Calc. Var. 3, 139–153 (1995)
© Springer-Verlag 1995

Elliptic equations in R^2 with nonlinearities in the critical growth range

D.G. de Figueiredo[1,*], O.H. Miyagaki[2,*], B. Ruf[3]

[1] IMECC-UNICAMP, Caixa Postal 6065, 13081 Campinas, S.P., Brasile
[2] Departamento de Matemática, Universidade Federal de Uberlândia, 38400 Uberlândia, M.G.
[3] Dipartimento di Matematica, Università degli Studi, I-20133 Milano, Italy

Received June 10, 1993 / Accepted January 3, 1994

1. Introduction

In this paper we study the solvability of problems of the type

$$(1.1) \qquad -\Delta u = f(x,u) \quad \text{in} \quad \Omega \quad , \quad u=0 \quad \text{on} \quad \partial\Omega,$$

where Ω is some bounded domain in R^2, and the function $f(x,s)$ has the maximal growth on s which allows to treat problem (1.1) variationally in $H_0^1(\Omega)$. More precisely, we treat the so-called subcritical case and also the critical case, which we define next.

We say that f has *subcritical growth* at $+\infty$ if for all $\alpha > 0$

$$(1.2) \qquad \lim_{t\to+\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = 0$$

and f has *critical growth* at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$(1.3) \lim_{t\to+\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = 0 \ , \ \forall\alpha > \alpha_0 \ ; \ \lim_{t\to+\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = +\infty \ , \ \forall\, \alpha < \alpha_0.$$

Similarly we define subcritical and critical growth at $-\infty$. This notion of criticality is motivated by the so-called Trudinger-Moser inequality [12,9] which says that if u is a $H_0^1(\Omega)$ function then the integral $\int e^{u^2}$ is finite.

Problems of the above type have been studied recently by several authors, Atkinson-Peletier [5], Carleson-Chang [7] , Adimurthi et al. [1], [2], [3], [4]. In this paper we improve the existence conditions in [2], and extend the results to the nonsymmetrical case and to more general nonlinearities. Also we propose a

* These authors have been partially supported by CNPq

unified approach by putting all questions in the framework of the by now classical Critical Point Theory as first developed in papers by Ambrosetti-Rabinowitz and Rabinowitz. For a complete reference of the results used here, see Rabinowitz [11; Theorems 2.2, 5.3 and 9.12]. As usual in the applications the hard points are the verification of conditions which allow the use of this Critical Point Theory, in particular the Palais-Smale condition.

In the results for nonlinearities with *critical growth* (Theorems 1.3 and 1.4 below) we give sufficient conditions for the existence of solutions. The proofs of these results follow the ideas introduced by H. Brezis and L. Nirenberg [6] in their pioneering work on the solvability of equations with critical growth in dimensions larger than 2. In fact, one observes that (as in their case) the functional under consideration satisfies the Palais-Smale condition only at certain levels. In order to assure that the constructed minimax levels are inside the Palais-Smale region we use test functions connected with the optimal Trudinger-Moser inequality (while Brezis- Nirenberg used test functions related to the optimal Sobolev imbedding).

For easy reference we state now conditions on f that will be assumed in all theorems below.

(H1) $$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is continuous, } f(x,0) = 0.$$

(H2) $\exists\ t_0 > 0\ ,\ \exists M > 0$ such that
$$0 < F(x,t) = \int_0^t f(x,s)ds \le M\,|f(x,t)|\ ,\quad \forall |t| \ge t_0, \forall x \in \Omega.$$

(H3) $$0 < F(x,t) \le \frac{1}{2}\ f(x,t)t\ ,\quad \forall\ t \in \mathbb{R} - \{0\}\ ,\quad \forall x \in \Omega.$$

Now we state the results which will be proved here. We denote by $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ the eigenvalues of $(-\Delta, H_0^1(\Omega))$. By "solution" in the theorems below we mean weak solution $u \in H_0^1(\Omega)$.

Theorem 1.1. (The subcritical case, local minimum at 0). *Assume (H1), (H2), (H3) and that f has subcritical growth at both $+\infty$ and $-\infty$. Furthermore suppose that*

(H4) $$\limsup_{t \to 0} \frac{2\,F(x,t)}{t^2}\ <\ \lambda_1\ , \text{ uniformly in } (x,t).$$

Then, problem (1.1) has a nontrivial solution. Moreover if $f(x,t)$ is an odd function in t, then (1.1) has infinitely many solutions.

Theorem 1.2. (The subcritical case, saddle at 0). *Assume (H1), (H2), (H3) and that f has subcritical growth at both $+\infty$ and $-\infty$. Furthermore suppose that*

(H5) $$\exists \delta > 0\ ,\ \exists\ \lambda_k \le \mu < \lambda_{k+1} \text{ such that } F(x,t) \le \frac{1}{2}\ \mu t^2\ ,$$
$$\forall x \in \Omega, \forall |t| \le \delta.$$

(H6) $$F(x,t) \geq \frac{1}{2} \lambda_k t^2 \qquad \forall x \in \Omega \ , \ \forall t \in \mathbb{R}.$$

Then, problem (1.1) has a nontrivial solution. Moreover, if instead of (H6) we assume that $f(x,t)$ is an odd function in t, then (1.1) has infinitely many solutions.

For the problem with critical growth, Adimurthi [2] showed that (1.1) has a solution provided f satisfies (among other conditions) the asymptotic hypothesis $\lim_{t\to\infty} f(x,t)te^{-\alpha_0 t^2} = +\infty$. The next two theorems improve and generalize this result. In order to state these theorems we introduce the following notations: for $0 \leq \varepsilon < 1$

$$M_\varepsilon = \lim_{n\to\infty} \int_0^1 ne^{n[(1-\varepsilon)^2 t^2 - t]} dt$$
$$\widehat{M} = \lim_{\varepsilon\to 0} M_\varepsilon$$

Numerical calculations indicate that $M_0 = 2$ and $\widehat{M} = 1$. We recall the concept of inner radius of a set Ω:

$$d =: \text{radius of the largest open ball} \subset \Omega.$$

Theorem 1.3. (The critical case, local minimum at 0). *Assume $(H1), (H2), (H3)$ and that f has critical growth at both $+\infty$ and $-\infty$. Furthermore assume $(H4)$ and*

(H7) $$\lim_{t\to+\infty} f(x,t)e^{-\alpha_0 t^2} t \geq \beta \ , \quad \beta > \frac{4}{(1+M_0)\alpha_0 d^2}$$

Then, problem (1.1) has a nontrivial solution.

Theorem 1.4. (The critical case, saddle point at 0). *Assume $(H1), (H2), (H3)$ and that f has critical growth at both $+\infty$ and $-\infty$. Furthermore assume $(H5), (H6)$ and*

(H8) $$\lim f(x,t)e^{-\alpha_0 t^2} t \geq \beta \ , \quad \beta > \frac{4}{\widehat{M}\alpha_0 d^2} \ ,$$

(H9) $\exists \delta > 0$ and $C > 0$ such that $|f(x,t)| \leq C|t|, \forall x \in \Omega, \forall |t| \leq \delta$.

Then problem (1.1) has a nontrivial solution.

Remarks on the conditions above. Conditions (H1) and (H2) imply an exponential growth in t for both $f(x,t)$ and $F(x,t)$, see relation (2.2) later on.
Condition (H3) is satisfied if we assume as in [2] that f is C^1 and that $f'(x,t) \geq f(x,t)t^{-1}$, for all $t \neq 0$. Indeed, such a condition implies that $f(x,t)t^{-1}$ is a nondecreasing function of t, and from this fact (H3) follows readily. Condition (H5) is satisfied if one assumes as in [2] that f is C^1 and

$$\lambda_k \leq \inf_{x\in\overline{\Omega}} f'(x,0) \leq \sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_{k+1}$$

In order to see this one uses the mean value theorem and the continuity of f' at $s = 0$. If one assumes in addition, that $f'(x,t) \geq f(x,t)t^{-1}$, for all $t \neq 0$, then condition (H6) follows. Finally condition (H9) is satisfied if f is C^1.

2. The variational formulation

We assume (H1), (H2) and the existence of positive constants C and β such that

$$(2.1) \qquad |f(x,t)| \leq C\, e^{\beta t^2} \qquad \forall\, x \in \Omega \;,\; \forall\, t \in \mathbb{R}.$$

(In particular, this is the case if f has subcritical or critical growth). Then

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(x,u)$$

is a C^1 functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$, and

$$< \Phi'(u), v >= \int \nabla u \nabla v - \int f(x,u)v \quad , \quad \forall v \in H_0^1(\Omega),$$

where $< .,. >$ denotes the inner product in H_0^1 (we also write $\|.\|$ for the corresponding H_0^1-norm). These statements follow from the fact that e^{v^2} is $L^1(\Omega)$ for all $v \in H_0^1(\Omega)$, see [12,9]. So in view of (2.1) we conclude that $f(x,u(x))$ is in $L^q(\Omega)$, for all $q > 1$, when $u \in H_0^1(\Omega)$.

It follows easily from (H1) and (H2) that
(i) there is a constant $C > 0$ such that

$$(2.2) \qquad F(x,t) \geq Ce^{\frac{1}{M}|t|} \qquad \forall \;\; |t| \geq t_0;$$

(ii) given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$(2.3) \qquad F(x,t) \leq \varepsilon f(x,t)t \qquad \forall x \in \Omega, \quad \forall |t| \geq t_\varepsilon.$$

Proposition 2.1. *Assume (H1), (H2), (H3) and (2.1). Then the functional Φ satisfies $(PS)_c$ for all $c < \frac{2\pi}{\beta}$.*

Corollary 2.1. *Assume (H1), (H2) and (H3). If f has subcritical growth at both $+\infty$ and $-\infty$ then Φ satisfies $(PS)_c$ for all $c \in \mathbb{R}$. If f has critical growth at both $+\infty$ and $-\infty$ with the same α_o , then Φ satisfies $(PS)_c$ for all $c \in (-\infty, \frac{2\pi}{\alpha_0})$.*

Proof of Proposition 2.1. Let $(u_n) \subset H_0^1(\Omega)$ be a Palais-Smale sequence, i.e.

$$(2.4) \qquad \frac{1}{2}\int |\nabla u_n|^2 - \int F(x,u_n) \to c$$

$$(2.5) \qquad \left|\int \nabla u_n \nabla v - f(x,u_n)v\right| \leq \varepsilon_n \|v\|, \qquad \forall v \in H_0^1(\Omega),$$

where $\varepsilon_n \to 0$. It follows from (2.4) using (2.3) that, for any $\varepsilon > 0$,

$$\frac{1}{2}\|u_n\|^2 \;\leq\; C + \int F(x,u_n) \leq C_\varepsilon + \varepsilon \int f(x,u_n)u_n,$$

and using (2.5) we obtain

$$\frac{1}{2}\|u_n\|^2 \leq C_\varepsilon + \varepsilon\|u_n\|^2 + \varepsilon_n\|u_n\|,$$

which implies that

$$(2.6) \qquad \|u_n\| \leq C \ , \quad \int f(x,u_n)u_n \leq C \quad \int F(x,u_n) \leq C.$$

Observe that from (H3) the two integrals above are nonnegative. Now we take a subsequence denoted again by (u_n) such that, for some $u \in H_0^1$, we have

$$u_n \rightharpoonup u \ \text{ in } H_0^1 \ ; \ u_n \to u \ \text{ in } L^q(\Omega), \ \forall\, q \geq 1 \ ; \ u_n(x) \to u(x) \ \text{ a.e. in } \Omega.$$

Next we assume the following result, which will be proved later.

Lemma 2.1. *$f(x,u_n) \to f(x,u)$ in $L^1(\Omega)$.*

It follows from (H2) and Lemma 2.1, using the generalized Lebesgue dominated convergence theorem, that $F(x,u_n) \to F(x,u)$ in $L^1(\Omega)$. From (2.4) and (2.5) we obtain

$$(2.7) \ \lim \|u_n\|^2 = 2(c + \int F(x,u)) \ ; \ \lim \int f(x,u_n)u_n = 2(c + \int F(x,u)).$$

Using (H3) and (2.7) we conclude that $c \geq 0$. So any Palais-Smale sequence approaches a nonnegative level. It follows from Lemma 2.1 and relation (2.5) that

$$\int \nabla u \nabla \psi = \int f(x,u)\psi \qquad \forall\ \psi \in D(\Omega).$$

Since $f(x,u) \in L^2(\Omega)$ we conclude that $u \in H^2(\Omega)$ and $-\Delta u = f(x,u)$ in the strong sense. Hence

$$\int |\nabla u|^2 = \int f(x,u)u \geq 2\int F(x,u).$$

So $\Phi(u) \geq 0$. Now we separate the proof into three cases.

Case 1. $c = 0$. If this is the case we have using (2.7)

$$0 \leq \Phi(u) \leq \liminf \Phi(u_n) = \int F(x,u) - \int F(x,u) = 0.$$

So $\|u_n\| \to \|u\|$ and then $u_n \to u$ in H_0^1. The proof is finished in this case.

Case 2. $c \neq 0$, $u = 0$. We show that this cannot happen for a Palais-Smale sequence. First we claim that, for some $q > 1$, we have

$$(2.8) \qquad \int |f(x,u_n)|^q \leq \ \text{const.}$$

Assume that (2.8) has been proved. Then using (2.5) with $v = u_n$ we have

(2.9) $$\left| \int |\nabla u_n|^2 - \int f(x,u_n)u_n \right| \le \varepsilon_n \|u_n\| \le C\varepsilon_n \ .$$

We estimate the second integral above using Hölder's inequality. And then from (2.8) and the fact that $u_n \to 0$ in $L^{q'}$, we conclude that $\|u_n\| \to 0$. This contradicts (2.7), which says, in this case , that $\|u_n\|^2 \to 2c \neq 0$. So it remains to prove (2.8). Since $u = 0$, it follows form (2.7) that, given $\varepsilon > 0$, $\|u_n\|^2 \le 2c+\varepsilon$, for large n. So we estimate the integral in (2.8) using (2.1)

(2.10) $$\int |f(x,u_n)|^q \le C \int e^{q\beta u_n^2} = C \int e^{q\beta \|u_n\|^2 (\frac{u_n}{\|u_n\|})^2}.$$

By the sharpened form of Trudinger's inequality proved by Moser [9], the integral in (2.10) is bounded, independently of n, if $q\beta\|u_n\|^2 < 4\pi$. But this will be indeed the case for $c < \frac{2\pi}{\beta}$, if we choose $q > 1$ sufficiently close to 1 and ε sufficiently small.

Case 3. $c \neq 0$, $u \neq 0$. In this case we claim that

(2.11) $$\Phi(u) = c.$$

If this is the case, it follows from (2.7) that $\|u_n\| \to \|u\|$ and the proof is also finished in this case. So it remains to prove (2.11). Assume by contradiction that $\Phi(u) < c$.
Then

(2.12) $$\|u\|^2 < 2(c + \int F(x,u))$$

Let $v_n = u_n/\|u_n\|$ and $v = u/\sqrt{2(c + \int F(x,u))}$. Since $v_n \rightharpoonup v$, $\|v_n\| = 1$, and $\|v\| < 1$, it follows by a result of Lions [8] that

(2.13) $$\sup \int e^{4\pi p v_n^2} < \infty \quad , \quad \forall p < \frac{1}{1 - \|v\|^2}.$$

Now we estimate the L^q-norm of $f(x,u_n)$ using (2.1)

$$\int |f(x,u_n)|^q \le C \int e^{q\beta\|u_n\|^2 v_n^2}$$

and this will be bounded if q and p can be chosen such that

$$q\beta\|u_n\|^2 \le 4\pi p < 4\pi \frac{c + \int F(x,u)}{c - \Phi(u)} = 4\pi \frac{1}{1 - \|v\|^2}.$$

This will be the case for large n if

$$\frac{\beta}{2\pi} < \frac{1}{c - \Phi(u)}$$

which is actually so, since $\Phi(u) \ge 0$ and $c < \frac{2\pi}{\beta}$.

Finally, using the fact just proved that $f(x,u_n)$ is bounded in some L^q, we see as in case 2 that $u_n \to u$ in $H_0^1(\Omega)$. This is impossible in view of (2.7) and (2.12). Thus, the proof of Proposition 2.1 is complete. □

Now we give the complete statement of lemma 2.1 and its respective proof.

Lemma 2.1. *Let (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x,u_n(x))$ and $f(x,u(x))$ are also L^1 functions. If*

$$\int |f(x,u_n(x))u_n(x)| \leq C_1$$

then $f(x,u_n)$ converges in L^1 to $f(x,u)$.

Proof. It suffices to prove $\int |f(x,u_n)| \to \int |f(x,u)|$, cf. [10, p.89]. Since $f(x,u(x)) \in L^1(\Omega)$ it follows that given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_A |f(x,u(x)| \leq \varepsilon \qquad \text{if} \qquad |A| \leq \delta \tag{2.14}$$

for all measurable subsets A of Ω. We use $|\cdot|$ to denote the Lebesgue measure. Next using the fact that $u \in L^1(\Omega)$ we find $M_1 > 0$ such that

$$|\{x \in \Omega : |u(x)| \geq M_1\}| \leq \delta. \tag{2.15}$$

Let $M = \max\{M_1, C_1/\varepsilon\}$. We write

$$\left| \int |f(x,u_n)| - \int |f(x,u)| \right| \leq I_1 + I_2 + I_3$$

and estimate each integral separately.

$$I_1 \equiv \int_{|u_n|\geq M} |f(x,u_n)| = \int_{|u_n|\geq M} \frac{f(x,u_n)u_n}{|u_n|} \leq \frac{C_1}{M} \leq \varepsilon.$$

By the choices made above we have

$$I_3 \equiv \int_{|u|\geq M} |f(x,u)| \leq \varepsilon$$

Next we claim that

$$I_2 \equiv \int_{|u_n|<M} |f(x,u_n)| - \int_{|u|<M} |f(x,u)| \longrightarrow 0 \text{ as } n \to \infty$$

Indeed, $g_n(x) \equiv |f(x,u_n(x))|\chi_{|u_n|<M} - |f(x,u(x))|\chi_{|u|<M}$ tends to 0 a.e. in Ω. Moreover $|g_n(x)| \leq |f(x,u(x))|$, if $|u_n| \geq M$ and $|g_n(x)| \leq C + |f(x,u(x))|$, if $|u_n(x)| < M$, where $C = \sup\{|f(x,t)| : x \in \overline{\Omega}, |t| < M\}$. So, the claim follows from the Lebesgue dominated convergence theorem. □

The next proposition concerns the behaviour of Φ at ∞.

Proposition 2.2. *Assume (H1) and (H2). Let Z be a finite dimensional subspace of $H_0^1(\Omega)$ spanned by L^∞ functions. Then Φ is bounded above in Z, and moreover, given $M > 0$ there is an $R > 0$ such that*

$$\Phi(u) \leq -M, \qquad \forall\, \|u\| \geq R \qquad u \in Z.$$

Proof. Given $u_0 \in Z$ with $\|u_0\|_{L^\infty} = 1$, let us define

$$\xi(t) = \Phi(tu_0) = \frac{t^2}{2}\int |\nabla u_0|^2 - \int F(x, tu_0) \qquad \forall\ t \in R.$$

It follows from (2.2) that, for $p > 2$, there is a constant $C > 0$ such that $F(x,t) \geq C|t|^p - C$ for all x and t. So

$$\xi(t) \leq \frac{t^2}{2}\|u_0\|^2 - C|t|^p \int |u_0|^p - C|\Omega|.$$

By the equivalence of norms in Z, we obtain

$$\xi(t) \leq \frac{t^2}{2}\|u_0\|^2 - C|t|^p\|u_0\|^p - C$$

which implies that $\xi(t) \to -\infty$ as $t \to \infty$. The result follows by compactness. □

Next we study the behavior of the functional Φ near $u = 0$.

Proposition 2.3. *Assume (H1), (H2), (H4) and condition (2.1). Then there exist $a > 0$ and $\rho > 0$ such that*

$$\Phi(u) \geq a \qquad \text{if} \qquad \|u\| = \rho.$$

Proof. From (H4) we have that there exist $\mu < \lambda_1$ and $\delta > 0$ such that

$$F(x,t) \leq \frac{1}{2}\ \mu\, t^2 \quad , \text{if} \quad |t| \leq \delta.$$

On the other hand, from (2.1) we obtain for $q > 2$

$$F(x,t) \leq Ce^{\beta t^2}|t|^q \quad , \quad \text{if} \quad |t| > \delta,$$

Putting these two estimates together we obtain

$$F(x,t) \leq \frac{1}{2}\ \mu\, t^2 + Ce^{\beta t^2}|t|^q \qquad \forall\ \ t \in \mathbb{R},$$

which implies

$$\begin{aligned} \Phi(u) &\geq \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\mu \int u^2 - C\int e^{\beta u^2}|u|^q \\ &\geq \frac{1}{2}(1 - \frac{\mu}{\lambda_1})\|u\|^2 - C\left(\int e^{\beta p u^2}\right)^{1/p}\left(\int |u|^{qp'}\right)^{1/p'} \end{aligned}$$

Next we observe that

$$\int e^{\beta p u^2} = \int e^{\beta p \|u\|^2 (\frac{u}{\|u\|})^2} < \quad \text{const}$$

if $\|u\| \leq \delta$, where $\beta p \delta < 4\pi$. So

$$\Phi(u) \geq \frac{1}{2}(1 - \frac{\mu}{\lambda_1})\|u\|^2 - C\|u\|^q$$

Now choose $\rho > 0$ as the point where the function $g(s) = \frac{1}{2}(1 - \frac{\mu}{\lambda_1})s^2 - Cs^q$ assumes its maximum. Take $a = g(\rho)$. □

The above proposition will be used in the proofs of Theorems 1.1 and 1.3. For the proofs of the other theorems we shall need the next result. Let us denote by V the subspace of $H_0^1(\Omega)$ generated by the eigenfunctions φ_j of $(-\Delta,\ H_0^1(\Omega))$ corresponding to the eigenvalues λ_j for $j = 1, \dots, k$. Let $W = V^{\perp}$.

Proposition 2.4. *Assume* $(H1)$, $(H2)$, $(H5)$ *and condition (2.1). Then, there exist* $a > 0$ *and* $\rho > 0$ *such that*

$$\Phi(u) \geq a \quad \text{if} \quad ||u|| = \rho \quad \text{and} \quad u \in W$$

Proof. The proof is completely analogous to the previous one; here we use the variational characterization of λ_{k+1}.

3. Proof of Theorems 1.1 and 1.2.

It follows from the hypotheses in both theorems that Φ satisfies $(PS)_c$ for all $c \in \mathbb{R}$, cf. Corollary 2.1. To finish the proof of Theorem 1.1 we use Propositions 2.2 and 2.3, and apply the Mountain Pass Theorem. To prove theorem 1.2 we apply the Generalized Mountain Pass Theorem. Proposition 2.4 and the following argument are used. Let $R > \rho$ and such that $\Phi(u) \leq 0$ for $||u|| \geq R$ and $u \in V \oplus \mathbb{R}\varphi_{k+1}$, see Proposition 2.2. Let $Q = \{v + s\varphi_{k+1} : ||v|| \leq R,\ \ 0 \leq s \leq R\}$ and ∂Q its relative boundary in $V \oplus \mathbb{R}\varphi_{k+1}$. Clearly $\Phi(u) \leq 0$ in $\partial Q \cap \{u : ||u|| \geq R\}$. For $u \in \partial Q \cap V$, we use $(H6)$ to see that $\Phi(u) \leq 0$. If the function $f(x,t)$ is odd in t, then we apply Theorem 9.12 of [11] in order to conclude the existence of infinitely many solutions. □

4. Proof of Theorem 1.3.

It follows from the assumptions that Φ satisfies $(PS)_c$ for all $c < \frac{2\pi}{\alpha_0}$, see Corollary 2.1. From $(H4)$ we see that Φ has a local minimum at 0, see Proposition 2.3. To conclude via the Mountain Pass Theorem it suffices to show that there is a $\omega \in H_0^1$, $||\omega|| = 1$, such $\max\,\{\Phi(t\omega) : t \geq 0\} < \dfrac{2\pi}{\alpha_0}$. For that matter we start by introducing the following functions

$$\overline{\omega}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2}, & 0 \leq |x| \leq \dfrac{1}{n} \\[2ex] \dfrac{\log \frac{1}{|x|}}{(\log n)^{1/2}}, & \dfrac{1}{n} \leq |x| \leq 1 \\[2ex] 0 & |x| \geq 1 \end{cases}$$

We see that $\overline{\omega}_n \in H_0^1(B_1(0))$ and $||\overline{\omega}_n|| = 1$ for all $n = 1, 2, \dots$ Here $B_1(0)$ denotes the ball of radius 1 centered at the origin of R^2. Next let d be the inner radius of Ω and $x_0 \in \Omega$ such that $B_d(x_0) \subset \Omega$. We then define the functions

$$\omega_n(x) = \overline{\omega}_n\left(\frac{x - x_0}{d}\right),$$

which are in $H_0^1(\Omega)$, $||\omega_n|| = 1$ and $\mathrm{supp}\omega_n \subset B_d(x_0)$.

We claim that there exists n such that

$$\max\{\Phi(t\omega_n) : t \geq 0\} < \frac{2\pi}{\alpha_0}.$$

Suppose by contradiction that this is not the case. So, for all n, this maximum (it is indeed a maximum, in view of Proposition 2.2) is larger or equal to $\frac{2\pi}{\alpha_0}$. Let $t_n > 0$ be such that

$$\max\{\Phi(t\omega_n) : t \geq 0\} = \Phi(t_n\omega_n) \geq \frac{2\pi}{\alpha_0}. \tag{4.1}$$

It follows readily from (4.1) and $(H3)$ that

$$t_n^2 \geq \frac{4\pi}{\alpha_0}. \tag{4.2}$$

Also at $t = t_n$, we have $\frac{d}{dt}\Phi(t\omega_n) = 0$, i.e.

$$t_n - \int f(x, t_n\omega_n)\omega_n = 0 \ ,$$

which implies that

$$t_n^2 \geq \int_{B_d(x_0)} f(x, t_n\omega_n)t_n\omega_n. \tag{4.3}$$

Now, it follows from $(H7)$ that given $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$f(x, s)s \geq (\beta - \varepsilon)e^{\alpha_0 s^2}, \ \ \forall s \geq s_\varepsilon.$$

So from (4.3) we obtain, for large n

$$\begin{aligned} t_n^2 \ &\geq \ (\beta - \varepsilon)\pi\frac{d^2}{n^2}e^{\alpha_0 t_n^2 \frac{1}{2\pi}\log n} = \\ &= \ (\beta - \varepsilon)\pi d^2 e^{2\log n(\frac{\alpha_0 t_n^2}{4\pi} - 1)} \end{aligned} \tag{4.4}$$

which implies readily that t_n is bounded. And moreover (4.4) together with (4.2) gives that $t_n^2 \to 4\pi/\alpha_0$.

Next let us estimate (4.3) more precisely. Let

$$A_n = \{x \in B_d(x_0) : t_n\omega_n(x) \geq t_\varepsilon\} \quad , \quad B_n = B_d(x_0)\backslash A_n \ ,$$

and break the integral in (4.3) into a sum of the integrals over A_n and B_n. Using (H7) we estimate (4.3):

$$t_n^2 \geq (\beta - \varepsilon)\int_{B_d(x_0)} e^{\alpha_0 t_n^2\omega_n^2} + \int_{B_n} f(x, t_n\omega_n)t_n\omega_n - (\beta - \varepsilon)\int_{B_n} e^{\alpha_0 t_n^2\omega_n^2} \tag{4.5}$$

Since $t_n\omega_n(x) < t_\varepsilon$ for $x \in B_n$, we see that the characteristic functions $\chi_{B_n} \to 0$ a.e. in $B_d(x_0)$ as $n \to \infty$. Hence, the two last integrals in (4.5) go to 0 as $n \to \infty$, in view of the Lebesgue Dominated Convergence Theorem. Passing to the limit in (4.5) we obtain

$$(4.6) \qquad \frac{4\pi}{\alpha_0} \geq (\beta - \varepsilon)\lim \int_{B_d(x_0)} e^{\alpha_0 t_n^2 \omega_n^2} \geq (\beta - \varepsilon)\lim \int_{B_d(x_0)} e^{4\pi\omega_n^2}$$

The last integral in (4.6), denote it by I_n, is evaluated as follows:

$$(4.7) \qquad I_n = d^2 \int_{B_1(0)} e^{4\pi\overline{\omega}_n^2} = d^2 \left\{ \frac{\pi}{n^2} e^{4\pi\frac{1}{2\pi}\log n} + 2\pi \int_{1/n}^{1} e^{4\pi\frac{1}{2\pi}\frac{(\log\frac{1}{r})^2}{\log n}} rdr \right\}$$

Changing variables in the integral in (4.7), $s = \log\left(\frac{1}{r}\right) / \log n$, we obtain

$$I_n = d^2 \left\{ \pi + 2\pi \log n \int_0^1 e^{2s^2\log n - 2s\log n} ds \right\}$$

So finally from (4.6) we get

$$\frac{4\pi}{\alpha_0} \geq (\beta - \varepsilon) d^2 \pi (1 + M_0) \quad , \quad \forall \varepsilon > 0,$$

which implies $\beta \leq 4/\alpha_0 d^2(1 + M_0)$, a contradiction to (H7). □

5. Proof of Theorem 1.4.

It follows from the assumptions that Φ satisfies $(PS)_c$ for all $c < \frac{2\pi}{\alpha_0}$, see Corollary 2.1. The proof is accomplished by the use of the Generalized Mountain Pass Theorem. For that matter we have to select a $\omega \in W$, $||\omega|| = 1$, such that for some $R > \rho$, [the ρ of Proposition 2.4], one has the set

$$Q = \{v + s\omega : ||v|| \leq R,\ v \in V,\ 0 \leq s \leq R\}$$

with the properties: (p_1) $\Phi|_{\partial Q} \leq 0$ and (p_2) $\Phi(u) < \frac{2\pi}{\alpha_0}$, $\forall u \in Q$. Property (p_1) follows easily from $(H6)$ and Proposition 2.2; in fact this part is true no matter which ω we choose in W. However, in order to get (p_2) we choose a sequence ω_n like in the proof of Theorem 1.3. We then need to show that there is $n > \rho$ such that

$$\max\{\Phi(v + s\omega_n) : ||v|| \leq n,\ 0 \leq s \leq n\} < \frac{2\pi}{\alpha_0}.$$

Assume, by contradiction, that this is not the case. So for all n this maximum is $\geq \frac{2\pi}{\alpha_0}$. Let $u_n = v_n + t_n\omega_n$ be the point where this maximum is achieved. So

$$(5.1) \qquad \Phi(u_n) \geq \frac{2\pi}{\alpha_0}$$

(5.2) $$||u_n||^2 - \int f(x, u_n)u_n = 0$$

where we have used the fact that the derivative of Φ, restricted to $V \oplus \mathbb{R}\omega_n$, is zero at u_n. A contradiction will be obtained after the proof of the following assertions.

Assertion 1. (v_n), (t_n) *are bounded sequences. See proof later.*

So we may assume that $v_n \to v_0$ and $t_n \to t_0$.

Assertion 2. $v_0 = 0$ *and* $t_0^2 = \frac{4\pi}{\alpha_0}$. *See proof later.*

In view of Assertion 2, the part of the integral in (5.2) over $\Omega \setminus B_d(x_0)$ goes to 0.

So

(5.3) $$\lim ||u_n||^2 = \lim \int_{B_d(x_0)} f(x, u_n)u_n$$

The integral in (5.3), denote it by I_n, can be estimated as in the previous theorem

(5.4) $$I_n \geq (\beta - \varepsilon) \int_{B_d(x_0)} e^{\alpha_0 u_n^2} + \int_{B_n} f(x, u_n)u_n - (\beta - \varepsilon) \int_{B_n} e^{\alpha_0 u_n^2}$$

Again the characteristic functions of B_n go to zero a.e, and consequently the last two integrals in (5.4) go to 0. Now given $\varepsilon > 0$, there is $c(\varepsilon)$ such that $u_n^2 \geq (1 - \varepsilon)t_n^2\omega_n^2 - c(\varepsilon)v_n^2$. Since $t_n^2 \to 4\pi/\alpha_0$, we get for large n

$$u_n^2 \geq (1 - \varepsilon)^2 \frac{4\pi}{\alpha_0} \omega_n^2 - c(\varepsilon)||v_n||_{L^\infty}^2 \; .$$

So the first integral in (5.4) is estimated from below by

(5.5) $$(\beta - \varepsilon)e^{-c(\varepsilon)||v_n||_{L^\infty}^2} \int_{B_d(x_0)} e^{(1-\varepsilon)^2 4\pi\omega_n^2}$$

The integral in (5.5), denote it by J_n, can be evaluated as follows

(5.6) $$J_n = d^2 \int_{B_1(0)} e^{(1-\varepsilon)^2 4\pi\overline{\omega}_n^2} = d^2 \left\{ \frac{\pi}{n^2} e^{(1-\varepsilon)^2 \log n} + 2\pi \int_{1/n}^1 e^{(1-\varepsilon)^2 2(\log \frac{1}{r})^2 (\log n)^{-1}} r \, dr \right.$$

Using a change of variables as before we prove that the second integral in (5.6) goes to $\pi\widehat{M}$. So $J_n \to \pi\widehat{M}$. Hence

(5.7) $$\lim J_n \geq (\beta - \varepsilon)d^2\pi\widehat{M}$$

On the other hand, for any ε,

$$\lim ||u_n||^2 \leq (1 + \varepsilon) \lim t_n^2 = (1 + \varepsilon)\frac{4\pi}{\alpha_0}$$

So it follows from (5.3) and (5.7) that

$$(1+\varepsilon)\frac{4\pi}{\alpha_0} \geq (\beta-\varepsilon)d^2\pi\widehat{M} \quad ,\forall\varepsilon > 0$$

which implies $\beta \leq 4/\alpha_0 d^2\widehat{M}$, a contradiction to (H8). □

Proof of Assertion 1. Given (t_n) and (v_n) as above, one of the following two possibilities has to hold:
(i) either there exists a constant $C_0 > 0$ such that $t_n/||v_n|| \geq C_0$, or
(ii) there are subsequences, denoted again by (t_n) and (v_n), such that $t_n/||v_n|| \to 0$.

First, let us assume that (i) holds. Then there is a constant $C > 0$ such that

$$||u_n|| \leq ||v_n|| + t_n \leq Ct_n \quad ,$$

which applied to (5.2) gives

$$C^2t_n^2 \geq ||u_n||^2 \geq \int_{B_{d/n}(x_0)} f(x,u_n)u_n \geq (\beta-\varepsilon)\int_{B_{d/n}(x_0)} e^{\alpha_0 u_n^2} \quad , \tag{5.8}$$

where (H8) was used. Let $m_n = \frac{1}{\sqrt{2\pi}}(\log n)^{1/2}$. So

$$u_n(x) = t_n m_n\left(\frac{v_n(x)}{t_n}\frac{1}{m_n}+1\right) \quad , \forall x \in B_{d/n}(x_0).$$

Hence, given $\varepsilon \in (0,1)$, we have $u_n(x) \geq (1-\varepsilon)t_n m_n$ for large n and $x \in B_{d/n}(x_0)$. Going on with the estimate in (5.8) we obtain

$$C^2t_n^2 \geq (\beta-\varepsilon)\frac{\pi}{n^2}e^{\alpha_0(1-\varepsilon)^2t_n^2m_n^2},$$

which can be written as

$$C^2t_n^2 \geq (\beta-\varepsilon)\pi d^2 e^{2\log n(\alpha_0(1-\varepsilon)^2t_n^2\frac{1}{4\pi}-1)}. \tag{5.9}$$

It follows readily that t_n is bounded. Consequently $||v_n||$ is also bounded in case (i).

Next, we assume that (ii) holds. Then, for large n, we have $t_n \leq ||v_n||$, which implies $||u_n|| \leq 2||v_n||$. Suppose by contradiction that $||v_n|| \to \infty$. As before, let $s_\varepsilon > 0$ be such that

$$f(x,s)s \geq (\beta-\varepsilon)e^{\alpha_0 s^2} \quad ,\forall s \geq s_\varepsilon.$$

It follows then from (5.2)

$$1 \geq \int_{u_n\geq s_\varepsilon}\frac{f(x,u_n)u_n}{||u_n||^2} \geq \frac{\beta-\varepsilon}{4}\int_{u_n\geq s_\varepsilon}\frac{e^{\alpha_0 u_n^2}}{||v_n||^2} \quad . \tag{5.10}$$

Now in order to estimate the exponent in the last integral above we observe that

$$\frac{u_n}{||v_n||}\chi_{u_n \geq t_\varepsilon} = \frac{v_n}{||v_n||} + \frac{t_n}{||v_n||}\omega_n - \frac{u_n}{||v_n||}\chi_{u_n < t_\varepsilon}$$

converges a.e. to $\widehat{v} \in V$ where $\widehat{v}$ is the limit of $v_n/||v_n||$ in H_0^1, and $||\widehat{v}|| = 1$. So using Fatou's lemma in (5.10) and recalling that we have assumed $||v_n|| \to \infty$, we come to a contradiction. So $||v_n|| \leq$ const. and, consequently, also t_n is bounded.

Before proving Assertion 2 we show the folllowing auxiliary result.

Lemma 5.1. $t_0^2 \geq 4\pi/\alpha_0$.

Proof. First observe that ω_n converges weakly to zero in H_0^1. So $||u_n||^2 \to t_0^2 + ||v_0||^2$. On the other hand, we observe that $u_n \to v_0$ in $L^1(\Omega)$ and the other hypotheses of Lemma 2.1 are satisfied. So $\int f(x, u_n) \to \int f(x, v_0)$, and consequently $\int F(x, u_n) \to \int F(x, v_0)$. Using these informations in (5.1) we obtain

$$\frac{2\pi}{\alpha_0} \leq \frac{1}{2}(t_0^2 + ||v_0||^2) - \int F(x, v_0)$$

Using (H6) we get

$$\frac{2\pi}{\alpha_0} \leq \frac{1}{2}(t_0^2 + ||v_0||^2) - \frac{1}{2}\lambda_k \int v_0^2 \leq \frac{1}{2}t_0^2$$

where we have used the variational characterization of λ_k.

Remark 5.1. We may assume that in every neighbourhood of $t = 0$, $F(x, t) \not\equiv \frac{1}{2}\lambda_k t^2$. Since otherwise $f(x, t) \equiv \lambda_k t$ in some neighbourhood of $t = 0$, and this would imply that $\varepsilon\varphi_k$ would be a solution of (1.1) for small $\varepsilon > 0$ and φ_k an eigenfunction of $-\Delta$ corresponding to the eigenvalue λ_k. Hence if $v_0 \not\equiv 0$ then we actually would have $t_0^2 > 4\pi/\alpha_0$.

Proof of Assertion 2. We work again with the alternative set in the Proof of Assertion 1. We first observe that in view of lemma 5.1 the alternative (ii) cannot hold. So let us assume (i). We conclude readily from (5.9) that

$$t_0^2 \leq \frac{4\pi}{\alpha_0(1-\varepsilon)^2} \Rightarrow t_0^2 \leq \frac{4\pi}{\alpha_0}.$$

This, in conjunction with Lemma 5.1, implies $t_0^2 = 4\pi/\alpha_0$.Finally, as a consequence of Remark 5.1 we conclude that $v_0 = 0$. □

References

1. Adimurthi: Existence of Positive Solutions of the Semilinear Dirichlet Problem with critical Growth for the n-Laplacian. Ann. Sc. Norm. Sup. Pisa **XVII** (1990) 393–413
2. Adimurthi: Existence Results for the Semilinear Dirichlet Problem with Critical Growth for the n-Laplacian (preprint)
3. Adimurthi, S.L. Yadava: Multiplicity Results for Semilinear Elliptic Equations in a Bounded Domain of R^2 involving critical exponent. Ann. Sc. Norm. Sup. Pisa **XVII** (1990) 481–504

4. Adimurthi, P.N. Srikanth, S.L. Yadava: Phenomena of Critical Exponent in R^2. Proc. Royal Soc. Edinb. **119A** (1991) 19–25
5. F.V. Atkinson, L.A. Peletier: Elliptic equations with critical growth. Math. Inst. Univ. Leiden, Rep 21 (1986)
6. H. Brezis, L. Nirenberg: Positive solutions of nonlinear elliptic equations involving critical sobolev exponents. Comm. Pure Appl. Math. **36** (1983) 437–477
7. L. Carleson, A. Chang: On the existence of an extremal function for an inequality of J. Moser. Bull. Sc. Math. **110** (1986) 113–127
8. P.-L. Lions: The Concentration–compactness principle in the Calculus of Variations. Part 1. Revista Iberoamericana **11** (1985) 185–201
9. J. Moser: A sharp form of an inequality by N. Trudinger. Ind. Univ. Math. J. **20** (1971) 1077–1092
10. H.L. Royden: Real Analysis. New York (1963)
11. P.H. Rabinowitz: Minimax methods in critical point theory with applications to differential equations. CBMS, no. 65 AMS (1986) 100
12. N.S. Trudinger: On imbedding into Orlicz spaces and some applications. J. Math. Mech. **17** (1967), 473–484