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EXTENSION THEOREMS FOR FREDHOLM AND INVERTIBILITY SYMBOLS

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This paper is a continuation of [GK3] where the theory of Invertibility Symbol in Banach algebras was developed. In the present paper we generalize these results for the case when the Invertibility Symbol is defined on a subalgebra of the Banach algebras. The difficulty which arises here in this more general case is connected with the fact that some elements of the subalgebra may have the inverses which do not belong to the subalgebra. This generalization of the theory allows us to study the Fredholm Symbols of linear operators. Applications to subalgebras generated by two idempotents and to algebras generated by singular integral operators are presented.

Introduction.

1. Let $\mathbb{B}$ be a Banach space over the field $\mathbb{C}$, $L(\mathbb{B})$-algebra of all linear bounded operators acting in $\mathbb{B}$, and $\mathcal{K}(\mathbb{B})$ its two-sided ideal of all compact operators. Recall that an operator $\mathbb{A}$ ($\in L(\mathbb{B})$) is a Fredholm operator if its image is closed ($\overline{\operatorname{Im} A} = \operatorname{Im} A$), dimKer $< \infty$ and dimKer $A^* < \infty$.

For any algebra Σ we denote by $\Sigma^{n\times n}$ the algebra of $n \times n$ matrices with entries from Σ. Let $\mathcal{A}$ be a subalgebra of $L(\mathbb{B})$. We say that $\mathcal{A}$ is an algebra with Fredholm Symbol of order n if there exists a set $\{\gamma_\tau\}$ $(\tau \in \mathcal{T})$ of homomorphisms $\gamma_\tau : \mathcal{A} \to \mathbb{C}^{\ell\times\ell}$ ($\ell = \ell(\tau) \leq n$) such that any operator $A \in \mathcal{A}$ is a Fredholm operator in $\mathbb{B}$ iff

$$(0.1) \qquad \det \gamma_\tau(A) \neq 0$$

for all $\tau \in \mathcal{T}$. We denote by $\mathrm{FS}(n, \mathbb{B})$ the class of algebras $\mathbb{A}$ which have Fredholm Symbols of order n.

2. Let $\tilde{\mathbb{A}}$ be a Banach algebra with identity e over the field $\mathbb{C}$, $G\tilde{\mathbb{A}}$-the group of all invertible elements of $\tilde{\mathbb{A}}$, $\mathbb{A}$ some subalgebra of $\tilde{\mathbb{A}}$ with identity e and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ a

set of representations of $\mathfrak{A}$ such that $\dim \nu_\tau \leq n$ for all $\tau \in \mathcal{T}$ and any element $a \in \mathfrak{A}$ is invertible in $\tilde{\mathfrak{A}}$ iff

$$\det \nu_\tau(a) \neq 0 \tag{1.2}$$

for all $\tau \in \mathcal{T}$.

In this case we say that the set $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ generates an Invertibility Symbol of order n for $\mathfrak{A}$ in an algebra $\tilde{\mathfrak{A}}$. We denote by $\mathrm{IS}(n, \tilde{\mathfrak{A}})$ the class of all algebras $\mathfrak{A}$ which have an Invertibility Symbol in $\tilde{\mathfrak{A}}$. In the case when $\mathfrak{A} = \tilde{\mathfrak{A}}$ we write $\mathfrak{A} \in \mathrm{IS}(n)$ instead of $\mathfrak{A} \in \mathrm{IS}(n, \mathfrak{A})$.

3. Let a set $\{\gamma_\tau\}$ $(\tau \in \mathcal{T})$ generate a Fredholm Symbol of order n of an algebra $\mathcal{A} \subset L(\mathbb{B})$. Using the same arguments as in **[GK3]** (example 1.1) one can suppose that for every $\tau \in \mathcal{T}$ the representation γ_τ is irreducible and hence $\mathrm{Im}\, \gamma_\tau = \mathbb{C}^{\ell \times \ell}$ $(\ell = \ell(\tau))$.

Set $\tilde{\mathfrak{A}} = L(\mathbb{B})/\mathcal{K}(\mathbb{B})$ and $\mathfrak{A} = \pi(\mathcal{A})$, where π is the canonical homomorphism $\pi : L(\mathbb{B}) \to \tilde{\mathfrak{A}}$. For any $\tau \in \mathcal{T}$ by ν_τ we denote the following representation of $\mathfrak{A}$:

$$\nu_\tau(\pi(A)) = \gamma_t(A_0),$$

where A_0 is some operator from $\pi(A)$. The representations ν_τ are well-defined, because $\gamma_\tau(T) = 0$ for all $T \in \mathcal{A} \cap \mathcal{K}(\mathbb{B})$. Indeed for every $\tau \in \mathcal{T}$ the set

$$\{\gamma_\tau(T) : T \in \mathcal{A} \cap \mathcal{K}(\mathbb{B})\}$$

is a two-sided ideal in the simple algebra $\mathbb{C}^{\ell \times \ell} = \mathrm{Im}\, \gamma_\tau$ and thus it consists only of $\gamma_\tau(T) = 0$.

It remains to note (**[GK]**; see also **[GK4]**) that $A \in F(\mathbb{B})$ iff $\pi(A) \in G\tilde{\mathfrak{A}}$. Hence $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ generates an Invertibility Symbol for $\mathfrak{A}$ in an algebra $\tilde{\mathfrak{A}}$. Vice versa if $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ generates an Invertibility Symbol for $\pi(\mathcal{A})$ in an algebra $\pi(L(\mathbb{B}))$, then the set $\{\gamma_\tau\}$ $(\tau \in \mathcal{T})$ of representations $\gamma_\tau(A) := \nu_\tau(\pi(A))$ generates a Fredholm Symbol for the algebra $\mathcal{A} \subset L(\mathbb{B})$.

In the paper **[GK3]** some extension theorems were proved for the Invertibility Symbol in the case $\tilde{\mathfrak{A}} = \mathfrak{A}$. In this paper we consider the more general case $\mathfrak{A} \subset \tilde{\mathfrak{A}}$ and as a corollary we obtain the corresponding extension theorems for Fredholm Symbol . The present paper consists of an Introduction, Chapter I (§§1-4) and Chapter II (§§1-4).

Chapter I starts with some preliminaries (§1). The necessary conditions of the existence of an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$ is established in §2. Some sufficient conditions are proved in §3. The Symbols extension problem from a dense subalgebra is considered in §4. Chapter II contains some applications of the results obtained in Chaper I. In §1 we show that any subalgebra $\mathfrak{A}$, generated by two idempotents, has an Invertibility Symbol of order 2 in the entire algebra $\tilde{\mathfrak{A}}$, even if $\mathfrak{A}$ is not inverse-closed in $\tilde{\mathfrak{A}}$. The explicit form of the symbol is presented. A certain generalization is considered in §2. Using these results we prove in §3 that the Banach algebra generated by singular integral operators with piecewise continuous coefficients on a simple contour Γ has a Fredholm Symbol of order 2, in the space $L_p(\Gamma, \rho)$ with an arbitrary weight ρ. The general case of an arbitrary weight ρ and nonsimple contour is considered in §4. Algebras of $n \times n$ matrices with entries from the algebras $\mathfrak{A}$ are also considered in §§1-4 of Chapter II.

The main results obtained in this paper were presented in the International Symposium, "Operator Equations and Numerical Analysis", September-October 1992, Gosen, Germany.

The present paper was already written when we received a preprint [**FRS**] in which the above-mentioned results of §3, Chaper II as well as the results of §1, Chaptre II (with some additional nonessential requirement) are also obtained.

CHAPTER I. EXTENSION OF SYMBOLS.

§1. Preliminaries.

Let $\mathfrak{A}$ be an algebra. Set

$$F_k(a_1, \dots, a_k) = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)},$$

where $a_1, \dots a_k \in \mathfrak{A}$ and σ runs through the symmetric group S_k. If for any element $a_1, \dots a_k \in \mathfrak{A}$ the polynomial identity $F_k(a_1, \dots a_k) = 0$ holds, we write $\mathfrak{A} \in \mathrm{PI}(F_k)$. By the well-known Amitsur-Levitzki theorem [**AL**] the algebra $\mathbb{C}^{n \times n}$ belongs to $\mathrm{PI}(F_{2n})$. From this theorem follows

Proposition 1.1. *Let $\{\nu_\tau\}$ $(\tau \in T)$ generate an Invertibility Symbol of order n for a subalgebra $\mathfrak{A}$ of a Banach algebra $\tilde{\mathfrak{A}}$. Then $\nu_\tau(F_{2n}(a_1, \dots, a_{2n})) = 0$ for any elements $b, a_1, \dots, a_{2n}$ from the algebra $\mathfrak{A}$ and the element $w = e + bF_{2n}(a_1, \dots, a_{2n})$ is invertible in $\tilde{\mathfrak{A}}$.*

For details see [**K**].

Denote by $\mathcal{R}$ the radical of algebra $\mathfrak{A}$, i.e. the set

$$\mathcal{R}(\mathfrak{A}) = \{r \in \mathfrak{A} : e - rx \in G\mathfrak{A} \text{ for all } x \in \mathcal{A}\}.$$

PROPOSITION 1.2 [**K**]. *An algebra* $\mathfrak{A}$ *has an Invertibility Symbol of order* n *(in itself) iff* $\mathfrak{A}/\mathcal{R}(\mathfrak{A}) \in \mathrm{PI}(F_{2n})$.

We call an element $x \in \mathfrak{A}$ limpotent if x is a limit of a sequence of quasinilpotent elements, but x itself is not a quasinilpotent element.

§2. Invertibility Symbols for Banach algebras and their subalgebras.

Let $\mathfrak{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$ with the same identity e. In this section we establish some connections between the existence of an Invertibility Symbol in an algebra $\mathfrak{A}$ and the existence of the Invertibility Symbol for $\mathfrak{A}$ in an algebra $\tilde{\mathfrak{A}}$.

THEOREM 1.1. *Let* $\mathfrak{A}$ *be a closed subalgebra of a Banach algebra* $\tilde{\mathfrak{A}}$. *Each of the following assertions* 1-5 *is necessary for the existence of an Invertibility Symbol for* $\mathfrak{A}$ *in the algebra* $\tilde{\mathfrak{A}}$.

1. *The quotient algebra* $\mathfrak{A}(\mathcal{R}(\mathfrak{A})$ *is an algebra with polynomial identity* F_{2n}.
2. $\mathfrak{A} \in \mathrm{IS}(n)$.
3. $\mathfrak{A}$ *does not have limpotent elements.*
4. *For any subalgebra* $\mathbb{K}$ *dense in* $\mathfrak{A}$ *any element* $a \in \mathfrak{A}\backslash G\mathfrak{A}$ *can be approximated with any precision by elements from* $\mathbb{K} \setminus G\mathfrak{A}$.
5. *For any subalgebra* $\mathbb{K}$ *dense in* $\mathfrak{A}$ *any element* $a \in \mathfrak{A}\backslash G\tilde{\mathfrak{A}}$ *can be approximated with any precision by elements from* $\mathbb{K} \setminus G\tilde{\mathfrak{A}}$.

PROOF: 1. Let $a_1, \dots, a_{2n}, x$ be some fixed elements in $\mathfrak{A}$ and $w := e + cx$, where

$$c := \sum_{\sigma \in S_{2n}} \operatorname{sgn} \sigma a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(2n)}.$$

Since $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$ there exists a set $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ of representation of $\mathfrak{A}$ which generates the Invertibility Symbol of order n for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$. By Proposition 1.1, $\det \nu_\tau(w - \lambda e) = (1-\lambda)^{\ell(\tau)} (\ell(\tau) = \dim \nu_\tau)$. It follows from here that the spectrum $\sigma(w, \tilde{\mathfrak{A}})$ of w in the algebra $\tilde{\mathfrak{A}}$ is a singleton $\lambda = 1$ and hence $\sigma(w, \tilde{\mathfrak{A}})$ coincides with the spectrum $\sigma(w, \mathfrak{A})$ of w in the subalgebra $\mathfrak{A}$. Thus, w is invertible in $\mathfrak{A}$ for any $x \in \mathfrak{A}$ and (by the definition of radical) $c \in \mathcal{R}(\mathfrak{A})$. This implies that $\mathfrak{A}/\mathcal{R}(\mathfrak{A}) \in \mathrm{PI}(F_{2n})$. The first assertion of the theorem is proved.

Taking into account theorem 1.1 of [**GK3**] it remains only to prove assertion 5. Let $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$. Assume that $a \in A \setminus G\tilde{\mathfrak{A}}$. Then there exists a representation $\nu \in \{\nu_\tau\}$ $(\tau \in \mathcal{T})$ such that $\det \nu(a) = 0$. If $a_n \in \mathbb{K}$ and $\|a_n - a\| \to 0$ then $\det \nu(a_n) \to 0$ and there exists a sequence of eigenvalues λ_n of the matrices $\nu(a_n)$ which tends to zero:

$\lambda_n \to 0$. Now for the elements $b_n := a_n - \lambda_n e$ we have: $b_n \in \mathbb{K}$ and $\|a - b_n\| \to 0$. Since $\det \nu(b_n) = 0,\ b_n \in \mathbb{K} \setminus G\tilde{\mathfrak{A}}$ and the theorem is proved.

The condition $\mathfrak{A} \in \mathrm{IS}(n)$ is not sufficient for the existence of the Invertibility Symbol for $\mathfrak{A}$ in an algebra $\tilde{\mathfrak{A}}$. This follows from an example which is given in [**K**] (example 14.2). For completeness we state it here.

EXAMPLE 1.1: Let $\mathfrak{A}_1$ be some Banach algebra with identity e and elements a, b such that $ab = e$ and $ba \neq e$. We set $\tilde{\mathfrak{A}} = \mathfrak{A}_1^{2\times 2}$ and

$$\mathfrak{A} = \left\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \right\} \subset \tilde{\mathfrak{A}},$$

where x belongs to the Banach algebra $\mathfrak{A}_b$, generated by one element b, $y \in \mathfrak{A}_a$ and $z \in \mathfrak{A}_1$. It is not difficult to check that $\mathfrak{A}$ is the Banach subalgebra of $\tilde{\mathfrak{A}}$, and every element x of the form

$$x = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \quad (z \in \mathfrak{A}_1)$$

belongs to the radical $\mathcal{R}(\mathfrak{A})$. The quotient algebra $\mathfrak{A}/\mathcal{R}(\mathfrak{A})$ is commutative and thus $\mathfrak{A} \in \mathrm{IS}(1)$. However, the algebra $\mathfrak{A}$ does not have any Invertibility Symbol in $\tilde{\mathfrak{A}}$. Let us first show that $\mathfrak{A} \notin \mathrm{IS}(1, \tilde{\mathfrak{A}})$. Suppose that $\mathfrak{A} \in \mathrm{IS}(1, \tilde{\mathfrak{A}})$ and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ is a set of one-dimensional representations which generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$. Since $ba \neq e$ the element

$$u := \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$$

is not invertible in $\tilde{\mathfrak{A}}$. Thus there exists a representation $\nu \in \{\nu_\tau\}$ $(\tau \in \mathcal{T})$ for which $\nu(u) = 0$. Since $\operatorname{Im} \nu$ is a commutative algebra

$$0 = \nu \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} = \nu \left(\begin{bmatrix} b & e \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} b & e \\ 0 & a \end{bmatrix} \right) =$$
$$= \nu \left(\begin{bmatrix} b & e \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & e \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \right) = \nu \begin{bmatrix} b & e \\ 0 & a \end{bmatrix}.$$

and hence the element

$$v := \begin{bmatrix} b & e \\ 0 & a \end{bmatrix}$$

is not invertible in $\tilde{\mathfrak{A}}$.

But, in fact, v is invertible in $\tilde{\mathfrak{A}}$:

$$\begin{bmatrix} b & e \\ 0 & a \end{bmatrix} \begin{bmatrix} a & -e \\ e - ba & b \end{bmatrix} = \begin{bmatrix} a & -e \\ e - ba & b \end{bmatrix} \begin{bmatrix} b & e \\ 0 & e \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}.$$

The contradiction shows that $\mathfrak{A} \notin \mathrm{IS}(1, \tilde{\mathfrak{A}})$. Moreover, $\mathfrak{A} \notin \mathrm{IS}(n, \tilde{\mathfrak{A}})$ for any $n \in \mathbb{N}$. This follows from the following lemma.

LEMMA 1.1. *Let $\mathfrak{A}$ be a Banach subalgebra in a Banach algebra $\tilde{\mathfrak{A}}$. If $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$ and $\mathfrak{A} \in \mathrm{IS}(m)$, for some $m < n$, then $\mathfrak{A} \in \mathrm{IS}(m, \tilde{\mathfrak{A}})$.*

PROOF: Let $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ be a set of representations which generates an Invertibility Symbol of order n for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$. Using the same arguments as in **[GK3]** (example 1.1) one can assume that $\mathrm{Im}\,\nu_\tau = \mathbb{C}^{\ell(\tau)\times\ell(\tau)}$. Let $a_1, \dots a_{2m}$ be any elements from $\mathfrak{A}$. Since $\mathfrak{A} \in \mathrm{IS}(m)$, the element $w = F_{2m}(a_1, \dots a_{2m})$ belongs to the radical $\mathcal{R}(\mathfrak{A})$ of an algebra $\mathfrak{A}$. It follows from here that for each $\tau \in \mathcal{T}$ the matrix $\nu_\tau(w)$ belongs to the radical of the simple algebra $\mathbb{C}^{(\ell(\tau)\times\ell(\tau)}$, i.e. $\nu_\tau(w) = 0$. So we have obtained that $\mathbb{C}^{\ell(\tau)\times\ell(\tau)} \in \mathrm{PI}(F_{2m})$. It is well-known (see, for example, **[H]**, lemma 6.3.1) that in this case $\ell(\tau) \leq m$. Thus $\mathfrak{A} \in \mathrm{IS}(m, \tilde{\mathfrak{A}})$.

§3. Inverse-closed and locally inverse-closed subalgebras.

Let $\mathfrak{A}$ be a Banach subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$. A subalgebra $\mathfrak{A}$ is inverse-closed in $\tilde{\mathfrak{A}}$ if any element a^{-1} inverse in an algebra $\tilde{\mathfrak{A}}$ to an element $a \in \mathfrak{A}$ belongs to $\mathfrak{A}$.

Let $\mathfrak{A} \in \mathrm{IS}(n)$ and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ generate an Invertibility Symbol in $\mathfrak{A}$. In this case the following two properties of this algebra are evident.

PROPOSITION 1.3. *If $\mathfrak{A}$ is inverse-closed in $\mathfrak{A}$ then $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$ and the set $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$.*

PROPOSITION 1.4. *If the set $\{\nu_\tau\}$ $(t \in \mathcal{T})$ generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$ then $\mathfrak{A}$ is inverse-closed in $\tilde{\mathfrak{A}}$.*

It is clear that there exist examples of algebras $\tilde{\mathfrak{A}}$ and their subalgebras $\mathfrak{A} \in \mathrm{IS}(n)$ such that $\mathfrak{A}$ is not inverse-closed in $\tilde{\mathfrak{A}}$, but $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$. Here we present a simple example. Let $\mathfrak{A}$ be a commutative subalgebra of $\tilde{\mathfrak{A}}$ and suppose that $\mathfrak{A}$ is not inverse-closed in $\tilde{\mathfrak{A}}$. Denote by A_1 the maximal commutative subalgebra of $\tilde{\mathfrak{A}}$ which contains $\mathfrak{A}$ $(\mathfrak{A} \subset \mathfrak{A}_1 \subset \tilde{\mathfrak{A}})$. Let $\{\nu_M\}$ be the set of all multiplicative functionals acting in $\mathfrak{A}_1$. By the well-known theorem of Gelfand (**[GRS]**) $\{\nu_M\}$ generates an Invertibility Symbol (of order 1) in an algebra $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is inverse-closed in $\tilde{\mathfrak{A}}$ the set $\{\nu_M\}$ also generates an Invertibility Symbol for $\mathfrak{A}_1$ in $\tilde{\mathfrak{A}}$. In particular, $\{\nu_M\}$ generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$. Thus $\mathfrak{A}$ has an Invertibility Symbol in an algebra $\tilde{\mathfrak{A}}$.

The class of commutative subalgebras are simple examples of *locally inverse-closed subalgebras* which we are going to introduce.

Let $\mathfrak{A}$ be a Banach subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$ and $\mathcal{C} = \mathcal{C}(\mathfrak{A})$ some subalgebra of the centre $Z(\mathfrak{A})$ of an algebra $\mathfrak{A}$.

Denote by Λ the commutator of the algebra $\mathcal{C}(\mathfrak{A})$ in $\tilde{\mathfrak{A}}$ i.e.

$$\Lambda = \{x \in \tilde{\mathfrak{A}} : ax = xa \text{ for every} a \in \mathcal{C}(\mathfrak{A})\}.$$

It is not difficult to check that Λ is a Banach algebra of $\tilde{\mathfrak{A}}$, $\mathcal{C}(\mathfrak{A})$ is a subalgebra of the centre $Z(\Lambda)$ of an algebra Λ, and Λ is inverse-closed in $\tilde{\mathfrak{A}}$.

Let $\mathbb{M} = \mathbb{M}(\mathcal{C})$ be the set of all maximal ideals of the algebra $\mathcal{C}$. For every ideal $M \in \mathbb{M}(\mathcal{C})$ we denote by J_M the smallest two-sided ideal in Λ which contains all elements of the form

$$\sum_{k=1}^{m} a_k b_k$$

where $a_k \in M$, $b_k \in \Lambda$ and $m \in \mathbb{N}$. If $J_M \neq \Lambda$ then J_M is a closed two-sided ideal in Λ. Denote by $\mathbb{M}_{\mathcal{C}}$ the set of all maximal ideals $M \in \mathbb{M}$ for which $J_M \neq \Lambda$.

For any element $a \in \Lambda$ we denote by a_M the element $a_M := a + J_M$ $(\in \Lambda/J_M)$. If Σ is a subalgebra of Λ then by Σ_M we denote image $\pi_M(\Sigma)$ where π is the canonical homomorphism $\pi_M : \Lambda \to \Lambda/J_M$.

Let us introduce the following definition. $\mathfrak{A}$ is a locally inverse-closed subalgebra in $\tilde{\mathfrak{A}}$ (relative to $\mathcal{C}$) if for any $M \in \mathbb{M}_{\mathcal{C}}$ the quotient algebra $\mathfrak{A}_M$ is an inverse-closed subalgebra in Λ_M.

THEOREM 1.2. *Let $\mathfrak{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$ and let $\mathfrak{A} \in \mathrm{IS}(n)$. If $\mathfrak{A}$ is a locally inverse-closed subalgebra of $\tilde{\mathfrak{A}}$, then $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$.*

For the proof of this theorem we need the following lemma.

LEMMA 1.2. *If $\mathfrak{A}_M \in \mathrm{IS}(n, \Lambda_M)$ for all $M \in \mathbb{M}_{\mathcal{C}}$ then $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$.*

PROOF: Let $\{\nu_\tau^M\}$ $(\tau \in \mathcal{T}(M))$ be a set of representations which generates an Invertibility Symbol for $\mathfrak{A}_M$ in Λ_M. For any $M \in \mathbb{M}_{\mathcal{C}}$ and $\tau \in \mathcal{T}(M)$ we denote by $\nu_{\tau,M}$ the following representation of the algebra $\mathfrak{A}$: $\nu_{\tau,M}(a) = \nu_\tau^M(a_M)$. The set $\{\nu_{\tau_M}\}$ $(M \in \mathbb{M}_{\mathcal{C}},\ \tau \in \mathcal{T}(M))$ generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$. Indeed, let $a \in \mathfrak{A}$. Since Λ is inverse-closed in $\tilde{\mathfrak{A}}$ the element a is invertible in $\tilde{\mathfrak{A}}$ iff a is invertible in Λ. By the well-known local principle [**A**], a is invertible in Λ iff a_M is invertible in Λ_M for all $M \in \mathbb{M}_{\mathcal{C}}$. Thus a is invertible in $\tilde{\mathfrak{A}}$ iff $\det \nu_{\tau,M}(a) \neq 0$ for all $M \in \mathbb{M}_{\mathcal{C}}$ and all $\tau \in \mathcal{T}(M)$. The lemma is proved.

PROOF OF THE THEOREM: Let $a_1, \ldots a_{2n} \in \mathfrak{A}$ and $c = F_{2n}(a_1, \ldots a_{2n})$. In the conditions of the theorem we assumed that $\mathfrak{A} \in \mathrm{IS}(n)$. By proposition 1.1 $c \in \mathcal{R}(\mathfrak{A})$. It is not difficult to check that in this case c_M belongs to the radical $\mathcal{R}(\mathfrak{A}_M)$ of the algebra $\mathfrak{A}_M$ $(= \pi(\mathfrak{A}))$, hence $\mathfrak{A}_M(\mathcal{R}(\mathfrak{A}_M) \in \mathrm{PI}(F_{2n})$ and by proposition 1.2 $\mathfrak{A}_M \in \mathrm{IS}(n)$. Since $\mathfrak{A}$ is locally inverse-closed in $\tilde{\mathfrak{A}}$, $\mathfrak{A}_M$ is inverse-closed in Λ_M. By proposition 1.3, $\mathfrak{A}_M \in \mathrm{IS}(n, \Lambda_M)$ for all $M \in \mathbb{M}_{\mathcal{C}}$ and by lemma 1.2, $\mathfrak{A} \in \mathrm{IS}(n, \tilde{\mathfrak{A}})$. The theorem is proved.

COROLLARY 1.1. *Let $\mathbb{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathbb{A}}$ and C a subalgebra of the centre of an algebra $\mathbb{A}$. If*

$$\sup_{M \in \mathbf{M}_C} \dim \mathbb{A}_M = m < \infty \tag{1.1}$$

then $\mathbb{A} \in \mathrm{IS}(n, \tilde{\mathbb{A}})$ for some $n \leq \sqrt{m}$.

Indeed, it follows from (1.1) that $\dim \mathbb{A}_M \leq m$. Hence (see, for example [**H**], lemma 6.2.2), $\mathbb{A}_M \in \mathrm{PI}(F_{m+1})$. By proposition 1.2, $\mathbb{A}_M \in \mathrm{IS}(k)$ for some k. Since $\dim \mathbb{A}_M < \infty$, the algebra $\mathbb{A}_M$ is inverse-closed in $\mathbf{\Lambda}_M$ and thus $\mathbb{A}_M \in \mathrm{IS}(k, \mathbf{\Lambda}_M)$ for all $M \in \mathbf{M}_C$. By lemma 1.2, $\mathbb{A} \in \mathrm{IS}(k, \tilde{\mathbb{A}})$. Let $\{\nu_\tau^M\}$ be a set of representations which generates an Invertibility Symbol for $\mathbb{A}_M$ in $\mathbf{\Lambda}_M$. We mentioned above that the representations ν_τ^M can be taken with the conditions $\mathrm{Im}\, \nu_\tau^M = \mathbb{C}^{\ell \times \ell}$. Since $\dim \mathrm{Im}\, \nu_\tau^M \leq \dim \mathbb{A}_M$ we get the inequality $\ell^2 \leq m$. Thus $\mathbb{A}_M \in \mathrm{IS}(\ell, \mathbf{\Lambda}_M)$ with some $\ell \leq \sqrt{m}$. Hence $\mathbb{A} \in \mathrm{IS}(n, \tilde{\mathbb{A}})$ with some $n \leq \sqrt{m}$.

COROLLARY 1.2. *Let $\mathbb{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathbb{A}}$ and $\mathbb{K}$ a dense subalgebra of $\tilde{\mathbb{A}}$. If $\mathbb{K}$ is an m-dimensional module over its centre, then $\mathbb{A} \in \mathrm{IS}(n, \tilde{\mathbb{A}})$ for some $n \leq \sqrt{m}$.*

Indeed in this case (1.1) holds.

§4. Extension of Symbols.

In this section the conditions are given for the possibility of an extension of Invertibility Symbols from dense subalgebras to their closures.

THEOREM 1.3. *Let $\mathbb{A}$ be a subalgebra of a Banach algebra $\tilde{\mathbb{A}}$, $\mathbb{K}$ a dense subalgebra of $\mathbb{A}$ and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ a set of continuous homomorphisms which generates an Invertibility Symbol of order n for $\mathbb{K}$ in an algebra $\tilde{\mathbb{A}}$.*

Under these conditions the following two assertions are equivalent:

1. *Any element $a \in \mathbb{A}$ is invertible in $\tilde{\mathbb{A}}$ iff*

$$\inf_{\tau \in \mathcal{T}} |\det \nu_\tau(a)| > 0 \tag{1.2}$$

2. *Any element $a \in \mathbb{A} \backslash G\tilde{\mathbb{A}}$ can be approximated with any precision by elements from $\mathbb{K} \setminus G\tilde{\mathbb{A}}$.*

For the particular case when $\mathbb{A} = \tilde{\mathbb{A}}$ this theorem was proved in [**GK3**]. The same arguments can be used for the proof of this theorem in the general case $\mathbb{A} \subset \tilde{\mathbb{A}}$.

COROLLARY 1.3. *Let $\mathbb{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathbb{A}}$, $\mathbb{K}$ a dense subalgebra of $\mathbb{A}$ and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ a set of representations which generates an Invertibility Symbol of order n for $\mathbb{K}$ in $\tilde{\mathbb{A}}$. If $\mathbb{A} \in \mathrm{IS}(m, \tilde{\mathbb{A}})$ for some $m \in \mathbb{N}$ then any element $a \in \mathbb{A}$ is invertible in $\tilde{\mathbb{A}}$ iff (1.2) holds.*

COROLLARY 1.4. *Let $\mathbb{A}$ be a closed subalgebra of a Banach algebra $\tilde{\mathbb{A}}$, $\mathbb{K}$ a dense subalgebra of $\mathbb{A}$ and $\{\nu_\tau\}$ $(\tau \in \mathcal{T})$ a set of representations which generates an Invertibility Symbol of order n for $\mathbb{K}$ in $\tilde{\mathbb{A}}$. If $\mathbb{K}$ is a finite-dimensional module over its centre then any element $a \in \mathbb{A}$ is invertible in $\tilde{\mathbb{A}}$ iff (1.2) holds.*

COROLLARY 1.5. *Let $\mathcal{A}_0$ be an algebra of linear bounded operators acting in a Banach space $\mathbb{B}$ and $\mathcal{A} = \overline{\mathcal{A}}_0$ the closure of $\mathcal{A}_0$ in an algebra $L(\mathbb{B})$. Suppose that*

1) *$\mathcal{A}$ has a Fredholm Symbol (of some order) in $\mathbb{B}$ and*
2) *a set of homomorphisms $\{\gamma_\tau\}$ $(\tau \in \mathcal{T})$ generates a Fredholm Symbol of order n for $\mathcal{A}_0$.*

Then any operator $A \in \mathcal{A}$ is a Fredholm operator in $\mathbb{B}$ iff

$$\inf_{\tau \in \mathcal{T}} |\det \gamma_\tau(A)| > 0.$$

CHAPTER II. EXAMPLES.

§1. Matrices with entries from an algebra generated by two idempotents.

Let $\tilde{\mathbb{A}}$ be some Banach algebra, $p, r \in \tilde{\mathbb{A}}$ and $p^2 = p$, $r^2 = r$. Denote by $\mathbb{K}$ the smallest (generally speaking nonclosed) subalgebra of $\tilde{\mathbb{A}}$ generated by p, r, and denote by $\mathbb{A}$ the closure of $\mathbb{K}$ in $\tilde{\mathbb{A}}$. It is shown in [W] (see also [GK3]) that the subalgebra $\mathcal{C}$ generated by one element $t = (p - r)^2$ is a subalgebra of the center of $\mathbb{K}$ and $\mathbb{K}$ is a four-dimensional module over $\mathcal{C}$. The algebra $\mathbb{K}^{m \times m}$ of matrices with entries from $\mathbb{K}$ is also a finite-dimensional ($4m^2$-dimensional) module over the subalgebra

$$\mathcal{C}_1 = \left\{ \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & a \end{bmatrix} \quad (a \in \mathcal{C}) \right\}$$

of the centre of $\mathbb{K}^{m \times m}$.

By corollary 1.2, $\mathbb{A}^{m \times m} \in \mathrm{IS}(2m, \tilde{\mathbb{A}})$ for every $m \in \mathbb{N}$.

In order to describe the Invertibility Symbol for $\mathbb{A}^{m \times m}$ in an algebra $\tilde{\mathbb{A}}^{m \times m}$ we need the following notations:

For any $\mu \in \mathbb{C}$ we define the mapping

$$g_\mu : \{e, p, r\} \to \mathbb{C}^{2\times 2} \tag{2.1}$$

by the equalities

$$g_\mu(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_\mu(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad g_\mu(r) = \begin{bmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{bmatrix} \tag{2.2}$$

with some fixed branch of the root $\sqrt{\mu(1-\mu)}$. By $f_k (f_k : \{e, p, r\} \to \mathbb{C};\ k \in \{0,1,2,3\})$ we denote the mappings, defined by the following equalities:

$$\begin{aligned} &f_0(e) = 1,\ f_0(p) = f_0(r) = 0; \quad f_1(e) = f_1(p) = 1,\ f_1(r) = 0; \\ &f_2(p) = 0,\ f_2(r) = f_2(e) = 1; \quad f_3(e) = f_3(p) = f_3(r) = 1. \end{aligned} \tag{2.3}$$

Let ν be some representations of $\mathfrak{A}$. By $\nu^{(m)}$ we denote the representation of the algebra $\mathfrak{A}^{m\times m}$ defined by the equality

$$\nu^{(m)}((a_{jk})_{j,k=1}^m) := (\nu(a_{jk}))_{j,k=1}^m \ (a_{jk} \in \mathfrak{A}). \tag{2.4}$$

By $\sigma(x, \tilde{\mathfrak{A}})$ we denote the spectrum of the element x in an algebra $\tilde{\mathfrak{A}}$.

THEOREM 2.1. *Let $\mathfrak{A}$ be a Banach subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$, generated by two idempotents p and r. Then*

1. *For any $m \in \mathbb{N}$ the algebra $\mathfrak{A}^{m\times m}$ has an Invertibility Symbol of order $2m$ in an algebra $\tilde{\mathfrak{A}}^{m\times m}$.*
2. *Let g_n and f_k be the mappings defined by (2.2) and (2.3). For any $\mu \in \sigma(prp, \tilde{\mathfrak{A}}) \setminus \{0,1\}$ and any $k \in \sigma(p+2r, \tilde{\mathfrak{A}}) \cap \{0,1,2,3\}$ it is possible to extend the mappings g_μ and f_k to homomorphisms of the algebra $\mathfrak{A}$ (we denote these extensions by the same letters g_μ and f_k).*
3. *The set of representations $\{g_\mu^{(m)}\}$ ($\mu \in \sigma(prp, \tilde{\mathfrak{A}}) \setminus \{0,1\}$) and $\{f_k^{(m)}\}$ ($k \in \sigma(p+2r) \cap \{0,1,2,3\}$) defined by (2.2)-(2.4) generates an Invertibility Symbol of order $2m$ for $\mathfrak{A}^{m\times m}$ in an algebra $\tilde{\mathfrak{A}}^{m\times m}$.*

The following two assertions give some additional information about the Invertibility Symbol .

1° If $0 \notin \sigma((e-p-r)^2, \tilde{\mathfrak{A}})$ and $1 \notin \sigma((e-p-r)^2, \tilde{\mathfrak{A}})$, then $\sigma(p+2r, \tilde{\mathfrak{A}}) \cap \{0,1,2,3\} = \emptyset$ and the Invertibility Symbol is generated only by $2m$-dimensional representations $\{g_\mu^{(m)}\}$ ($\mu \in \sigma(prp, \tilde{\mathfrak{A}}) \setminus \{0,1\}$).

2° If $0 \in \sigma(prp, \tilde{\mathfrak{A}})$, $1 \in \sigma(prp, \tilde{\mathfrak{A}})$ and these points are not isolated points of $\sigma(prp, \tilde{\mathfrak{A}})$, then the set $\{g_\mu^{(m)}\}$ ($\mu \in \sigma(prp)$) also generates an Invertibility Symbol .

In the second case, all the m-dimensional representations $f_k^{(m)}$ belong to the set of representations which generates the Invertibility Symbol in an implicit form:

$$g_0^{(m)} = f_1^{(m)} \dot{+} f_2^{(m)} \quad \text{and} \quad g_1^{(m)} = f_0^{(m)} \dot{+} f_3^{(m)}.$$

The proof of Theorem 2.1 as well as of the additional assertions 1°, 2° coincides with the proof of the theorem 2.1 and corresponding additional assertions in **[GK3]** for the case $\tilde{\mathfrak{A}} = \mathfrak{A}$.

§2. Generalization.

Let Σ be a commutative closed subalgebra of a Banach algebra $\tilde{\mathfrak{A}}$; p and r two idempotents which belong to the commutant of Σ and $\tilde{\mathfrak{A}}$, the Banach subalgebra of $\tilde{\mathfrak{A}}$ generated by Σ, p and r. Set $\mathcal{C} = \Sigma$. For any maximal ideal $M \in \mathsf{M}_{\mathcal{C}}$ the quotient algebra $\mathfrak{A}_M$ $(= \mathfrak{A} + J_M)$ is isomorphic to the subalgebra of Λ_M generated by two idempotents p_M and r_M. In the previous section we showed that $\mathfrak{A}_M^{m\times m} \in \mathrm{IS}(2m, \Lambda_M^{m\times m})$ for any $m \in \mathbb{N}$. Hence (see corollary 1.1) $\mathfrak{A}^{m\times m} \in \mathrm{IS}(2m, \tilde{\mathfrak{A}}^{m\times m})$. Moreover, let M be an arbitrary maximal ideal from $M_{\mathcal{C}}$ and $\{\nu_\tau^M\}$ $(\tau \in \mathcal{T}(M))$ the set of all two- and one-dimensional representations which generates an Invertibility Symbol for $\mathfrak{A}_M$ in Λ_M, then the set $\{\nu_{\tau,M}\}$ defined by the equalities $\nu_{\tau,M}(a) := \nu_\tau^M(a_M)$ $(M \in \mathsf{M}_{\mathcal{C}}, \tau \in \mathcal{T}(M))$ generates an Invertibility Symbol for $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$ the set of representations $\{\nu_{\tau,M}^{(m)}\}$ $(M \in \mathsf{M}_{\mathcal{C}}, \tau \in \mathcal{T}(M))$ generates an Invertibility Symbol for $\mathfrak{A}^{m\times m}$ in $\tilde{\mathfrak{A}}^{m\times m}$.

§3. An algebra generated by singular integral operators with piecewise continuous coefficients on the simple contour.

Let Γ be a simple closed oriented contour on the complex plane $\mathbb{C}$, $1 < p < \infty$ and $\rho : \Gamma \to \mathbb{R}_+$ a weight function such that the operator

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau) d\tau}{\tau - t}$$

is bounded in the weighted space $L_p(\Gamma, \rho)$.

Let $t_1, t_2, \ldots, t_{2s}$ be some different points on Γ which divides Γ to $2s$ arcs $\Gamma_1, \ldots, \Gamma_{2s}$. By $PC(\Gamma)$ we denote all functions $a : \Gamma \to \mathbb{C}$ piecewise continuous on Γ and continuous on $\Gamma \setminus \{t_1, \ldots, t_{2s}\}$. Let $\mathcal{A}$ denote the Banach subalgebra of $L(L_p(\Gamma, \rho))$ generated by the operator S_Γ and all operators R_a of multiplication by functions $a \in PC(\Gamma)$.

An algebra $\mathcal{A}$ can be considered as an algebra generated by an operator S_Γ, all operators R_c of multiplicaions by <u>continuous</u> functions c and an operator R_χ of multiplication by the characteristic function χ of the set $\Gamma_1 \cup \Gamma_3 \cup \cdots \cup \Gamma_{2s-1}$.

Consider the quotient algebra $\tilde{\mathbb{A}} = L(L_p(\Gamma,\rho))/\mathcal{K}$. Recall that $\mathcal{K} = \mathcal{K}(L_p(\Gamma,\rho))$ is the two-sided ideal of all compact operators acting in the space $\mathbb{B} = L_p(\Gamma,\rho)$. For any operator $A \in L(\mathbb{B})$ by $\hat{A}$ we denote the element $\hat{A} = A + \mathcal{K} \in \tilde{\mathbb{A}}$.

So we have a Banach subalgebra $\mathbb{A}$ of a Banach algebra $\tilde{\mathbb{A}}$ and $\mathbb{A}$ is generated by a commutative subalgebra $\Sigma = \{\hat{R}_c : c \in C(\Gamma)\}$ and two idempotents: $\hat{R}_x$ and $\hat{P} = (\hat{I} + \hat{S}_\Gamma)/2$ which belong to the commutator of an algebra Σ. Such algebras were discussed in the previous section. It is shown here that $\mathbb{A} \in \mathrm{IS}(2,\tilde{\mathbb{A}})$ and for any $m \in \mathbb{N}$, $\mathbb{A}^{m\times m} \in \mathrm{IS}(2m,\tilde{\mathbb{A}}^{m\times m})$. Thus an algebra $\mathcal{A}$ has a Fredholm Symbol of order 2 and for any $m \in \mathbb{N}$ the matrix algebra $\mathcal{A}^{m\times m}$ has a Fredholm Symbol of order $2m$. For the space $L_p(\Gamma,\rho)$ with the weights of the form

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\beta_k} \tag{2.5}$$

this statement as well as the explicit form of a Fredholm Symbol was obtained in [**GK1**]. In the case of general weight, the explicit form of a Fredholm Symbol for a nonclosed algebra $\mathcal{A}_0^{m\times m}$ dense in $\mathcal{A}^{m\times m}$ is given in [**Sp**]. We proved above that $\mathcal{A}^{m\times m} \in \mathrm{FS}(2m, L_p(\Gamma,\rho))$. By theorem 1.3 (see corollary 1.5) the symbol constructed in [**Sp**] for the dense subalgebra $\mathcal{A}_0^{m\times m}$ can be extended to the algebra $\mathcal{A}^{m\times m}$.

The results obtained in this section conclude the investigations of the structure of Banach algebras generated by singular integral operators with continuous and piecewise continuous coefficients on the simple contour in the spaces $L_p(\Gamma,\rho)$. The beginning of this theory was stated in [**G**].

§4. An Algebra generated by singular integral operators with piecewise continuous coefficients on a nonsimple contour..

Let Γ be a closed non-simple curve which consists of a finite number of simple arcs Γ_j without common interior points. If the point $z \in \Gamma$ belongs to n arcs Γ_j then z is a node of the multiplicity n. Let $1 < p < \infty$ and $\rho : \Gamma \to \mathbb{R}_+$ be a weight function such that the operator

$$S_\Gamma \varphi(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t}$$

is bounded in the space $\mathbb{B} = L_p(\Gamma,\rho)$ with the norm

$$\|f\|_{L_p(\Gamma,\rho)} = \|\rho^{\frac{1}{p}} f\|_{L_p(\Gamma)}.$$

In this section we denote by $\mathcal{A}$ the Banach subalgebra of $L(\mathbb{B})$ generated by an operator S_Γ, all operators R_c of multiplication by continuous functions c and all operators $R\chi_j$ of multiplication by characteristic functions χ_j of the arcs Γ_j.

THEOREM 2.2. *Let n be the maximal number of mutliplicities of the nodes of contour Γ. Then*

1. *For any $m \in \mathbb{N}$, the algebra $\mathcal{A}^{m\times m}$ has a Fredholm Symbol of order mn in the space $L_p^m(\Gamma, \rho)$.*
2. *Let $\{\gamma_\tau\}$ $(t \in \mathcal{T})$ be a set of representations of some subalgebra $\mathcal{F}$ dense in $\mathcal{A}^{m\times m}$ and*

$$\sup\{\dim \gamma_\tau : \tau \in \mathcal{T}\} < \infty.$$

If the set $\{\gamma_\tau\}$ $(\tau \in \mathcal{T})$ generates a Fredholm Symbol for the algebra $\mathcal{F}$ then any operator $A \in \mathcal{A}^{m\times m}$ is a Fredholm operator in $L_p^m(\Gamma, \rho)$ iff

$$\inf_{\tau\in\mathcal{T}} |\gamma_\tau(A)| < \infty.$$

PROOF: Let $\mathcal{A}_0$ be the nonclosed subalgebra of $L(L_p(\Gamma, \rho))$ generated by the operator S_Γ, all operators R_c $(c \in \mathcal{A}(\Gamma))$ and all operators $R\chi_j$; $\mathcal{A}$ the closure of $\mathcal{A}_0$; $\mathfrak{A} = L(L_p(\Gamma, \rho))/\mathcal{K}(L_p(\Gamma, \rho))$; $\tilde{\mathfrak{A}} = \mathcal{A}/\mathcal{K}(L_p(\Gamma, \rho))$; $\mathcal{C}$ the subalgebra of the centre $Z(\mathfrak{A})$ of $\mathfrak{A}$ which consists of all operators $\hat{R}_c = Rc + \mathcal{K}$ with $c \in C(\Gamma)$; $\mathbf{\Lambda}$ the commutator of the algebra $\mathcal{C}$ in $\tilde{\mathfrak{A}}$; M_z $(z \in \Gamma)$ the maximal ideal in $\mathcal{C}$ defined by

$$M_z = \{\hat{R}_c \in \mathcal{C} : c(z) = 0\};$$

J_z the minimal two-sided ideal in $\mathbf{\Lambda}$ which contains M_z; $\mathbf{\Lambda}_z$ the quotient algebra: $\mathbf{\Lambda}_z = \mathbf{\Lambda}/J_z$ $(:= \pi_z(\mathbf{\Lambda}))$; $\mathfrak{A}_z = \pi_z(\mathfrak{A})$ and for all $A \in \mathbf{\Lambda}$ by A_z we denote the element $A + J_z \in \mathbf{\Lambda}_z$.

In order to prove that $\mathcal{A}^{m\times m} \in \mathrm{FS}(mn, \mathbb{B}^{m\times m})$ it is enough to show that $\mathcal{A}_z^{m\times m} \in \mathrm{IS}(mn, \mathbf{\Lambda}_z^{m\times m})$ for all $z \in \Gamma$ (see lemma 1.2). Set $\mathbb{K}_z = \pi_z(\mathcal{A}_0)$. It is clear that $\mathbb{K}_z$ is a dense subalgebra of $\mathfrak{A}_z$. By theorem 1.2 (see corollary 1.2) in order to prove that $\mathfrak{A}_z^{m\times m} \in \mathrm{IS}(mn, \mathbf{\Lambda}_z^{m\times m})$ it is enough to show that the algebra $\mathbb{K}_z$ is an N-dimensional module (with $N \leq n^2$) over some subalgebra of the centre of an algebra $\mathbb{K}_z$.

Finally, in order to prove that $\mathcal{A}^{m\times m} \in \mathrm{FS}(mn, L_p^m(\Gamma, \rho))$ it is enough to present $N+2$ operators $X, Y, W_1, \ldots, W_N$ from $\mathcal{A}_0$ with the following properties:

1° $XB - BX \in \mathcal{K}(L_p(\Gamma, \rho))$ for all $B \in \mathcal{A}_0$

and

2° Every operator $A \in \mathcal{A}_0$ can be represented in the form

$$A = \sum_{j,k=1}^{N} f_{jk}(X) W_j Y W_k + T, \tag{2.6}$$

where $f_{jk}(X)$ are polynomials of X and the operator T has the following properties: there exists a neighborhood $u(z) \subset \Gamma$ of the point z and a function $h(z) \in C(\Gamma)$ such that $h(t) = 1$ for all $t \in u(z)$ and the operator $T_1 : dR_hT$ is a compact operator in the space $L_p(\Gamma, \rho)$. In other words, T is locally compact in the point z.

We are going to present such operators $X, Y, W_1, \ldots, W_N$ (with $N = N(z)$).

If z is not a node of the contour Γ we can take $N = 2$, $X = Y = I$, $W_1 = I$ and $W_2 = S_\Gamma$.

Let z be a node of the contour Γ. Since Γ is a closed contour it splits the extended complex plane into two open sets F^+ and F^- such that Γ constitutes the boundary for both of them ([**GK4**], p.15). It follows from here that the node z is the intersection of $2r$ arcs $\ell_1, \ldots, \ell_{2r}$, half of them $\ell_1, \ell_3, \ldots, \ell_{2r-1}$ are directed away from the point z and $\ell_2, \ell_4, \ldots, \ell_{2r}$ are directed to the point z. Set

$$\varphi_i(t) = \chi_{\ell_i}(t),\ P_\Gamma = (I + S_\Gamma)/2,\ Q_\Gamma = I - P_\Gamma,\ Y = \sum_{i=1}^{r} (\varphi_{2i-1} P_\Gamma + \varphi_{2i} Q_\Gamma),$$
$$W_{2i} = (\varphi_{2i} + \varphi_{2i+1}) Q_\Gamma,\ W_{2i-1} = (\varphi_{2i-1} + \varphi_{2i}) P_\Gamma$$

and

$$X = \sum_{i=1}^{2r} W_i Y W_i.$$

In order to show that these operators satisfy the required conditions we will first prove that

$$\mathcal{A}_0 \cap \mathcal{K}(L_p(\Gamma, \rho)) = \mathcal{A}_0 \cap \mathcal{K}(L_2(\Gamma)). \tag{2.7}$$

We start with the case $p \neq 2$. In this case we set $\mathbb{B}_s = L_s\left(\Gamma, \rho^{\frac{s-2}{p-2}}\right)$ $(1 < s < \infty)$. In particular, $\mathbb{B}_p = L_p(\Gamma, \rho)$ and $\mathbb{B}_2 = L_2(\Gamma)$ (here $s = 2$ and $p \neq 2$). Since an operator S_Γ is bounded in the space $L_p(\Gamma, \rho)$ it is bounded in the space $\mathbb{B}_s$ for all s from some ε-neighborhood $\Delta(\varepsilon)$ of the segment $\Delta = [\min(p, p(p-1)^{-1}), \max(p, p(p-1)^{-1}]$. This statement for $s \in \Delta$ was obtained in [**S**] and for $s \in \Delta(\varepsilon)$ in [**HMW**]. It is clear that the algebra $\mathcal{A}_0$ does not depend on s. Moreover, if $T \in \mathcal{A}_0$ and T is compact in one of the spaces $\mathbb{B}_s$ $(s \in \Delta(\varepsilon))$, T is compact in every $\mathbb{B}_s$ $(s \in \Delta(\varepsilon))$, [**Kra**]. Thus for the case $p \neq 2$, assertion (2.7) holds. In the case $p = 2$ the proof is the same. One must only consider the scale of spaces $\mathbb{B}_{2,\tau} = L_2(\Gamma, \rho^\tau)$ with τ from some ε-neighborhood of the segment $\Delta = [-1, 1]$ instead of the scale $\mathbb{B}_s$ considered above. Thus assertion (2.7) is proved and it follows from (2.7) that one can restrict ourselves to the space $L_2(\Gamma)$ when checking the required properties of the operators $X, Y, W_1, \ldots, W_{2r}$ in the space $L_p(\Gamma, \rho)$.

In the space $L_2(\Gamma)$ the algebra $\mathcal{A}$ is well investigated (see [**GK2**,5], [**C**], [**RS**]). It has a Fredholm Symbol , it is semisimple and the explicit form of the Symbol is known. The operator T in this case is locally compact in the point z iff the local value of the Symbol of the operator T in the point $t = z$ is zero (i.e. a zero matrix). Using, for example, the notations from [**GK2**] one can see that $\gamma_{z,\mu}(X) = \mu E$ (E is an identity matrix), $\gamma_{z,\mu}(W_i) = (a_{jk})_{j,k=1}^{2r}$, where $a_{ii} = 1$ and $a_{jk} = 0$ for $(j,k) \neq (i,i)$, $Y = (y_{jk}(\mu))_{j,k=1}^{2r}$ and for any operator $A \in \mathcal{A}_0$

$$\gamma_{z,\mu}(A) = (f_{jk}(\mu) y_{jk}(\mu))_{j,k=1}^{2r} = \gamma_{z,\mu}\left(\sum_{j,k=1}^{2r} f_{jk}(X) W_j Y W_k\right),$$

where f_{jk} are some polynomials. Thus for the operator

$$T = A - \sum_{j,k=1}^{2r} f_{jk}(X) W_j Y W_k$$

we have $\gamma_{z,\mu}(T) = 0$ for all $0 \leq \mu \leq 1$. The first assertion of the theorem is proved. The second assertion follows from the first assertion and theorem 1.3.

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