COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 12 (2) (1992) 125–134

COLORINGS AND ORIENTATIONS OF GRAPHS

N. ALON* and M. TARSI

Received June 21, 1989
Revised August 29, 1989

Bounds for the chromatic number and for some related parameters of a graph are obtained by applying algebraic techniques. In particular, the following result is proved: If G is a directed graph with maximum outdegree d, and if the number of Eulerian subgraphs of G with an even number of edges differs from the number of Eulerian subgraphs with an odd number of edges then for any assignment of a set $S(v)$ of $d+1$ colors for each vertex v of G there is a legal vertex-coloring of G assigning to each vertex v a color from $S(v)$.

1. Introduction

A subdigraph H of a directed graph D is called *Eulerian* if the indegree $d_H^-(v)$ of every vertex v of H in H is equal to its outdegree $d_H^+(v)$. Note that we do not assume that H is connected. H is *even* if it has an even number of edges, otherwise, it is *odd*. Let $EE(D)$ and $EO(D)$ denote the numbers of even and odd Eulerian subgraphs of D, respectively. (For convenience we agree that the empty subgraph is an even Eulerian subgraph.) Our main result is the following:

Theorem 1.1. *Let $D=(V,E)$ be a digraph. For each $v \in V$, let $S(v)$ be a set of $d_D^+(v)+1$ distinct integers, where $d_D^+(v)$ is the outdegree of v. If $EE(D) \neq EO(D)$ then there is a legal vertex-coloring $c: V \to \mathbb{Z}$ such that $c(v) \in S(v)$ for all $v \in V$.*

Corollary 1.2. *Let G be an undirected graph. If G has an orientation D satisfying $EE(D) \neq EO(D)$ in which the maximum outdegree is d, then G is $(d+1)$-colorable. In particular, if the maximum outdegree is d and D contains no odd directed (simple) cycle then G is $(d+1)$-colorable.*

Since a complete graph G on $d+1$ vertices has such an orientation (i.e., the acyclic orientation), the upper bound $d+1$ is sharp. In case the orientation D is acyclic, the conclusion of the theorem and that of the corollary can be easily proved by induction. The general case seems much more complicated and our only proof for it is algebraic.

Any upper bound to the chromatic number of a graph supplies a lower bound to its independence number. The resulting estimates are stated in the following two corollaries.

* Research supported in part by a United States–Israel BSF Grant and by a Bergmann Memorial Grant.

AMS Subject Classification codes (1991): 05 C 15, 05 C 20

Corollary 1.3. *Let G be an undirected graph on n vertices. If G has an orientation D satisfying $EE(D) \neq EO(D)$ in which the maximum outdegree is d, then G has an independent set of size at least $n/(d+1)$. In particular, if the maximum outdegree is d and D contains no odd directed (simple) cycle then G has an independent set of size at least $n/(d+1)$.*

Note that the estimate $n/(d+1)$ is sharp, as shown by any union of vertex-disjoint complete graphs on $d+1$ vertices each. As is the case in Corollary 1.2 the assertion of the last corollary can be easily proved by induction in the special case when D is acyclic. However, we do not have a non-algebraic proof for the general case.

Corollary 1.4. *Let G be an undirected graph on a set $V = \{v_1, \ldots, v_n\}$ of n vertices, and suppose it has an orientation D satisfying $EE(D) \neq EO(D)$. Let $d_1 \geq d_2 \ldots \geq d_n$ be the ordered sequence of outdegrees of the n vertices of D. Then, for every k, $n > k \geq 0$, G has an independent set of size at least $\lceil (n-k)/(d_{k+1}+1) \rceil$.*

Theorem 1.1 actually deals with the notion of choosability of graphs, introduced and studied in [9] and [15]. Let $G = (V, E)$ be an undirected graph and let $f : V \to \mathbb{Z}^+$ be a function which assigns to each vertex $v \in V$ a positive integer $f(v)$. We say that G is *f-choosable* if for every choice of sets $S(v)$ of integers, where $|S(v)| = f(v)$ for all $v \in V$ there is a legal coloring $c : V \to \mathbb{Z}$ such that $c(v) \in S(v)$ for all $v \in V$. In particular, if G is f-choosable for the constant function f given by $f(v) = k$ for each $v \in V$ we say that G is k-choosable. Theorem 1.1 is clearly a statement concerning the choosability of graphs.

Our paper is organized as follows:

In Section 2 we present the proofs of Theorem 1.1 and its three Corollaries. This Theorem is applied in Section 3 to obtain results concerning the choosability of graphs. The final Section 4 contains some concluding remarks and open problems.

2. The Proof of the Main Result

Our method resembles the one we applied in [3]. (See also [1] for a similar approach.) We start with the following simple lemma.

Lemma 2.1. *Let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables over the ring of integers $\mathbb{Z}$. Suppose that for $1 \leq i \leq n$ the degree of P as a polynomial in x_i is at most d_i and let $S_i \subset \mathbb{Z}$ be a set of $d_i + 1$ distinct integers. If $P(x_1, x_2, \ldots, x_n) = 0$ for all n-tuples $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$ then $P \equiv 0$.*

Proof. We apply induction on n. For $n = 1$ the lemma is simply the assertion that a nonzero polynomial of degree d_1 in one-variable can have at most d_1 distinct zeros. Assuming the lemma holds for $n-1$ we prove it for n, $(n \geq 2)$. Given a polynomial $P = P(x_1, \ldots, x_n)$ and sets S_i satisfying the hypotheses of the lemma, let us write P as a polynomial in x_n, i.e., $P = \sum_{i=0}^{d_n} P_i(x_1, \ldots, x_{n-1}) x_n^i$, where each P_i is a polynomial with x_j-degree bounded by d_j. For each fixed $(n-1)$-tuple $(x_1, \ldots, x_{n-1}) \in S_1 \times S_2 \times \ldots \times S_{n-1}$, the polynomial in x_n obtained from P by substituting the values of $x_1, \ldots, x_{n-1}$ vanishes for all $x_n \in S_n$ and is thus identically 0. Therefore $P_i(x_1, \ldots, x_{n-1}) = 0$ for all $(x_1, \ldots, x_{n-1}) \in S_1 \times \ldots \times S_{n-1}$ and hence, by the induction hypothesis, $P_i \equiv 0$ for all i, implying that $P \equiv 0$. This completes the induction and the proof of the lemma. ∎

The *graph polynomial* $f_G(x_1, x_2, \ldots, x_n)$ of an undirected graph $G = (V, E)$ on a set $V = \{v_1, \ldots, v_n\}$ of n vertices is defined by $f_G(x_1, x_2, \ldots, x_n) = \prod\{(x_i - x_j) : i < j,\ \{v_i, v_j\} \in E\}$. This polynomial is studied by Li and Li in [12], following a previous related work by Graham, Li and Li ([11]). An analogous polynomial for certain linear matroids is considered in [3]. It is not too difficult to see that the coefficients of the monomials that appear in the standard representation of f_G as a linear combination of monomials can be expressed in terms of the orientations of G. For each oriented edge $e = (v_i, v_j)$ of G, define its weight $w(e)$ by $w(e) = x_i$ if $i < j$ and $w(e) = -x_i$ if $i > j$. The weight $w(D)$ of an orientation D of G is defined to be the product $\prod w(e)$, where e ranges over all oriented edges e of D. Clearly $f_G = \sum w(D)$, where D ranges over all orientations of G. This is simply because each term in the expansion of the product $f_G = \prod\{(x_i - x_j) : i < j,\ \{v_i, v_j\} \in E\}$ corresponds to a choice of the orientation of the edge $\{v_i, v_j\}$ for each edge $\{v_i, v_j\}$ of G. Let us call an oriented edge (v_i, v_j) of G *decreasing* if $i > j$. An orientation D of G is called *even* if it has an even number of decreasing edges; otherwise, it is called *odd*. For non-negative $d_1, d_2, \ldots, d_n$, let $DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ denote, respectively, the sets of all even and odd orientations of G in which the outdegree of the vertex v_i is d_i, for $1 \le i \le n$. By the last paragraph, the following lemma holds.

Lemma 2.2. *In the above notation*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \ge 0} (|DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)|) \prod_{i=1}^{n} x_i^{d_i}. \qquad ∎$$

Consider, now, a fixed sequence $d_1, \ldots, d_n$ of nonnegative integers and let D_1 be a fixed orientation in $DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$. For any orientation $D_2 \in DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$ let $D_1 \oplus D_2$ denote the set of all oriented edges of D_1 whose orientation in D_2 is in the opposite direction. Since the outdegree of every vertex in D_1 is equal to its outdegree in D_2, it follows that $D_1 \oplus D_2$ is an Eulerian subgraph of D_1. Moreover, $D_1 \oplus D_2$ is even as an Eulerian subgraph iff either both D_1 and D_2 are even or both are odd. The mapping $D_2 \to D_1 \oplus D_2$ is clearly a bijection between $DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$ and the set of all Eulerian subgraphs of D_1. In case D_1 is even, it maps even orientations to even (Eulerian) subgraphs and odd orientations to odd subgraphs. Otherwise, it maps even orientations to odd subgraphs and odd orientations to even subgraphs. In any case,

$$\Big| |DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)| \Big| = |EE(D_1) - EO(D_1)|$$

where $EE(D_1)$ and $EO(D_1)$ denote, as in Section 1, the numbers of even and odd Eulerian subgraphs of D_1, respectively. Combining this with Lemma 2.2 we obtain the following.

Corollary 2.3. *Let D be an orientation of an undirected graph $G = (V, E)$ on a set $V = \{v_1, \ldots, v_n\}$ of n vertices. For $1 \le i \le n$, let $d_i = d_D^+(v_i)$ be the outdegree of*

v_i in D. Then the absolute value of the coefficient of the monomial $\prod_{i=1}^{n} x_i^{d_i}$ in the standard representation of $f_G = f_G(x_1, \ldots, x_n)$ as a linear combination of monomials is $|EE(D) - EO(D)|$. In particular, if $EE(D) \neq EO(D)$ then this coefficient is not zero. ∎

Proof of Theorem 1.1. Let $D = (V, E)$ be a digraph on the set of vertices $V = \{v_1, \ldots, v_n\}$ and let $d_i = d_D^+(v_i)$ be the outdegree of v_i. Suppose that $EE(D) \neq EO(D)$. For $1 \leq i \leq n$, let $S_i \subset \mathbb{Z}$ be a set of $d_i + 1$ distinct integers. We must show that there is a legal vertex-coloring $c : V \to \mathbb{Z}$ such that $c(v_i) \in S_i$ for all $1 \leq i \leq n$. Suppose this is false and there is no such coloring. Let G be the underlying undirected graph of D and let $f_G = f_G(x_1, \ldots, x_n)$ be its polynomial. The assumption that the required coloring does not exist is equivalent to the statement:

(2.1) $\qquad f_G(x_1, \ldots, x_n) = 0$ for every n-tuple $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$.

For each i, $1 \leq i \leq n$, let $Q_i(x_i)$ be the polynomial

$$Q_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

Observe that

(2.2) $\qquad$ If $x_i \in S_i$ then $Q_i(x_i) = 0$, i.e., $x_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij} x_i^j$.

Let $\bar{f}_G$ be the polynomial obtained from f_G by writing f_G as a linear combination of monomials and replacing, repeatedly, each occurrence of $x_i^{f_i}$, $(1 \leq i \leq n)$, where $f_i > d_i$, by a linear combination of smaller powers of x_i, using the relations (2.2). The resulting polynomial $\bar{f}_G$ is clearly of degree at most d_i in x_i for each $1 \leq i \leq n$. Moreover, $\bar{f}_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$, since the relations (2.2) hold for these values of $x_1, \ldots, x_n$. Therefore, by (2.1), $\bar{f}_G(x_1, \ldots, x_n) = 0$ for every n-tuple $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$ and hence, by Lemma 2.1 $\bar{f}_G \equiv 0$. However, by Corollary 2.3, the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in f_G is nonzero, since, by assumption, $EE(D) \neq EO(D)$. Since the degree of each x_i in this monomial is d_i, the relations (2.2) will not effect it. Moreover, as the polynomial f_G is homogeneous and each application of the relations (2.1) strictly reduces degree, the process of replacing f_G by $\bar{f}_G$ will not create any new scalar multiples of $\prod_{i=1}^{n} x_i^{d_i}$. Thus, the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in $\bar{f}_G$ is equal to its coefficient in f_G and is not 0. This contradicts the fact that $\bar{f}_G \equiv 0$. Therefore, our assumption was false and there is a legal coloring $c : V \to \mathbb{Z}$ satisfying $c(v_i) \in S_i$ for all $1 \leq i \leq n$, as needed. ∎

Proof of Corollary 1.2. Let $D = (V, E)$ be an orientation of a graph G such that $EE(D) \neq EO(D)$, in which the maximum outdegree is d. By Theorem 1.1, with $S(v) = \{1, \ldots, d+1\}$ for all $v \in V$, G is $(d+1)$-colorable, as needed.

In case D contains no odd directed simple cycle, it cannot contain odd Eulerian subgraphs at all. This is because every Eulerian subgraph H is a union of edge-disjoint directed simple cycles, and if H is odd then at least one of these cycles must be odd. Thus, in this case $EO(D) = 0$ and as always $EE(D) \geq 1$ (because the empty subgraph is an even Eulerian subgraph) $EO(D) \neq EE(D)$. It thus follows from the previous paragraph that G is $(d+1)$-colorable, in this case too. ∎

Proof of Corollary 1.3. This is an immediate consequence of Corollary 1.2, since in any legal $(d+1)$-coloring of a graph on n vertices the largest color class contains at least $\lceil n/(d+1) \rceil$ vertices. ∎

Proof of Corollary 1.4. Renumber the vertices so that $d_D^+(v_i) = d_i$. By Theorem 1.1 there is a legal coloring $c : V \to \mathbb{Z}$ of G such that $1 \leq c(v_i) \leq d_i + 1$ for all $1 \leq i \leq n$. For each k, $0 \leq k < n$, the colors of $v_{k+1}, \ldots, v_n$ all lie in $\{1, 2, \ldots, d_{k+1} + 1\}$ and hence the largest color class has size at least $\lceil (n-k)/(d_{k+1}+1) \rceil$. ∎

Remark 2.4. The assertion of Theorem 1.1 (and hence that of Corollaries 1.2, 1.3 and 1.4) is trivial in the special case that D is acyclic. This is because in this special case there is a vertex v with 0-indegree. By induction we can obtain a coloring of the desired type of $D - v$ and this coloring can be trivially extended to the needed coloring of D. This argument also supplies a polynomial time algorithm for finding the desired coloring. The general case seems more difficult. Note that already the case of no odd directed cycle is certainly more general than the acyclic one as it implies, e.g., that an even cycle is 2-colorable and 2-choosable (by orienting it cyclically), although it has no acyclic orientation with maximum outdegree less than 2.

As observed by J. A. Bondy, R. Boppana and A. Siegel [4] Theorem 1.1 (and hence also Corollary 1.2 and Corollary 1.3) for the special case that D has no odd directed cycle follows easily from Richardson's Theorem (cf., e.g., [5]). Richardson's Theorem states that any digraph with no odd directed cycles has a kernel, i.e., an independent set such that every vertex outside it has an edge to a neighbor in it. Thus we can fix some color x in the union of the sets $S(v)$, apply Richardson's Theorem to the induced subgraph of D on the set of all vertices v that contain x in their lists $S(v)$, color the vertices in the kernel by x (which will not be used again), and apply induction. Note that Richardson's proof is algorithmic. This argument does not seem to imply the general statement of Theorem 1.1 and its corollaries.

Remark 2.5. An alternative proof of Corollary 1.2 can be deduced from a result of Kleitman and Lovász (cf. [12]). They proved that if the chromatic number of a graph $G = (V, E)$ is at least k, then f_G lies in the ideal generated by the polynomials f_H, where H is a complete graph on some subset of cardinality k of V. Notice that if $K \subset V$ is the set of vertices of such an H then $f_H = \prod\{(x_p - x_q) : v_p, v_q \in K, p < q\}$. However, this product is exactly the Vandermonde's determinant $\det(x_p^i)_{p \in K, 0 \leq i < k}$ and each term in the expansion of this determinant contains a variable of degree $k-1$. Therefore, if G is not $d+1$ colorable, every monomial with a non-zero coefficient in f_G has a variable of degree at least $d+1$. But under the assumptions of Corollary 1.2, Corollary 2.3 implies that the monomial $\prod_{i=1}^{n} x_i^d$ appears with a non-zero coefficient in f_G, and this monomial has no variable of degree $d+1$ or more. Therefore, G is $(d+1)$-colorable, as needed. A similar derivation of Corollary 1.3 from the main result of Li and Li [12], that characterizes the ideal generated by the polynomials f_G where G ranges over all graphs on n vertices with independence number at most k can be also given.

It is worth noting that Gessel [10] describes a combinatorial proof for the expression of the Vandermonde's determinant as a product, by associating every term in the expansion of the product with a turnament, i.e., with an orientation of a complete graph. Lemma 2.2 is a simple modification of Gessel's arguments to general graphs.

Remark 2.6. Our method can be easily applied to prove the following Nullstellensatz-type result, which is similar to Theorem 1.1. Since the proof is similar to that of Theorem 1.1 we omit it.

Proposition 2.7. *Let $G = (V, E)$ be a graph, and let $f_G = f_G(x_1, \dots, x_n)$ be its polynomial. For every integer k the following statements are equivalent:*

(i) *G is not k-colorable.*

(ii) *There is a set S of k distinct (complex) numbers such that $f_G(x_1, \dots, x_n) = 0$ for every $x_1, \dots, x_n \in S$.*

(iii) *For every set S of k distinct (complex) numbers $f_G(x_1, \dots, x_n) = 0$ for every $x_1, \dots, x_n \in S$.*

(iv) *There is a set S of k distinct (complex) numbers such that the polynomial f_G belongs to the ideal generated by the n polynomials $Q_i(x_i) = \prod_{s\in S}(x_i - s)$, $(1 \le i \le n)$.*

(v) *For every set S of k distinct (complex) numbers the polynomial f_G belongs to the ideal generated by the n polynomials $Q_i(x_i) = \prod_{s\in S}(x_i - s)$, $(1 \le i \le n)$.*

(vi) *The polynomial f_G, considered as a polynomial in the ring of polynomials in the $n+k$ variables $x_1, \dots, x_n$, $z_1, \dots, z_k$ over the complex numbers, belongs to the ideal generated by the n polynomials $Q_i = \prod_{1\le j\le k}(x_i - z_j)$, $(1 \le i \le n)$.*

(vii) *For every set S of k not necessarily distinct (complex) numbers the polynomial f_G belongs to the ideal generated by the n polynomials $Q_i(x_i) = \prod_{s\in S}(x_i - s)$, $(1 \le i \le n)$.*

3. k-Choosable Graphs

For a graph $G = (V, E)$, define $L(G) = \max(|E(H)|/|V(H)|)$, where $H = (V(H), E(H))$ ranges over all subgraphs of G. Thus $L(G)$ is simply a half of the maximum value of the average degree of a subgraph of G. The following simple lemma appears in [2] (and, probably, in other places as well). The proof we present here follows [14].

Lemma 3.1. *A graph $G = (V, E)$ has an orientation D in which every outdegree is at most d if and only if $L(G) \le d$.*

Proof. If there is such an orientation D, then, for any subgraph H of G

$$|E(H)| = \sum_{v\in V(H)} d_H^+(v) \le d|V(H)|$$

and hence $|E(H)|/|V(H)| \leq d$. Thus $L(G) \leq d$. Conversely, suppose $L(G) \leq d$. Let F be the bipartite graph on the classes of vertices A and B, where $A = E$ and B is a union of d disjoint copies $V_1, V_2, \ldots, V_d$ of V. Each member $e = \{u, v\}$ of E is joined by edges in F to the d copies of u and to the d copies of v in B. We claim that F contains a matching of size $|A| = |E|$. Indeed, if $E' \subseteq E$ is a set of edges of a subgraph H of G whose vertices are all endpoints of members of E', then in F, E' has $d|V(H)|$ neighbours. By the definition of $L(G)$: $|E'|/|V(H)| \leq L(G) \leq d$ and hence $d|V(H)| \geq |E'|$. Therefore, by Hall's theorem, the desired matching exists. We can now orient each edge of G from the vertex to which it is matched. This gives an orientation D of G with maximum outdegree $\leq d$ and completes the proof of the lemma.

Recall that a graph $G = (V, E)$ is k-choosable if for any assignment of sets $S(v) \subset \mathbb{Z}$ of cardinality k for each vertex $v \in V$ there is a proper coloring $c : V \to \mathbb{Z}$ of G satisfying $c(v) \in S(v)$ for each $v \in V$.

Theorem 3.2. *Every bipartite graph G is $(\lceil L(G) \rceil + 1)$-choosable.*

Proof. Put $d = \lceil L(G) \rceil$. By Lemma 3.1 there is an orientation D of G in which the maximum outdegree is at most d. Since D contains no odd directed cycles (and in fact no odd cycles at all), $EE(D) \neq EO(D)$ and the result follows from Theorem 1.1. ∎

Remark 3.3. The assumption that G is bipartite is essential, since if $G = K_n$ is the complete graph on n vertices then $L(G) = (n-1)/2$ and clearly K_n is not k-choosable for $k < n$. Moreover, Theorem 3.2 is sharp in the sense that for every k there is a bipartite graph G satisfying $L(G) \leq k$, which is not k- choosable. Indeed, let G be the complete bipartite graph on the classes of vertices A and B where $|A| = k^k$ and $|B| = k$. Clearly $L(G) \leq k$, since if H is the induced subgraph of G on $A' \cup B'$, where $A' \subseteq A$ and $B' \subseteq B$ then $|E(H)| = \sum_{a \in A'} d_H(a) \leq k|A'| \leq k|V(H)|$. In order to show that G is not k-choosable, put $B = \{b_1, \ldots, b_k\}$ and define $S(b_i) = \{k(i-1)+1, k(i-1)+2, \ldots, ki\}$. Enumerate the vertices of A so that $A = \{a_{i_1, i_2, \ldots, i_k} : 1 \leq i_j \leq k\}$. Now define

$$S(a_{i_1, i_2, \ldots, i_k}) = \{i_1, k + i_2, 2k + i_3, \ldots, (k-1)k + i_k\}.$$

Clearly, there is no legal coloring of G in which $c(v) \in S(v)$ for each $v \in A \cup B$. Indeed, assuming c is such a coloring, there are $1 \leq i_1, \ldots, i_k \leq k$ such that $c(b_j) = (j-1)k + i_j$ for $1 \leq j \leq k$. But then there is no value in $S(a_{i_1, i_2, \ldots, i_k})$ which is distinct from the colors of all its neighbours and hence c is not a legal coloring.

An immediate corollary of Theorem 3.2 is the following result, which solves one of the open problems raised in [9].

Corollary 3.4. *Every bipartite planar graph G is 3-choosable.*

Proof. $L(G) \leq 2$, since any bipartite (simple) planar graph on r vertices contains at most $2r - 4$ edge. ∎

Corollary 3.4 is also sharp, since $K_{2,4}$ is bipartite and planar, and is not 2-choosable, by the discussion in Remark 3.3.

It is interesting to note that in general $L(G)$ may be much larger than the degree of choosability of G. For example, if $k > 1$ and G is the complete bipartite graph $K_{n,n}$

on the classes of vertices A and B, where $|A| = |B| = n = 2^{k-1}$ then $L(G) = 2^{k-2}$ and still G is k-choosable. To see this suppose we are given a set $S(v) \subset \mathbb{Z}$ of cardinality k for each $v \in A \cup B$. Put $S = \bigcup_{v \in V} S(v)$ and let $S = S_A \cup S_B$ be a random partition of S into two disjoint classes obtained by assigning each $s \in S$ independently either to S_A or to S_B with equal probability. Let us call a vertex $a \in A$ *bad* if $S(a) \cap S_A = \emptyset$. Similarly, a vertex $b \in B$ is called bad if $S(b) \cap S_B = \emptyset$. Since the probability that a fixed vertex is bad is precisely 2^{-k}, the expected number of bad vertices is 1. However, since for some partitions (e.g., $S_A = S$, $S_B = \emptyset$) there are $2^{k-1} > 1$ bad vertices there is at least one partition (S_A, S_B) with no bad vertices. We can now choose, for each $a \in A$, $c(a) \in S(a) \cap S_A$ and for each $b \in B$, $c(b) \in S(b) \cap S_B$ and obtain a legal coloring satisfying $c(v) \in S(v)$ for each vertex v. Hence G is k-choosable, as claimed.

One of the most fascinating problems concerning choosability is a conjecture of Jeff Dinitz (cf. [8]), which asserts that the line graph of $K_{m,m}$ is m-choosable. A more appealing formulation of this conjecture is the following: Given an arbitrary m by m array of m-sets, it is always possible to choose one element from each set, keeping the chosen elements distinct in every row and distinct in every column. This conjecture is, of course, trivial for $m \leq 2$, and, as mentioned in [7], it has been verified by a surprisingly hard case by case analysis for $m = 3$. Applying our Theorem 1.1 we can reduce the conjecture to a certain problem. For each m by m Latin square L (on the symbols $1, 2, \ldots, m$) we can define a weight $w(L) \in \{\pm 1\}$ as the product of the signs of the $2m$ permutations appearing in the rows and columns of L and show that if $\sum w(L) \neq 0$, where L ranges over all $m \times m$ Latin squares then Dinitz Conjecture holds for m. Unfortunately, for every odd m, the above sum is always 0, and hence this method cannot yield a proof of the conjecture for odd m. We do believe, however, that the sum is nonzero for every even m, but at the moment we are unable to prove it. Still, it is trivial to check that the sum is nonzero for $m = 4$, and with a computer we also checked that it is nonzero for $m = 6$. Therefore, the conjecture is true for $m = 4$ and $m = 6$; both cases are probably too difficult to be checked directly by a case by case analysis. We omit the details, and hope to return to this subject in the future.

4. Concluding Remarks and Open Problems

1) Corollary 1.2 generalizes the well known result that if every induced subgraph of a graph G has a vertex of degree at most d then G is $(d+1)$-colorable (cf., e.g., [6]), which is equivalent to the special case of the Corollary in which the orientation D is acyclic.
2) There are several known results that reveal the connection between the chromatic number of a graph G and its orientations. For example, it is known that the chromatic number of G is equal to the minimum, over all orientation D of G of the maximum length of a simple directed path in D. This minimum is always obtained by an acyclic orientation, but may be obtained by other orientations too (cf., e.g., [6]). It is also known (see [13]) that the number of acyclic orientations of a graph G is equal to the absolute value of its chromatic polynomial evaluated at -1. Theorem 1.1 and Corollary 1.2 form another example of a result that connects chromatic numbers and orientations.

3) There are simple examples of directed graphs $D = (V(D), E(D))$ that contain odd directed cycles and still satisfy $EE(D) \neq EO(D)$. (Two directed odd cycles sharing an edge is one such example.) Therefore, the condition that D has no odd directed cycles (which implies that $EE(D) \neq EO(D)$) is a strictly stronger condition than the one $EE(D) \neq EO(D)$ and hence the first part of, e.g., Corollary 1.2, is stronger than its last part. The advantage of the simpler condition that there is no odd directed cycle is that it can be easily checked in polynomial time; one way to do so is to observe that there is no directed odd cycle iff no odd power A^k of the adjacency matrix of D ($k \leq |V(D)|$) has a positive entry in its diagonal.
4) It would be interesting to find a non-algebraic proof of Theorem 1.1 and its Corollaries.

Acknowledgement. We would like to thank J. A. Bondy, R. Boppana and A. Siegel for helpful comments.

References

[1] N. Alon, S. Friedland, and G. Kalai: Regular subgraphs of almost regular graphs, *J. Combinatorial Theory*, Ser. B **37** (1984), 79–91.
[2] N. Alon, C. McDiarmid, and B. Reed: Star Arboricity, to appear.
[3] N. Alon, and M. Tarsi: A nowhere zero point in linear mappings, *Combinatorica* **9** (1989), 393–395.
[4] J. A. Bondy, R. Boppana, and A. Siegel: Private communication.
[5] C. Berge: *Graphs and Hypergraphs*, Dunod, Paris, 1970.
[6] B. Bollobás: *Extremal Graph Theory*, Academic Press, New York, 1978.
[7] A. Chetwynd, and R. Häggkvist: A note on list colorings, *J. Graph Theory* **13** (1989), 87–95.
[8] P. Erdős: Some old and new problems in various branches of combinatorics, *Congressus Numerantium* **23** (1979), 19–37.
[9] P. Erdős, A. Rubin, and H. Taylor: Choosability in graphs, *Congressus Numerantium* **26** (1979), 125–157.
[10] I. Gessel: Tournaments and Vandermonde's determinant, *J. Graph Theory* **3** (1979), 305–307.
[11] R. L. Graham, S.-Y. R. Li, and W.-C. W. Li: On the structure of t-designs, *SIAM J. Alg. Disc. Meth.* **1** (1980), 8–14.
[12] S.-Y. R. Li, and W.-C. W. Li: Independence numbers of graphs and generators of ideals, *Combinatorica* **1** (1981), 55–61.
[13] R. P. Stanley: Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.
[14] M. Tarsi: On the decomposition of a graph into stars, *Discrete Math.* **36** (1981), 299–304.

[15] V. G. VIZING: Coloring the vertices of a graph in prescribed colors (in Russian), *Diskret. Analiz.* **29**, *Metody Discret. Anal. v. Teorii Kodov i Shem* **101** (1976), 3–10.

N. Alon

Department of Pure Mathematics
School of Mathematical Sciences
Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Tel Aviv, Israel
noga@math.tau.ac.il

M. Tarsi

Department of Computer Science
School of Mathematical Sciences
Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Tel Aviv, Israel
tarsi@math.tau.ac.il