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HYPERINVARIANT SUBSPACES FOR BILATERAL WEIGHTED SHIFTS

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Let H be a complex Hilbert space with the orthonormal basis $\{e_n\}_{n\in Z}$ and let U be an invertible bilateral weighted shift defined by

$$Ue_n = \rho_n e_{n+1} \quad \rho_n > 0, \quad n \in Z.$$

The main result of this paper (Theorem 1) is a reduction of the hyperinvariant subspace problem (see [4], Question 23, p. 109) to the case when the sequences $\{(\rho_0 \ldots \rho_{k-1})^{1-k}\}^{\infty}_{k-1}$, $\{(\rho_{-k} \ldots \rho_{-1})^{1/k}\}^{\infty}_{k=1}$ are both convergent to either $|U|_{sp}$ or $|U^{-1}|^{-1}_{sp}$. Note that if U is not invertible then by [4], Corollary (a), p. 91, U has a proper hyperinvariant subspace. The technique we shall use, bears some similarity with a part of Scott Brown's technique [1], but its essence is an analysis of real sequences. A byproduct of this technique will be a new proof of Theorem 6.2, [2].

Let $\delta^{(n)} = \{\delta^{(n)}_k\}_{k\in Z}$ be defined by $\delta^{(n)}_k = 0$, $k \neq n$, $\delta^{(n)}_n = 1$ and put $\ell^0(Z) = \ell m\{\delta^{(n)}\}_{n\in Z}$. If $\ell^1(Z)$ denotes the Banach space of all summable complex sequences endowed with the norm

$$||s||_1 = \sum_{k\in Z} |s_k|, \quad s = \{s_k\}_{k\in Z} \in \ell^1(Z),$$

then obviously $\ell^0(Z)$ is a linear manifold in $\ell^1(Z)$.

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Further $a = \{a_n\}_{n \in Z}$ will denote a sequence of positive numbers such that

$$a_n ||U^n|| \geq 1, \quad \sup_{k \in Z} \frac{a_{k+n}}{a_k} < \infty , \quad n \in Z.$$

For every f, g $\in$ H define the sequence of complex numbers

$$L^a(f,g) = \{L_n^a\ (f,g)\}_{n \in Z} , \quad L_n^a\ (f,g) = a_n \ \langle U^n f, g\rangle .$$

Since we have

$$L^a(f,g) = 0 \quad \Longleftrightarrow \quad \langle U^n f, g\rangle = 0, \quad n \in Z$$

and by [4], Corollary (6), p. 91, rationally invariant subspaces of U are hyperinvariant, if $L^a(f,g) = 0$, $f \not\equiv 0$, $g \not\equiv 0$ then $c\ell m\{U^n f\}_{n \in Z}$ is a proper hyperinvariant subspace of U.

Let us put $H' = \ell m\{e_n\}_{n \in Z}$. It is plain that whenever $f,g \in H'$, $f \not\equiv 0$, $g \not\equiv 0$ we have $L^a(f,g) \not\equiv 0$ but $L^a(f,g) \in \ell^0(Z)$ and in particular $L^a(f,g) \in \ell^1(Z)$.

Let $n \in Z$, $0 < \eta < 1$ be given. Because $a_n ||U^n|| \geq 1$, the set $\{k \in Z : ||L^a(e_k, e_{k+m}, ||_1 \geq \eta\}$ will be non-empty and we may define

$$|L^a|_{n,\eta} = \inf\{||L^a(e_0,e_k)||_1 + ||L^a(e_k,e_0)||_1 : \\ ||L^a(e_k,e_{k+n})||_1 \geq \eta\} .$$

Consider the following two possibile properties for L^a.

The (*) *- property: For every* f, g $\in$ H', m ≥ 0, $\epsilon > 0$ *there exist* u,v $\in$ H' *such that*

$$||u||^2 = ||v||^2 \leq ||L^a(f,g)||_1, \quad ||L^a(f+u,g+v)||_1 < \varepsilon,$$

$$\langle u,e_k\rangle = \langle v,e_k\rangle = 0 \ , \quad -m \leq k \leq m.$$

The (**) *- property:* $|L^a|_{n,\eta} = 0$ *for every* $n \in Z$, $0 < \eta < 1$.

PROPOSITION 1. *If* L^a *has the* (*) *- property then* U *has a proper hyperinvariant subspace.*

PROOF. Suppose that L^a has the (*) - property and let $0 < \varepsilon < 1$ be given. Since L^a is linear in its first argument we can find $f_0, g_0 \in H'$, $f_0' \neq 0$, $g_0' \neq 0$ such that $||L^a(f_0,g_0)||_1 < \varepsilon^2$. Because L^a has the (*) - property we can find by induction two orthogonal sequences $\{f_k\}_{k=1}^{\infty} \subset H'$, $\{g_k\}_{k=1}^{\infty} \subset H'$ such that

$$||f_{k+1}||^2 = ||g_{k+1}||^2 < \varepsilon^{k+1},$$

$$||L^a(\sum_{j=0}^{k} f_j, \sum_{j=0}^{k} g_j)||_1 < \varepsilon^{2(k+1)}, \quad k \geq 0,$$

If we put $x = \sum_{j=0}^{\infty} f_j$, $y = \sum_{j=0}^{\infty} g_j$ then $x \neq 0$ and for every $n \in Z$

$$|L_n^a(x,y)| = \lim_{k\to\infty} |L_n^a(\sum_{j=0}^{k} f_j, \sum_{j=0}^{k} g_j)| \leq \lim_{k\to\infty} ||L_n^a(\sum_{j=0}^{k} f_j, \sum_{j=0}^{k} g_j)||_1 = 0.$$

It follows $L^a(x,y) = 0$ and as we observed before U will have a proper hyperinvariant subspace.

PROPOSITION 2. (**) - *property implies* (*) - *property.*

PROOF. Suppose that L^a has the (**) - property and let $f,g \in H'$, $m \geq 0$, $\varepsilon > 0$, $0 < \eta < 1$ be given. Since $L^a(f,g) \in \ell^0(Z)$ we may suppose that we have $L_k^a(f,g) = 0$, $|k| > m$. Let u, v be of the form $u = \sum_{k=-m}^{m} \alpha_k e_{n_k}$, $v = -\sum_{k=-m}^{m} \alpha_k e_{n_k+k}$, where $\alpha_k^2 = \eta L_k^a(f,g)(L_k^a(e_{n_k}, e_{n_k+k}))^{-1}$ and $n_k \notin [-m,m]$, $n_k+k \notin [-m,m]$, $n_k \neq n_j$, $n_k + k \neq n_j + j$, $k \neq j$. If we put

$$u_{-m} = f,\ v_{-m} = g,\ u_k = f + \sum_{j=-m}^{k-1} \alpha_j e_{n_j},\ v_k = g - \sum_{j=-m}^{k-1} \bar{\alpha}_j e_{n_j+j},$$

$k \in [-m+1,m]$, then an easy computation shows that we have

$$L^a(f+u, g+v) = (1-\eta) L^a(f,g) + \sum_{k=-m}^{m} \alpha_k (L^a(e_{n_k}, v_k) - L^a(u_k, e_{n_k+k})).$$

Let $f', g' \in H'$ be given. The assumption $|L^a|_{n,\eta} = 0$ easily implies that the set $\sigma_{n,\eta} = \{j \in Z : ||L^a(e_j, e_{j+n})|| \geq \eta\}$ is infinite and

$$\varliminf_{j\in\sigma_{n,\eta}} (||L^a(e_0,e_j)||_1 + ||L^a(e_j,e_0)||_1) = 0.$$

Since we have

$$\sup_j \frac{a_{j+k}}{a_j} < \infty, \quad ||U^{-1}||^{-1} \leq \rho_j \leq ||U||$$

we can check the relations

$$\sup_j \frac{||L^a(e_p, e_{j+n})||_1}{||L^a(e_0,e_{j+n})||_1} < \infty \quad \sup_j \frac{||L^a(e_j,e_q)||_1}{||L^a(e_j,e_0)||_1} < \infty, \ p,\ q \in Z.$$

This implies

$$\varliminf_{j\in\sigma_{n,\eta}} (||L^a(e_p,e_{j+n})||_1 + ||L^a(e_j,e_q)||_1) = 0$$

and because f' and g' are finite combinations of vectors in the basis we also infer

$$\varliminf_{j\in\sigma_{n,\eta}} (||L^a(f',e_{j+n})||_1 + ||L^a(e_j,g')||_1) = 0$$

These observations allow us to determine successively $n_{-m},\dots,n_m$ such that

$$||L^a(e_{n_k},e_{n_k+k})||_1 \geq \eta,\ ||L^a(e_{n_k},v_k)||_1 + ||L^a(u_k,e_{n_k+k})||_1 < 1-\eta,$$

$k \in [-m,m]$, consequently

$$||u||^2 = ||v||^2 = \sum_{k=-m}^{m} |\alpha_k|^2 \leq \sum_{k=-m}^{m} |L_k^a(f,g)| = ||L^a(f,g)||_1$$

Because $1 - \eta$ can be made arbitrarily small the proof is concluded.

Let us put

$$r_+ = \underline{\lim}_{k\to\infty} (\rho_0 \cdots \rho_{k-1})^{1/k}, \quad R_+ = \overline{\lim}_{k\to\infty} (\rho_0 \cdots \rho_{k-1})^{1/k},$$

$$r_- = \underline{\lim}_{k\to\infty} (\rho_{-k} \cdots \rho_{-1})^{1/k}, \quad R_- = \overline{\lim}_{k\to\infty} (\rho_{-k} \cdots \rho_{-1})^{1/k},$$

$$r = |U^{-1}|_{sp}^{-1} \qquad , \quad R = |U|_{sp} .$$

By [4], Theorem 9, p. 67, we know that we have $\sigma(U) = \{\lambda \in C : r \le |\lambda| \le R\}$ and obviously $[r_-, R_-] \cup [r_+, R_+] \subset [r,R]$.

LEMMA 1. *Suppose* $r_+ < R_+$, $a_n = R_+^{-n}$, $n \ge 0$, $a_n = r_+^{-n}$, $n < 0$ *and for every* $m \ge 1$, $0 < \eta < 1$ *put*

$$\alpha_{m,\eta}^+ = \{k \ge 1 : ||L^a(e_0, e_{k+m})|| \ge \eta^{k+m}, \ ||L^a(e_0,e_k)||_1 \le \eta^k\}.$$

If $r_+ < \eta R_+$ *then* $\alpha_{m,\eta}^+$ *is infinite.*

PROOF. Assume $r_+ < \eta R_+$. Using the relations

$$\frac{r_+}{R_+} = \underline{\lim}_{k\to\infty} \frac{(\rho_0 \cdots \rho_{k-1})^{1/k}}{R_+} = \underline{\lim}_{k\to\infty} ||L^a(e_0,e_k)||_1^{1/k} < \eta$$

we derive that the set $\alpha_\eta^+ = \{k \ge 1 : ||L^a(e_0,e_k)||_1 \le \eta^k\}$ is infinite. If $\alpha_{m,\eta}^+$ is finite we can find $k_0 \ge 1$ such that $k_0 + mj \in \alpha_\eta^+$, $j \ge 0$. Because every $k \ge k_0$ is of the form $k = k_0 + mj_k + p_k$, $j_k \ge 0$, $0 \le p_k \le m - 1$ we deduce

$$||L^a(e_0,e_k)||_1 \le ||L^a(e_0,e_{k-p_k})||_1 \left(\frac{||U||}{R_+}\right)^{p_k} \le \eta^{k-p_k} \left(\frac{||U||}{R_+}\right)^{p_k}$$

and this implies

$$1 = \overline{\lim}_{k\to\infty} \frac{(\rho_0 \cdots \rho_{k-1})^{1/k}}{R_+} = \overline{\lim}_{k\to\infty} ||L^a(e_0,e_k)||_1^{1/k} \le \eta.$$

This contradiction concludes the proof.

LEMMA 2. *If* $r_+ < R_+$ *and* $a_n = R_+^{-n}$, $n \ge 0$, $a_n = r_+^{-n}$, $n < 0$, *then* L^a *has the (**)-property.*

PROOF. Let $m \ge 1$, $0 < \eta < 1$ be given and let $0 < c < 1$

such that $r_+ < cR_+$, $\eta \le c^m$. By Lemma 1 we know that $\alpha^+_{m,c}$ is infinite and using the obvious relations

$$c^{k+m} \le ||L^a(e_0,e_{k+m})||_1 \le ||L^a(e_0,e_k)||_1(\frac{||U||}{R_+})^m \le c^k(\frac{||U||}{R_+})^m ,$$

$k \in \alpha^+_{m,c}$ and

$$||L^a(e_0,e_k)||_1||L^a(e_k,e_0)||_1 = (\frac{r_+}{R_+})^k , \quad k \ge 1,$$

we derive

$$\lim_{k\in\alpha^+_{m,c}} ||L^a(e_0,e_k)||_1^{1/k} = c < 1, \quad \lim_{k\in\alpha^+_{m,\eta}} ||L^a(e_k,e_0)||_1^{1/k} = \frac{r_+}{cR_+} < 1$$

and in particular

$$\lim_{k\in\alpha^+_{m,c}} (||L_a(e_0,e_k)||_1 + ||L^a(e_k,e_0)||_1) = 0.$$

Since we also have

$$||L^a(e_k,e_{k+m})||_1 = \frac{||L^\alpha(e_0,e_{k+m})||_1}{||L^a(e_0,e_k)||_1} \ge c^m, \ k \in \alpha^+_{m,c}$$, we deduce

$|L^a|_{m,c^m} = 0$. But because obviously $|L^a|_{m,\eta} \le |L^a|_{m,c^m}$, we proved that we have $|L^a|_{m,\eta} = 0$.

Let V be the bilateral weighted shift defined by $Ve_k = \rho_k^{-1} e_{k+1}$, $k \in Z$ and let $b = \{b_k\}_{k\in Z}$ be defined by $b_k = r_+^k$, $k \ge 0$, $b_k = R_+^k$, $k < 0$. For every $f, g \in H$ define $M^b(f,g)$ by

$$M^b(f,g) = \{M_k^b(f,g)\}_{k\in Z} , \quad M_k^b(f,g) = b_k \ \langle V^k f,g\rangle.$$

Then it is easy to check the relation $|M^b|_{m,\eta} = |L^a|_{-m,\eta}$ and by the first part of the proof, $|L^a|_{-m,\eta} = 0$. Since obviously $|L^a|_{0,\eta} \le |L^a|_{m,\eta}$, we deduce $|L^a|_{n,\eta} = 0$, for all $n \in Z$ and this concludes the proof.

LEMMA 3. *If* $r_- = R_- = r_+ = R_+ = t$, $r < t < R$, $a_n = R^{-n}$, $n \ge 0$, $a_n = r^{-n}$, $n < 0$ *then* L^a *has the* (**)-*property*.

PROOF. Let $m \geq 1$, $0 < \eta < 1$ be given and let $\tilde{U}$ denote the image of U in the Calkin algebra. Using the obvious relations

$$R = |U|_{sp} = |\tilde{U}|_{sp} , \quad ||\tilde{U}^m|| = \overline{\lim_{|k|\to\infty}} ||U^m e_k|| ,$$

we derive

$$\overline{\lim_{|k|\to\infty}} ||L^a(e_k, e_{k+m})||_1 = \overline{\lim_{|k|\to\infty}} \frac{||U^m e_k||}{R^m} = \frac{||\tilde{U}^m||}{R^m} \geq 1.$$

Since by our assumptions we can easily check that we have

$$\overline{\lim_{|k|\to\infty}} (||L^a(e_0, e_k)||_1 + ||L^a(e_k, e_0)||_1) = 0,$$

we duduce $|L^a|_{m,\eta} = 0$. The rest of the proof imitates the last part of the proof of Lemma 2.

The reduction of the hyperinvariant subspace problem for U, mentioned at the beginning, will appear as a consequence of the following:

THEOREM 1. *Suppose that* $[r_-, R_-] \cup [r_+, R_+]$ *is not a singleton set included in* $\{r\} \cup \{R\}$. *Then* U *has a proper hyperinvariant subspace.*

PROOF. If $r_+ < R_+$ we apply Lemma 2, Proposition 2 and Proposition 1. If $r_- < R_-$ we reduce to the preceeding case by a unitary equivalence. If $r_- = R_- \neq r_+ = R_+$, then by [4], Theorem 9, p. 71, we have $\sigma_p(U) \cup \sigma_p(U^*) \neq \phi$, and this obviously implies that U has a proper hyperinvariant subspace. The leftover possibility is $r_- = R_- = r_+ = R_+ = t$, $r < t < R$ and then we apply Lemma 3, Proposition 2 and Proposition].

A positive result on hyperinvariant subspaces for U is presented in [2], §6. We end this note with a new proof of [2], Theorem 6.2.

THEOREM 2 (Chevreau - Pearcy - Shields). *If* $\sigma(U) = \{\lambda \in C : ||U^{-1}||^{-1} \leq |\lambda| \leq ||U||\}$ *then* U *has a proper hyperinvariant subspace.*

PROOF. Applying Theorem 1 we can reduce to the following nontrivial case

$$r_- = R_- = r_+ = R_+ = r < R = ||U|| = 1.$$

If we put $H_+ = c\ell m\{e_k\}_{k\geq 0}$, $H_- = c\ell m\{e_k\}_{k<0}$, then the decomposition $H = H_+ + H_-$ determines the matrix representation

$$U = \begin{vmatrix} U_+ & * \\ 0 & U_- \end{vmatrix}.$$

Since obviously $\max\{|\tilde{U}_+|_{sp}, |\tilde{U}_-|_{sp}\} = 1$, making a unitary equivalence and passing to the adjoint, if necessary, we shall assume $|\tilde{U}_+|_{sp} = 1$. Put $a_n = 1$, $n \geq 0$, $a_n = r^{-n}$, $n < 0$. For every $n \in Z$ we have, as in the proof of Lemma 3, $\overline{\ell im}_{k\to\infty} ||L^a(e_k, e_{k+n})||_1 = 1$. But using the relations

$$||L^a(e_0,e_k)||_1 + ||L^a(e_k,e_0)||_1 = \rho_0\cdots\rho_{k-1} + \frac{r^k}{\rho_0\cdots\rho_{k-1}},$$

$k \geq 1$, $r \leq \rho_k \leq 1$, $\overline{\lim}_{k\to\infty} \rho_k = 1$, $\lim_{k\to\infty} (\rho_0\cdots\rho_{k-1})^{1/k} = r$, we deduce $|L^a|_{\eta,\eta} = 0$, for every $0 < \eta < 1$. To conclude the proof we apply Proposition 2 and Proposition 1.

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