sigma_y)^{n_b} (\beta J_b)^{n_b} / n_b!$$

The substitution of this expansion in (2.3) gives:

$$Z_\Lambda = \sum_{\mathbf{n}} w(\mathbf{n}) \, \mathrm{tr}_\Lambda \prod_{x \in \Lambda} \sigma_x^{\sum_{b \ni x} n_b},$$

where $\mathbf{n}$ varies over all the functions which assign an integer to each bond of the lattice, and

$$w(\mathbf{n}) = \prod_b (\beta J_b)^{n_b} / n_b! \tag{2.4}$$

[Notice that bonds with $J_b = 0$ can be ignored, since for such bonds the weights $w(\mathbf{n})$ produce the restriction: $n_b = 0$.] The average (tr_Λ) in Z_Λ is easy to carry out. For each $\mathbf{n}$ its value is either 0 or 1, the latter being the case only if $\sum_{b \ni x} n_b$ is even *for each* x. Thus

$$Z_\Lambda = \sum_{\partial \mathbf{n} = \emptyset} w(\mathbf{n}), \tag{2.5}$$

where $\emptyset$ is the empty set and

$$\partial \mathbf{n} = \left\{ x \in \Lambda \,|\, (-1)^{\sum_{b \ni x} n_b} = -1 \right\}. \tag{2.6}$$

It is convenient to regard each $\mathbf{n}$ as a configuration of *flux numbers*, representing a current through the lattice bonds, and $\partial \mathbf{n}$ as the set of *sources*

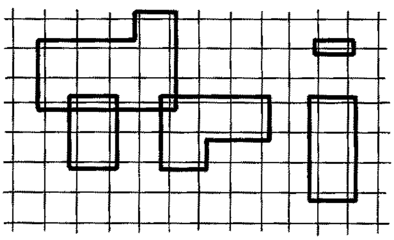

Fig. 1. A possible representation of $\mathbf{n}$ with $\partial \mathbf{n} = \emptyset$

($\equiv$sinks) for the corresponding current. Specifically, it is easy to see that each $\mathbf{n}$ with $\partial\mathbf{n}=\emptyset$ can be obtained (without uniqueness) by summing the fluxes of a collection of eddy currents, which are closed loops formed by lattice bonds, allowing the loops to repeat any bond (Fig. 1). For infinite systems one should include currents which extend to ∞.

For correlation functions, i.e. expectation values [in the sense of (2.2)] of $\sigma_A = \prod_{x\in A} \sigma_x$, $A \subset \Lambda$, one may follow, mutatis-mutandis, the derivation of (2.5), to obtain

$$\langle\sigma_A\rangle_\Lambda = \sum_{\partial\mathbf{n}=A} w(\mathbf{n}) \Big/ \sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n}) . \tag{2.7}$$

Similarly, the $\langle - \rangle_{\Lambda,+}$ state is given by

$$\langle\sigma_A\rangle_{\Lambda,+} = \sum_{\partial\mathbf{n}\setminus\partial\Lambda=A} w(\mathbf{n}) \Big/ \sum_{\partial\mathbf{n}\setminus\partial\Lambda=\emptyset} w(\mathbf{n}) . \tag{2.8}$$

The quantity $-\frac{1}{\beta}\ln Z_\Lambda$ is usually regarded as the free energy of the spin system. Were we to regard it instead as the free energy of the system of (eddy) currents, (2.7) would tell us that $\langle\sigma_A\rangle = \exp(-\beta\delta_A F)$, where $\delta_A F$ is the increase in the free energy of the system of currents due to the insertion of sources at the points of A.

Notice that configurations $\mathbf{n}$ with $\partial\mathbf{n}=\{x,y\}$ (in a finite system) can always be represented as sums of fluxes of a current which runs between x and y and a system of eddy currents (Fig. 2). The same is true for configurations with $\partial\mathbf{n}\setminus\partial\Lambda=\{x,y\}$, provided one regards two current lines which connect to $\partial\Lambda$ as connected (i.e. regard $\partial\Lambda$ as a single point).

For low values of β, the combined weight of configurations which have long current lines connecting to any given point, is small. This observation leads to high temperature expansions which, for example, prove that for translation invariant interactions over $\Lambda=\mathbb{Z}^d$ of finite range

$$\langle\sigma_x\sigma_y\rangle \leqq e^{-m|x-y|}, \tag{2.9}$$

with $m=m(\beta)>0$, provided $\beta<\beta_c$. (We define $m(\beta)$ and β_c, by optimizing the above inequalities; a simple rederivation of (2.9) and of the continuity of $m(\beta)$ [4] is given in [5].) Such expansions suffer from problems of convergence even below β_c. As could be expected, this phenomenon is related to the formation of long current lines. In the next section we shall make such a relation precise.

In general, our approach to the study of Ising systems is to utilize the properties of ensembles of currents, restricted by $\partial\mathbf{n}=\emptyset$, with the probability weights $w(\mathbf{n})/Z_\Lambda$. As already stated, we shall not discuss here the limiting states of currents on infinite lattices; these can be defined and are of interest (– exhibiting a different form of symmetry breaking at *high* temperatures, $\beta<\beta_c$, see Appendix II). Rather, we derive properties of correlations in finite volumes and rely on the continuity as $\Lambda\uparrow\mathbb{Z}^d$.

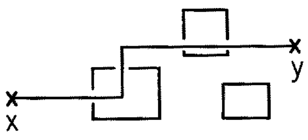

Fig. 2. A representation of a current with $\partial\mathbf{n}=\{x,y\}$

3. The Long Range Order as a Phenomenon of Percolation

The representation (2.7) of spin correlations can be used in a number of ways. In this section we introduce a method which leads to a simple geometric interpretation of a variety of quantities of the spin system. The resulting formalism offers an insight into the critical behavior in high and low dimensions, and provides simple means to prove various properties of the model. Another method to extract some of this information from (2.7) is discussed in Sect. 9.

For a given configuration of (integer) flux numbers $\mathbf{m}=\{m_b\}$ let us decompose the set of sites of Λ to clusters which are connected by bonds with nonvanishing m_b, and let $\mathscr{C}_{\mathbf{m}}(x)$ be the connected cluster containing the site $x\in\Lambda$. Thus the condition that $\mathbf{m}$ connects x with y is expressed by $\mathscr{C}_{\mathbf{m}}(x)=\mathscr{C}_{\mathbf{m}}(y)$ or, equivalently, by $\mathscr{C}_{\mathbf{m}}(x)\ni y$.

We shall consider duplicated systems of two independent currents ($\equiv$flux numbers) $\mathbf{n}_1$, $\mathbf{n}_2$ each of which has a specified set of sources $\partial\mathbf{n}_i=A_i$, and a *probability* distribution where the probability of $\mathbf{n}$ is proportional to $w(\mathbf{n})$. We denote the probability of an event F in such an ensemble by $\mathrm{Prob}(F|A_1,A_2)\equiv\mathrm{Prob}\left(F\,\middle|\begin{matrix}\partial\mathbf{n}_1=A_1\\ \partial\mathbf{n}_2=A_2\end{matrix}\right)$ – the latter being the notation used in [1]. For instance, when $A_1=A_2=\emptyset$ the probability of a pair of sourceless configurations is $w(\mathbf{n}_1)w(\mathbf{n}_2)Z_\Lambda^2$ [see (2.5)]. In other ensembles the probability of an *admissible* pair $\mathbf{n}_1$, $\mathbf{n}_2$ is

$$\begin{aligned}\mathrm{Prob}(\{\mathbf{n}_1,\mathbf{n}_2\}|A_1,A_2)&=w(\mathbf{n}_1)w(\mathbf{n}_2)\Big/\sum_{\partial\mathbf{n}_i'=A_i}w(\mathbf{n}_1')w(\mathbf{n}_2')\\ &=w(\mathbf{n}_1)w(\mathbf{n}_2)/[Z_\Lambda^2\langle\sigma_{A_1}\rangle\langle\sigma_{A_2}\rangle]\end{aligned}\tag{3.1}$$

[by (2.5) and (2.7)].

In some instances we shall refer to ensembles which consist of more than two independent current systems $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\ldots$. This would be manifested in the notation by listing the source constraints – in the right order. By $\mathrm{Prob}(K)$ – where K is a *condition*, we refer, of course, to the measure of current configurations in which the condition K is satisfied.

The first examples of the geometric identities mentioned above are described by the following result.

Proposition 3.1. *For all* $x,y,z\in\Lambda$

$$\langle\sigma_x\sigma_y\rangle^2=\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)=\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(y)|\emptyset,\emptyset),\tag{3.2}$$

and

$$\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle=\langle\sigma_x\sigma_z\rangle\,\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\ni y|\{x,z\},\emptyset),\tag{3.3}$$

where $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)$ *is defined as above with* $\mathbf{m}=\mathbf{n}_1+\mathbf{n}_2$ *(i.e.* $m_b=n_{1,b}+n_{2,b}$ *on each bond).*

Remark. Graphically, (3.2) is suggested by the fact that adding two currents with sources at $\{x,y\}$ one obtains a sourceless current which necessarily links x with y. Similarly, adding two current lines linking $\{x,y\}$ and $\{y,z\}$ one obtains a single line, which necessarily passes through y as it links the sources $\{x,z\}$ (Fig. 3). Remarkably, the weights are so adjusted that the correspondence is precise.

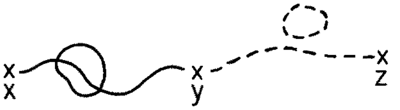

Fig. 3. The cancellation of sources in Eq. (3.3)

The proof of Proposition 3.1 is based on the principle expressed in the very useful Lemma 3.2, which follows from the following combinatorial identity.

Lemma 3.1. *Let* $\mathbf{m}=\{m_b\}$ *be a configuration of integer flux numbers, and* $A, B\subset \Lambda$. *If there exists* $\mathbf{k}=\{k_b\}$ *such that* $0\leqq \mathbf{k}\leqq \mathbf{m}$ *(i.e.* $0\leqq k_b\leqq m_b$ $\forall b$*) and* $\partial\mathbf{k}=A$ *then*

$$\sum_{\substack{0\leqq \mathbf{n}\leqq m\\ \partial\mathbf{n}=B}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{0\leqq \mathbf{n}\leqq \mathbf{m}\\ \partial\mathbf{n}=A\Delta B}} \binom{\mathbf{m}}{\mathbf{n}}, \tag{3.4}$$

where $\mathbf{n}$ *varies over integer flux configurations,* $\binom{\mathbf{m}}{\mathbf{n}}=\prod_b \binom{m_b}{n_b}$, and $A\Delta B$ is the "symmetric difference" $A\cup B\backslash A\cap B$.

Proof. With the given $\mathbf{m}$ we associate a graph $\mathscr{M}$ which consists of lines connecting the lattice sites, with m_b, $b=\{x,y\}$, lines connecting each pair $\{x,y\}$. The assumption implies that $\mathscr{M}$ has a subgraph $\mathscr{K}$ with $\partial\mathscr{K}=A$, where $z\in\partial\mathscr{K}$ iff the number of edges of $\mathscr{K}$ containing z is odd.

The left side of (3.4) equals the number of subgraphs $G\subset\mathscr{M}$ such that $\partial G=B$, whereas the right side counts $G'\subset\mathscr{M}$ such that $\partial G'=A\Delta B$. However the two families of graphs are isomorphic under the relation $G\mapsto G'=G\Delta\mathscr{K}$, since $\partial(G\Delta\mathscr{K})=\partial G\Delta\partial\mathscr{K}$ and $(G\Delta\mathscr{K})\Delta\mathscr{K}=G$. □

Remark. Lemma 3.1 appears already in [6], where it was used to derive the GHS inequality. Its power, however, has not been fully utilized, possibly because the probabilistic content of the results has not been exposed.

Lemma 3.2. *For a finite* Λ, *let* $\{x,y\}$, $A\subset\Lambda$ *and let* $\mathscr{I}$ *be a collection of flux configurations over bonds of* Λ. *Then*

$$\sum_{\substack{\partial\mathbf{n}_1=A\\ \partial\mathbf{n}=\{x,y\}\\ \mathbf{n}_1+\mathbf{n}_2\in\mathscr{I}}} w(\mathbf{n}_1)w(\mathbf{n}_2)= \sum_{\substack{\partial\mathbf{n}_1=A\Delta\{x,y\}\\ \partial\mathbf{n}_2=\emptyset\\ \mathbf{n}_1+\mathbf{n}_2\in\mathscr{I}}} w(\mathbf{n}_1)w(\mathbf{n}_2)X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)=\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(y)], \tag{3.5}$$

where $X[-]$ *is a characteristic function which takes the value* 1 *if the condition in* $[-]$ *is satisfied, and* 0 *otherwise.*

Proof. Let us change the summation variables to $\mathbf{m}=\mathbf{n}_1+\mathbf{n}_2$, $\mathbf{n}=\mathbf{n}_2$. One obtains

$$\sum_{\substack{\partial\mathbf{n}_1=A\\ \partial\mathbf{n}_2=\{x,y\}\\ \mathbf{n}_1+\mathbf{n}_2\in\mathscr{I}}} w(\mathbf{n}_1)w(\mathbf{n}_2)= \sum_{\substack{\partial\mathbf{m}=A\Delta\{x,y\}\\ \mathbf{m}\in\mathscr{I}}} w(\mathbf{m}) \sum_{\substack{0\leqq\mathbf{n}\leqq\mathbf{m}\\ \partial\mathbf{n}=\{x,y\}}} \binom{\mathbf{m}}{\mathbf{n}}, \tag{3.6}$$

where we used the facts that

$$w(\mathbf{n}_1)w(\mathbf{n}_2)\equiv\prod_b\left[\frac{(\beta J_b)^{n_{1,b}}}{n_{1,b}!}\right]\prod_b\left[\frac{(\beta J_b)^{n_{2,b}}}{n_{2,b}!}\right]=w(\mathbf{n}_1+\mathbf{n}_2)\binom{\mathbf{n}_1+\mathbf{n}_2}{\mathbf{n}_1}$$

and

$$\partial(\mathbf{n}_1+\mathbf{n}_2)=\partial\mathbf{n}_1 \triangle \partial\mathbf{n}_2 .$$

Observe now that the configurations $\mathbf{m}$ can be divided into the two following classes:

i) Configurations for which $\mathscr{C}_{\mathbf{m}}(x) \not\ni y$. For such $\mathbf{m}$

$$\sum_{\substack{0 \leqq \mathbf{n} \leqq \mathbf{m} \\ \partial n=\{x,y\}}} \binom{\mathbf{m}}{\mathbf{n}} = 0 . \tag{3.7}$$

Since if there is a configuration $\mathbf{n} \leqq \mathbf{m}$ with $\partial\mathbf{n}=\{x,y\}$ then $\mathscr{C}_{\mathbf{m}}(x) \supset \mathscr{C}_{\mathbf{n}}(x) \ni y$; a contradiction!

ii) Configurations for which $\mathscr{C}_{\mathbf{m}}(x)=\mathscr{C}_{\mathbf{m}}(y)$. For such configurations the conditions of Lemma 3.1 are satisfied with $A=B=\{x,y\}$ (and $A \triangle B=\emptyset$), therefore

$$\sum_{\substack{0 \leqq \mathbf{n} \leqq \mathbf{m} \\ \partial\mathbf{n}=\{x,y\}}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{0 \leqq \mathbf{n} \leqq \mathbf{m} \\ \partial\mathbf{n}=\emptyset}} \binom{\mathbf{m}}{\mathbf{n}} . \tag{3.8}$$

Substituting (3.7) and (3.8) in (3.6), and then changing back to the variables $\mathbf{n}_1=\mathbf{m}-\mathbf{n}$, $\mathbf{n}_2=\mathbf{n}$ we obtain

$$\begin{aligned}\sum_{\substack{\partial\mathbf{n}_1=A \\ \partial\mathbf{n}_2=\{x,y\} \\ \mathbf{n}_1+\mathbf{n}_2 \in \mathscr{I}}} w(\mathbf{n}_1)w(\mathbf{n}_2) &= \sum_{\substack{\partial\mathbf{m}=\emptyset \\ \mathscr{C}_{\mathbf{m}}(x)=\mathscr{C}_{\mathbf{m}}(y) \\ \mathbf{m}\in\mathscr{I}}} w(\mathbf{m}) \sum_{\substack{0 \leqq \mathbf{n} \leqq \mathbf{m} \\ \partial\mathbf{n}=\emptyset}} \binom{\mathbf{m}}{\mathbf{n}} \\ &= \sum_{\substack{\partial\mathbf{n}_1=A\triangle\{x,y\} \\ \partial\mathbf{n}_2=\emptyset \\ \mathbf{n}_1+\mathbf{n}_2\in\mathscr{I}}} w(\mathbf{n}_1)w(\mathbf{n}_2) X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)=\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)] . \quad \square\end{aligned}$$

Proof of Proposition 3.1. Let us square (2.7) and apply Lemma 3.2, with $A=\{x,y\}$ (and $A\triangle\{x,y\}=\emptyset$):

$$\begin{aligned}\langle\sigma_x\sigma_y\rangle^2 &= \sum_{\substack{\partial\mathbf{n}_1=\{x,y\} \\ \partial\mathbf{n}_2=\{x,y\}}} w(\mathbf{n}_1)w(\mathbf{n}_2) \Big/ \sum_{\substack{\partial\mathbf{n}_1=\emptyset \\ \partial\mathbf{n}_2=\emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2) \\ &= \sum_{\substack{\partial\mathbf{n}_1=\emptyset \\ \partial\mathbf{n}_2=\emptyset}} w(n_1)w(n_2) X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)=\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(y)] \Big/ \sum_{\substack{\partial\mathbf{n}_1=\emptyset \\ \partial\mathbf{n}_2=\emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2) .\end{aligned}$$

Since the last numerator is a partial sum of terms which appear in the denominator, the ratio is the probability that the condition $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)=\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(y)$ is satisfied in the duplicate ensemble (see 3.1 and the explanation preceding it). This proves (3.2).

For $\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle$ one repeats the above argument. The difference is in the constraint on $\mathbf{n}_1$ in the numerator, which here is $\partial\mathbf{n}_1=\{x,z\}$ $(\equiv\{x,y\}\triangle\{y,z\})$. Adjusting the denominator one arrives at (3.3). $\square$

Proposition 3.1 has a number of implications:

i) It is well known that Ising models with a translation invariant interaction on $\mathbb{Z}^d$, $d \geqq 2$, exhibit Long Range Order at low temperatures (i.e. high β). This is described by a *uniform* lower bound:

$$\langle\sigma_x\sigma_y\rangle \geqq \text{const} > 0 . \tag{3.9}$$

Equation (3.2) identifies the onset of the Long Range Order (in *any* Ising model) as a phenomenon of *percolation* – of the set of bonds on which $\mathbf{n}_1+\mathbf{n}_2 \neq 0$, in the duplicate system of sourceless currents. By percolation we mean here the formation of infinite clusters of connected bonds, which have positive density.

A priori, the last statement may be stronger than the occurrence of infinite clusters [7]. However, it follows from our next observation that for the translation-invariant nearest-neighbor models on $\mathbb{Z}^d$, $d>4$, the two forms of percolation coincide [the same should certainly be true for $d=2$, by other (yet incomplete) arguments].

ii) Equation (3.2) implies the following simple formula for the expected size of the connected cluster.

$$E(|\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)||\emptyset,\emptyset)=\sum_y \langle\sigma_x\sigma_y\rangle^2, \tag{3.10}$$

where we denote by $E(-|A_1,A_2)$ the expectation value with respect to the measure (3.1). To prove (3.10) one just has to express:

$$|\mathscr{C}_{\mathbf{m}}(x)|=\sum_y X_{x,y}(\mathbf{m}), \tag{3.11}$$

with

$$X_{x,y}(\mathbf{m})=\begin{cases}1 & \mathscr{C}_{\mathbf{m}}(x)=\mathscr{C}_{\mathbf{m}}(y)\\ 0 & \mathscr{C}_{\mathbf{m}}(x)\neq\mathscr{C}_{\mathbf{m}}(y),\end{cases}$$

substitute it in the left side of (3.10), and evaluate using (3.2).

The quantity $\sum_y \langle\sigma_x\sigma_y\rangle^2$ which appears in (3.10), plays an important role in our arguments. It may also be expressed by the Fourier transform, $\hat{G}(p)=\sum_x e^{ipx}\langle\sigma_0\sigma_x\rangle$, as:

$$\sum\langle\sigma_o\sigma_x\rangle^2=(2\pi)^{-d}\int_{[-\pi,\pi]^d} d^dp|\hat{G}(p)|^2$$

(Plancherel identity). In this form, for the nearest-neighbor model it is controlled (as noted in [8]) by the infrared bound of Fröhlich et al. [9]:

$$\hat{G}(p)\leqq\frac{1}{2\beta\sum_1^d[2\sin^2(p_i/2)]} \tag{3.12}$$

valid for any β at which there is no long-range-order [otherwise (3.12) is modified by a $\delta(p)$ term]. Thus, if $d>4$ and

$$\beta<\bar{\beta}_c\equiv\inf\left\{\beta>0\,\middle|\,\inf_{x\in\mathbb{Z}^d}\langle\sigma_o\sigma_x\rangle>0\right\}(\geqq\beta_c)$$

then

$$E(|\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)||\emptyset,\emptyset)\leqq\frac{1}{2\beta(2\pi)^d}\int_{[-\pi,\pi]^d} d^dp\frac{1}{\sum_{i=1}^d[2\sin^2(p_i/2)]}<\infty. \tag{3.13}$$

[It can be shown, using continuity arguments, that (3.13) is valid also for $\beta=\beta_c-0$.] In other words, we see that *above four dimensions the expected size of the cluster remains uniformly bounded up to the percolation threshold* (!).

iii) Since a probability is never larger than 1, (3.3) implies, by inspection, the special case of the second Griffiths inequality

$$\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle \leqq \langle\sigma_x\sigma_z\rangle .$$

Furthermore, having the probabilistic interpretation

$$\frac{\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle}{\langle\sigma_x\sigma_z\rangle} = \mathrm{Prob}\,(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\ni y|\{x,z\},\emptyset),$$

we may look for lower bounds on this quantity. For instance, by a similar argument to the one made for (3.10)

$$\sum_{y\in B}\frac{\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle}{\langle\sigma_x\sigma_z\rangle} = E(|B\cap\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)|\,|\{x,z\},\emptyset) \tag{3.14}$$

for any set of sites, $B\subset\Lambda$. If B *separates* the sites x, z, in the sense that every path from x to z along bonds with $J_b\neq 0$ has to intersect B, then $|B\cap\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)|\geqq 1$ for any $(\mathbf{n}_1,\mathbf{n}_2)$ in the above ensemble. (Since $\mathbf{n}_1$ connects x with z.) This implies the Simon inequality [5]:

$$\sum_{y\in B}\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle \geqq \langle\sigma_x\sigma_z\rangle . \tag{3.15}$$

(For finite range lattice models (3.14) has the nontrivial consequence that if the correlations decay *faster* than $\dfrac{1}{|x-y|^{d-1}}$ then the decay is exponential.)

The above is a typical example, in the sense that our method offers a *unified derivation* of a variety of "classical" *correlation inequalities* (FKG is the exception). These can be presented as "evident by inspection" once the expressions are transformed by *identities*. Occasionally we would be able to obtain *opposite* bounds by bringing geometric considerations into account (Sects. 5 and 12 and Appendix II).

iv) The reason that in (3.2) the probability that two sites are connected is the *square* of $\langle\sigma_x\sigma_y\rangle$ is that the statement refers to ensembles of sourceless "eddy currents." If the two sites are connected by such a current then, when properly defined, there would be *two* paths connecting these points.

It may be desirable to have a simpler probabilistic interpretation for $\langle\sigma_x\sigma_y\rangle$ itself. For this purpose, consider an ensemble of currents with $\partial\mathbf{n}_1=\{x,z\}$, $\partial\mathbf{n}_2=\emptyset$. Here $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)$ looks like a random path connecting x with z, *augumented* by sourceless "current loops." As we have seen above, for the nearest-neighbor lattice models in $\mathbb{Z}^d$, $d>4$, these current loops are not very large at any $\beta\leqq\bar{\beta}_c$. The probability for such a path to "visit y" is given by (3.3). The mean value of this probability, in a translation invariant system where z is averaged with the weights

$\langle\sigma_x\sigma_z\rangle \Big/ \left[\sum_{z'}\langle\sigma_x\sigma_{z'}\rangle\right]$ (i.e. over distances of the order of the correlation length), is:

$$\sum_z \frac{\langle\sigma_x\sigma_z\rangle}{\left[\sum_{z'}\langle\sigma_x\sigma_{z'}\rangle\right]} \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x) \ni y \mid \{x,z\},\emptyset)$$
$$= \sum_z \langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle \Big/ \sum_{z'}\langle\sigma_x\sigma_{z'}\rangle = \langle\sigma_x\sigma_y\rangle, \tag{3.16}$$

where we used (3.3) and the translation invariance.

Heuristically speaking, when $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)$ is approximable by a simple random walk with a "weak" self interaction, as we expect the case to be if $d>4$, $\beta \leqq \bar{\beta}_c$, one could expect the above probability to be similar to the well-known law for the corresponding Markovian random walk, i.e.

$$\langle\sigma_x\sigma_y\rangle \approx \left(\frac{1}{-\Delta+m^2}\right)(x,y), \quad \text{and at} \quad \beta=\beta_c\text{:} \quad \langle\sigma_0\sigma_x\rangle \approx \frac{1}{|x|^{d-2}}.$$

While this has not yet been fully proven, in the following sections we shall demonstrate the power of a similar intuition.

4. A Heuristic Explanation of the Gaussian Structure of Continuum Limits in $d \geqq 4$ Dimensions

Probability Theory is a Measure Theory – with a Soul

Mark Kac

We now focus on translation invariant systems over $\mathbb{Z}^d$. An infrared bound like (3.12), which holds for the nearest-neighbor interaction, is expected to be valid also for systems with finite range interactions (i.e. such that $J_{x-y}=0$ if $|x-y|>R$ for some $R<\infty$). As we have seen, it implies that for temperatures above the threshold of the long-range-order (i.e. $\beta \leqq \bar{\beta}_c$) the sourceless current loops would be "sparse," if the dimension is $d>4$. Consider now $\langle\sigma_{x_1} \ldots \sigma_{x_{2n}}\rangle$ for $2n$ widely separated points (the odd correlations vanish). It is expressed in (2.7) by a sum over currents with sources at these sites. For each such current one may organize the sources into n *linked pairs*. If the (long) linking currents intersect, the pairing is not unique. However, we have just seen that for $d>4$, *even at* $\beta=\bar{\beta}_c(-0)$, the system does not favor long currents, except of course those imposed by the separate sources. Furthermore, it may be expected, on the basis of analogy with the behavior of paths of the "Brownian motion," that for $d>4$ these long currents miss each other.

As an aside, let us recall that the probability for two independent Wiener paths (in $\mathbb{R}^d$) to intersect is zero, above four dimensions, [10] (see also Appendix I). This probability is positive for $d<4$, $d=4$ being the critical dimension. At $d=4$ the intersection probability is zero, however the probability that two paths (defined for $t\in[0,\infty)$) come to within distance $\varepsilon \geqq 0$, is one for each $\varepsilon \neq 0$. The intersection properties as well as the $|x|^{2-d}$ law for the hitting probability which was mentioned in Sect. 3) are better understood if one observes that the Wiener paths have "dimension" *two*. It is also expected that "mild" self interaction of random paths (e.g. self avoidance for random walks on a lattice) does not modify their long range behavior (since the path's positions at two distant times would only seldom be close). If, in the picture described above, the effects of distant currents factorize, we should obtain

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle = \sum_{\text{pairings}} \langle \sigma_{x_{i_1}} \sigma_{x_{j_1}} \rangle \dots \langle \sigma_{x_{i_n}} \sigma_{x_{j_n}} \rangle + \text{correction}, \tag{4.1}$$

with a small correction due to interaction (e.g. intersection) of long currents. Indeed we shall prove that, at least for the nearest-neighbor interactions, for $d>4$ the correction is insignificant at large separations. Removal of the "correction" term from (4.1) leaves it in the form of the *Wick identities* which characterize expectation values of (centered) *Gaussian variables.*

The statements about the long distance behavior are best formulated in terms of the *rescaled correlation functions.* Although small, the correlations at large distances are significant if the correlation length,

$$\xi = m^{-1}$$

is large [see (2.9)]. Since ξ diverges as $\beta \to \beta_c$, it is natural to consider

$$S_{2n}^{\text{cont}}(x_1, \dots, x_{2n}) = \lim_{\substack{\beta \to \beta_c \\ \alpha, \eta \to \infty}} \alpha^{2n} \langle \sigma_{[x_1 \eta]} \dots \sigma_{[x_{2n} \eta]} \rangle, \tag{4.2}$$

where $x_i \in \mathbb{R}^d$, and $[x_i \eta] \in \mathbb{Z}^d$ are defined by the integral parts of the components of $x_i \eta$. In the limit (4.2) α and η are varied in a way which ensures (weak) convergence of, say, $S_2(x, y)$. The picture we described above suggests that the limits (4.2) describe *Gaussian fields*, if $d>4$.

Remark. Since the lattice points in the limit (4.2) are widely separated, one may regard S_n^{cont} as describing the correlations of *averages* of Ising spins, taken over blocks which are large, yet $o(\eta)$. This explains the transition from a discrete *single-site* distribution to a continuous one.

The above considerations also strongly suggest that in low dimensions, where random paths do intersect (this certainly is the case in $d=2$) the scaling limits are *not* Gaussian.

In the next sections we shall prove similar assertions, and in Part II we shall consider analogous limits to (4.2) which have been used in the attempts to construct *field theories* over the *continua* $\mathbb{R}^d$.

5. An Upper Bound on $|U_4|$ for Ising Models

For $n=2$ the correction in (4.1) is, by definition, the truncated two point function (for the one-phase regime, $\beta > \bar{\beta}_c$)

$$U_4(x_1, \dots, x_4) \equiv \langle \sigma_{x_1} \dots \sigma_{x_4} \rangle - [\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle]. \tag{5.1}$$

Lemma 3.2 permits us to perform *exact cancellations.* The result can be described as follows

Proposition 5.1. *For any (finite) ferromagnetic system*

$$U_4(x_1, \dots, x_4) = -2\langle \sigma_{x_1} \dots \sigma_{x_4} \rangle \operatorname{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \ni x_2, x_3, x_4 | \{x_1, \dots, x_4\}, \emptyset)$$
$$= -2\langle \sigma_{x_1}\sigma_{x_2} \rangle \langle \sigma_{x_3}\sigma_{x_4} \rangle \operatorname{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)$$
$$= \mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_3) \,|\, \{x_1, x_2\}, \{x_3, x_4\}) . \tag{5.2}$$

(Notice that both expressions refer to probabilities that all four points are connected.)

Remark. If $x_1 = x_2$, by $\partial \mathbf{n}_1 = \{x_1, x_2\}$ – or $\{x_1, x_2, x_3, x_4\}$, one should understand $\partial \mathbf{n}_1 = \emptyset$, or, correspondingly, $\{x_3, x_4\}$.

Proof. The claim is trivially true if $x_1 = x_2$ or $x_1 = x_3$. We assume therefore that the four points are *distinct.* By (2.7) and (3.5)

$$Z^2 U_4(x_1, \dots, x_4) = \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, \dots, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2)$$
$$- \left[\sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2\} \\ \partial \mathbf{n}_2 = \{x_3, x_4\}}} + \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_3\} \\ \partial \mathbf{n}_2 = \{x_2, x_4\}}} + \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_4\} \\ \partial \mathbf{n}_2 = \{x_2, x_3\}}} \right] w(\mathbf{n}_1) w(\mathbf{n}_2)$$
$$= \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2, x_3, x_4\} \\ \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) [1 - X_{x_1, x_2}(\mathbf{n}_1 + \mathbf{n}_2)$$
$$- X_{x_1, x_3}(\mathbf{n}_1 + \mathbf{n}_2) - X_{x_1, x_4}(\mathbf{n}_1 + \mathbf{n}_2)] ,$$

$X_{x,y}(\mathbf{m})$ being defined in (3.11). For configurations $\mathbf{n}_1 + \mathbf{n}_2$ in which the source x_1 is connected to exactly one other source, we obtain for the above expression $[1 - X - X - X] = 0$. The only other possibility is that x_1 is connected to all the other three sources, since no single source can be left disconnected from the rest. In that case, $[1 - X - X - X] = -2$. Therefore

$$Z^2 U_4(x_1, \dots, x_4) = -2 \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, \dots, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \ni x_2, x_3, x_4]$$
$$= -2 \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2\} \\ \partial \mathbf{n}_2 = \{x_3, x_4\}}} w(\mathbf{n}_1) w(\mathbf{n}_2) X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) = \mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_3)] ,$$

where the last equality follows by Lemma 3.2, with

$$\mathscr{I} = \{\mathbf{m} | \mathscr{C}_{\mathbf{m}}(x_1) \ni x_2, x_3, x_4\} .$$

Dividing each of the above expressions by the corresponding unrestricted sum over $(\mathbf{n}_1, \mathbf{n}_2)$, one arrives at (5.1) and (5.2). □

Proposition 1 implies, in particular, the *Lebowitz inequality* [11], $U_4 \leqq 0$, and a simple lower bound:

$$0 \geqq U_4(x_1, \dots, x_4) \geqq -2\langle \sigma_{x_1}\sigma_{x_2} \rangle \langle \sigma_{x_3}\sigma_{x_4} \rangle . \tag{5.3}$$

Glimm and Jaffe used an even mildly weaker lower bound (i.e. upper bound on $|U_4|$) to derive *finite* upper bounds for the renormalized coupling constants [12]. [In Sect. 15, (5.3) is applied for a related result.]

Furthermore, since we now have an exact expression for U_4, which is of definite sign, we are able to obtain significantly improved *lower* and *upper* bounds. These are expressed by Propositions 5.2–5.4.

Proposition 5.2. *For any (finite) ferromagnetic system*

$$2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_3)\neq\emptyset \,|\, \{x_1,x_2\},\{x_3,x_4\}) \leqq |U_4(x_1,\ldots,x_4)| \tag{5.4}$$

and

$$|U_4(x_1,\ldots,x_4)| \leqq 2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle \cdot \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_3)\neq\emptyset \,|\, \{x_1,x_2\},\{x_3,x_4\},\emptyset), \tag{5.5}$$

where the last probability refers to a system of *three* independent currents, and is defined by an extension of (3.1).

Remark. The conceptual (and technical) advantage of Proposition 5.2, over Proposition 5.1, is that in each case it refers to probabilities of intersection of two random sets (current-connected clusters) which are now *independent* of each other.

Proof. (5.4) is a trivial consequence of (5.2) [since $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\supset\mathscr{C}_{\mathbf{n}_1}(x)$ etc.]. (In Sect. 9 we prove that for a proper pairing this lower bound is in fact correct up to a lesser factor than $\frac{3}{2}$.)

To prove (5.5) let us assume that the four points are distinct (the other cases are obvious). To shorten the notation we represent x_i by i, $i=1,\ldots,4$. A comparison with (5.2) shows that we have to demonstrate that

$$\begin{aligned}&\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\cap\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(3)\neq\emptyset \,|\, \{1,2\},\{3,4\},\emptyset)\\ &\quad\leqq \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(1)\cap\mathscr{C}_{\mathbf{n}_2}(3)\neq\emptyset \,|\, \{1,2\},\{3,4\},\emptyset)\end{aligned}$$

or, equivalently, that

$$\begin{aligned}&\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\cap\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(3)=\emptyset \,|\, \{1,2\},\{3,4\},\emptyset)\\ &\quad\geqq \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(1)\cap\mathscr{C}_{\mathbf{n}_2}(3)=\emptyset \,|\, \{1,2\},\{3,4\},\emptyset)\\ &\quad\equiv \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\cap\mathscr{C}_{\mathbf{n}_3}(3)=\emptyset \,|\, \{1,2\},\emptyset,\{3,4\})\end{aligned} \tag{5.6}$$

(exchanging $\mathbf{n}_2\leftrightarrow\mathbf{n}_3$).

Explicitly, (5.6) is the following statement (obtained after multiplying it by Z_Λ^3).

$$\begin{aligned}&\sum_{\substack{\partial\mathbf{n}_1=\{1,2\}\\ \partial\mathbf{n}_2=\{3,4\}\\ \partial\mathbf{n}_3=\emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2)w(\mathbf{n}_3)X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\not\ni x_3]\\ &\geqq \sum_{\substack{\partial\mathbf{n}_1=\{1,2\}\\ \partial\mathbf{n}_2=\emptyset\\ \partial\mathbf{n}_3=\{3,4\}}} w(\mathbf{n}_1)w(\mathbf{n}_2)w(\mathbf{n}_3)X[\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\cap\mathscr{C}_{\mathbf{n}_3}(3)=\emptyset].\end{aligned} \tag{5.7}$$

To compare the two terms, let us consider the partial sums which correspond to fixed:

i) $\mathbf{n}_1$

ii) $\mathbf{m}\stackrel{\text{def}}{=}\mathbf{n}_2+\mathbf{n}_3$,

iii) $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)$.

Notice that the two sums have identical source constraints on $\mathbf{m}$. By the construction used in the proof of Lemma 3.1 – with each $\mathbf{m}$ we associate a graph $\mathscr{M}$. The corresponding partial sum for the *left-hand-side* (LHS) of (5.7) is then $w(\mathbf{n}_1)w(\mathbf{m})\times$ the number of subgraphs $G\subset\mathscr{M}$, corresponding to $\mathbf{n}_2$, which are consistent with the condition iii) and have the sources $\partial G=\{3,4\}$. On the *right-hand-side* (RHS) one counts the sourceless graphs $G'\subset\mathscr{M}$ whose complement satisfies the condition imposed by:

iv) $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)\cap\mathscr{C}_{\mathbf{n}_3}(3)=\emptyset$

and has the sources $\partial\mathscr{M}\backslash G=\{3,4\}$.

For each of the abovementioned graphs $\mathscr{M}$ there is a subgraph $\gamma\subset\mathscr{M}$ describing a path from x_3 to x_4, i.e. $\partial\gamma=\{3,4\}$, which *avoids* $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)$. The graph γ may be chosen in a way which depends only on $\mathbf{n}_1$ and the set $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(1)$. Repeating the symmetric difference argument which was used in the proof of Lemma 3.1, we associate with each graph G which is counted in the RHS the graph $G'=\gamma\Delta G$ which contributes the LHS. Not all the relevant graphs would be obtained this way, since the constraint iv) plays a role only for the RHS, however the mapping $G\to G'$ is invertible. Thus (5.7) holds even at the level of the above partial sums.

A side remark: (5.7) can be also proved by an alternative argument which uses the Griffiths inequality. □

The upper bound (5.5) leads to the following new *correlation inequality:*

Proposition 5.3. *In a ferromagnetic system, for every four points*

$$|U_4(x_1,\ldots,x_4)|\leqq 2\sum_y\langle\sigma_{x_1}\sigma_y\rangle\langle\sigma_{x_2}\sigma_y\rangle\langle\sigma_{x_3}\sigma_y\rangle\langle\sigma_{x_4}\sigma_y\rangle\,. \tag{5.8}$$

Proof. Let us first prove (5.8) for finite systems, where the right side of (5.5) is already well defined. For $(\mathbf{n}_1,\mathbf{n}_2)$ for which $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_2)\neq\emptyset$, the size of the intersection is certainly not less than 1. Therefore

$$\begin{aligned}
&\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_2)\neq\emptyset|\{x_1,x_2\},\{x_3,x_4\},\emptyset)\\
&\leqq E(|\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_2)|\,|\{x_1,x_2\},\{x_3,x_4\},\emptyset)\\
&\leqq E(|\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathscr{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_2)|\,|\{x_1,x_2\},\{x_3,x_4\},\emptyset,\emptyset)\\
&=\sum_y\frac{\langle\sigma_{x_1}\sigma_y\rangle\langle\sigma_{x_2}\sigma_y\rangle}{\langle\sigma_{x_1}\sigma_{x_2}\rangle}\,\frac{\langle\sigma_{x_3}\sigma_y\rangle\langle\sigma_{x_4}\sigma_y\rangle}{\langle\sigma_{x_3}\sigma_{x_4}\rangle}\,.
\end{aligned}\tag{5.9}$$

In the last step we assumed $\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle\neq 0$ and used (3.11) and (3.3). Substituting this result in (5.5) we obtain (5.8).

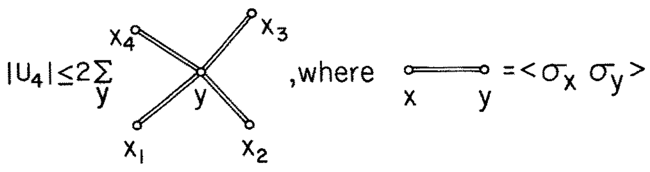

Fig. 4. A schematic representation of (5.8)

For infinite systems, one defines the states $\langle - \rangle_\Lambda$ by limits of the restriction of J to finite regions $\Lambda' \subset \Lambda$. For such limits (5.8) follows by the monotone convergence theorem, since $\langle \sigma_A \rangle_{\Lambda'}$ are *increasing* as $\Lambda' \uparrow \Lambda$. □

Remark. It is important to keep track of where does one lose significant factors. It seems that a major loss in (5.8) occurs in the step (5.9). However, an analogy with the study of intersection properties of the simple random walk suggests that this tactic is *quite efficient for* $d>4$. We expect (5.8) to be a bad bound for $d<4$ (this is obviously the case if $d=1$) and marginal (i.e. overestimating by an infinite factor, but still somewhat useful) for $d=4$, see Appendix I.

6. The Vanishing of the Renormalized Coupling Constant in $d>4$ Dimensions

We postpone to Sect. 13 the analysis of the scaling limits (4.2), where it would be carried out in a more general context (see however a remark on the subject in [1]). Here we shall consider a very instructive implication of Proposition 5.3 for the *renormalized coupling constant*, g, which is a scale-invariant measure of the "interaction" of two currents, i.e. deviation of the model from the Gaussian-like ("free-field") behavior. It is defined as

$$g = \overline{|U_4|}/(\chi^2 \xi^d), \tag{6.1}$$

where

$$\overline{|U_4|} = \sum_{x_2, x_3, x_4} |U_4(O, x_2, x_3, x_4)|$$

[see the discussion of (4.1) and (5.1)], and χ is the magnetic *susceptibility*

$$\chi = \sum_x \langle \sigma_o \sigma_x \rangle . \tag{6.2}$$

Proposition 5.3 implies that generally

$$\overline{|U_4|} \leqq \chi^4 , \tag{6.3}$$

reducing the problem of bounding g to that of the study of only the *two*-point function, which is better understood. In fact, by the consequences of reflection positivity, for the *nearest-neighbor* models,

$$\chi \leqq \beta^{-1} \xi^2 \left(1 + O\left(\frac{1}{\xi} 2\right)\right) \tag{6.4}$$

for $\beta < \beta_c$ (see [13, 14]).

Combining (6.3) with (6.4) we have:

$$g \leqq \beta^{-2} / \xi^{(d-4)} , \tag{6.5}$$

for the nearest-neighbor models at inverse-temperatures $\beta < \beta_c$.

Therefore, in *more than four dimensions* $g \to 0$ when $\xi \to \infty$. (The divergence of ξ in the limit $\beta \to \beta_c$, is proven in [4, 5].)

7. Proof of the Mean-Field Value for the Critical Exponent γ, for $d>4$

The critical exponent γ has been defined by the expected power law:

$$\chi(\beta) \approx t^{-\gamma} \tag{7.1}$$

for $\beta<\beta_c$, $t=(\beta_c-\beta)/\beta_c$.

Generally, $\gamma \geqq 1$ (Glimm and Jaffe [15]). In the simple mean-field approximation, $\gamma=1$. We shall now prove that 1 is indeed the *exact* value of γ, in translation-invariant models (over $\mathbb{Z}^d$), with $|J| \equiv \sum_x J_{o,x} < \infty$, for which

$$\sum_x \langle \sigma_o \sigma_x \rangle^2_{\beta_c - o} < \infty, \tag{7.2}$$

with

$$\beta_c = \sup\{\beta | \chi(\beta) < \infty\}. \tag{7.3}$$

This class of models includes the *nearest-neighbor* interaction in *more that four dimensions*, for which (7.3) is equivalent to our previous definition of β_c – via (2.9) [4, 5], and (7.2) follows from the infrared bound (3.12). That fact has already been used by Sokal [8] to prove the mean-field behavior (i.e. finiteness) of the specific heat, at β_c-0, in $d>4$ dimensions. To avoid a new symbol, for the rest of this section we adopt (7.3) as the definition of β_c.

Proposition 7.1. *If*[1] *in translation invariant ferromagnetic model (with* $|J|<\infty$*)* (7.2) *is satisfied for the* β_c *defined by* (7.3), *then for any* $o<\beta<\beta_c$

$$[\varepsilon |J| \beta_c]^{-1} t^{-1} \geqq \chi(\beta) \geqq [|J|\beta_c]^{-1} t^{-1} \tag{7.4}$$

with some $\varepsilon=\varepsilon(J)>0$, *and* $t=(\beta_c-\beta)/\beta_c$. *Consequently:*

$$\gamma \equiv -\lim_{\beta\to\beta_c} \frac{\ln \chi(\beta)}{\ln(\beta_c-\beta)} = 1. \tag{7.5}$$

Remark. A formal differentiation of the Gibbs formula (2.2) gives

$$\begin{aligned}\frac{d}{d\beta}\langle \sigma_o\, \sigma_x\rangle &= \frac{1}{2}\sum_{u\neq v} J_{u,v}[\langle \sigma_o \sigma_x\rangle \langle \sigma_u \sigma_v\rangle - \langle \sigma_o \sigma_x\rangle \langle \sigma_u \sigma_v\rangle] \\ &= \sum_{u\neq v} J_{u,v}[\langle \sigma_o \sigma_u\rangle \langle \sigma_v \sigma_x\rangle - \tfrac{1}{2}|U_4(O,x,u,v)|]\end{aligned} \tag{7.6}$$

(using the symmetry $J_{u,v}=J_{v,u}$, and the fact that $U_4 \leqq 0$). Summing over x one obtains:

$$\frac{d}{d\beta}\chi = |J|\chi^2 - \sum_{u,v,x} \tfrac{1}{2} J_{u,v} |U_4(O,x,u,v)|, \tag{7.7}$$

where $|J| = \sum_y J_{O,y}$. Since (7.6) is a standard expression, we shall mention only briefly how its validity is established for *infinite systems.* For this purpose, it is convenient to use the *integrated version* of (7.6) in the finite-system approximation of $\chi(\beta_1)-\chi(\beta_2)$. The monotonicity properties of $\langle \sigma_A \rangle_{\Lambda,\beta}$, as a function of Λ and β, and the finiteness of $\chi(\beta)$ for $\beta<\beta_c$, allow then an application of the dominated

---

1 This result is further simplified in [29], where it is shown that $\chi(\beta) \leqq t^{-1}|J|\beta_c/[1+|J|\beta_c \sum \langle \sigma_o \sigma_x \rangle^2_{\beta_c - 0}]$ (note added in proof)

convergence theorem. These arguments prove (7.6) for $\beta \leqq \beta_c$, $\frac{d}{d\beta}$ being interpreted as the derivative *from below* when $\beta = \beta_c$.

To illuminate the stochastic-geometric context of Proposition 7.1, we invoke (5.2) and rewrite (7.7) as follows:

$$\begin{aligned} -\frac{d\chi^{-1}}{d\beta} &= \chi^{-2}\frac{d\chi}{d\beta} \\ &= |J| \sum_{\substack{x \\ u \neq v}} \frac{\langle \sigma_o \sigma_u \rangle J_{u,v} \langle \sigma_v \sigma_x \rangle}{|J| \chi^2} \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) = \emptyset | \{O, u\}, \{v, x\}, \emptyset) \\ &\overset{\mathrm{def}}{=} |J| \overline{\mathrm{Prob}}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(v) = \emptyset), \end{aligned} \tag{7.8}$$

By $\overline{\mathrm{Prob}}$ we denote here the normalized probability in a duplicate system of currents whose sources are also subject to the averaging procedure, as described in the second expression ($\mathbf{n}_3$ is, at this point, *superfluous*).

Strictly speaking, so far we made sense of expressions like (7.8) only for finite-system approximants. Since the bounds on ε which would emerge are uniform in Λ, we ignore this minor detail in our notation.

Replacing the probability in (7.8) by 1, one obtains the universal upper bound on $|d\chi^{-1}/d\beta|$ of Glimm and Jaffe [15]. However, for (7.4) to be satisfied *this probability should not vanish.* For instance, it would not vanish if the probability for two currents to avoid each other is uniformly positive, even if their sources are close. For simple random walks this is indeed the case above four dimensions.

Proof of Proposition 7.1. Let $B_{r,u} = \{y \in \mathbb{R}^d \,|\, |y-u| \leqq r\}$. Then

$$\overline{\mathrm{Prob}}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathscr{C}_{\mathbf{n}_3}(v) = \emptyset) = \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathscr{C}_{\mathbf{n}_3}(v) \backslash B_{r,u} = \emptyset) \varepsilon_1(r),$$

$\varepsilon_1(r)$ being the conditional probability that the two clusters are disjoint, *given* that outside of $B_{r,u}$ they have no point in common. It is easy, albeit cumbersome, to prove that for each r, $\varepsilon_1(r) > 0$.

By the same argument as in (5.9):

$$\begin{aligned} &\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_3}(u) \cap \mathscr{C}_{\mathbf{n}_2}(v) \backslash B_{r,u} \neq \emptyset \,|\, \{O, u\}, \{v, x\}, \emptyset) \\ &\leqq \sum_{y \in \mathbb{Z}^d \backslash B^r} \frac{\langle \sigma_o \sigma_y \rangle \langle \sigma_u \sigma_y \rangle \langle \sigma_v \sigma_y \rangle \langle \sigma_x \sigma_y \rangle}{\langle \sigma_o \sigma_u \rangle \langle \sigma_v \sigma_x \rangle}. \end{aligned}$$

Let us now substitute this in the previous equality. By the translation invariance one obtains cancellation of χ^2 terms. The result is:

$$\begin{aligned} &\overline{\mathrm{Prob}}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathscr{C}_{\mathbf{n}_3}(v) = \emptyset) \\ &\geqq \varepsilon_1(r) \Bigg[1 - \sum_{\substack{k \\ |Z| > r}} J_{o,k} \langle \sigma_o \sigma_z \rangle \langle \sigma_z \sigma_k \rangle / |J| \Bigg] \\ &\geqq \varepsilon_1(r) \Bigg[1 - \Big(\sum_{|Z| > r} \langle \sigma_o \sigma_z \rangle^2\Big)^{1/2} \Big(\sum_x \langle \sigma_o \sigma_x \rangle^2\Big)^{1/2}\Bigg] \equiv \varepsilon_1(r) \varepsilon_2(r) \end{aligned} \tag{7.9}$$

(for the last expression we used the Schwartz inequality for each k). If (7.2) is satisfied then $\varepsilon_2(r) > 0$ for r large enough. With such r, let us substitute (5.6) and

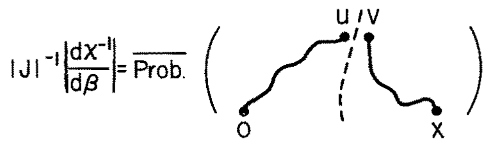

Fig. 5. A graphical representation of the relation (7.8)

(7.9) in (7.8). We obtain:

$$|J| \geqq -\frac{d\chi^{-1}}{d\beta} \geqq \varepsilon|J| \quad (\text{for } \beta < \beta_c), \tag{7.10}$$

with $\varepsilon = \varepsilon_1(r)\varepsilon_2(r) > 0$.

The upper bound on $|d\chi^{-1}/d\beta|$ (which holds also for finite systems at any β) implies that

$$\chi^{-1}(\beta) \xrightarrow[\beta\to\beta_c]{} 0,$$

β_c being defined (for this proposition only) by (7.3). Thus (7.10) directly implies (7.4). □

8. Proof of Hyperscaling in Two Dimensions

In *low* dimensions the critical exponents of the Ising model are no longer given by the simple mean-field approximation. It is also known, from the exact solution, that the scaling limit of the two dimensional nearest-neighbor Ising model is *not* Gaussian. We shall now offer a simple explanation of these facts, and prove that a "universal" relation – known as *hyperscaling* [16], is indeed satisfied by the critical exponents in two dimensional models (of the type considered here). Hyperscaling means the vanishing of the dimension-dependent combination of critical-exponents which describes the behavior of g – the renormalized coupling constant defined by (6.1).

The abovementioned properties of the spin models are related, by Proposition 5.1, to the question of how typical is it for random currents to intersect. The key point is now that in two dimensions intersection is quite natural, and sometimes even hard to avoid. Quantifying it, we obtain a simple *lower bound* on g. Hyperscaling follows from this lower bound and the upper bound of Glimm and Jaffe [12], – or actually a new variant of the latter.

Although limited, the following result is presented as a specially simple example.

Proposition 8.1. *In the nearest-neighbor Ising model on* $\mathbb{Z}^2$, *the spin correlations for the sites* $x_1 = -x_3 = (x, 0)$, $x_2 = -x_4 = (0, x)$ *satisfy*

$$|U_4(x_1, x_2, x_3, x_4| \geqq \langle \sigma_{x_1}\sigma_{x_3}\rangle \langle \sigma_{x_2}\sigma_{x_4}\rangle . \tag{8.1}$$

Proof. Let Γ_{x_2,x_4} be the collection of curves in the punctured plane $\mathbb{R}^2/\{x_2, x_4\}$ which are homotopic (i.e. deformable) to the segment

$$[x_1, x_2] = \{\lambda x_1 + (1-\lambda)x_2 | \lambda \in [0, 1]\} .$$

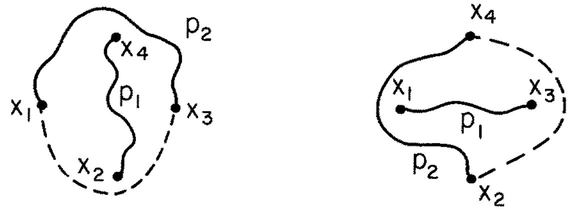

Fig. 6. The non-intersecting current configurations in $d=2$ dimensions

We denote by P_1 and P_2 the probabilities of the following two events:

$P_1[P_2]=\text{Prob}(\mathscr{C}_{\mathbf{n}_1}(x_2)\not\ni\{x_2,x_4\}$, and any [no] path linking x_1 and x_3 along neighboring sites in $\mathscr{C}_{\mathbf{n}_1}(x_1)$ is in $\Gamma_{x_2,x_4}\mid\{x_1,x_3\},\{x_2,x_4\})$.

Clearly $P_1+P_2\leqq 1$. A simple analysis of the two possibilities in which $\mathscr{C}_{\mathbf{n}_1}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_3)=\emptyset$ (Fig. 6) shows that

$$\begin{aligned}\text{Prob}(\mathscr{C}_{\mathbf{n}_1}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_2)=\emptyset|\{x_1,x_3\},\{x_2,x_4\})\\ \leqq 2P_1P_2\equiv[(P_1+P_2)^2-(P_1-P_2)^2]/2\leqq\tfrac{1}{2},\end{aligned}\tag{8.2}$$

where we used the similarity of $\mathscr{C}_{\mathbf{n}_1}(x_2)$ to $\mathscr{C}_{\mathbf{n}_2}(x_2)$, which follows by either rotation or reflection symmetry.

Substituting (8.2) in (5.4), we obtain

$$\begin{aligned}|U(x_1,\dots,x_4)|&\geqq 2\langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_2}\sigma_{x_4}\rangle\text{Prob}(\mathscr{C}_{\mathbf{n}_1}(x_1)\cap\mathscr{C}_{\mathbf{n}_2}(x_2)\neq\emptyset|\{x_1,x_3\},\{x_2,x_4\})\\ &\geqq\langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_2}\sigma_{x_4}\rangle,\end{aligned}$$

thus proving the claim (8.1). □

Remark. Equation (8.1) clearly extends also to other quadruples $\{x_1,\dots,x_4\}$ with a two-fold symmetry. While it suggests that in the continuum limit $U_4\neq 0$, it is not conclusive, since it applies only to exceptional points in $(\mathbb{R}^2)^4$.

The next result is more general. We refer in it to the modified notions of the correlation length, ξ_ϕ, and the renormalized coupling constant, g_ϕ, which are defined as follows

$$\xi_\phi^\phi=\sum_x\langle\sigma_o\sigma_x\rangle|x|^\phi/\chi,\tag{8.3}$$

and

$$g_\phi=\overline{|U_4|}/[\chi^2\xi_\phi^d].\tag{8.4}$$

We also denote $\|x\|_\infty=\max(|x^{(1)}|,|x^{(2)}|,\dots)$, and define the *range* of an interaction by

$$R(J)=\sup\{\|x\|_\infty|x\in\mathbb{Z}^d,J_{o,x}\neq 0\}.\tag{8.5}$$

Proposition 8.2. *For any finite range interaction in $d=2$ dimensions, which has all the translation and reflection invariance of $\mathbb{Z}^2$, and $\beta<\beta_c$,*

$$g_1\geqq 1/[2R(J)^2].\tag{8.6}$$

Furthermore, if in the limit (4.2) S_2 *is rotation invariant, and*

$$\limsup_{\eta\to\infty}\int dx(1+|x|^{1+\varepsilon})S_2^{(\alpha,\eta)}(x)<\infty$$

for some $\varepsilon>0$, *then*

$$\lim_{\eta\to\infty} g_1 \geqq 4/[\pi R(J)^2]. \tag{8.7}$$

Remark. The symmetry which we demand of J excludes interactions which are effectively one dimensional, in which case (8.6) is not satisfied.

An upper bound on g_1 is given by Proposition 15.1.

In the proof of Proposition 8.2 we shall use the following result:

Lemma 8.1. *Let* $A=\{x_1,\ldots,x_I\}$, $B=\{y_1,\ldots,y_J\}$ *be two sequences of points in* $\mathbb{Z}^2$, *such that*

$$\|x_{i+1}-x_i\|_\infty, \quad \|y_{j+1}-y_j\|_\infty \leqq R \quad \forall 1\leqq i\leqq I-1, \quad 1\leqq j\leqq J-1. \tag{8.8}$$

Let $A+B\overset{\text{def}}{=}\{x+y|x\in A, y\in B\}$. *Then*

$$|A+B|\geqq |(x_I-x_1)\wedge(y_J-y_1)|/R^2, \tag{8.9}$$

where $|A|$ *is the number of points in the set A, and* $\wedge$ *is the skew product.*

Proof. Let $\hat{x}=x_I-x_1$, $\hat{y}=y_J-y_1$, and $A_k=A+\{k\hat{x}\}$, $B_l=B+\{l\hat{y}\}$. Clearly $|A_k+B_l| =|A+B|$, and thus

$$\begin{aligned}|A+B| &\geqq \frac{1}{KL}\left|\bigcup_{\substack{k=1,\ldots,K\\ l=1,\ldots,L}}(A_k+B_l)\right|\\ &\geqq |\hat{x}\wedge\hat{y}| \lim_{K,L\to\infty}\frac{1}{KL|\hat{x}\wedge\hat{y}|}\left|\bigcup_{k=1}^{K} A_k+\bigcup_{l=1}^{L} B_l\right| \overset{\text{def}}{=} |\hat{x}\wedge\hat{y}| f(A,B).\end{aligned}$$

$f(A,B)$ is the density, in $\mathbb{Z}^2$, of the set $\bigcup_{k=-\infty}^{\infty} A_k+\bigcup_{l=-\infty}^{\infty} B_l$.

Viewing this set as the union of the translates of one doubly-infinite sequence, $\bigcup A_k$, by the other, $\bigcup B_l$, it is clear that, as a consequence of (8.8), any translate of the cube $[0,R)\times[0,R)$ contain at least one point of this set. Therefore

$$f(A,B)\geqq 1/R^2,$$

which implies (8.9). □

Proof of Proposition 8.2. By (5.4)

$$\begin{aligned}&\sum_{1,2,3}|U_4(O,1,2,3)| = \sum_{x,y,z}|U_4(O,x,y,y+z)|\\ &\geqq 2\sum_{x,z}\langle\sigma_o\sigma_x\rangle\langle\sigma_o\sigma_z\rangle\sum_y \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1}(O)\cap\mathscr{C}_{\mathbf{n}_2}(y)\neq\emptyset\,|\,\{O,x\},\{y,y+z\}).\end{aligned} \tag{8.10}$$

However, using the translation invariance,

$$\begin{aligned}&\sum_y \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1}(O)\cap\mathscr{C}_{\mathbf{n}_2}(y)\neq\emptyset\,|\,\{O,x\},\{y,y+z\})\\ &=\sum_y \mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1}(O)\cap(y+\mathscr{C}_{\mathbf{n}_2}(O))\neq\emptyset\,|\,\{O,x\},\{O,z\})\\ &=E(|\mathscr{C}_{\mathbf{n}_1}(O)-\mathscr{C}_{\mathbf{n}_2}(O)|\,|\,\{O,x\},\{O,z\})\geqq|x\wedge z|/R(J)^2.\end{aligned} \tag{8.11}$$

The last steps follow by performing the sum over y before the average in $(\mathbf{n}_1, \mathbf{n}_2)$, and then applying Lemma 8.1. Notice that $\mathscr{C}_{\mathbf{n}_1}(O)$, and $\mathscr{C}_{\mathbf{n}_2}(O)$, necessarily contain paths from O to x, and from O to z, which satisfy (8.8). Therefore:

$$\begin{aligned}\overline{|U_4|} &\geqq 2 \sum_{x,z} \langle\sigma_o\sigma_x\rangle \langle\sigma_o\sigma_z\rangle |x \wedge z|/R(J)^2 \\ &\geqq 2 \sum_{x,z} \langle\sigma_o\sigma_x\rangle \langle\sigma_o\sigma_z\rangle (x^{(1)}z^{(2)} - x^{(2)}z^{(1)}) \operatorname{sgn} x^{(1)} \operatorname{sgn} z^{(2)}/R(J)^2 \\ &= 2 \sum_{x,z} \langle\sigma_o\sigma_x\rangle \langle\sigma_o\sigma_z\rangle |x^{(1)}| |z^{(2)}|/R(J)^2\end{aligned}$$

(using the reflection symmetries)

$$\geqq \chi^2 \xi_1^2/[2R(J)^2],$$

which proves (8.6).

To prove the better bound (8.7) one may rewrite the first inequality as follows

$$g_1 \geqq \frac{\int dx \int dz S_2^{(\alpha,\eta)}(x) S_2^{(\alpha,\eta)}(z) |x \wedge z|}{\int dx \int dz S_2^{(\alpha,\eta)}(x) S_2^{(\alpha,\eta)}(z) |x| |z|} \cdot \frac{2}{R(J)^2}, \tag{8.12}$$

and use the (unique) value of the ratio for rotation invariant functions S_2. (The continuity of the integrals at $\eta = \infty$ is ensured by the assumed bound.) □

Corollary 8.1. *Under the main assumptions of Proposition 8.1, if the limit* (4.3) *exists (for S_n regarded as densities of measures on $\mathbb{R}^{2n}$) then the limiting theory is not a free field.*

Sketch of the Proof. It suffices to prove that for some bounded $A \subset \mathbb{R}^2$

$$\int_{\substack{x, y \in A \\ z \in \mathbb{R}^2}} dx dy dz |U_4^{\text{cont}}(O, x, y, z)| \neq 0. \tag{8.13}$$

One is tempted to use (8.11) to deduce that, similarly to (8.12), (8.13) is bounded below by

$$2 \int_{A \times A} dx dy S_2^{\text{cont}}(x) S_2^{\text{cont}}(y) |x \wedge y|/R^2.$$

Unfortunately, the continuity of the integrals is not automatic, since z is integrated over all of $\mathbb{R}^2$, while we assumed only local convergence of S_2. This problem may be overcome by restricting the range of z in (8.13) to $\{z | \operatorname{dist}(z, A) \leqq \operatorname{diam}(A)\}$. One needs then a correspondingly improved version of (8.11). We claim that this can be derived, using the symmetry of J, but we shall omit here the details.

Corollary 8.2. (Hyperscaling in $d=2$). *If, under the main assumptions of Proposition 8.2,*

$$\chi \approx t^{-\gamma}, \quad \xi_1 \approx t^{-\nu_1}, \quad \overline{|U_4|} \approx t^{-(\gamma + 2\Delta)} \tag{8.14}$$

for $t = (\beta_c - \beta)/\beta_c > 0$ $\left(\text{in the sense that } \gamma = \lim_{t \downarrow 0} \ln \chi/|\ln t|, \text{ etc.}\right)$ then

$$d\nu_1 - 2\Delta_4 + \gamma = 0. \tag{8.15}$$

Proof. Equation (8.15) is an immediate consequence of the fact that, as $t \to 0$, g_1 remains uniformly bounded above and below, (8.6). For the upper bound we need an extension of the results of Glimm and Jaffe [12] and Schrader [17], who proved it for g and g_ϕ, $\phi > d$. Such a bound on $g_{d/2}$ has recently been proven by Lieb and Sokal [18]. In Sect. 15 we also derive it, using a very simple argument. □

9. A Random-Walk Representation

In the above discussion we have related the coupling in Ising models to intersection of random currents. It is also useful to express such a relation in terms of random walks, which in concept are simpler than the clusters $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}$.

Our starting point is the expression (2.7) and the observation that, in a finite system, if $\partial\mathbf{n} = \{x, y\}$ then any nonrepeating path along bonds with *odd* values of n which starts at x, eventually reaches y. We construct a random walk by the following iterative procedure, in which it is assumed that we are already given a (finite) flux configuration $\mathbf{n}$, with $\partial\mathbf{n} = \{x, y\}$.

i) The walk starts at x.

ii) At each step choose a bond at random, with equal probabilities among all the bonds with $J_b \neq 0$ which link to the present site and which were not traversed before.

iii) Check the parity of $\mathbf{n}$ for the chosen bond. If n_b is odd, the random walk *traverses* the bond, moving to its other end. If n_b is even, no move is made but the bond is recorded as *attempted.*

iv) The walk stops upon the first hit of y.

Before formally presenting the random walk expansion which is thus generated, let us state our main result.

We denote by $\hat{\mathscr{C}}_{\mathbf{n}}(x, y)$ the (random) set of sites visited by the above walk. Clearly

$$\hat{\mathscr{C}}_{\mathbf{n}}(x, y) \subset \mathscr{C}_{\mathbf{n}}(x) . \tag{9.1}$$

Proposition 9.1. *In a finite system with ferromagnetic pair interaction let*

$$\begin{aligned} &V(x_1, \ldots, x_4) \\ &= \langle \sigma_{x_1}\sigma_{x_2} \rangle \langle \sigma_{x_3}\sigma_{x_4} \rangle \operatorname{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}(x_1, x_2) \cap \hat{\mathscr{C}}_{\mathbf{n}_2}(x_3, x_4) \neq \emptyset \,|\, \{x_1, x_2\}, \{x_3, x_4\}) \\ &\quad + 2 \text{ permutations}, \end{aligned}$$

where the probability is an average over both the ensemble of random currents and the random walk variables. Then, for any $x_1, \ldots, x_4 \in \Lambda$,

$$\tfrac{2}{3} V(x_1, \ldots, x_4) \leqq |U_4(x_1, \ldots, x_4)| \leqq V(x_1, \ldots, x_4) . \tag{9.2}$$

For the sake of concreteness, we use the following terminology.

Definition 9.1. i) A *step sequence* is a sequence of pairs of sites and bonds, $\omega = \{(x_j, b_j)\}_{j=1}^k$ with $b_j \ni x_j$, whose bonds are separated into two given, mutually exclusive, sets – B_t and B_a, each bond of B_t appearing only once in the sequence. With a slight abuse of notation we shall refer to the sets of bonds as $B_t(\omega)$, $B_a(\omega)$, and denote $B(\omega) = B_t(\omega) \cup B_a(\omega)$.

ii) A *path* is a step sequence in which $b_j=\{x_j, x_{j+1}\}$ whenever $b_j \in B_t$ (we then regard the bond as *traversed*), and $x_{j+1}=x_j$ when $b_j \in B_a$ (in which case we regard the bond as merely *attempted*).

iii) $\Omega(x, y)$ is the collection of paths from x to y, i.e. $\omega=\{(x_j, b_j)\}_{j=1}^k$ with $x_1=x$, $b_k=\{x_{k-1}, y\}$, for which in addition $x_j \neq y$ for $1 \leq j \leq k$.

The compatibility of a step sequence ω with a flux configuration $\mathbf{n}$ is expressed by a function defined as follows

$$\theta(\omega, \mathbf{n})=\begin{cases}1 & \text{if } \begin{cases}(-1)^{n_b}=-1 & \forall b \in B_t(\omega)\\ (-1)^{n_b}=+1 & \forall b \in B_a(\omega)\end{cases}\\ 0 & \text{otherwise}.\end{cases}$$

For a fixed flux configuration $\mathbf{n}$, with $\partial\mathbf{n}=\{x, y\}$, the probability that the above mentioned random walk follows a path $\omega \in \Omega(x, y)$, with $\theta(\omega, \mathbf{n})=1$, is

$$q(\omega)=\prod_{j=1}^{k} (\text{the number of bonds containing } x_k \text{ which are not among } \{b_1, \ldots, b_{k-1}\} \cap B_t(\omega), \text{ and for which } J_b \neq 0)^{-1}. \tag{9.3}$$

We therefore have the following identity, details of whose proof we leave to the reader.

Lemma 9.1. *Let* $\mathbf{n}$ *be a flux configuration over a finite system, with* $\partial\mathbf{n}=\{\mathbf{x}, \mathbf{y}\}$. *Then*

$$\sum_{\omega \in \Omega(x, y)} \theta(\omega, \mathbf{n}) q(\omega)=1. \tag{9.4}$$

As a corollary we have the following random-walk representation of the correlation function.

Proposition 9.2. *In a finite system with a ferromagnetic pair interaction*

$$\langle\sigma_x \sigma_y\rangle=\sum_{\omega \in \Omega(x, y)} g(\omega) \tag{9.5}$$

with $g(\omega)=q(\omega) s(\omega) z(B(\omega))$; $s(\omega)$ *and* $z(B)$ *being defined for each step sequence* ω *and a set of bonds* B, *as*

$$s(\omega)=\prod_{b \in B_t(\omega)} \tanh(\beta J_b) \tag{9.6}$$

and

$$z(B)=\sum_{\substack{\partial\mathbf{n}=\emptyset \\ n_b \text{ is even } \forall b \in B}} w(\mathbf{n})/Z. \tag{9.7}$$

Proof. (9.5) is obtained by substituting the decomposition of identity (9.4) in (2.7), and noting that for each $\omega \in \Omega_{x, y}$

$$\sum_{\partial\mathbf{n}=\{x, y\}} \theta(\omega, \mathbf{n}) w(\mathbf{n})/Z=s(\omega) z(B(\omega)). \quad \square \tag{9.8}$$

To express a correlation function of a higher order, $\langle\sigma_{x_1} \ldots \sigma_{x_n}\rangle$ we consider a random walk which is generated by starting from x_1 and proceeding, as before, until the first hit of either of the sites $\{x_2, \ldots, x_4\}$. After that, a second walk is started at one of the remaining sources; specifically – the one with the lowest index (with respect to some arbitrary but fixed order of the sites). This construction leads to the following definition and result.

Definition 9.2. i) Paths $\omega_1, \ldots, \omega_n$ are *compatible* (in that order) if for any $1 \leqq i \neq j \leqq k$:

$$B_t(\omega_i) \cap B(\omega_j) = \emptyset \tag{9.9}$$

and if, for each k, the end points of $\omega_k, \ldots, \omega_n$ are not visited by $\omega_1, \ldots, \omega_{k-1}$.

ii) The *composition* of compatible paths is defined by composing the sequences of steps, and is denoted by $\omega_1 \circ \omega_2 \circ \ldots$.

iii) The composition of an ordered collection of sets of paths $\Omega_1, \ldots, \Omega_k$ is

$$\Omega_1 \circ \ldots \circ \Omega_k = \{\omega_1 \circ \ldots \circ \omega_k | \{\omega_1, \ldots, \omega_k\} \text{ are compatible and } \omega_j \in \Omega_j\}.$$

Proposition 9.3. *In a finite system with a ferromagnetic pair interaction*

$$\langle \sigma_{x_1} \ldots \sigma_{x_{2n}} \rangle = \sum_{T \in \mathscr{P}(\{1, \ldots, 2n\})} \sum_{\omega \in \Omega_1(T) \circ \ldots \circ \Omega_n(T)} g(\omega), \tag{9.10}$$

where $\mathscr{P}(\{1, \ldots, 2n\})$ is the collection of pairings of $\{1, \ldots, 2n\}$. Each pairing is represented by a permutation, T, such that for all $1 \leqq k < l \leqq n$: $Tk < Tl$ and $T(k+n) > T(k)$ (e.g. $Tk \equiv k$) – the k^{th} pair being $\{Tk, T(k+n)\}$. Correspondingly,

$$\Omega_k(T) = \Omega(x_{Tk}, x_{T(k+n)})$$

is the collection of paths which link the k^{th} pair of sites, starting at x_{Tk}.

The following lemma expresses another useful property. Since it is an identity, we again leave the proof to the reader.

Lemma 9.2. *In a finite system, for any step sequence ω_1 – which does not visit $\{x, y\}$,*

$$\sum_{\omega \in \{\omega_1\} \circ \Omega(x, y)} g(\omega) = g(\omega_1) \langle \sigma_x \sigma_y \rangle'_{\omega_1}, \tag{9.11}$$

where $\langle \sigma_x \sigma_y \rangle'_{\omega_1}$ is the correlation in a system obtained by setting the interaction to zero on all the bonds in $B(\omega_1)$. [Hint: It is useful to *reduce* the paths in $\Omega(x, y)$ by removing (via partial summation) the attempts along the bonds of $B_a(\omega_1)$.]

Remark. The substitution of (9.12) in (9.11) leads to yet another derivation of the *Lebowitz inequality* (using the Griffiths inequality: $\langle \sigma\sigma \rangle' \leqq \langle \sigma\sigma \rangle$).

A key ingredient for the proof of Proposition 9.1 is the following lemma.

Lemma 9.3. *Let ω_1, ω_2 be a pair of compatible step sequences, then*

i) $$s(\omega_1 \circ \omega_2) = s(\omega_1) s(\omega_2),$$

ii) $$q(\omega_1 \circ \omega_2) \geqq q(\omega_1) q(\omega_2).$$

Furthermore, for any two sets of bonds B_1, B_2

iii) $$Z(B_1 \cup B_2) \geqq Z(B_1) Z(B_2),$$

iv) *and if $B_1 \subset B_2$, then*

$$Z(B_1) \geqq Z(B_2).$$

In particular:

$$g(\omega_1 \circ \omega_2) \geqq g(\omega_1) g(\omega_2). \tag{9.12}$$

Proof. i) is obvious, and ii) follows by the observation that for $q(\omega_1 \circ \omega_2)$ each factor in (9.3) is not smaller than the corresponding factor in $q(\omega_1)q(\omega_2)$. In fact, we have an equality unless the sets of sites visited by the two walks intersect.

Next, we argue that iv) is valid as a consequence of the fact that increasing B only reduces the collection of terms which contribute in (9.7).

In view of iv), to prove iii) it suffices to demonstrate it for *disjoint* sets. Let us denote

$$K_i = \sum_{b=\{x,y\}\in B_i} J_b \sigma_x \sigma_y .$$

Then, assuming $B_1 \cap B_2 = \emptyset$,

$$\begin{aligned} z(B_1 \cup B_2) &= \prod_{b\in B_1\cup B_2} \cosh(\beta J_b) \frac{\operatorname{tr}\exp\left(\sum_{b\equiv\{x,y\}\notin B_1\cup B_2} J_b\sigma_x\sigma_y\right)}{\operatorname{tr}\exp\left(\sum_{b\equiv\{x,y\}} J_b\sigma_x\sigma_y\right)} \\ &= \prod_{b\in B_1\cup B_2} \cosh(\beta J_b) 1/\langle e^{K_1+K_2}\rangle'_{B_1\cup B_2}, \end{aligned} \tag{9.13}$$

where $\langle - \rangle'_{B_1\cup B_2}$ represents the Gibbs state obtained by setting the interaction to zero over all the bonds in $B_1 \cup B_2$. Similarly one derives:

$$z(B_1) = \prod_{b\in B_1} \cosh(\beta J_b) \langle e^{K_2}\rangle'_{B_1\cup B_2} / \langle e^{K_1+K_2}\rangle'_{B_1\cup B_2}$$

and a corresponding expression for $z(B_2)$.

Combining these expressions, we see that iii) is equivalent to the following statement.

$$\langle e^{K_1+K_2}\rangle' \geqq \langle e^{K_1}\rangle' \langle e^{K_2}\rangle' . \tag{9.14}$$

This however is true by the Griffiths inequality (which is applicable after the exponentials are expanded into power series). □

We shall now return to:

Proof of Proposition 9.1. i) *The lower bound.* Recalling that $\hat{\mathscr{C}}_{\mathbf{n}_1}(x_1, x_2) \subset \mathscr{C}_{\mathbf{n}_1}(x_1)$, we see that if

$$\hat{\mathscr{C}}_{\mathbf{n}_1}(x_1, x_2) \cap \hat{\mathscr{C}}_{\mathbf{n}_2}(x_3, x_4) \neq \emptyset$$

then the four sources are necessarily connected, i.e. in the same cluster – $\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)$. Therefore Proposition 5.1 [specifically, Eq. (5.2)] implies that, up to the factor 2, *each* of the three terms of V and hence also their mean, is a lower bound for $|U_4|$.

ii) *The upper bound.* By the Lebowitz inequality [see (5.3) and a remark in this section], we know that $|U_4| = -U_4$. We seek therefore a *lower* bound on $\langle \sigma_{x_1} \dots \sigma_{x_4} \rangle$. Such a bound is obtained by an application of Lemma 9.2 in the representation provided by Proposition 9.3.

Explicitly, using (9.10), (9.12) and then (9.5)

$$\begin{aligned}\langle\sigma_{x_1}\ldots\sigma_{x_4}\rangle &= \sum_{T\in\mathscr{P}(\{1,\ldots,4\})}\ \sum_{(\omega_1,\omega_1)\in\Omega_1(T)\circ\Omega_2(T)} g(\omega_1\circ\omega_2)\\ &\geqq \sum_{T\in\mathscr{P}(\{1,\ldots,4\})}\ \sum_{\substack{(\omega_1,\omega_2)\in\Omega_1\times\Omega_2\\ \textit{compatible}\text{ pairs}}} g(\omega_1)g(\omega_2)\\ &=\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle+2\text{ permutations}\\ &\quad-\sum_{T\in\mathscr{P}(\{1,\ldots,4\})}\ \sum_{\substack{(\omega_1,\omega_2)\in\Omega_1\times\Omega_2\\ \textit{un}\text{compatible}}} g(\omega_1)g(\omega_2)\,. \end{aligned} \tag{9.15}$$

Thus

$$\begin{aligned}(0\leqq)-U_4(x_1,\ldots,x_4) &\leqq \sum_{T\in\mathscr{P}(\{1,\ldots,4\})}\ \sum_{\substack{(\omega_1,\omega_2)\in\Omega_1(T)\times\Omega_2(T)\\ \textit{un}\text{compatible}}} g(\omega_1)g(\omega_2)\\ &\leqq V(x_1,\ldots,x_4)\,. \end{aligned} \tag{9.16}$$

The last inequality follows from the simple observation that a *necessary* condition for the *un*compatibility of two paths, is that the corresponding walks visit a common site. The total sum over *such* paths is exactly $V(x_1,\ldots,x_4)$, as follows from the discussion preceding Proposition 9.2. □

Thus, some of the results discussed earlier can be derived without the use of the identity expressed by Lemma 3.2 – using instead a random walk representation. A related expansion was used by Fisher [19] to derive bounds on the critical temperature. For us, the key property is the fact that the interaction of the random paths is *attractive* as long as the paths do not intersect. Similar arguments were used by Fröhlich [3], in the context of a different expansion. Here we enjoy the advantage of the availability of the two methods, due to which we obtain in Proposition 9.1 both *upper* and *lower* bounds. We shall use this combination in Sect. 12.

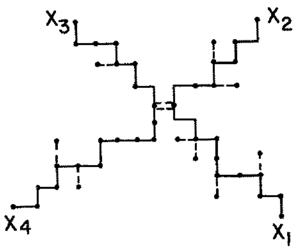

Fig. 7. Random paths which contribute to $\langle\sigma_{x_1}\ldots\sigma_{x_4}\rangle$

Part II. ϕ^4 Fields

10. The ϕ^4 Field Theory as a System of Block Spins

The ϕ_d^4 field theory is described by its Schwinger functions,

$$S_n(x_1,\ldots,x_n)=\langle\phi(x_1)\ldots\phi(x_n)\rangle\,, \tag{10.1}$$

which are (or would have been) averages with respect to the formal measure

$$\prod_{x\in\mathbb{R}^d} d\phi(x)\exp\left[-\int\left(\frac{\tilde{A}}{2}|\nabla\phi(x)|^2+\tilde{B}\phi^2(x)+\frac{\tilde{\lambda}}{4!}\phi^4(x)\right)dx\right]/\mathrm{Norm}. \qquad (10.2)$$

(See [19–21] for a preliminary discussion of the structure of the corresponding measure space.)

In order to make sense of (10.2) it is natural to consider its lattice approximations, obtained by partitioning the continuum to cubic cells whose centers form the lattice $\mathbb{L}=a\mathbb{Z}^d$, $a\to 0$, and replacing the field $\phi(x)$ by variables ϕ_x, $x\in\mathbb{L}$, associated with the lattice sites. Substituting the corresponding "Riemann sum" for the action integral in (10.2), one obtains a well defined system of lattice variables, whose correlations (in the infinite-volume limit) define the Schwinger functions

$$S_n^{(a)}(x_1,\dots,x_n)=\langle\phi_{[x_1]_a}\dots\phi_{[x_n]_a}\rangle, \qquad (10.3)$$

$[x]_a$ being the lattice site closest to x.

The question now is whether there is a method of adjusting the bare parameters $\tilde{A}$, $\lambda_0\equiv\tilde{\lambda}/\tilde{A}^2$, $\tilde{B}$, as $a\to 0$, so that the Schwinger functions (regarded as densities of measures) attain a limit which describes an interacting field, i.e. one which is not Gaussian. (An analytic continuation would then lead to Wightman functions of a quantum field theory.)

For $d=2,3$ such a construction was carried out, see [19, 21], and references therein. (However some issues, related to the very strongly coupled theory in $d=3$ dimensions, have not yet been clarified.)

For $d>4$ we answer the above question in negative, for limits derived from the single-phase regime.

Proposition 10.1. *In $d>4$ dimensions any limit obtained by the above procedure, in which $S_2^{(a)}(0,x)\xrightarrow[|x|\to\infty]{}0$ for every $a>0$, and $S_2(0,x)\equiv\lim_{a\to 0}S_2^{(a)}(0,x)$ is locally integrable, inevitably describes a Gaussian field.*

The proof, and a discussion, of this result is given in Sect. 13 – *which is directly accessible.* In order to set the stage for the proof of the key inequalities which are used there, we shall now describe a relation of the lattice ϕ^4 field with a system of Ising spins.

Explicitly, the above lattice action is

$$\mathscr{A}=-\frac{1}{2}\sum_{x,y\in\mathbb{L}}J_{x,y}|\phi_x\phi_y+\sum_{x\in\mathbb{L}}\left(\hat{B}\phi_x^2+\frac{\hat{\lambda}}{4!}\phi_x^4\right), \qquad (10.4)$$

with

$$J=\tilde{A}a^{d-2}, \quad \hat{B}=\tilde{B}a^d+2d\tilde{A}a^{d-2} \quad \text{and} \quad \hat{\lambda}=\tilde{\lambda}a^d.$$

Thus the ϕ^4 lattice system is a collection of variables with the continuous *a priori* distribution

$$\varrho_0(d\phi_x)=\exp\left[-\left(\hat{B}\phi_x^2+\frac{\hat{\lambda}}{4!}\phi_x^4\right)\right]d\phi_x/\mathrm{Norm} \qquad (10.5)$$

and a pair interaction similar to (2.1). The Ising model may be recovered from the above system in the strong coupling limit: choosing $\hat{B}$ so that

$$\varrho_0(d\phi_x)=\exp\left[-\frac{\hat{\lambda}}{4!}(\phi_x^2-1)^2\right]d\phi_x/\text{Norm},$$

and letting $\hat{\lambda}\to\infty$. However, there is also a converse relation. This is based on the Simon and Griffiths [22] representation of the *a priori* measure (10.5) as the limiting distribution, for $N\to\infty$, of the "block-spin" variable

$$\phi_x=(2N/\hat{\lambda})^{1/4}N^{-1}\sum_{\alpha=1}^{N}\sigma_x^{(\alpha)}, \tag{10.6}$$

where $\sigma_x^{(\alpha)}$ are Ising spins with the mean-field Hamiltonian

$$H_x=-[1-\hat{B}(\hat{\lambda}N/2^3)^{-1/2}](2N)^{-1}\sum_{\alpha,\delta=1}^{N}\sigma_x^{(\alpha)}\sigma_x^{(\delta)}. \tag{10.7}$$

Our strategy is to use the representation (10.6) to derive bounds on the Schwinger functions in which N does not appear, and then let $N\to\infty$.

We shall not refer to any of the details in (10.7), except for the basic fact that one may view the ϕ^4 lattice field as describing the "block spins" of an underlying system of Ising spins, which are organized into blocks with a *ferromagnetic* interaction which is independent of the intrablock parameter (α) (see Fig. 8). We shall therefore always assume that N is so large that $[1-\hat{B}(\hat{\lambda}N/2^3)^{-1/2}]\geqq 0$.

The above representation is of course vaguely reminiscent of the deeper relation of Ising systems (near T_c) to a ϕ^4 field theory, which is *postulated* in the Landau-Ginzburg theory of phase transitions.

We shall ignore here the question of the infinite-volume limit. One may use standard arguments to show that for the lattice field this limit exists, at any fixed bare parameters with $\hat{\lambda}$, $J>0$, and is continuously approached as $N\to\infty$. Alternative organization of the argument is to say that for finite systems our derivation would be complete, and the resulting bounds hold in any limit at which the Schwinger functions are continuous (as functions of the domain).

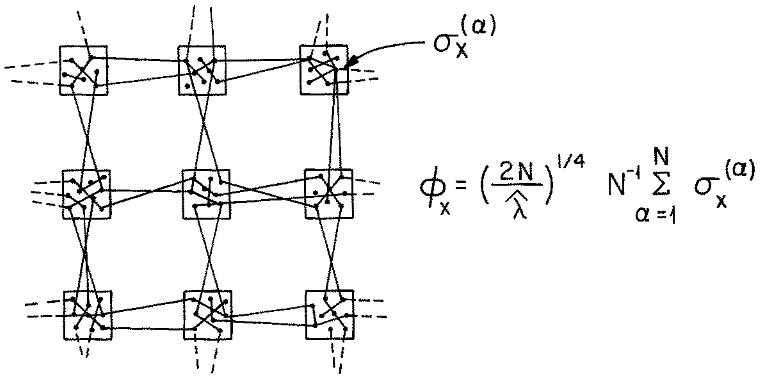

Fig. 8. A system of block spins which generates the ϕ^4 field

11. An Upper Bound on $|U_4|$ for ϕ^4 Fields

Extending (5.1) let us denote

$$U_4^{(a)}(x_1, \dots, x_4) = S_4^{(a)}(x_1, \dots, x_4) - [S_2^{(a)}(x_1, x_2) S_2^{(a)}(x_3, x_4) + S_2^{(a)}(x_1, x_3) S_2^{(a)}(x_2, x_4) + S_2^{(a)}(x_1, x_4) S_2^{(a)}(x_2, x_3)], \tag{11.1}$$

with the function $S_2^{(a)}$ defined in Sect. 10.

The results of this section are valid for any lattice approximation, however their most interesting implications are about the continuum limit. We therefore use the continuum notation in the following statement of the main result.

Proposition 11.1. *For any lattice* ϕ^4 *field, with* $\tilde{A}, \tilde{\lambda} > 0$ *and the lattice spacing* a,

$$0 \leqq -U_4^{(a)}(x_1, \dots, x_4) \leqq G \int d^d y S_2^{(a)}(x_1, y) S_2^{(a)}(x_2, y) S_2^{(a)}(x_3, y) S_2^{(a)}(x_4, y) \tag{11.2}$$

with

$$G = 3(2d)^2 \tilde{A}^2 a^{d-4} [1 - \exp(-C_d \lambda_0 a^{4-d})] e^{2ma} \quad (\leqq C' e^{2ma} \tilde{A}^2 \lambda_0), \tag{11.3}$$

where

$$\lambda_0 = \tilde{\lambda} / \tilde{A}^2$$

and

$$C_d = \left[\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} d^d p \frac{1}{\sum_1^d 4 \sin^2(p_i/2)} \right]^2 \Big/ 2 \underset{(d>2)}{<} \infty .$$

The quantity λ_0 is the (dimensionless) *bare coupling constant* – in a common field theoretical nomenclature. The *mass* m is measured in the units of the continuum, being defined by maximizing the constant μ for which

$$S_2^{(a)}(0, x) \leqq \text{const} \, e^{-\mu \|x\|_\infty}, \quad \text{as} \quad |x| \to \infty . \tag{11.4}$$

Lest one doubt the relevance of m, which as defined by (11.4) may seem to reflect only the properties of S_2 for hyperlarge x, let us point out that

$$S_2(0, x+y) \leqq S_2(0, x) e^{-m|y|}$$

for x, y directed along one of the principal axes (we return to this point in Sect. 13).

We now switch back to the lattice notation, which will be used until the final step in the proof of Proposition 11.1, at the end of this section. The major ingredient in that proof is the following correlation inequality, which extends Proposition 5.3.

Proposition 11.2. *For the lattice* ϕ^4 *system described by* (10.4)

$$0 \leqq -U_4(x_1, \dots, x_4) \leqq 3 \sum_{y \in \mathbb{L}} \langle \phi_{x_1} \phi_y \rangle \dots \langle \phi_{x_4} \phi_y \rangle (1 - e^{-\lambda \langle \phi^2 \rangle_0^2 / 2}) / \langle \phi^2 \rangle_0^2 , \tag{11.5}$$

where $\langle - \rangle_0$ *is the expectation value with respect to the (decoupled) measure* (10.5).

To prove it, we consider the system of Ising spins which was introduced in Sect. 10. The sites, which can be thought of as "microscopic" on the scale of the lattice $\mathbb{L}$, are now parametrized by $(x,\alpha)\in\mathbb{L}\times\{1,\dots,N\}$. Correspondingly, the criterion provided by Theorem 9.1 refers to the probability that two random trajectories $\hat{\mathscr{C}}_{\mathbf{n}_1}, \hat{\mathscr{C}}_{\mathbf{n}_2}$ not only visit the same block, but also intersect inside.

For $y\in\mathbb{L}$, let

$$\mathscr{B}_y=\{(y,\alpha)|1\leqq\alpha\leqq N\}$$

denote the y^{th} block of sites. We then have the following bound on the probability of it being visited by a random path.

Lemma 11.1.

$$\begin{aligned}&\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}}((x,\alpha),(z,\alpha'))\cap\mathscr{B}_y\neq\emptyset\,|\,\{(x,\alpha),(z,\alpha')\})\\&\leqq\langle\phi_x\phi_y\rangle\langle\phi_y\phi_z\rangle[\langle\phi_x\phi_z\rangle\langle\phi^2\rangle_0]^{-1}\left[\sum_{\delta,\gamma=1}^{N}\langle\sigma_x^{(\delta)}\sigma_z^{(\gamma)}\rangle/(N^2\langle\sigma_x^{(\alpha)}\sigma_z^{(\alpha')}\rangle)\right],\end{aligned}\tag{11.6}$$

$\langle - \rangle_0$ *being defined as in* (11.5).

Remark. The last factor on the right hand side of (11.6) is 1 if $x\neq z$, due to the local symmetry of the interaction. For large N it is not significant even when $x=z$, since by (10.6) (and the Griffiths inequality)

$$\begin{aligned}&\sum_{\delta,\gamma=1}^{N}\langle\sigma_x^{(\delta)}\sigma_x^{(\gamma)}\rangle/(\mathrm{N}^2\langle\sigma_x^{(\alpha)}\sigma_x^{(\alpha')}\rangle)\\&\qquad\leqq 1+(\sqrt{N\lambda/2}\langle\phi^2\rangle_0-1)^{-1}=1+O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}\tag{11.7}$$

The last term is $O\left(\frac{1}{\sqrt{N}}\right)$ for *fixed bare parameters*. When handled explicitly, the above factor is typically *canceled* in the bounds on U_4. However we lose nothing by regarding it as $1+O\left(\frac{1}{\sqrt{N}}\right)$.

Proof. Let us assume that $y\notin\{x,z\}$, for otherwise (11.6) is trivial, and consider the following ratio

$$R=\mathrm{Prob}(\hat{\mathscr{C}}_n((x,\alpha),(z,\alpha'))\cap\mathscr{B}_y\neq\emptyset\,|\{(x,\alpha),(z,\alpha')\})\langle\sigma_x^{(\alpha)}\sigma_z^{(\alpha')}\rangle/\langle\sigma_x^{(\alpha)}\sigma_y^{(1)}\rangle.$$

As explained in Sect. 9,

$$R=\sum_{\substack{\omega\in\Omega((x,\alpha),(z,\alpha'))\\ \hat{\mathscr{C}}(\omega)\cap\mathscr{B}_y\neq\emptyset}}g(\omega)\Big/\sum_{\omega\in\Omega((x,\alpha),(y,1))}g(\omega),\tag{11.8}$$

where

$$\hat{\mathscr{C}}(\omega)=\bigcup_{b\in B_t(\omega)}b$$

is the set of sites *visited* by the corresponding random walk. We claim that

$$R\leqq\langle\sigma_y^{(1)}\sigma_z^{(\alpha')}\rangle/\langle\sigma_y^{(2)}\sigma_y^{(1)}\rangle_0.\tag{11.9}$$

To prove (11.9), let ω_1 denote the part of the path ω obtained by stopping it at the *first time* $\mathscr{B}_y$ is hit. More specifically, ω_1 consists of the step sequence interrupted after reaching the first *traversed* bond which intersects $\mathscr{B}_y$.

It is easy to see that, regarding ω_1 as a function of ω, for each admissible $\hat{\omega}$

$$\sum_{\substack{\omega\in\Omega((x,\alpha),(z,\alpha'))\\ \mathscr{C}(\omega)\cap\mathscr{B}_y\neq\emptyset}} \delta_{\omega_1,\hat{\omega}}g(\omega)=g(\hat{\omega})\langle\sigma_y^{(\delta)}\sigma_z^{(\alpha')}\rangle_{\hat{\omega}}$$

and

$$\sum_{\omega\in\Omega((x,\alpha),(y,1))} \delta_{\omega_1,\hat{\omega}}g(\omega)=g(\hat{\omega})\langle\sigma_y^{(\delta)}\sigma_y^{(1)}\rangle'_{\hat{\omega}}, \tag{11.10}$$

where we denote by (y,δ) the site in $\mathscr{B}_y$ which is hit by $\hat{\omega}$, and by $\langle - \rangle'_{\hat{\omega}}$ the state obtained by setting J to zero on the bonds of $\mathscr{B}(\hat{\omega})$.

Furthermore, we have the following consequences of the Griffiths inequality

$$\begin{aligned}&\langle\sigma_y^{(\delta)}\sigma_z^{(\alpha')}\rangle'_{\hat{\omega}}\leqq\langle\sigma_y^{(\delta)}\sigma_z^{(\alpha')}\rangle=\langle\sigma_y^{(1)}\sigma_z^{(\alpha')}\rangle,\\ &\langle\sigma_y^{(\delta)}\sigma_y^{(1)}\rangle'_{\hat{\omega}}\geqq\langle\sigma_y^{(\delta)}\sigma_y^{(1)}\rangle_0\geqq\langle\sigma_y^{(2)}\sigma_y^{(1)}\rangle_0,\end{aligned} \tag{11.11}$$

The last expression in each line is justified by the local symmetry of the interaction. Equations (11.10) and (11.11) prove (11.9), since R is shown to be a ratio of two sums whose terms can be matched so that the bound is obeyed pairwise.

Inequality (11.9) may be restated as follows:

$$\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}}\cap\mathscr{B}_y\neq\emptyset)\leqq\langle\sigma_x^{(\alpha)}\sigma_y^{(1)}\rangle\langle\sigma_y^{(1)}\sigma_z^{(\alpha')}\rangle/[\langle\sigma_x^{(\alpha)}\sigma_z^{(\alpha')}\rangle\langle\sigma_y^{(1)}\sigma_y^{(2)}\rangle_0]\,. \tag{11.12}$$

The symmetry in the inter-block parameter, and the homogeneity of the right hand side of (11.17) almost permit to substitute there the fields ϕ for the spins σ, by the relation (10.6). However some care has to be exercised, since

$$\langle\sigma_y^{(\delta)}\sigma_y^{(\gamma)}\rangle_0\neq\langle\sigma_y^{(1)}\sigma_y^{(2)}\rangle_0 \quad \text{if} \quad \delta=\gamma\,.$$

Nevertheless, a slightly more attentive analysis of the above argument shows that the replacement of σ_y by ϕ_y is justified back at (11.9). The improved inequality is equivalent to (11.6). □

Actually, we need to estimate the probability that the two random paths not only hit a given block, but that they truly intersect inside. For this we shall use the following result on the probability of intersection of random sets.

Lemma 11.2. *Assume that for each integer N we are given a probability distribution of pairs of subsets $A_N, D_N\subset\{1,\dots,N\}$, for which the two sets are independent, and which is invariant under the permutations of $\{1,\dots,N\}$. If*

$$\begin{aligned}&\mathrm{Prob}(A_N\neq\emptyset)\leqq a_0\\ &\mathrm{Prob}(D_N\neq\emptyset)\leqq d_0,\end{aligned} \tag{11.13}$$

and if the expected values of the sets' sizes obey

$$(\langle|A_N|\rangle/\sqrt{N},\langle|D_N|\rangle/\sqrt{N})\leqq(\bar{a},\bar{d}), \tag{11.14}$$

componentwise with $\bar{a},\bar{d}<\infty$, then

$$\limsup_{N\to\infty}\mathrm{Prob}(\{A_N\cap D_N\neq\emptyset\})\leqq a_0d_0[1-e^{-\bar{a}\bar{d}/(a_0d_0)}]\,. \tag{11.15}$$

Proof. For specified values of $|A_N|$, $|D_N|$ the probability distribution of A_N, D_N is still invariant under independent permutations. If $|A_N|+|D_N|\leqq N$, simple combinatorics yield

$$\mathrm{Prob}(A_N\cap D_N=\emptyset \,|\, |A_N|,|D_N|)=\frac{(N-|A_N|)!\,(N-|D_N|)!}{(N-|A_N|-|D_N|)!\,N!}\equiv f_N(a_N,d_N), \quad (11.16)$$

where

$$(a_N,d_N)=\left(\frac{|A_N|}{\sqrt{N}},\frac{|D_N|}{\sqrt{N}}\right).$$

Using the Stirling formula,

$$M!=e^{M(\ln M-1)}\sqrt{2\pi M}\left(1+O\left(\frac{1}{M}\right)\right),$$

it is easy to see that for each $a,d<\infty$

$$f_N(a,d)\to e^{-ad}. \quad (11.17)$$

The convergence is locally uniform, in the sense that

$$\lim_{N\to\infty}\sup_{a,d\leqq\hat{a}}\{|f_N(a,d)-e^{-ad}|\}=0$$

for each $\hat{a}<\infty$.

The random distributions of A_N, D_N induce probability distributions for the variables a_N, d_N. For each $\bar{a},\bar{d}<\infty$, the space of probability measures on $\mathbb{R}^2_+$ which satisfy (11.14) is compact in the weak topology. Therefore the uniform boundedness of f_N ($\leqq 1$), and its local convergence, imply that

$$\lim_{N\to\infty}|\mathrm{Prob}(A_N\cap D_N=\emptyset)-\langle e^{-a_Nd_N}\rangle|=0. \quad (11.18)$$

We need therefore an upper bound on $\langle 1-e^{-a_Nd_N}\rangle$. To derive it, let X be the function

$$X(a)=\begin{cases}1 & a\neq 0,\\ 0 & a=0.\end{cases}$$

Then

$$\langle 1-e^{-a_Nd_N}\rangle=\langle X(a_Nd_N)\rangle\frac{\langle X(a_Nd_N)(1-e^{-a_Nd_N})\rangle}{\langle X(a_Nd_N)\rangle},$$

the last ratio representing the normalized average conditioned on $a_Nd_N\neq 0$. Applying the Jensen inequality to *this* average, we obtain

$$\langle 1-e^{-a_Nd_N}\rangle\leqq\langle X(a_Nd_N)\rangle\{1-\exp[-\langle a_Nd_N\rangle/\langle X(a_Nd_N)\rangle]\}. \quad (11.19)$$

The function

$$f(x,y)=x(1-e^{-y/x})$$

is monotone increasing in each component. Applying this monotonicity, and the bounds (11.13) and (11.14), to (11.19) we obtain

$$\langle 1-e^{-a_Nd_N}\rangle\leqq a_0d_0[1-e^{-\bar{a}\bar{d}/(a_0d_0)}].$$

The assertion (11.15) follows now by combining the last inequality with (11.18). □

We are now ready to prove the correlation inequality (11.5).

Proof of Proposition 11.2. We shall use the setup explained after the statement of the proposition. To shorten the notation we denote $\mathbf{x}_i=(x_i,\alpha_i)$. By Proposition 9.1, and (10.6),

$$0\leqq -U_4(x_1,\ldots,x_4)\leqq \limsup_{N\to\infty}\langle\phi_{x_1}\phi_{x_2}\rangle\langle\phi_{x_3}\phi_{x_4}\rangle\frac{1}{N^4}\sum_{\alpha_1,\ldots,\alpha_4=1}^{N}$$
$$\cdot\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}(\mathbf{x}_1,\mathbf{x}_2)\cap\hat{\mathscr{C}}_{\mathbf{n}_2}(\mathbf{x}_3,\mathbf{x}_4)\neq\phi\,|\,\{\mathbf{x}_1,\mathbf{x}_2\},\{\mathbf{x}_3,\mathbf{x}_4\})$$
$$+2\text{ permutations}. \tag{11.20}$$

Restating the principle which was used in Proposition 5.3 we have, with the obvious abbreviations,

$$\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}\cap\hat{\mathscr{C}}_{\mathbf{n}_2}\neq\phi)\leqq\sum_y\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}\cap\hat{\mathscr{C}}_{\mathbf{n}_2}\cap\mathscr{B}_y\neq\phi). \tag{11.21}$$

To bound a term in the last sum [with the source assignment spelled in (11.20)], let $A_N=\hat{\mathscr{C}}_{\mathbf{n}_1}\cap\mathscr{B}_y$ and $D_N=\hat{\mathscr{C}}_{\mathbf{n}_2}\cap\mathscr{B}_y$. We are now in the situation which was assumed in Lemma 11.2. Furthermore, Lemma 11.1, and (11.7), provide us with the values

$$a_{\text{o}}=\frac{\langle\phi_{x_1}\phi_y\rangle\langle\phi_{x_2}\phi_y\rangle}{\langle\phi_{x_1}\phi_{x_2}\rangle\langle\phi^2\rangle_{\text{o}}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right),$$
$$d_{\text{o}}=\frac{\langle\phi_{x_3}\phi_y\rangle\langle\phi_{x_4}\phi_y\rangle}{\langle\phi_{x_3}\phi_{x_4}\rangle\langle\phi^2\rangle_{\text{o}}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right). \tag{11.22}$$

For $\bar{a}$ we use the following bound

$$\begin{aligned}E(|\hat{\mathscr{C}}_{\mathbf{n}_1}\cap\mathscr{B}_y|\,|\{\mathbf{x}_1,\mathbf{x}_2\})&=\sum_{\alpha=1}^{N}\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}\ni(y,\alpha)|\{\mathbf{x}_1,\mathbf{x}_2\})\\&\leqq\sum_{\alpha=1}^{N}\mathrm{Prob}(\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)\ni(y,\alpha)|\{\mathbf{x}_1,\mathbf{x}_2\})\\&=\sum_{\alpha=1}^{N}\frac{\langle\sigma_{x_1}^{(\alpha_1)}\sigma_y^{(\alpha)}\rangle\langle\sigma_{x_2}^{(\alpha_2)}\sigma_y^{(\alpha)}\rangle}{\langle\sigma_{x_1}^{(\alpha_1)}\sigma_{x_2}^{(\alpha_2)}\rangle}\\&=\sqrt{N\hat{\lambda}/2}\,\frac{\langle\phi_{x_1}\phi_y\rangle\langle\phi_{x_2}\phi_y\rangle}{\langle\phi_{x_1}\phi_{x_2}\rangle}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\\&\equiv\sqrt{N}\,\bar{a}\end{aligned} \tag{11.23}$$

[the inequality could have also been justified by (11.10)]. We used here (3.3) and (10.6), and the $O\left(\frac{1}{\sqrt{N}}\right)$ is explained in conjunction with (11.7). A similar expression is obtained for $\bar{d}$, by replacing (x_1,x_2) in (11.23) by (x_3,x_4).

We can now invoke Lemma 11.2, which implies that

$$\limsup_{\substack{N\to\infty\\ \alpha_1,\ldots,\alpha_4\leqq N}}\mathrm{Prob}(\hat{\mathscr{C}}_{\mathbf{n}_1}\cap\hat{\mathscr{C}}_{\mathbf{n}_2}\cap\mathscr{B}_y\neq\phi|\{\mathbf{x}_1,\mathbf{x}_2\},\{\mathbf{x}_3,\mathbf{x}_4\})$$
$$\leqq\frac{\langle\phi_{x_1}\phi_y\rangle\ldots\langle\phi_{x_4}\phi_y\rangle}{\langle\phi_{x_1}\phi_{x_2}\rangle\langle\phi_{x_3}\phi_{x_4}\rangle\langle\phi^2\rangle_{\text{o}}^2}\,(1-e^{-\hat{\lambda}\langle\phi^2\rangle_{\text{o}}^2/2}). \tag{11.24}$$

The claimed (11.5) is obtained by the substitution of (11.24) in (11.20). The exchange of $\limsup_{N\to\infty}$ with the summation is justified by an implication of Lemma 11.1, which is that up to a constant factor the terms of the final sum dominate the summands for each N. The resulting contribution is the same for each of the three pairings of the sources. Hence the factor 3. □

Remark. Equation (11.5) is a significantly improved version of the inequality (13) claimed in [1]. However, the factor 3 was prematurely reported there as 2. With the factor 2 the inequality would have been a direct generalization of (5.8), which (as our formalism makes clear) is saturated under certain conditions.

We close this section by proving Proposition 11.1 which, essentially, is a recast of Proposition 11.2 in the continuum notation, with $\langle\phi^2\rangle_0$ replaced by quantities which may be more recognizable.

Proof of Proposition 11.1. Equation (11.5) acquires the form of (11.2) upon the substitution of $S^{(a)}(x,y)$ and $a^{-d}\int dy$ for $\langle\phi_x\phi_y\rangle$ and $\sum_y$ [with our convention that $S^a(x,y)$ is constant as y varies over a lattice cell]. However instead of G we obtain the following constant G':

$$G'=3a^{-d}[1-\exp^{(-\hat{\lambda}\langle\phi^2\rangle_0^2/2)}]/\langle\phi^2\rangle_0^2. \tag{11.25}$$

To relate G to G' we need both an upper and a lower bound on $J\langle\phi^2\rangle_0$ in the single-phase regime. The former is obtained by the Griffiths inequality and the "infrared bound" [9], which imply, assuming there is no long range order, that

$$\langle\phi^2\rangle_0\leqq\langle\phi^2\rangle\leqq J^{-1}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]}d^dp\left[\sum_{i=1}^{d}4\sin^2(p_i/2)\right]^{-1}. \tag{11.26}$$

[If there is long range order then (11.2) is trivially rapid.]

Thus the term in the exponent in G' may be replaced as follows

$$\hat{\lambda}\langle\phi^2\rangle_0^2/2\leqq\frac{\hat{\lambda}}{J^2}C_d=\frac{\tilde{\lambda}a^d}{\tilde{A}^2a^{2(d-2)}}C_d=C_d\lambda_0a^{4-d}, \tag{11.27}$$

with the notation defined in (10.4) and (11.3).

For the lower bound we recall the result of Sokal [23]:

$$2dJ\langle\phi^2\rangle_0\geqq e^{-ma}, \tag{11.28}$$

m being the mass defined on the scale of the continuum.

For completeness let us just remark that (11.28) is derived using the following consequence of Lebowitz inequality:

$$\frac{d}{dJ}S^{(a)}(x,y)\leqq\sum_{z,z'}S^{(a)}(x,z)\delta_{|z-z'|,a}S^{(a)}(z',y), \tag{11.29}$$

which implies the pointwise bound:

$$(0\leqq)S^{(a)}(x,y)\leqq G_J(x,y), \tag{11.30}$$

where $G_J(x, y)$ is the solution of the system

$$\begin{cases} \dfrac{d}{dJ} G_J(x,y) = \sum_{z,z'} G_J(x,z)\delta_{|z-z'|,a} G_J(z',y) \\ G_0(x,y) = \langle\phi^2\rangle_0 \delta_{x,y}. \end{cases} \tag{11.31}$$

This shows that

$$S^{(a)}(x,y) \leqq [-J\triangle + (\langle\phi^2\rangle_0^{-1} - 2dJ)]^{-1}, \tag{11.32}$$

where $\triangle(\approx a^2\Delta)$ is the lattice second-difference operator. Equation (11.28) is an easy consequence of (11.32).

Thus, by (11.28),

$$a^{-d}/\langle\phi^2\rangle_0^2 \leqq a^{-d}(2d)^2 J^2 e^{2ma} = (2d)^2 a^{d-4} \tilde{A}^2 e^{2ma}. \tag{11.33}$$

The substitution of (11.28) and (11.33) in (11.25) shows that $G' \leqq G$, which proves Proposition 11.1. □

12. Bounds for the Higher Correlation Functions

In the next section we shall demonstrate that under certain conditions, in particular when $d>4$, continuum limits of the ϕ^4 field theory are Gaussian, i.e. generalized free fields. This amounts to proving that *Wick identities* [(13.4), below] are obeyed by *all* the Schwinger functions S_n, $n=4,6,\ldots$ (in the even theory $S_{2k+1}\equiv 0$). For the systems at hand, it was shown in Newman [24] (using the "Lee-Yang property") that it suffices to prove that for *some* n the *fully truncated* correlation function vanishes, e.g. to show that $U_4\equiv 0$. We shall now derive some relations which manifest a similar equivalence in an *explicit* form.

We regard the "Gaussian component" of a Schwinger function to be given by:

$$G_{2n}(x_1,\ldots,x_{2n}) = \sum_{T\in\mathscr{P}(\{1,\ldots,2n\})} \prod_{k=1}^{n} S_2(x_{Tk}, x_{T(k+n)}),$$

where the sum is over *pairings*, as explained in (9.10). For Ising models, $S_2(x,y)$ is to be replaced by $\langle\sigma_x\sigma_y\rangle$.

Gaussian fields are characterized by the following *Wick identities*

$$S_{2n} \equiv G_{2n} \qquad n=2,3,\ldots. \tag{12.1}$$

For ϕ^4 fields and Ising systems, with the interactions discussed above, we have the *Newman inequality* [25]:

$$S_{2n}(x_1,\ldots,x_{2n}) \leqq G_{2n}(x_1,\ldots,x_{2n}) \tag{12.2}$$

of which the Lebowitz inequality is a particular case, since

$$U_4 \equiv S_4 - G_4.$$

We shall now obtain an estimate on the difference in (12.2).

Proposition 12.1. *In a* ϕ^4 *field theory described by* (9.4), *or a ferromagnetic Ising system* (12.1), *let*

$$R_{2n}(x_1, \ldots, x_{2n}) = \sum_{1 \leq j < k < l < m \leq n} |U_4(x_j, \ldots, x_m)| G_{2n-4}(\ldots, \underset{\wedge}{x_j}, \ldots, \underset{\wedge}{x_k}, \ldots, \underset{\wedge}{x_l}, \ldots, \underset{\wedge}{x_m}, \ldots),$$

where $\wedge$ *indicates an omitted site. Then*

$$\frac{2}{n(n-1)} R_{2n}(x_1, \ldots, x_{2n}) \leqq |S_{2n}(x_1, \ldots, x_{2n}) - G_{2n}(x_1, \ldots, x_{2n})| \leqq \tfrac{3}{2} R_{2n}(x_1, \ldots, x_{2n}). \tag{12.3}$$

Before proving this result let us state an obvious implication, part of which, as mentioned above, was already derived by very different means, [24].

Corollary 12.1. *If, in the continuum limit,* (12.1) *is obeyed for some n, e.g. if* $U_4 \equiv 0$, *then it is satisfied for all n, and the "theory" is Gaussian.*

Remark. For the completeness of the argument let us offer a simple rederivation of the *Newman inequality.*

The representation (10.6) makes it clear that it suffices to prove (12.2) for finite systems of Ising spins, with a general two point ferromagnetic interaction. Furthermore, we assume that the $2n$ points $x_1, \ldots, x_{2n}$ are all distinct. The cases with coincidental points are easily reduced to this situation. Using Lemma 3.2 we then obtain:

$$\begin{aligned}
&\sum_{k=1}^{n} S_2(x_1, x_k) S_{2n-2}(\underset{\wedge}{x_1}, \ldots, \underset{\wedge}{x_k}, \ldots) \\
&\quad = E(|\mathscr{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \cap \{x_2, \ldots, x_{2n}\}| \,|\, \{x_1, \ldots, x_{2n}\}, \emptyset)\, S_{2n}(x_1, \ldots, x_{2n}) \\
&\quad \geqq S_{2n}(x_1, \ldots, x_{2n}),
\end{aligned} \tag{12.4}$$

where the inequality is derived by the observation that x_1 is necessarily linked to at least one other source. Inequality (12.2) is obtained by a simple iteration of (12.4) (I am grateful to R. Graham for pointing this out to me). □

Proof of Proposition 12.1. Let us use the representation (10.6), making the same assumptions as in the proof contained in the last remark.

i) *The upper bound.* By Proposition 9.3, and Lemma 9.3, we have

$$\begin{aligned}
S_{2n}(x_1, \ldots, x_{2n}) &\geqq \sum_{T \in \mathscr{P}(\{1, \ldots, 2n\})} \; \sum_{\omega_1 \circ \ldots \circ \omega_n \in \Omega_1(T) \circ \ldots \circ \Omega_n(T)} g(\omega_1) \ldots g(\omega_n) \\
&= \sum_T \Bigg[\sum_{(\omega_1, \ldots, \omega_n) \in \Omega_1 \times \ldots \times \Omega_n} g(\omega_1) \ldots g(\omega_n) \\
&\qquad - \sum_{\substack{(\omega_1, \ldots, \omega_n) \in \Omega_1 \times \ldots \times \Omega_n \\ \text{not all compatible}}} g(\omega_1) \ldots g(\omega_n) \Bigg] \\
&\geqq G_{2n}(x_1, \ldots, x_{2n}) - \sum_{1 \leqq a < b \leqq n} \sum_T \sum_{\substack{(\omega_1, \ldots, \omega_n) \in \Omega_1(T) \times \ldots \times \Omega_n(T) \\ (\omega_a, \omega_b) \text{ not compatible}}} g(\omega_1) \ldots g(\omega_n).
\end{aligned} \tag{12.5}$$

The last inequality takes into account the fact that there may be more than a single pair of incompatible ω's.

Arranging the final expression in (12.5) in terms of the vertices of ω_a and ω_b one obtains:

$$(0\leqq)[G_{2n}-S_{2n}](x_1,\ldots,x_{2n}) \leqq \sum_{1\leqq j<k<l<m\leqq 2n}\ \sum_{T'\in\mathscr{P}(\{x_j,x_k,x_l,x_m\})}\ \sum_{(\omega_1,\omega_2)\in\Omega_1(T')\times\Omega_2(T')} g(\omega_1)g(\omega_2) \cdot G_{2n-4}(\ldots,\hat{x}_j,\ldots,\hat{x}_k,\ldots,\hat{x}_l,\ldots,\hat{x}_m,\ldots). \tag{12.6}$$

However, by the *lower* bound in Proposition 9.1 [i.e. in (9.2)], and the second inequality in (9.16),

$$\sum_{T'\in\mathscr{P}(\{x_j,x_k,x_l,x_m\})}\ \sum_{\substack{(\omega_1,\omega_2)\in\Omega_1(T')\times\Omega_2(T')\\ \text{not compatible}}} g(\omega_1)g(\omega_2)\leqq\tfrac{3}{2}|U_4(x_j,x_k,x_l,x_m)|. \tag{12.7}$$

Substituting (12.7) in (12.6) we obtain the *upper* bound of (12.3).

Remark. By a slightly more careful analysis one can also obtain the following upper bound on the "error" in (12.4):

$$(0\leqq)\sum_{k=1}^{n} S_2(x_1,x_k)S_{2n}(\hat{x}_1,\ldots,\hat{x}_k,\ldots)-S_{2n}(x_1,\ldots,x_{2n}) \leqq\tfrac{3}{2}\sum_{1=j<k<l<m\leqq 2n}|U_4(x_j,x_k,x_l,x_m)|S_{2n-4}(x_1,\ldots,x_k,\ldots,x_l,\ldots,x_m,\ldots). \tag{12.8}$$

The iteration of (12.8) leads to a slightly improved version of the upper bound in (12.3).

i) *The lower bound.* There is a *combinatorial* identity, regarding the pairings of $\{x_1,\ldots,x_{2n}\}$ $n\geqq 2$, which is simply expressed as follows

$$G_{2n}(x_1,\ldots,x_{2n})=\frac{2}{n(n-1)}\sum_{1\leqq j<k<l<m\leqq 2n} G_4(x_j,x_k,x_l,x_m) \cdot G_{2n-4}(\ldots,\hat{x}_j,\ldots,\hat{x}_k,\ldots,\hat{x}_l,\ldots,\hat{x}_m,\ldots). \tag{12.9}$$

The proof of (12.9) is obvious, once one observes the fact that the number of pairings of $\{x_1,\ldots,x_{2n}\}$ is $\prod_{k=1}^{n}(2n-2k+1)$.

Applying the same identity to the representation (9.10), we obtain, in the notation explained there,

$$S_{2n}(x_1,\ldots,x_{2n})=\frac{2}{n(n-1)}\sum_{1\leqq j<k<l<m\leqq 2n}\ \sum_{\substack{T'\in\mathscr{P}(\{j,k,l,m\})\\ T''\in\mathscr{P}(\{1,\ldots,\hat{j},\ldots,\hat{k},\ldots,\hat{l},\ldots,\hat{m},\ldots,2n\})}} \cdot \sum_{\omega_1\circ\ldots\circ\omega_n\in\Omega_1(T')\circ\Omega_2(T')\circ\Omega_1(T'')\circ\ldots\circ\Omega_{n-2}(T'')} g(\omega_1\circ\ldots\circ\omega_n). \tag{12.10}$$

By the argument of Lemma 9.2, for each $\omega_1 \circ \omega_2 \in \Omega_1(T') \circ \Omega_2(T')$

$$\sum_{T'' \in \mathscr{P}(\{1,\ldots,\hat{j},\ldots,\hat{k},\ldots,\hat{l},\ldots,\hat{m},\ldots,2n\})} \sum_{\substack{\omega \in \Omega_1(T'') \circ \ldots \circ \Omega_{n-2}(T'') \\ \text{compatible with } \omega_1 \circ \omega_2}} g(\omega_1 \circ \omega_2 \circ \omega)$$

$$= g(\omega_1 \circ \omega_2) S'_{2n-4}(\ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, \hat{x}_l, \ldots, \hat{x}_m, \ldots)_{\omega_1 \circ \omega_2} \leqq g(\omega_1 \circ \omega_2) S_{2n-4}, \quad (12.11)$$

where $S'(-)_{\omega_1 \circ \omega_2}$ represents the value of the multiple correlation in a system with reduced interactions, as in (9.11), and the last step is by the Griffiths inequality.

Substituting (12.11) in (12.10), and reapplying (9.10) and (12.2), we obtain the following simple inequality:

$$S_{2n}(x_1, \ldots, x_{2n}) \leqq \frac{2}{n(n-1)} \sum_{1 \leqq j < k < l < m \leqq 2n} S_4(x_j, x_k, x_l, x_m) \cdot G_{2n-4}(\ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, \hat{x}_l, \ldots, \hat{x}_m, \ldots). \quad (12.12)$$

The subtraction of (12.12) from (12.9) leads to the *lower* bound in (12.3). □

13. Proof of the Triviality of the (Single Phase) Continuum Limits of ϕ_d^4, for $d > 4$

Combining the results of the two preceding sections, we shall now prove the triviality of the class of limits described by Proposition 10.1

To repeat the procedure, we are considering Schwinger functions which are constructed as limits of infinite-volume lattice approximants:

$$S_n(x_1, \ldots, x_n) = \underset{\substack{a \to 0 \\ \tilde{A}, \tilde{B}, \tilde{\lambda}}}{w\text{-lim}}\, S_n^{(a)}(x_1, \ldots, x_n). \quad (13.1)$$

The "bare" parameters $\tilde{A}$, $\tilde{B}$, $\tilde{\lambda}$ are adjusted as $a \to 0$, the only restrictions being imposed are that for each a

$$S_2^{(a)}(0, x) \xrightarrow[|x| \to \infty]{} 0 \quad (13.2)$$

and the local integrability of $S_2(0, x)$:

$$\int_{|x| \leqq \varepsilon} dx S_2(0, x) < \infty \quad (13.3)$$

for some $\varepsilon > 0$.

Remark. For $d \geqq 2$, and any $a > 0$, the lattice fields have a phase transition, where $m \to 0$. In essence, the continuum limits are constructed by approaching the "critical manifold" in the parameter space, since it is there that correlation functions become significant for separations which are large on the scale of the lattice spacing (corresponding to finite distances in the continuum limit). For large values of $J(a)$, at fixed $\hat{B}(a)$, $\hat{\lambda}(a)$ [see (10.4)], the lattice fields exhibit long-range-order, due to a symmetry-breaking (of $\phi \to -\phi$). Expression (13.2) is a restriction on the bare parameters, confining them to the single-phase region. The study of the limits obtained by approaching the "critical manifold" from the symmetry-broken regime is a very interesting problem which would not be resolved here. The two classes of limits need not be the same.

As a background for our result, let us summarize some known features, first of the two point function (in the configuration space):

i) $S_2^{(a)}(0,x) \geqq 0$, and $S_2^{(a)}$ is decreasing as function of the coordinates of $x = \{x_1, \ldots, x_d\}$ (the Schrader-Messager-Miracle Sole inequality [26], see also Appendix II). The same is true in any coordinate system in which the lattice has a reflection-symmetry plane. As noted in [14], this implies that S_2 has the following regularity property:

$$S_2^{(a)}(0,x) \geqq S_2^{(a)}(0,y) \qquad (\geqq 0) \tag{13.4}$$

whenever $|x| \leqq |y| d^{-1}$.

ii) Although the "mass", m, was defined by the asymptotic decay of $S_2(0,x)$, being the gap in the spectrum of the "transfer-matrix" its presence has also local implications. Specifically, let $x_n = x + ny$, where x, y are two vectors oriented in the same direction along a principal axis. Then, by a reflection-positivity argument, $S_2^{(a)}(0,x_{n+1})/S_2^{(a)}(0,x_n)$ is non-decreasing in n. Combined with (11.4), this implies that

$$\frac{S_2^{(a)}(0,x+y)}{S_2^{(a)}(0,x)} \leqq e^{-m|y|} \tag{13.5}$$

$\left(\text{the inequality being saturated by } \lim_{x\to\infty}\right)$.

iii) The infrared bound [(3.12) – with β replaced by $\tilde{A}$], of Fröhlich et al. [9], has the following manifestation in the configuration space (Sokal [14])

$$S_2^{(a)}(0,x) \leqq \frac{C(d)}{\tilde{A}|x|^{d-2}}, \tag{13.6}$$

for any $a > 0$, assuming there is no long-range-order.

Thus we learn that if a limit (3.2) exists, in "any" weak sense (e.g. regarding S_2 as a density of a measure), and $S_2 \neq 0$ then

$$\limsup_{a\to 0} \tilde{A}(a) < \infty. \tag{13.7}$$

And if, furthermore, $S_2(0,x)$ is not infinite over a region of the form $\{x \mid |x| \leqq \varepsilon (\geqq 0)\}$, including here the case $S_2(0,x) = \text{const}\,\delta(x)$, then *necessarily* [by (13.5) and (13.4)]

$$\limsup_{a\to 0} m(a) \leqq m < \infty, \tag{13.8}$$

m being defined by $S_2(0,x)$.

Corresponding statements can be shown for limits formulated in the momentum space, where the spectral representation of $S_2(0,x)$ plays a fundamental role.

As for the n-point functions, S_n, their singularities, and zeroes, are controlled by those of the two point function; since pointwise

$$\prod_{k=1}^{n} [2(n-k)+1]^{-1} G_{2n} \leqq S_{2n} \leqq G_{2n}$$

(Griffiths and Newman inequalities).

We shall now turn our attention to the question of triviality of the limits.

Proof of Proposition 10.1. Let us assume that a field is constructed by the procedure explained above, (13.1) and (13.2) being satisfied. As previously proven by Newman [24], and shown explicitly by Proposition 12.1, the field is Gaussian if and only if $U_4 \equiv 0$.

A bound on U_4 is provided by Proposition 11.1, where a crucial term for our discussion is the factor a^{d-4} in G. Taking into account (13.7) and (13.8), we see that for dimensions $d>4$:

$$\lim_{a\to 0} G=0 \quad [\text{as } O(a^{d-4})]\,. \tag{13.9}$$

By compactness, and the local integrability (13.3), for any $R<\infty$

$$\int_{|y|\leq R} d^d y S_2^{(a)}(x,y)\ldots S_2^{(a)}(x_4,y) \tag{13.10}$$

is *continuous* as $a\to 0$, and the limit is finite (for non-coincidental points $x_1,\ldots,x_4$). Thus (13.9) almost proves that $U_4\equiv 0$; except that one still has to control the contribution from *large y*. However, choosing R large, the remainder can be made arbitrarily small, since for $R>R_0$:

$$\begin{aligned}&\limsup_{a\to 0} G \int_{|y|\geq R} d^d y S_2^{(a)}(x_1,y)\ldots S_2^{(a)}(x_4,y)\\ &\leq 3(2d)^2 K(R_0) \int_{|y|\geq R} d^d y \frac{C(d)^2}{|y-x_1|^{d-2}|y-x_2|^{d-2}} \xrightarrow[\binom{R\to\infty}{d>4}]{} 0\end{aligned} \tag{13.11}$$

[using (13.6)], where

$$K(R_0)=\limsup_{a\to 0}\sup\{S_2^{(a)}(0,x)\,|\,|x|>R_0\}<\infty$$

is finite by the local convergence of $S_2^{(a)}$, and the regularity condition (13.4). □

14. Some Comments on ϕ_4^4

The bounds on the interaction in ϕ_4^4 fields which were discussed in the previous sections yield only partial answers for the question of possible constructions of non-trivial ϕ_4^4 fields. The situation in four dimensions is very interesting both on physical and mathematical grounds: physically – since the Schwinger functions are a stepping stone for the construction of the Wightman functions of a scalar field theory in the Minkowski space-time, mathematically – since four seems to be a critical dimension. Nevertheless, even the above bounds do yield some information about ϕ_4^4.

A key argument in our proof of the triviality of the constructive limits for ϕ_d^4, $d>4$, was provided by the vanishing of the factor

$$G=3(2d)^2\tilde{A}^2 a^{d-4}[1-\exp(-C_d\lambda_0 a^{4-d})]e^{2ma} \tag{14.1}$$

which multiplies the "tree diagram" integral in (11.2). As we saw, in addition to this, one needs some control on the possible divergence of the integral itself, the only worrisome part being the contribution from large y. That problem does not present itself in limits for which

$$\liminf_{a\to 0} m(a) > 0, \tag{14.2}$$

due to (13.5) and (13.4). (One may also hope to resolve this issue by an entirely different argument – keeping in mind the fact that we are really bounding a *probability* of intersection.)

Returning to the prefactor, let us point out that

$$\lim_{a\to 0} G = 0 \tag{14.3}$$

under either one of the following three conditions, assuming the necessary bounds (13.7) and (13.8) are satisfied.

i) $$d > 4.$$

This case was discussed in the previous section.

ii) $$\lambda_0 \equiv \tilde{\lambda}/\tilde{A}^2 \to 0, \quad d > 2. \tag{14.4}$$

The proof is by an application of the inequality $1 - e^{-x} \leqq x$ in (14.1). (The restriction $d > 2$ is presumably just due to a removable technicality.) This result may sound as "Gaussian limits are Gaussian," since λ_0 is the "bare coupling constant" – $\lambda_0 = 0$ corresponding, for $a > 0$, to a manifestly Gaussian system [see (10.4)]. Nevertheless, the statement is not an explicit tautology, since it expresses independence of all the other adjustments of the bare parameters in the continuum limit (except for the constraints we have discussed).

iii) $$d = 4, \quad \tilde{A} \to 0. \tag{14.5}$$

In a work which extends the results reported in [1] to the two component ϕ^4 field theory Fröhlich [3] obtained, using independently derived arguments, a similar bound to (11.2) (except for the term involving λ_0), thus proving the conditions i) and iii) in that generality. As he pointed out in this context, the asymptoticity of the perturbation theory would have implied that the condition (14.5) is satisfied!

Therefore, there are now at least two possible directions, which if followed successfully could provide a key argument for the extension of Proposition 11.1 to four dimensions.

i) Show that $\tilde{A} \to 0$ is a necessary condition for the existence of the limit (13.1), if either the *bare* coupling constant, λ_0, or the *renormalized* coupling constant, g, are kept away from zero. This calls for an improvement (say in the "ultraviolet region") of the "infrared bound," or at least for a proof that

$$\lim_{|x|\to 0} S_2(0,x)|x|^2 = \infty \quad (d=4) \tag{14.6}$$

[see (13.6)].

ii) Improve the bound (11.2). I am reasonably convinced that this should be possible for $d = 4$ (drawing encouragement from the analysis in [27]).[2]

Let me end these comments by calling the attention again to the point made in Sect. 13, that even if the above routes can be followed, a very interesting question

2 An improvement which should be significant for $d=4$ is derived in [29]

would remain with respect to the possible construction of fields by limits of a symmetry-broken phase. Some very preliminary, non-perturbative, results can be found in the work of Graham [28].

15. More on Absolute Bounds on the Coupling

In this anticlimax we return to an observation made by Glimm and Jaffe [12] that renormalized coupling constants in a ϕ^4 field theory obey uniform upper bounds, which depend only on the dimension.

As a dimensionless measure of the physical interaction one may define the coupling constant(s)

$$g_{(\phi)} = |\overline{U_4}|/[\chi^2 \xi^d_{(\phi)}], \tag{15.1}$$

where

$$|\overline{U_4}| = \int dx_2 dx_3 dx_4 |U_4(x_1, \ldots, x_4)|$$
$$\chi = \int dx S_2(0, x)$$

and $\xi = m^{-1}$ is the "correlation length" which is alternatively expressed by

$$\xi_\phi = [\int dx S_2(0, x)|x|^\phi/\chi]^{1/\phi}.$$

For lattice approximations, and for Ising models, we extend the functions U_4 and S_2 to the continuum – the extension being constant on the lattice cells.

In addition to the Glimm and Jaffe [12] bound on g, g_ϕ were bounded by Schrader [17] – for $\phi > d$. This result was recently extended by Lieb and Sokal [18] to $\phi \geqq \frac{d}{2}$. We shall now present a simple derivation of a bound on $g_{d/2}$ and, in effect show that what seemed to be a difficult borderline case is perhaps the most natural one. Our special interest in $g_{d/2}$ derives, however, from the fact that in two dimensions we get a simple *lower* bound on g_1 (Sect. 8), we then need the upper bound for a proof of hyperscaling.

The only ingredient which is used for the following result is the inequality:

$$|U_4(x_1, \ldots, x_4)| \leqq 2S_2(x_1, x_2)S_2(x_3, x_4), \tag{15.2}$$

which by (5.3) is satisfied for general Ising models, and hence, by the Griffiths-Simon representation (10.6), for ϕ^4 fields (all in the single-phase regime).

Notice that the factor which is lost in (15.2) is the probability of intersection of the random currents – which is all the interaction in these models. Nevertheless, one still obtains the following bound.

Proposition 15.1. *For the ϕ^4 fields, and translation invariant Ising models with ferromagnetic pair interactions, in the single-phase regime*

$$g_{d/2} \equiv |\overline{U_4}|/[\int dx S_2(0, x)|x|^{d/2}]^2 \leqq 12\pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right) \tag{15.3}$$

(at the critical point, where $\chi = \infty$ and possibly $|\overline{U_4}| = \infty$, $g_{d/2}$ is defined by introducing factors $e^{-\varepsilon|x|}$, and letting $\varepsilon \to 0$).

Proof. As a consequence of the permutation symmetry of $U_4(x_1, \ldots, x_4)$ we have:

$$\frac{1}{4!}\int dx_2 dx_3 dx_4 |U_4(0, x_2, x_3, x_4)|$$
$$= \frac{1}{4}\int dxdydz|U_4(0,x,z,y+z)|X[|z| \leqq \min\{|x|,|y|,|y+z-x|,|x-z|,|y+z|\}]. \tag{15.4}$$

Equation (15.4) is apparent if one realizes that both sides represent the average, per unit volume, of $|U_4|$ evaluated at random quadruples of sites. The integral on the left hand side corresponds to randomly labeling the sites of each quadruple as $(0, x_2, x_3, x_4)$, whereas on the right hand side $\{0, z\}$ are reserved to denote the ends of the *shortest* segment, the other points being labeled by x and $z+y$.

Substituting (15.2) in (15.4) (and using the translation invariance) we obtain

$$|\overline{U_4}| \leqq \frac{2 \cdot 4!}{4} \int_{|z| \leqq \sqrt{|x||y|}} dxdydz S_2(0,x) S_2(0,y)$$
$$= \frac{12 \cdot \pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} [\int dx S_2(0,x)|x|^{d/2}]^2 \tag{15.5}$$

which proves (15.3) $\left[\pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right)\right.$ being the volume of the unit ball; the analogous bound for the expression obtained by replacing $|x|$ with $\|x\|_1$ is $\left.12 \cdot 2^d/d!\right]$. □

Remark. Equation (15.2) is slightly stronger than the bound used in [12, 17]. To recover a bound on g from the above result, one may use the inequality:

$$\xi \geqq \text{const}\, \xi_{d/2} \tag{15.6}$$

derived in [14] from the spectral representation of S_2.

Appendix I. The Criticality of $d=4$ Dimensions from a Geometrical Point of View

The lever which is applied in our study of the Ising models and ϕ^4 fields is a representation in which the correlations are being mediated by a system of random currents. Various features of the critical behavior in these models are directly related to the probabilities of intersection, under various circumstances, of currents which link widely separated sources. Faced with such a stochastic-geometric system, it is quite instructive to search for a simple setup in which similar problems can be formulated, and try to extract a general lesson from its solution.

An example which we shall consider here is provided by the paths of Brownian motion (the *Wiener process*). The motion can be viewed as the continuum limit of the random walk, on a lattice of spacing a with $a \to 0$. However it is possible, and is even quite advantageous, to define the process directly in the continuum. A key fact is the continuity of the motion, as function of the time, which allows one to ask for the probability of intersection of two independent paths. Although the paths are parametrized by $\mathbb{R}_+$ (the time) one should not be misled into thinking that their intersection properties are similar to those of simple polygonal lines. These, generically, do not intersect already in three dimensions. For Brownian motion the situation is described by the following celebrated result.

Proposition I.1. (Kakutani and Dvoretsky et al. [10].) *Let ω_i, $i=1,2$, be the trajectories of two independent Brownian motions in $\mathbb{R}^d$, which start at x_1 and, correspondingly, at x_2. Then, for any region $\Lambda \subset \mathbb{R}^d$,*

$$\mathrm{Prob}(\{\omega_1 \cap \omega_2 \cap \Lambda \neq \{x_1\} \cap \{x_2\}\}) \quad \text{is} \quad \begin{cases} =0 & d \geqq 4 \\ >0 & d<4. \end{cases} \tag{I.1}$$

(That is, the probability of having a point in common, except for the starting point if $x_1 = x_2$, vanishes if $d \geqq 4$.) For $\Lambda = \mathbb{R}^d$ the probability is one if $d<4$.

The critical dimension for *triple* intersection is $d=3$. To reconcile all this with a simple intuition one may regard the Brownian paths as ("almost") two dimensional (in fact, that is their Hausdorff dimension).

For $d>4$, the proof of Proposition I.1 is quite simple and instructive. The following is a variant of an argument shown to me by T. Spencer. The main input is the hitting probability – which may be viewed as an analog of the two point function.

Lemma I.1. *For any $y \in \mathbb{R}^d$, and $a>|y|$, the probability that a Brownian motion (in $\mathbb{R}^d$), which starts at 0, hits the ball of radius a centered at x, is*

$$\mathrm{Prob}(\{\mathrm{dist}(\omega, y) \leqq a\}) = \left(\frac{a}{|y|}\right)^{d-2}. \tag{I.2}$$

(The proof is easy, once one observes that the probability is necessarily a *harmonic* function of x.)

Proof of Proposition I.1, for $d>4$. Let us assume that Λ is compact, and at finite distance from x_1, x_2, the general case follows by the countable additivity. For a specified pair of paths ω_1, ω_2

$$V_a(\omega_1, \omega_2) = \int_\Lambda dy X[\mathrm{dist}(\omega_i, y) \leqq a; i=1,2]$$

is the volume of points which are within the distance a from both paths. Here ($X[-]$ is a characteristic function, which is 1 if the condition $[-]$ is satisfied and 0 otherwise.) If there is a point of intersection in Λ then, ignoring a trivial boundary effect,

$$V_a(\omega_1, \omega_2) \geqq |B_a|,$$

$|B_a|$ being the volume of a ball of radius a. In fact,

$$\lim_{a \to 0} \frac{V_a(\omega_1, \omega_2)}{|B_a|} = \mathrm{card}(\omega_1 \cap \omega_2 \cap \Lambda).$$

The point now is, that it is very easy to evaluate the expected value of the integrand in $V_a(\omega_1, \omega_2)$, which is just the product of two (independent) hitting probabilities, (I.2). Thus, (using Fatou's lemma)

$$\begin{aligned}\operatorname{Prob}(\{\omega_1\cap\omega_2\cap\Lambda\neq\emptyset\}) &\leqq E(\operatorname{card}(\omega_1\cap\omega_2\cap\Lambda))\\ &\leqq \liminf_{a\to 0}\frac{1}{|B_a|}\int_\Lambda dy\left(\frac{a}{|y-x_1|}\right)^{d-2}\left(\frac{a}{|y-x_2|}\right)^{d-2}\\ &=\lim_{a\to 0} a^{d-4}\frac{1}{|B_1|}\int_\Lambda dy\frac{1}{|y-x_1|^{d-2}}\frac{1}{|y-x_2|^{d-2}}\\ &=0,\quad \text{for}\quad d>4. \end{aligned}\tag{I.3}$$

Proof completed. □

Notice that the key inequality which is used in the above argument is a bound on the *probability* of intersection in terms of the intersection's "coarse grained" *size*. A moral to be learned is that – while such a bound is grossly inefficient for $d<4$ – it is *very effective* for $d>4$, and *marginal* for $d=4$, where at least it is finite.

In the work in this paper we apply the tactic suggested by the above analysis. The pivotal factor a^{d-4} emerges again in (6.5) and (11.3).

Another important hint which (I.3) offers us is the suggestion that perhaps we should *not* expect the argument described in this paper to be conclusive for the case of a massless ϕ_4^4 field. After all, even for Brownian motion in $\mathbb{R}^4$, (I.3) is only a marginal upper bound which misses a crucial, logarithmically weak, factor. The analysis, in [10], of this case utilizes potential theory and other arguments which at this point have no analog in the situation we encounter. It may therefore be very desirable to have a more rudimentary (yet simple) proof of Theorem I.1 for four dimensions. Such arguments would be presented in [27].

Finally, to impress upon the reader the critical nature of $d=4$ dimensions, let us describe another approach to the problem of intersection of Brownian paths. Since we are dealing with the question of whether the probability is strictly zero, or not, it suffices to settle the issue for Λ the unit ball, and x_1, x_2 having a uniform distribution on the sphere $\partial\Lambda$ – rather than being fixed.

Let $\{\mathscr{C}_i\}_{i=1}^{\infty}$ be a partition of Λ, formed by a collection of open disjoint balls which exhaust the volume, i.e. $\left|\Lambda-\bigcup_{i=1}^{\infty}\mathscr{C}_i\right|=0$. By scaling it down we may use this partition to further populate each of its elements, $\mathscr{C}_i$, by a collection of second generation balls. Iterating this construction, we obtain a hierarchical collection of partitions. The *cheese* is then defined as the union of the remainders which are left out at the various steps (i.e., what is left after all the "holes" are removed).

We shall now use this hierarchy of partitions to associate with each pair of paths ω_1, ω_2 a *random tree*.

First, let $n(\omega_1, \omega_2)$ be the number of the elements of the partition $\{\mathscr{C}_i\}$ which are intersected by both paths. With each such ball, $\mathscr{C}_{i_k} 1\leqq k\leqq n(\omega_1,\omega_2)$, we associate a vertex which is then linked to a fixed point, 0.

Next, from each vertex we draw a number of lines, each corresponding to a second generation ball, of the subpartition of $\mathscr{C}_{i_k}$, which is intersected by both paths (see Fig. 9).

By iteration, we obtain a "tree" which may, or may not, have an infinite "ascending" path. Each such path corresponds to a point of the intersection

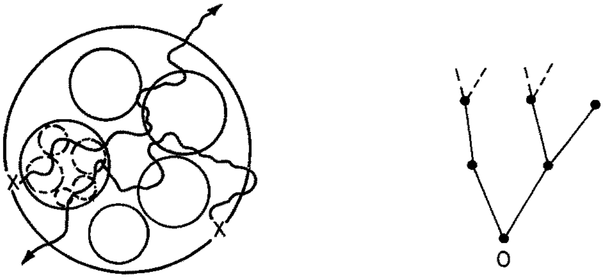

Fig. 9. A random-tree formulation of the intersection problem

$\omega_1 \cap \omega_2 \cap \Lambda$ which does not fall in the cheese (this having Lebesgue measure zero, may be ignored here).

Hence we have converted the problem of intersection to one of *percolation*, on a random tree. Using stopping-time arguments, it is easy to see that, given that a fixed sphere (in Λ) is doubly hit – the number of elements of the corresponding subpartition which are hit by both paths has the *same* distribution as $n(\omega_1, \omega_2)$.

Were we to ignore the existing *interaction* between the numbers of lines which are drawn at the various vertices, we would obtain a solvable model. At *this approximation* the answer is determined by a single quantity – the expected value $E(n(\omega_1, \omega_2))$. The (classical) answer being:

$$\text{there is percolation of } E(n(\omega_1, \omega_2)) > 1,$$

and (II.2)

$$\text{no percolation if } E(n(\omega_1, \omega_2)) \leqq 1.$$

The following result manifests therefore that $d=4$ is indeed a critical case.

Proposition I.2 [27]. *Let $E(-)$ represent the average over independent paths (ω_1, ω_2) whose starting points have the uniform distribution over the sphere $\partial\Lambda$. Then, in $d=4$ dimensions,*

$$E(n(\omega_1, \omega_2)) = 1 \quad (!)$$

for any partition of Λ into balls $\{\mathscr{C}_i\}$, as described above.

An analogous result holds for any other shape which can be "packed" efficiently. In d-dimensions, for successively refined partitions,

$$E(n(\omega_1, \omega_2)) = O(a^{d-4}).$$

Acknowledgements. I wish to thank E. H. Lieb, J. Fröhlich, E. Nelson, A. Sokal, and T. Spencer for valuable discussions. I am also very grateful to numerous colleagues and friends for the encouragement and stimulation which I enjoyed during this work. Some of the writing took place, under ideal conditions, at Université D'Aix-Marseille II and C.N.R.S. at Luminy, whose members I would like to thank for their kind hospitality.

References

1. Aizenman, M.: Phys. Rev. Lett. **47**, 1 (1981) [and a contribution in Mathematical problems in theoretical physics. (Proceedings, Berlin 1981). Berlin, Heidelberg, New York: Springer (to appear)
2. Symanzik, K.: Euclidean quantum field theory. In: Local quantum theory. Jost, R. (ed.). New York: Academic Press 1969
3. Fröhlich, J.: Nucl Phys. B**200**, [FS4] 281 (1982)
4. McBryan, O.A., Rosen, J.: Commun. Math. Phys. **51**, 97 (1976)
5. Simon, B.: Commun. Math. Phys. **77**, 111 (1980)
6. Griffiths, R., Hurst, C., Sherman, S.: J. Math. Phys. **11**, 790 (1970)
7. Newman, C., Schulman, L.: J. Stat. Phys. **26**, 613 (1981)
8. Sokal, A.D.: Phys. Lett. **71**A, 451 (1979)
9. Fröhlich, J., Simon, B., Spencer, T.: Commun. Math. Phys. **50**, 79 (1976)
10. Kakutani, S.: Proc. Japan Acad. **20**, 648 (1944)
Dvoretsky, A., Erdös, P., Kakutani, S.: Acta Sci. Math. (Szeged) **12**, 75 (1950)
11. Lebowitz, J.: Commun. Math. Phys. **35**, 87 (1974)
12. Glimm, J., Jaffe, A.: Ann. Inst. Henri Poincaré A **22**, 97 (1975)
13. Glimm, J., Jaffe, A.: Commun. Math. Phys. **51**, 1 (1976)
14. Sokal, A.D.: Ann. Inst. Henri Poincaré (to appear)
15. Glimm, J., Jaffe, A.: Phys. Rev. D**10**, 536 (1974)
16. Widom, R.: J. Chem. Phys. **43**, 3892 (1965)
Kadanoff, L.P., et al.: Rev. Mod. Phys. **39**, 395 (1967)
Fisher, M.: Rep. Prog. Phys. **30**, 615 (1967), and references therein
17. Schrader, R.: Phys. Rev. B, **14**, 172 (1976)
18. Lieb, E.H., Sokal, A.D.: In preparation
19. Constructive quantum field theory. Velo, G., Wightman, A.S. (eds.). Berlin, Heidelberg, New York: Springer 1973
20. Simon, B.: The $P(\phi)_2$ euclidean (quantum) field theory. Princeton, NS: Princeton University Press 1974
21. Glimm, J., Jaffe, A.: Quantum physics. Berlin, Heidelberg, New York: Springer 1981
22. Simon, B., Griffiths, R.: Commun. Math. Phys. **33**, 145 (1973)
23. Sokal, A.D.: Mean-field bounds and correlation inequalities. J. Stat. Phys. (to appear)
24. Newman, C.: Commun. Math. Phys. **41**, 1 (1975)
25. Newman, C.: Z. Wahrscheinlichkeitstheorie **33**, 75 (1975)
26. Schrader, R.: Phys. Rev. B**15**, 2798 (1977)
Messager, A., Miracle-Sole, S.: J. Stat. Phys. **17**, 245 (1977)
27. Aizenman, M.: On brownian motion in $d=4$ dimensions (in preparation)
28. Graham, R.: Correlation inequalities for the truncated two-point function of an Ising ferromagnet (J. Stat. Phys., to appear) and: An improvement of the GHS inequality for the Ising ferromagnet (J. Stat. Phys., to appear)
29. Aizenman, M., Graham, R.: A bound on the renormalized coupling in the φ_d^4 field theory in $d=4$ dimensions (in preparation)

Communicated by A. Jaffe

Received March 22, 1982; in revised form April 21, 1982